Summary lecture 1

- Examined toy QM model with SUSY.
- Showed how naive discretization fails – lattice theory picks up new U.V contributions - fine tuning problem.
- Avoid by correcting action with counterterm computed at 1-loop.
- Generic way to deal with $D < 4$ theories (and maybe even $\mathcal{N} = 1$ in 4d with eg DWF)
- Better. Find modified action which is invariant under linear combination of SUSYs.
- New symmetry $Q^2 = 0$ and action Q-exact $S = Q\Lambda$. Twisted formulation

Two dimensions

- How can we generalize previous construction to field theory ?
- In QM needed 2 SUSYs. Also single scalar field and two real fermions. Try simplest 2D with these features: $\mathcal{N} = 2$ Wess-Zumino model ...

$$
\begin{aligned}
S_{\rm WZ} &= \int d^2x\, \partial_\mu\phi\partial_\mu\overline{\phi} + W'(\phi)W'(\overline{\phi}) + \overline{\psi}\gamma_\mu\partial_\mu\psi + \\
&+ \overline{\psi}\left(\frac{1}{2}\left(1+\gamma_5\right)W''(\phi) + \frac{1}{2}\left(1-\gamma_5\right)W''(\overline{\phi})\right)\psi
\end{aligned}
$$

Nicolai Map

Fermion operator can be rewritten (chiral basis):

$$M_{\mathrm{F}} = \begin{pmatrix} W''(\phi) & \partial_{\overline{z}} \\ \partial_z & W''(\overline{\phi}) \end{pmatrix}$$

Associated Nicolai Map is

$$\mathcal{N} = \partial_{\overline{z}}\overline{\phi} + W'(\phi)$$

Action $\mathcal{N}\overline{\mathcal{N}}$ again differs by cross term from continuum (total derivative)

Lattice action

$$S_L = \sum_x \mathcal{N}\overline{\mathcal{N}} + \overline{\omega}\left(\Delta^s_{\bar{z}}\lambda + W''_L(\phi)\omega\right) + \overline{\lambda}\left(\Delta^s_z\omega + W''_L(\overline{\phi})\lambda\right)$$

where $\psi = \begin{pmatrix} \omega \\ \lambda \end{pmatrix}$, $\overline{\psi} = \begin{pmatrix} \overline{\omega} \\ \overline{\lambda} \end{pmatrix}$.

Symmetric difference $\Delta^s_z = \Delta^s_1 + i\Delta^s_2$.

Doublers removed via:

$$W'_L(\phi) = W'(\phi) + \frac{1}{2}D^+_\mu D^-_\mu \phi$$

Equivalent in 1D

Supersymmetries

The single exact SUSY follows by analogy to QM:

$$
\begin{aligned}
Q\phi &= \omega \\
Q\overline{\phi} &= \lambda \\
Q\lambda &= 0 \\
Q\omega &= 0 \\
Q\overline{\omega} &= \overline{\mathcal{N}} \\
Q\overline{\lambda} &= \mathcal{N}
\end{aligned}
$$

Note that $Q^2 = 0$ with E.O.M
Homework Problem. Check this invariance.

Additional SUSYs

In continuum this model has 3 additional SUSYs.
Go back to QM:
Continuum bosonic action same if $\mathcal{N}$ replaced by
$\hat{N} = \Delta^{-}\phi - P'$.
Equal to transpose of original fermion op.
Fermion action invariant if intercharge $\psi \rightarrow \overline{\psi}$ – Gives 2nd SUSY!
Similar for WZ model: 3 other ways of writing $\mathcal{N}$ – corresponding to $\phi \rightarrow \overline{\phi}$ and flipping sign.
Three new fermion ops. Make fermion act invariant by exchanging $\omega \rightarrow \lambda$ and $\overline{\omega} \rightarrow \overline{\lambda}$ etc.

Twisted/topological form

Not hard to verify that action can be rewritten (see QM) as

$$S_L = Q \sum_x \overline{\omega} \left(\mathcal{N} + \frac{1}{2} B \right) + \overline{\lambda} \left(\overline{\mathcal{N}} + \frac{1}{2} \overline{B} \right)$$

with additional $B, \overline{B}$

$$\begin{aligned} Q\overline{\omega} &= \overline{B} \\ Q\overline{\lambda} &= B \\ QB &= 0 \\ Q\overline{B} &= 0 \end{aligned}$$

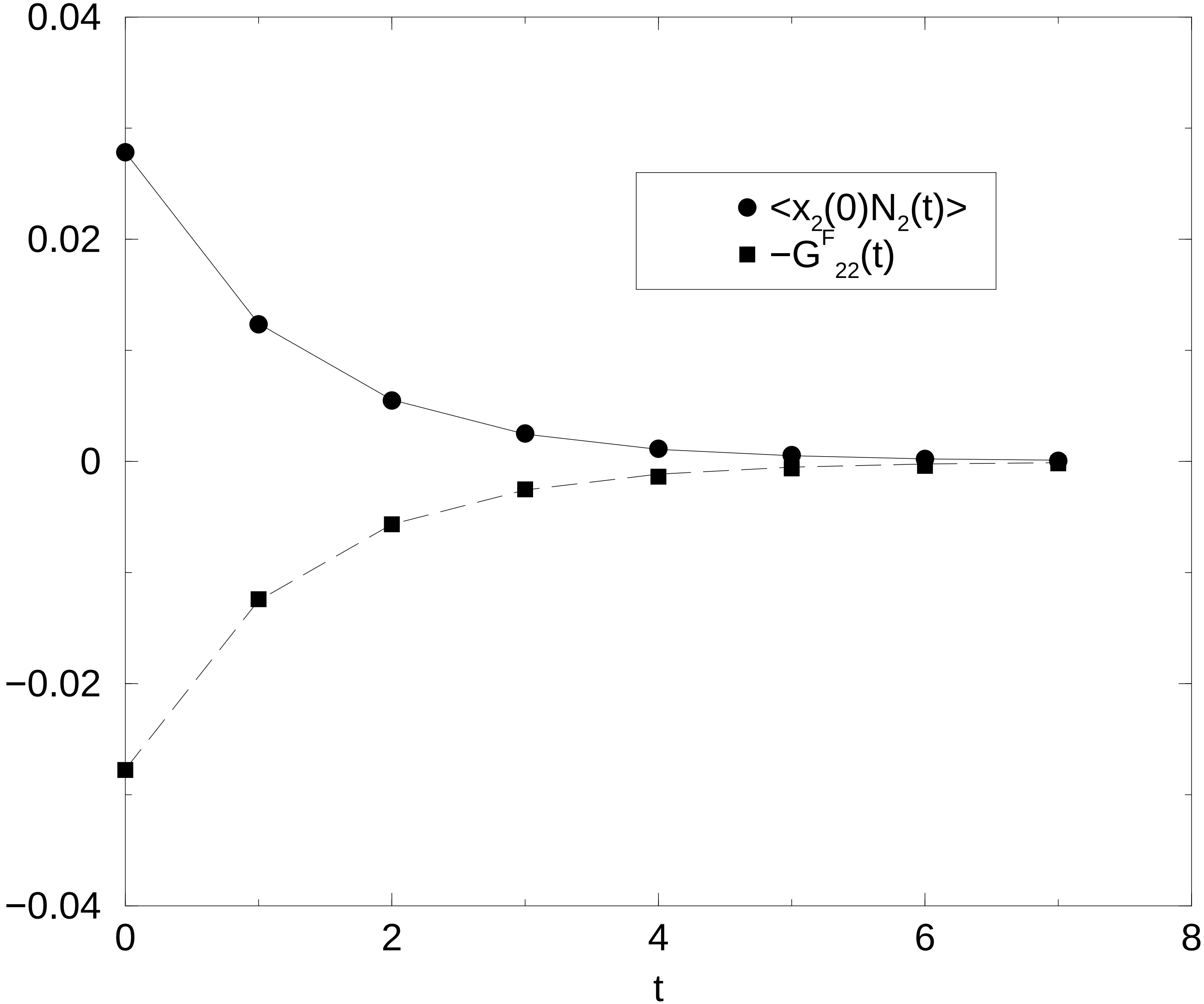
Ward identities

Again, existence of exact SUSY leads to Ward identities:

$$< S_B >= \frac{1}{2} N_{\rm d.o.f}$$

L	$< S_B >$	$\frac{1}{2} N_{dof}$
4	31.93(6)	32
8	127.97(7)	128
16	512.0(3)	512
32	2046(3)	2048

WZ Ward identity



Twisting in 2D

Process of exposing this nilpotent supercharge can be made systematic
Fields transform as spinors under $SO(2)_R$ and $SO(2)_{\rm rot}$
Choose to decompose fields under twisted symmetry

$$SO(2)' = \mathrm{Diag}\left(SO(2)_R \times SO(2)_{\rm rot}\right)$$

Regard spinors/supercharges as matrices
Decompose on products of gamma matrices

$$q = QI + Q_\mu \gamma_\mu + Q_{12}\gamma_1\gamma_2$$

Scalars, vectors, tensors
Note $\mathrm{Tr}(\Gamma^i\Gamma^j) = 2\delta^{ij}$ where $\Gamma^i = (I, \gamma_\mu, \gamma_5$

Algebra

Original supersymmetry algebra $\{q,q\}=\gamma_\mu p_\mu$

$$\begin{aligned}\{Q,Q\} &= \{Q_{12},Q_{12}\}=\{Q,Q_{12}\}=\{Q_\mu,Q_\nu\}=0\\ \{Q,Q_\mu\} &= p_\mu \qquad \{Q_{12},Q_\mu\}=\epsilon_{\mu\nu}p_\nu\end{aligned}$$

Note that p_μ is Q-variation of something.
Makes it plausible that entire $T_{\mu\nu}$ is Q-exact.
(why?: $p_\mu p_\nu = Q\Lambda_\mu Q\Lambda_\nu = Q(\Lambda_\mu Q_\nu)$)

Hence twisted theories have Q-exact actions! (remember $T_{\mu\nu}=\frac{\delta S}{\delta g_{\mu\nu}}$)

Kähler-Dirac fermions

Typical Dirac action can be rewritten as:

$$S_{\mathrm{F}} = \mathrm{Tr}\left[\overline{\Psi}\gamma_\mu\partial_\mu\Psi\right]$$

where

$$\Psi = \frac{\eta}{2}I + \psi_\mu\gamma_\mu + \chi_{12}\gamma_5$$

Components $(\eta/2, \psi_\mu, \chi_{12})$ consitute a Kähler-Dirac field In real (why real?) components:

$$S_{\mathrm{F}} = \frac{1}{2}\eta\partial_\mu\psi_\mu + \chi_{12}\left(\partial_1\psi_2 - \partial_2\psi_1\right)$$

Kähler-Dirac action. Basis indep. Geometrical treatment of fermions.

KD treatment equivalent to twisting

Discretize KD equations

Can discretize such geometrical actions without introducing doubles (Rabin, Joos)

- Replace $\partial_\mu \to \Delta^+_\mu$ in curl
- Replace $\partial_\mu \to \Delta^-_\mu$ in div.

Prohibits doubles!

$$S_F = \begin{pmatrix} \eta/2 & \chi_{12} \end{pmatrix} \begin{pmatrix} \Delta^-_1 & \Delta^-_2 \\ -\Delta^+_2 & \Delta^+_1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Relation to staggered fermions

Natural to place scalars on sites, vectors on links and tensors on diagonal links.
Introduce a lattice with half spacing – all fields now site fields.
Free KD action reduces to staggered action – fermion phases now arise from alternate use of Δ^-, Δ^+.
Usual U(1) symmetry: rotate $\psi_{1,2} \rightarrow e^{\alpha}\, \psi_{1,2}$ and $\eta, \chi_{12} \rightarrow e^{-\alpha}\, \eta, \chi_{12}$.
Homework problem 4. Check this

Wess-Zumino using KD fermions

If we choose

$$\begin{aligned}\omega &= \frac{1}{2}\eta + i\chi_{12} \\ \lambda &= \psi_1 + i\psi_2\end{aligned}$$

Find KD action

$$\omega^\dagger D_{\overline{z}}\lambda + \mathrm{h.c}$$

Technically uses self-dual parts of KD field

Duality $f_p \xrightarrow{*} f_{d-p}$ with

$$*f_{\mu_1\ldots\mu_{d-p}} = \epsilon_{\mu_1\ldots\mu_d} f_{\mu_{d-p+1}\ldots\mu_d}$$

$$P^+ = \frac{1}{2}(I + i*)$$