Introduction to Lattice Supersymmetry

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Motivation

- Motivation: SUSY theories - cancellations between fermions/bosons – soft U.V behavior. Higgs light $m_H^2 \sim log(\Lambda)$
- More tractable analytically – toy models for understanding confinement and chiral symmetry breaking
- Key component of string theory – remove tachyon of bosonic string.
- Generalizations of AdSCFT – $(p+1)$ SYM and type II strings with Dp-branes.
- Usually assume symmetry holds at high energy – must break nonperturbatively at low energies – lattice.

Problems

- Extension of Poincare symmetry: $\{Q, \overline{Q}\} = \gamma.P$. Broken by lattice.
- Equivalently: Leibniz rule does not hold for difference operators
- Fermion doubling - $n_B \neq n_F$. Wilson terms break SUSY.
- Consequence: Naively discretized classical action breaks SUSY. Effective action picks up (many) SUSY violating operators. Generically some of relevant. Couplings must be fine tuned as $a \to 0$.

Solutions

- Just do it.
  - Certain simple cases eg. $\mathcal{N}=1$ SYM in 4D single counterterm with Wilson fermions.
  - For $D<4$ finite number of divergences occuring at small numbers of loops.
- For special class of theories can find novel discretizations which preserve one or more SUSY's exactly. Two approaches:
  - Orbifold methods. SYM case.
  - Twisted formulations. Equivalent ...?

Overview

- Motivation/Problems.
- Witten's SUSYQM. Naive discretization. Fine tuning. Modification to maintain exact SUSY.
- Nicolai maps. Topological/twisted field theory interpretation.
- Generalizations. SYMQM, sigma models.
- Lifting to 2D. Wess Zumino models. Twisting in 2D. Kähler-Dirac fermions. $\mathcal{N}=2$ SYM.
- Lifting to 4D. $\mathcal{N}=4$ SYM.
- Orbifold constructions

Witten's SUSYQM

$$S = \int dt \, \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}P'(\phi)^2 + \frac{1}{2}\psi_i \frac{d\psi_i}{dt} + i\psi_1\psi_2 P''(\phi)$$

Invariant under 2 SUSYs:

$$\begin{array}{rclrcl} \delta_A\phi &=& \psi_1\epsilon_A & \delta_B\phi &=& \psi_2\epsilon_B \\ \delta_A\psi_1 &=& \frac{d\phi}{dt}\epsilon_A & \delta_B\psi_1 &=& -iP'\epsilon_B \\ \delta_A\psi_2 &=& iP'\epsilon_A & \delta_B\psi_2 &=& \frac{d\phi}{dt}\epsilon_B \end{array}$$

Homework Problem 1: verify these invariances

Continuum variation

Find:

$$\delta_A S = \int dt \; i\epsilon \left(P' \frac{d\psi_2}{dt} + \frac{d\phi}{dt} P'' \psi_2 \right)$$

Need to integrate by parts and use Leibniz to get zero.
Problem for lattice.
Notice that $\delta_A^2 = \delta_B^2 = \frac{d}{dt}$ acting on any field.
Example of SUSY algebra since $H = \frac{d}{dt}$.

Naive discretization

Place fields on sites of (periodic) 1D lattice. Replace $\int dt \rightarrow \sum_t a$ and replace $\frac{d}{dt}$ by (doubler free) backward difference.

$$a\Delta^-_\mu f_x = f(x) - f(x-\mu)$$

Now find:

$$\delta_A S_L = \sum_t i\epsilon \left(P'\Delta^-\psi_2 + \Delta^-\phi P''\psi_2\right)$$

where rescaled x by $a^{\frac{1}{2}}$. Naively invariant as $a \rightarrow 0$. But expect radiative corrections ....

Boson/fermion masses - naive

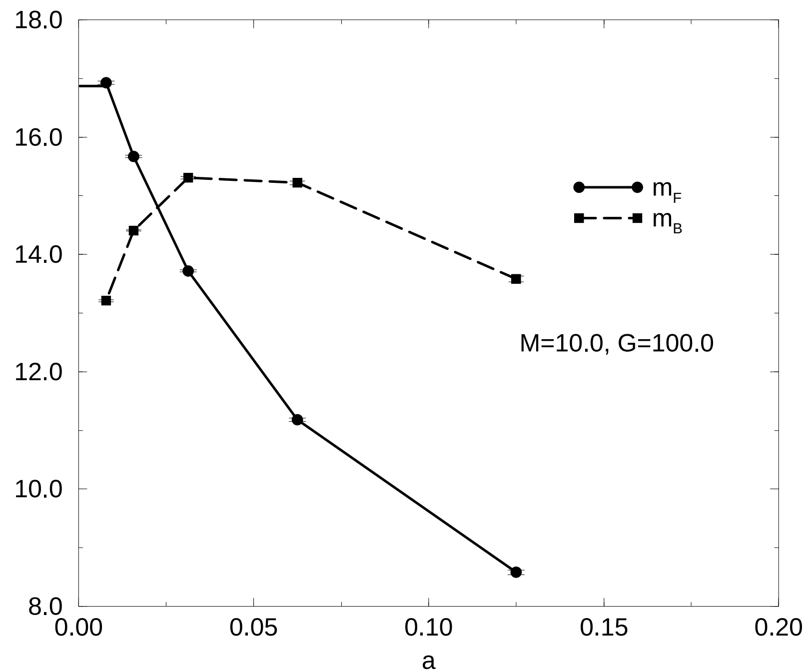

Figure 1: $P' = m\phi + g\phi^3$, $m = 10.0$, $g = 100.0$

Radiative corrections

Points to note:

- Any Feynman graph which is convergent in U.V can be discretized naively (Reisz theorem).
- Restrict attention to superficially divergent continuum graphs.
- In previous example only one of these. One loop fermion contribution to boson mass.

Radiative corrections II

Continuum:

$$\Sigma_{\rm cont} = 6g \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dp}{2\pi} \frac{-ip+m}{p^2+m^2}$$

Actually convergent ($p \to -p$ symmetry)

$$\Sigma_{\rm cont} = 6g\left(\frac{1}{\pi}\tan^{-1}\frac{\pi}{2ma}\right) \sim 6g\left(\frac{1}{2}+\mathcal{O}(ma)\right)$$

Lattice result is

$$\Sigma_{\rm latt} = \frac{6g}{L}\sum_{k=0}^{L-1}\frac{-2i\sin\left(\frac{\pi k}{L}\right)e^{i\left(\frac{\pi k}{L}\right)}+ma}{\sin^2\left(\frac{\pi k}{L}\right)+(ma)^2} \to 6g!$$

Radiative corrections III

- If take $a \to 0$ after doing sum get twice the result!
- Homework Problem 2. Convince yourself of this!
- $D^{-} = D^{s} + \frac{1}{2}m_W$. Would be doublers have mass $O(1/a)$ and make an additional contribution to integral (don't decouple from small loops)
- Restore SUSY need to add counterterm

$$S_L \to S_L + \sum_t 3g\phi^2$$

SUSY broken but regained now as $a \to 0$.

Intuitive argument

Consider using lattice derivative:

$$D^s + \frac{r}{2} m_W$$

Doubler mass $M = m + 2r/a$. Consider limit where $r << 1$. Then $ma << Ma << 1$. Lattice integral is approx:

$$\begin{aligned} \Sigma &= \int_{\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dp}{2\pi} \frac{m}{p^2 + m^2} + \int_{\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dp}{2\pi} \frac{M}{p^2 + M^2} \\ &= \frac{1}{\pi} \left(\tan^{-1} \frac{\pi}{2ma} + \tan^{-1} \frac{\pi}{2Ma} \right) \end{aligned}$$

Masses - counterterm corrected

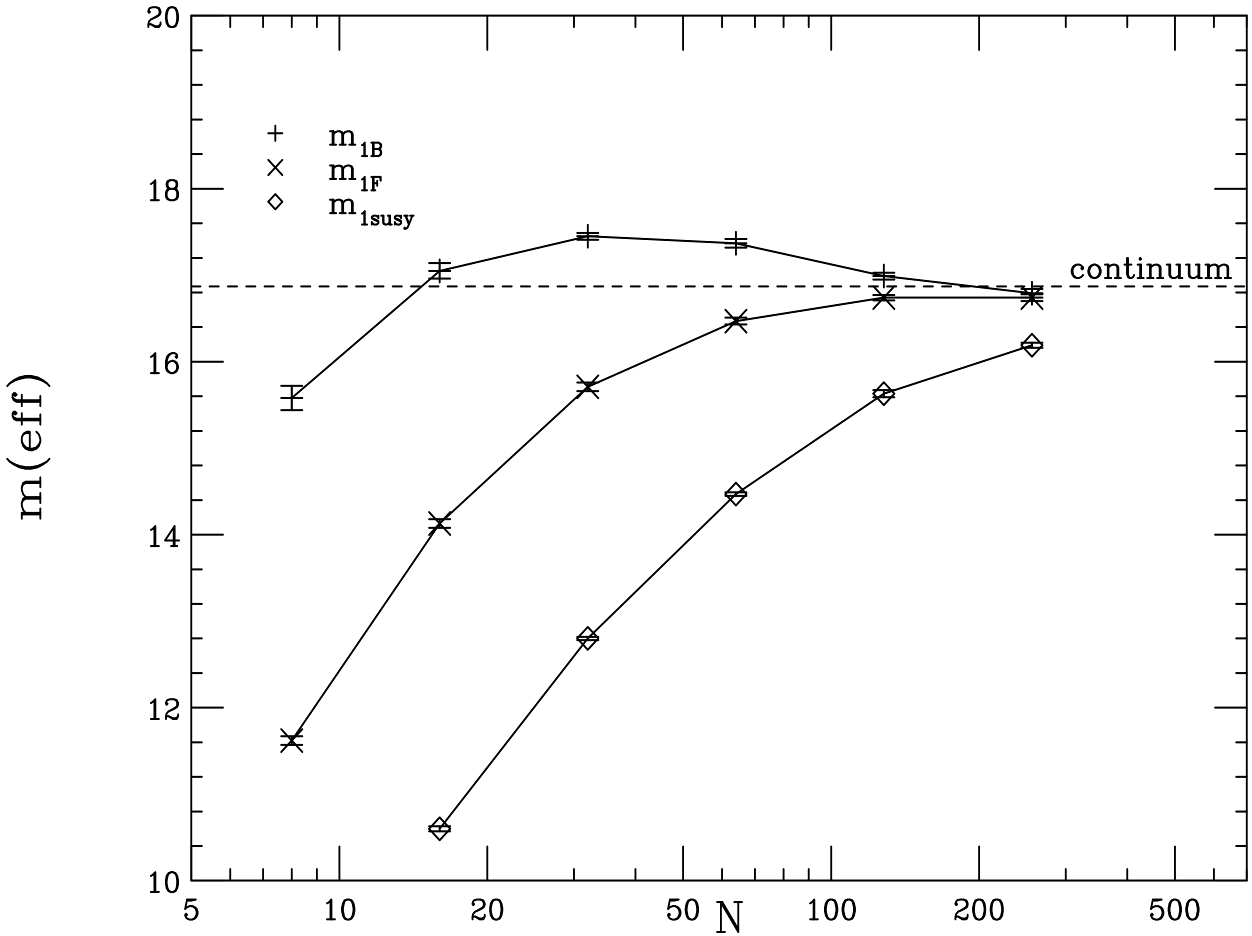

Figure 2: $P' = m\phi + g\phi^3$, $m = 10.0$, $g = 100.0$

Exact SUSY

Can do better. Find combination of SUSY's that can be preserved on lattice.
Notice that:

$$\delta_A S_L = -i\delta_B \sum_x P' \Delta^- \phi \qquad \delta_B S_L = i\delta_A \sum_x P' \Delta^- \phi$$

Thus

$$(\delta_A + i\delta_B)\, S_L = -\,(\delta_A + i\delta_B)\, O \qquad \text{where } O = \sum_t P' \Delta^- \phi$$

So can find δS_{exact} of form

$$S_L^{\text{exact}} = \sum_t \frac{1}{2}(\Delta^- \phi)^2 + \frac{1}{2}P'^2 + P' \Delta^- \phi + \overline{\psi}(\Delta^- + P'')\psi$$

Exact SUSY II

Where

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$$
$$\overline{\psi} = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2)$$

and the new supersymmetry acts:

$$\delta\phi = \psi\epsilon$$
$$\delta\psi = 0$$
$$\delta\overline{\psi} = (\Delta^-\phi + P'(\phi))\epsilon$$

Notice: $\delta^2 = 0$ now. No translations.

$$S_L^b = \sum_x \left(\Delta^-\phi + P'(\phi)\right)^2$$

Masses - exact SUSY

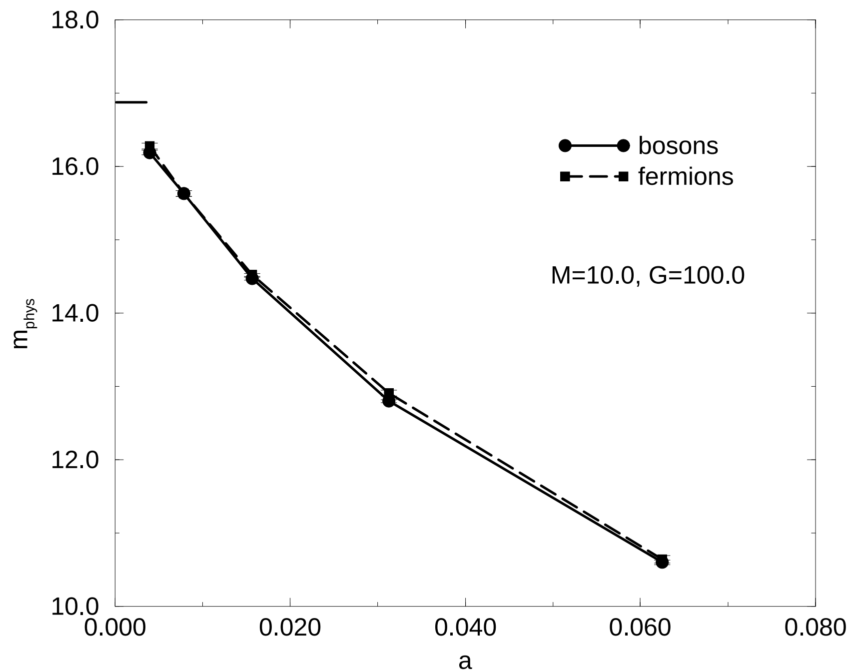

Figure 3: Boson and fermion masses vs lattice spacing for supersymmetric action

Nicolai map

Partition function:

$$Z = \int D\phi D\psi D\overline{\psi} e^{-S} = \int D\phi \det\left(\Delta^{-} + P''\right) e^{-S_B}$$

Change variables to $\mathcal{N} = \Delta^{-}\phi + P'(\phi)$ Jacobian is $\det\left(\frac{\partial \mathcal{N}_i}{\partial \phi_j}\right)$.

Cancels fermionic determinant!

$$Z = \int \prod_i d\mathcal{N}_i e^{-\mathcal{N}_i^2}$$

Details of $P(\phi)$ disappeared! Z is a topological invariant.

Simple argument: $< S_B >= \frac{1}{2} N_{\rm d.o.f}$

Ward identities

Classical invariance of action replaced by relationships between correlation functions of form

$$< \delta O >= 0$$

Choosing $O = \overline{\psi}_x \phi_y$ we find

$$< \overline{\psi}_x \psi_y > + < (\Delta^- \phi + P')_x \phi_y >= 0$$

Expect other SUSY $\bar{\delta} = \frac{1}{\sqrt{2}}(\delta_A - i\delta_B)$ broken.

Restored in continuum limit without fine tuning.

Ward identities II

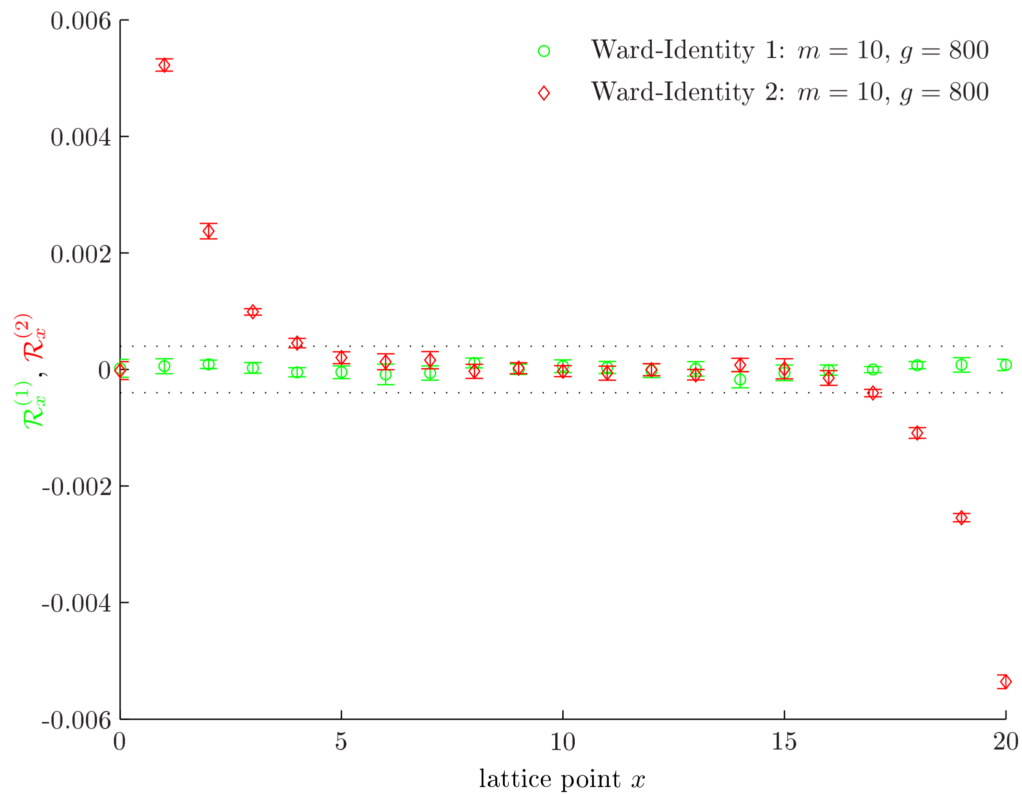

Figure 4: $m = 10.0$, $g = 800.0$, from Kaestner et al.

Topological field theory

Actually $\delta^2 = 0$ for the field $\overline{\psi}$ only by using EOM.
Can render symmetry nilpotent off-shell by introducing auxiliary field

$$\begin{aligned} Q\phi &= \psi \\ Q\psi &= 0 \\ Q\overline{\psi} &= B \\ QB &= 0 \end{aligned}$$

Note: absorbed ϵ into variation δ and renamed it Q. Also

$$S_L^b = \sum_x -B(\Delta^- \phi + P') - \frac{1}{2}B^2$$

TQFT II

Remarkably:

$$S_L = Q\sum_x \overline{\psi}(-\Delta^- \phi - P' - \frac{1}{2}B)$$

The action is Q-exact. Like BRST ?
Consider bosonic model with $S(\phi) = 0$. Invariant under $\phi \rightarrow \phi + \epsilon$ –topological symmetry.
Quantize: pick gauge function $\mathcal{N} = 0$ and introduce Fadeev-Popov factor

$$Z = \int D\phi \det(\frac{\partial \mathcal{N}}{\partial \phi}) e^{-\frac{1}{2\alpha}\mathcal{N}^2(\phi)}$$

Interpret ψ,$\overline{\psi}$ as ghost fields ($\alpha = 1$) recover our model!

Moral of the story

- Fine tuning problems can be handled in $D < 4$ by perturbative lattice cals.
- Even $D = 4$ $\mathcal{N} = 1$ using chirally improved actions.
- In some cases can do better – find combinations of the supersymmetries in (some) SUSY models which are nilpotent.
- (Twisted) reformulations are closely connected to construction of TQFT. Actions are Q-exact. Easy to translate to lattice.
- Simulation can be done with (R)HMC algs. and good agreement with theory