Schiff Theorem and Nuclear Schiff Moment [1]

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[1] LA-UR-07-1711

Outline

EDM Status

In units of *e–cm*, selected EDM limits are:

Particle	EDM limit	System	SM Prediction	New Physics
e	1.9×10^{-27}	^{205}Tl atom	10^{-38}	10^{-27}
μ	1.1×10^{-19}	rest frame $\vec{E}$	10^{-35}	10^{-22}
τ	3.1×10^{-16}	$e^+e^- \rightarrow \tau^+\tau^-\gamma$	10^{-34}	10^{-20}
p	6.5×10^{-23}	TlF molecule	10^{-31}	10^{-26}
n	2.9×10^{-26}	UCN	10^{-31}	10^{-26}
^{199}Hg	2.1×10^{-28}	atom cell	10^{-33}	10^{-28}

- Most precise measurements are taken in neutral systems.
- Results for charged particles (except μ) are inferred.

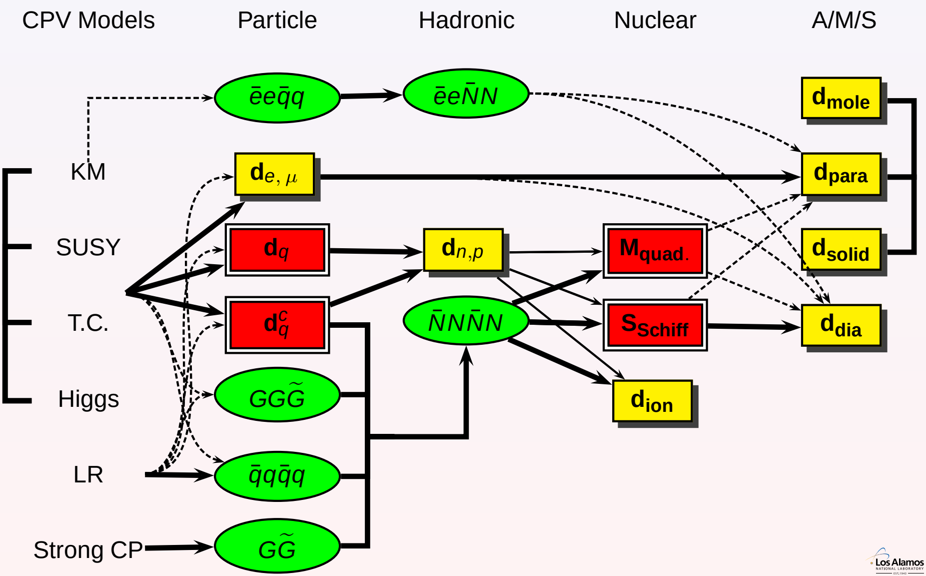
New-Generation EDM Searches

A non-exhaustive list:

Leptonic EDMs		Hadronic EDMs	
System	Group	System	Group
Cs (trapped)	Penn St.	n (UCN)	SNS
Cs (trapped)	Texas	n (UCN)	ILL
Cs (fountain)	LBNL	n (UCN)	PSI
YbF (beam)	Imperial	n (UCN)	Munich
PbO (cell)	Yale	^{199}Hg (cell)	Seattle
HBr^+ (trapped)	JILA	^{129}Xe (liquid)	Princeton
PbF (trapped)	Oklahoma	^{225}Ra (trapped)	Argonne
GdIG (solid)	Amherst	^{213,225}Ra (trapped)	KVI
GGG (solid)	Yale/Indiana	^{223}Rn (trapped)	TRIUMF
muon (ring)	J-PARC	deuteron (ring)	BNL?

- All leptonic searches except μ target at d_e (indirectly inferred).
- Most of them are subject to the shielding effects.

The Road Map


CPV Models
Particle
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Nuclear
A/M/S
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d_{mole}
KM
$d_{e,\mu}$
d_{para}
SUSY
d_q
$d_{n,p}$
$M_{quad.}$
d_{solid}
T.C.
d_q^c
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S_{Schiff}
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Higgs
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LR
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Strong CP
$G\tilde{G}$

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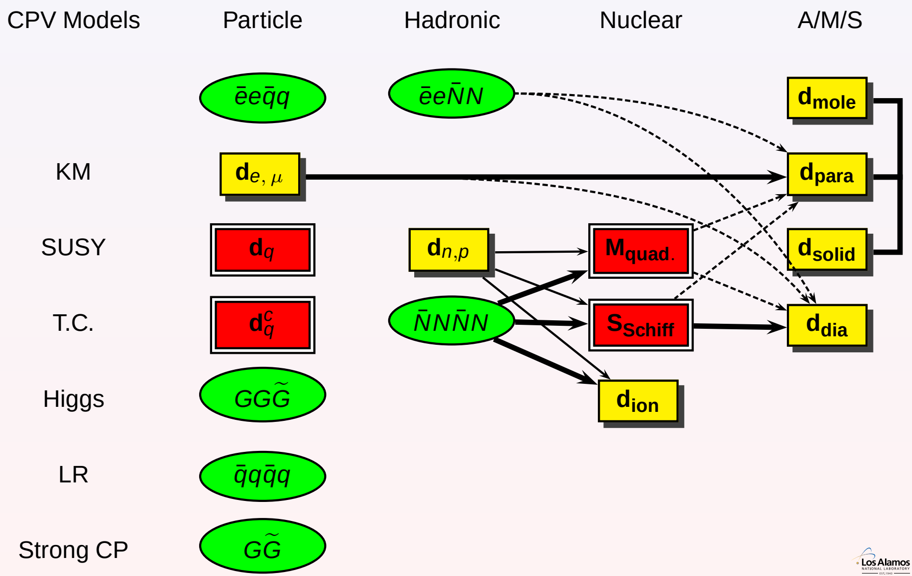
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Theorem

For a NR system made up of point, charged particles which interact electrostatically with each other and with an arbitrary external field, the shielding is complete. (Schiff, 63)

- Classical picture: The re-arrangement of constituent charged particles in order to keep the whole system stationary.
- Quantum-Mechanical description: Schiff (63), Sandars (68), Feinberg (77), Sushkov, Flambaum, and Khriplovich (84), Engel, Friar, and Hayes (00), Flambaum and Ginges (02) ...

What this implies for atoms? (molecules? solids? neutron?)

- The measurability of atomic EDMs is severely constrained.
- One has to look for the loopholes (Schiff ff 63, Sandars 68) in
  - relativistic effects (electron)
  - finite-size effects (nucleus)
  - magnetic interactions (electron–nucleus)
  - non-EM exotica such as $\not{P}\not{T}$ electron-nucleon interaction

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The Atom under Detection (weak $E^{(\mathrm{ext})}$)

The whole atomic EDM consists of:

1. Intrinsic EDMs of electrons and nucleus
   - d_e is elementary.
   - d_{nuc} has contributions from $d_{n,p}$ and $\not{P}\not{T}$ NN interaction, parametrized by $\tilde{C}^{\mathrm{had}}$.
2. Polarization effects by the $\not{P}\not{T}$ electron–nucleus interactions $\tilde{V}_{e-\mathrm{nuc}}$
   - Nuclear excitations are much less effective since $\Delta E_e / \Delta E_{\mathrm{nuc}} \sim 10^{-6}$.
   - $\tilde{V}_{e-\mathrm{nuc}}$ contains leptonic, semi-leptonic, and hadronic $\not{P}\not{T}$ sources.

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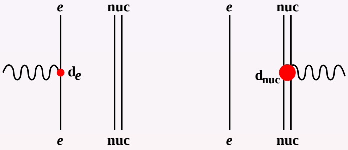

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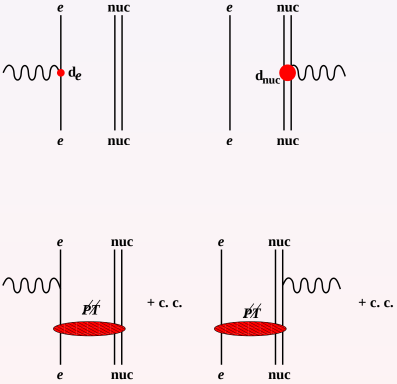
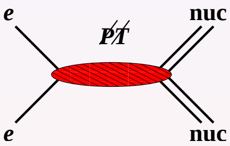
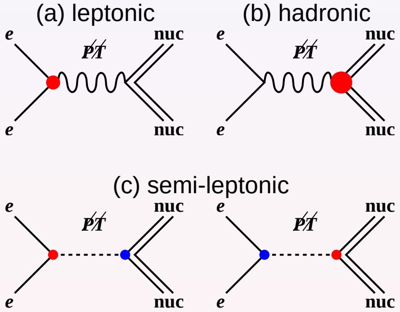


Contributions to $\not{P}\not{T}$ Electron–Nucleus Interaction

- Red vertices denote the $\not{P}\not{T}$ couplings: (a) d_e, (b) $d_{\rm nuc}$, (c) $\kappa_e^{\rm PS}$, $\kappa_N^{\rm PS}$ etc.





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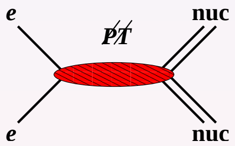
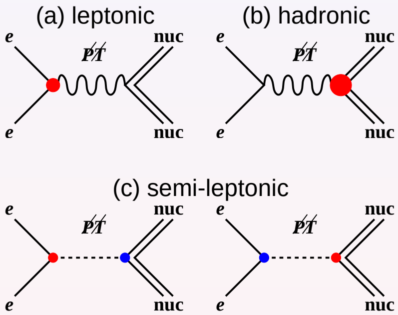
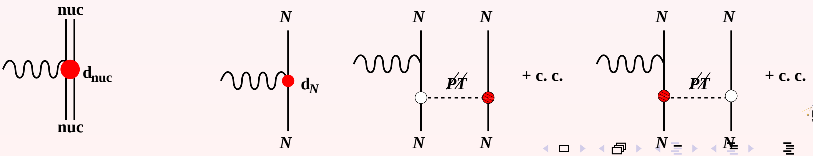




- What is inside of diagram (b) which involves nuclear EDM?



Formalism: Hamiltonian

The full atomic Hamiltonian:

$$H_{\text{atom}} = H_0 + H_I$$

- The unperturbed term describes the atom (PT)

$$H_0 = \sum_{i=1}^{Z} \left\{ (\beta_i m_e + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_i) + \frac{\alpha}{2} (\phi_i^{(e)} - \boldsymbol{\alpha}_i \cdot \boldsymbol{A}_i^{(e)}) - \alpha (\phi_i^{(\text{nuc})} - \boldsymbol{\alpha}_i \cdot \boldsymbol{A}_i^{(\text{nuc})}) \right\} + H_{\text{nuc}}$$

  - Index i denotes the ith electron.
  - $\phi_i^{(e)}$ and $\phi_i^{(\text{nuc})}$ are the Coulomb potentials from other electrons and nucleus (total charge, quadrupole etc.)
  - $\boldsymbol{A}_i^{(e)}$ and $\boldsymbol{A}_i^{(\text{nuc})}$ are the vector potentials from other electrons (Breit interaction) and nucleus (hyperfine interaction etc.)
  - To keep the most general form, just assume we have the solution, either w/o the conventional factorization of electron and nuclear wave functions.

$$H_0 |\Psi\rangle = E |\Psi\rangle \approx (E_e + E_{\text{nuc}}) |\psi_e\rangle \otimes |\psi_{\text{nuc}}\rangle$$

- The perturbed term contains the following:

$$H_I = V_e^{(\mathrm{ext})} + V_{\mathrm{nuc}}^{(\mathrm{ext})} + \tilde{V}_e^{(\mathrm{ext})} + \tilde{V}_{\mathrm{nuc}}^{(\mathrm{ext})} + \tilde{V}_e^{(\mathrm{int})} + \tilde{V}_{\mathrm{nuc}}^{(\mathrm{int})}$$

  - $V_e^{(\mathrm{ext})}$ and $V_{\mathrm{nuc}}^{(\mathrm{ext})}$ are EM interactions of electrons and nucleus with the external field, $O(E^{(\mathrm{ext})})$

$$V_e^{(\mathrm{ext})} = -\alpha \sum_{i=1}^{Z} \left(\phi_i^{(\mathrm{ext})} - \boldsymbol{\alpha}_i \cdot \boldsymbol{A}_i^{(\mathrm{ext})} \right) \quad V_{\mathrm{nuc}}^{(\mathrm{ext})} = \alpha \left(Z\, \phi_0^{(\mathrm{ext})} - \boldsymbol{\mu}_{\mathrm{nuc}} \cdot \boldsymbol{B}_0^{(\mathrm{ext})} \right) + \ldots$$

  - $\tilde{V}_e^{(\mathrm{ext})}$ and $\tilde{V}_{\mathrm{nuc}}^{(\mathrm{ext})}$ are $\not{P}\not{T}$ EM interactions of electrons and nucleus with the external field, $O(\tilde{G}_F E^{(\mathrm{ext})})$

$$\tilde{V}_e^{(\mathrm{ext})} = -\alpha \sum_{i=1}^{Z} d_e\, \beta_i \left(\boldsymbol{\sigma}_i \cdot \boldsymbol{E}_i^{(\mathrm{ext})} + i\, \boldsymbol{\alpha}_i \cdot \boldsymbol{B}_i^{(\mathrm{ext})} \right) \quad \tilde{V}_{\mathrm{nuc}}^{(\mathrm{ext})} = -\alpha\, \boldsymbol{d}_{\mathrm{nuc}} \cdot \boldsymbol{E}_0^{(\mathrm{ext})} + \ldots$$

  - $\tilde{V}_e^{(\mathrm{int})}$ and $\tilde{V}_{\mathrm{nuc}}^{(\mathrm{int})}$ are $\not{P}\not{T}$ EM interactions between electrons and nucleus, $O(\tilde{G}_F)$

$$\tilde{V}_e^{(\mathrm{int})} = -\alpha \sum_{i=1}^{Z} d_e\, \beta_i \left(\boldsymbol{\sigma}_i \cdot \boldsymbol{E}_i^{(\mathrm{int})} + i\, \boldsymbol{\alpha}_i \cdot \boldsymbol{B}_i^{(\mathrm{int})} \right) \quad \tilde{V}_{\mathrm{nuc}}^{(\mathrm{int})} = -\alpha \sum_{i=1}^{Z} \left(\widetilde{\phi}_i^{(\mathrm{nuc})} - \boldsymbol{\alpha}_i \cdot \widetilde{\boldsymbol{A}}_i^{(\mathrm{nuc})} \right)$$

Formalism: Multipole Expansion

The spherical multipole expansion for nuclear potential relies on

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} = \sum_{J\geq 0,M} \frac{4\pi}{2J+1}\frac{1}{x^{2J+1}}\left[\underbrace{1}_{(a)} + \underbrace{\theta(y-x)\left(\frac{x^{2J+1}}{y^{2J+1}}-1\right)}_{(b)}\right]\mathcal{Y}_J^{M*}(\boldsymbol{x})\,\mathcal{Y}_J^{M}(\boldsymbol{y})$$

- $\boldsymbol{x}$ and $\boldsymbol{y}$ are electronic and nuclear coordinates.
- $\mathcal{Y}_J^M(\boldsymbol{r}) \equiv r^J\, Y_J^M(\hat{r})$ is the solid harmonics.
- (a) terms define the normal long-ranged multipoles (field point $\boldsymbol{x}$ away from source points $\boldsymbol{y}$).
- (b) terms define the local multipoles (field point $\boldsymbol{x}$ inside source points $\boldsymbol{y}$), this leads to contact interactions.
- Notations:

$$\hat{C}_J^M = \int d\boldsymbol{y}\,(a)\,\mathcal{Y}_J^M(\boldsymbol{y})\hat{\rho}(\boldsymbol{y}) \quad \hat{M}_J^M = \int d\boldsymbol{y}\,(a)\,[\mathcal{Y}(\boldsymbol{y})\otimes\hat{\boldsymbol{j}}(\boldsymbol{y})]_J^M$$

$$\hat{\mathcal{C}}_J^M(x) = \int d\boldsymbol{y}\,(b)\,\mathcal{Y}_J^M(\boldsymbol{y})\hat{\rho}(\boldsymbol{y}) \quad \hat{\mathcal{M}}_J^M(x) = \int d\boldsymbol{y}\,(b)\,[\mathcal{Y}(\boldsymbol{y})\otimes\hat{\boldsymbol{j}}(\boldsymbol{y})]_J^M$$

Schiff Theorem in a Nutshell

The energy shift of order $O(\tilde{G}_F E^{(\text{ext})})$ contains:

1. 1st-order in perturbation expansion

$$\Delta E^{(1)} = \langle \Psi_0 | \widetilde{V}_{d_e}^{(\text{ext})} + \widetilde{V}_{d_\text{nuc}}^{(\text{ext})} | \Psi_0 \rangle$$

2. 2nd-order in perturbation expansion

$$\Delta E^{(2)} = \sum_n \frac{-1}{E_0 - E_n} \left\{ \langle \Psi_0 | V_e^{(\text{ext})} | \Psi_n \rangle \langle \Psi_n | \widetilde{V}_{d_e}^{(\text{int})} + \widetilde{V}_{d_\text{nuc}}^{(\text{int})} | \Psi_0 \rangle + \text{c.c.} \right\}$$

The trick to sum $\Delta E^{(1)}$ and $\Delta E^{(2)}$ is to re-write

$$\widetilde{V}_{d_e}^{(\text{int})} + \widetilde{V}_{d_\text{nuc}}^{(\text{int})} = [\tilde{O}_{d_e} + \tilde{O}_{d_\text{nuc}} \, , \, H_0] + \ldots \, ,$$

so one can apply the closure to the commutator term and get

$$\Delta E = \langle \Psi_0 | \widetilde{V}_{d_e}^{(\text{ext})} + \widetilde{V}_{d_\text{nuc}}^{(\text{ext})} - [V_e^{(\text{ext})} \, , \, \tilde{O}_{d_e} + \tilde{O}_{d_\text{nuc}}] | \Psi_0 \rangle + \ldots$$

Why? Precision consideration!

Shielding of Electron EDM

The key relation is

$$\widetilde{V}_{d_e}^{(\text{int})} = \sum_{i=1}^{Z} [-d_e\,\beta\,\boldsymbol{\sigma}_i \cdot \boldsymbol{\nabla}_i\,,\, H_0] + 2\,d_e\,\beta\,i\,\gamma_5 \left(\boldsymbol{p}_i^2 + \alpha\,\boldsymbol{A}_i^{(\text{int})} \cdot \boldsymbol{p}_i\right)$$

Therefore

$$\sum_{i=1}^{Z} [V_e^{(\text{ext})}\,,\, -d_e\,\beta\,\boldsymbol{\sigma}_i \cdot \boldsymbol{\nabla}_i] = -\widetilde{V}_{d_e}^{(\text{ext})} + \ldots$$

so $\widetilde{V}_{d_e}^{(\text{ext})}$ is completely canceled.

- The residual interaction contains γ_5 which connects the large and small components of a Dirac wave function, so vanishes in the NR limit.
- The omitted "…" terms are non-vanishing when there exists external vector potential and of the relativistic order $Z^2\alpha^2$.

Shielding of Nuclear EDM

The key relation is

$$\widetilde{V}^{(\mathrm{int})}_{d_{\mathrm{nuc}}} = -\alpha \sum_{i=1}^{Z} \boldsymbol{d}_{\mathrm{nuc}} \cdot \frac{\boldsymbol{x}}{x^3} = \sum_{i=1}^{Z} \left[-\frac{\boldsymbol{d}_{\mathrm{nuc}}}{Z} \cdot \boldsymbol{\nabla}_i \, , \, H_0 \right] - \left[-\frac{\boldsymbol{d}_{\mathrm{nuc}}}{Z} \cdot \boldsymbol{\nabla}_i \, , \, \Delta H_0 \right]$$

Therefore

$$\sum_{i=1}^{Z} \left[V_e^{(\mathrm{ext})} \, , \, -\frac{\boldsymbol{d}_{\mathrm{nuc}}}{Z} \cdot \boldsymbol{\nabla}_i \right] = -\widetilde{V}^{(\mathrm{ext})}_{d_{\mathrm{nuc}}} + \ldots$$

so $\widetilde{V}^{(\mathrm{ext})}_{d_{\mathrm{nuc}}}$ is completely canceled.

- The residual interaction depends on ΔH_0 which involves interactions with nuclear multipoles $\mathcal{C}_0, C_2, \mathcal{C}_2$ etc. (vanish in the point-like limit) and $M_1, \mathcal{M}_1, M_3$ etc. (which are not electrostatic).
- The omitted "…" terms are non-vanishing when the external vector potential exhibits the property that $\nabla_i A_j^{(\mathrm{ext})} + \nabla_j A_i^{(\mathrm{ext})} \neq 0$.

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Notations

For the following discussion, we adopt the notations:

1. 1st-order residual

$$\Delta E^{(1)} = \langle \Psi_0 | \, \hat{\mathcal{O}}_e^{(\text{ext})} + \hat{\mathcal{O}}_{\text{nuc}}^{(\text{ext})} \, | \Psi_0 \rangle$$

2. 2nd-order residual

$$\Delta E^{(2)} = \sum_n \frac{-1}{E_0 - E_n} \left\{ \langle \Psi_0 | \, V_e^{(\text{ext})} \, | \Psi_n \rangle \langle \Psi_n | \, \hat{\mathcal{O}}_e^{(\text{int})} + \hat{\mathcal{O}}_{\text{nuc}}^{(\text{int,el})} + \hat{\mathcal{O}}_{\text{nuc}}^{(\text{int,mag})} \, | \Psi_0 \rangle + \text{c.c.} \right\}$$

As most EDM measurements exploit the atomic s–p transitions, for $\tilde{V}_{e-\text{nuc}}$ being mostly short-ranged, we will just concentrate on $\hat{\mathcal{O}}$'s with $\Delta L_e = 1$.

Relativistic Effects of Electron

The residual interactions (Lindroth et al., 89) are

$$\hat{\mathcal{O}}_e^{(\mathrm{ext})} = 2\,i\,d_e \sum_{i=1}^{Z} \beta_i\,\gamma_5\,\alpha\,\boldsymbol{A}_i^{(\mathrm{ext})}\cdot\boldsymbol{p}_i$$

$$\hat{\mathcal{O}}_\mathrm{e}^{(\mathrm{int})} = 2\,i\,d_e \sum_{i=1}^{Z} \beta_i\,\gamma_5 \left(\boldsymbol{p}_i^2 + \alpha\,\boldsymbol{A}_i^{(\mathrm{int})}\cdot\boldsymbol{p}_i\right)$$

Another equivalent way (Sandars, 68) is re-writing the Dirac β matrix in EDM interaction as $1{+}(\beta-1)$, and applying the closure only to the first term "1". And this gives

$$\hat{\mathcal{O}}_e^{(\mathrm{ext})} = d_e \sum_{i=1}^{Z} \left\{(1-\beta_i)(\boldsymbol{\sigma}_i\cdot\boldsymbol{E}_i^{(\mathrm{ext})} + i\,\boldsymbol{\alpha}_i\cdot\boldsymbol{B}_i^{(\mathrm{ext})}) + 2\,\boldsymbol{\alpha}_i\cdot\boldsymbol{A}_i^{(\mathrm{ext})}\times\boldsymbol{p}_i\right\}$$

$$\hat{\mathcal{O}}_\mathrm{e}^{(\mathrm{int})} = d_e \sum_{i=1}^{Z} \left\{(1-\beta_i)(\boldsymbol{\sigma}_i\cdot\boldsymbol{E}_i^{(\mathrm{int})} + i\,\boldsymbol{\alpha}_i\cdot\boldsymbol{B}_i^{(\mathrm{int})}) + 2\,\boldsymbol{\alpha}_i\cdot\boldsymbol{A}_i^{(\mathrm{int})}\times\boldsymbol{p}_i\right\}$$

- The advantage of the former: purely one-body, when the Breit interaction is ignored.
- The advantage of the latter: $(1-\beta)$ is manifestly of order $O(\alpha^2)$ as it connects only two small components.

Finite-Size Effects of Nucleus

The residual interactions are

$$\hat{\mathcal{O}}^{(\mathrm{ext})}_{\mathrm{nuc}} = \frac{\alpha}{Z} \sum_{i=1}^{Z} [\boldsymbol{d}_{\mathrm{nuc}} \cdot \boldsymbol{\nabla}_i \,,\, \boldsymbol{\alpha}_i \cdot \boldsymbol{A}^{(\mathrm{ext})}_i] + [\boldsymbol{\alpha}_i \cdot \boldsymbol{\nabla}_i \,,\, \boldsymbol{d}_{\mathrm{nuc}} \cdot \boldsymbol{A}^{(\mathrm{ext})}_i]$$

$$\hat{\mathcal{O}}^{(\mathrm{int,el})}_{\mathrm{nuc}} = -\frac{4\pi\alpha}{Z x^2} \Big\{ Z\, Y_1(\hat{x}) \otimes \frac{1}{3} \mathcal{C}_1(x)$$

$$+ Y_1(\hat{x}) \odot \frac{1}{\sqrt{3}} \left[\boldsymbol{d}_{\mathrm{nuc}} \otimes \left(\mathcal{C}_{0>}(x) - \frac{\sqrt{2}}{x^2} \mathcal{C}_{2<}(x) \right) \right]^{\mathrm{sym}}_1$$

$$- \left(Y_0(\hat{x}) \otimes [\boldsymbol{d}_{\mathrm{nuc}} \,,\, \mathcal{C}_0(x)] + Y_2(\hat{x}) \otimes \left[\boldsymbol{d}_{\mathrm{nuc}} \,,\, \frac{1}{5x^2} (C_2 + \mathcal{C}_2(x)) \right] \right) \cdot (x\, \boldsymbol{\nabla}^{\mathrm{sym}}) \Big\}$$

- $\hat{\mathcal{O}}^{(\mathrm{int,el})}_{\mathrm{nuc}}$ is the most general form of the Schiff moment interaction.
- The 1st line is the local nuclear EDM ($\mathcal{C}_1$) interaction.
- The 2nd and 3rd line comes from $[\boldsymbol{d}_{\mathrm{nuc}} \cdot \boldsymbol{\nabla}_x \,,\, C_J\, F(x)]$; the former involves $[\boldsymbol{\nabla}_x \,,\, F(x)]$ and the second involves $[\boldsymbol{d}_{\mathrm{nuc}} \,,\, C_J]$.
- To order of $O(v/c)$ and $O(\tilde{G}_F)$, the 3rd line vanishes.

Schiff Operator: For Atomic s–p Transitions

To get a more compact form of the Schiff operator and compare with literature, consider the following case of atomic s–p transitions (Flambaum & Ginges, 02).

1. Expand the atomic radial wave functions (short-distance) in polynomials

$$u_s(x) = \sum_{k\geq 0} a_k\, x^k \qquad u_p(x) = \sum_{k\geq 1} b_k\, x^k$$

2. Recast the matrix element

$$\langle \Psi_p |\, \hat{\mathcal{O}}^{(\mathrm{int,el})}_{\mathrm{nuc}}\, | \Psi_s \rangle \equiv -4\pi\alpha\, \langle \Psi_p |\, \hat{S}_L \odot \boldsymbol{\nabla}\, \delta^{(3)}(\boldsymbol{x})\, | \Psi_s \rangle$$

3. We get

$$\hat{S}_L = \int d\boldsymbol{y}\, \hat{\rho}(\boldsymbol{y}) \sum_{k\geq 1} \frac{c_k}{c_1} \frac{y^{k+1}}{(k+1)(k+4)} \Big\{ \boldsymbol{y} - \frac{(k+4)}{3Z} \left(\boldsymbol{d}_{\mathrm{nuc}} - \frac{2(k+1)\sqrt{2}\,\pi}{(k+4)} [\boldsymbol{d}_{\mathrm{nuc}} \otimes Y_2(\hat{y})]^{\mathrm{sym}}_1 \right) \Big\}$$

4. Typical finite-size suppression factor $\langle y^2\rangle / \langle x^2 \rangle \sim \mathrm{fm}^2/a_0^2 \sim 10^{-9}$.

Schiff Moment: An Approximation

To the leading order $k = 1$

$$\langle \hat{\mathbf{S}}_L \rangle \approx \frac{1}{10} \left\{ \langle y^2 \mathbf{y} \rangle - \frac{5}{3Z} \left(\langle \boldsymbol{d}_{\rm nuc} \otimes y^2 \rangle - \frac{4\sqrt{2\pi}}{5} \langle \boldsymbol{d}_{\rm nuc} \otimes y^2 Y_2(\hat{y})]_1 \rangle \right) \right\}$$

Compared with the conventional form

$$\langle \hat{\mathbf{S}} \rangle \approx \frac{1}{10} \left\{ \langle y^2 \mathbf{y} \rangle - \frac{5}{3Z} \langle \boldsymbol{d}_{\rm nuc} \rangle \otimes \langle y^2 \rangle \right\}$$

two major differences are

1. The quadrupole term:
   Vanish if assuming a spherical charge density (not too different).
2. The matrix elements $\langle \boldsymbol{d}_{\rm nuc} \otimes y^2 \rangle$ vs. $\langle \boldsymbol{d}_{\rm nuc} \rangle \otimes \langle y^2 \rangle$:
   These are not equal in general since $\sum_n |\psi_n\rangle \langle \psi_n| \neq |\psi_0\rangle \langle \psi_0|$ is needed for the former evaluation (dramatically different!).

- For deuteron, $\hat{S}_L$ is purely one-body (c.m. factored out):
  Three terms contribute as $1 : -5/3 : -4/3$ (non-fac.) vs. $1 : -0.59 : -0.071$ (fac.).
- How about other nuclei?

What a Difference?!

Schiff theorem is a quantum-mechanical description of the screening dynamics, so the essential commutation relation

$$[\boldsymbol{d}_{\text{nuc}} \cdot \boldsymbol{\nabla}_x \, , \, C_J \, F(x)]$$

has to be realized on a operator level. Hence $\boldsymbol{d}_{\text{nuc}}$ should be taken as a q-number.

- $\boldsymbol{d}_{\text{nuc}}$ and C_J form a composite operator.
- Besides $[\boldsymbol{\nabla}_x \, , \, F(x)]$, one also has $[\boldsymbol{d}_{\text{nuc}} \, , \, C_J]$.
- Traditionally, $\boldsymbol{d}_{\text{nuc}}$ is taken as a c-number when deriving the Schiff theorem (not by Schiff himself), which naturally leads to

$$\langle \boldsymbol{d}_{\text{nuc}} \otimes C_J \rangle = \langle \boldsymbol{d}_{\text{nuc}} \rangle \otimes \langle C_J \rangle \quad [\boldsymbol{d}_{\text{nuc}} \, , \, C_J] = 0 \, .$$

- Traditional logic, but done it right: treating nucleus as an elementary particle (no internal excitation, frozen dynamics upon shielding), i.e. $\boldsymbol{d}_{\text{nuc}} = \langle \boldsymbol{d}_{\text{nuc}} \rangle \hat{I}$ and $C_J = \langle C_J \rangle \hat{I}_J$

$$\langle \hat{I} \otimes \hat{I}_J \rangle = \langle \hat{I} \rangle \otimes \langle \hat{I}_J \rangle \quad [\hat{I} \, , \, \hat{I}_{J=\text{even}}] = 0 \, .$$

How does the nucleus appear to electrons upon shielding?

Electron-Nucleus Magnetic Interaction

The residual interactions are

$$\hat{\mathcal{O}}^{(\mathrm{int,mag})}_{\mathrm{nuc}} = -\frac{4\pi\alpha}{Z\,x^3}\Bigg\{ Z\,[Y_2(\hat{x})\otimes\boldsymbol{\alpha}]_2 \odot \frac{1}{5}\,(M_2 + \mathcal{M}_2(x))$$

$$- [Y_2(\hat{x})\otimes\boldsymbol{\alpha}]_2 \odot \left[\boldsymbol{d}_{\mathrm{nuc}} \otimes \left(\sqrt{\frac{3}{10}}\,(M_1 + \mathcal{M}_{1>}(x)) + \sqrt{\frac{8}{15}}\,\mathcal{M}_{1<}(x)\right)\right]_2^{\mathrm{sym}}$$

$$+ [Y_1(\hat{x})\otimes\boldsymbol{\alpha}]_1 \odot \left[\boldsymbol{d}_{\mathrm{nuc}}\,,\, \frac{1}{3}\,(M_1 + \mathcal{M}_1(x))\right] \cdot (x\,\boldsymbol{\nabla}^{\mathrm{sym}})\Bigg\}$$

- The 1st line is the magnetic quadrupole M_2.
- The 2nd and 3rd line comes from $[\boldsymbol{d}_{\mathrm{nuc}} \cdot \boldsymbol{\nabla}_x\,,\, M_J\,F(x)]$; the former involves $[\boldsymbol{\nabla}_x\,,\, F(x)]$ and the second involve $[\boldsymbol{d}_{\mathrm{nuc}}\,,\, M_J]$. [2]
- Discarding the local moments $\mathcal{M}_J$'s, there are still long-range interactions which are not screened as in the electrostatic case.
- The suppression factor is of hyperfine splitting, on the order of $\alpha^2 m_e/m_N \sim 10^{-7}$. Thus, this would not be less important than the electrostatic case if M_2 is available.

[2] Schiff had the 3rd line for H ($I = 1/2$). If $\boldsymbol{d}_{\mathrm{nuc}}$ were a c-number, he would not have gotten this term.

In conclusion:

- The Schiff theorem is derived at the operator level in the most general fashion. The Schiff operator we got is different from existing literature. For a deuteron, the difference is huge, and check on nuclei of great interests like Hg, Xe, Ra ...etc. should be carried out.

Thank You!