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INSTITUTE OF NUCLEAR & PARTICLE PHYSICS

First Order Relativistic Three-Body Scattering

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Office of Nuclear Physics

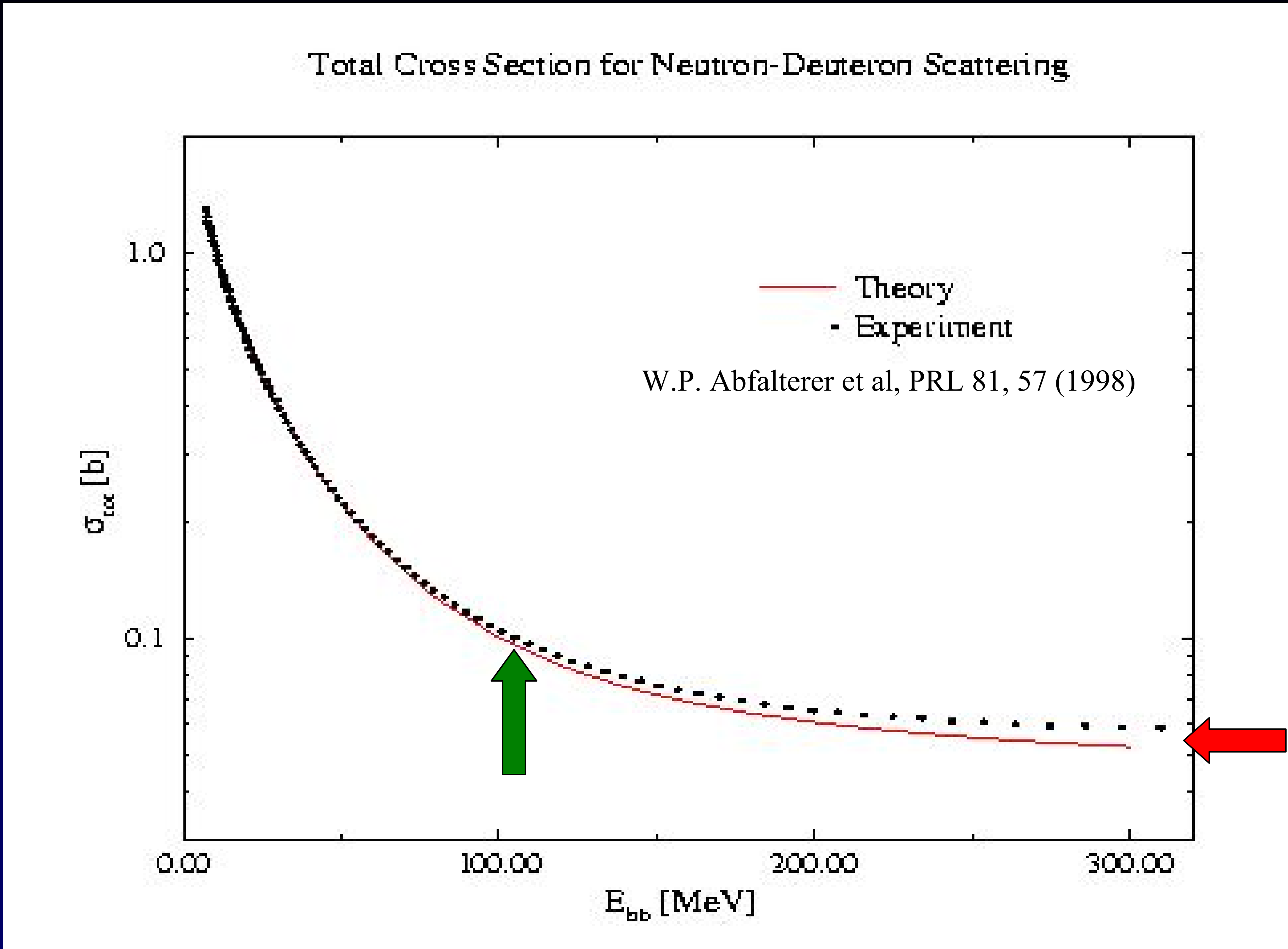
Exploring Nuclear Matter - Quarks to Stars

3 Nucleons: Binding Energy of ^3H

NN Model	E_t [MeV]
Nijm I	-7.73
Nijm II	-7.64
AV18	-7.65
CD-Bonn	-8.00
Experiment	-8.48

Discrepancy in E_t :

- **3NF**
- **Relativistic Effects**



Total Cross Section for Neutron-Deuteron Scattering
Theory
Experiment
W.P. Abfalterer et al, PRL 81, 57 (1998)
1.0
0.1
σ [b]
0.00
100.00
200.00
300.00
E [MeV]

Three Nucleon Observables:

- **Three-Body Forces:**
  - Needed to get the binding energies of ^{3}H and ^{3}He
  - General practice:
    - Model for 3N force (TM' and Urbana most common)
    - Adjust parameters to fit ^{3}H
  - Describe <u>bulk properties</u> (bound states & cross sections) Reasonably well
  - χPT: up to N^{4}LO - 2N & 3N forces consistent

- **Relativistic Effects:**
  - Bound state: Effect ~ 0.5 MeV & sign under debate
  - Scattering

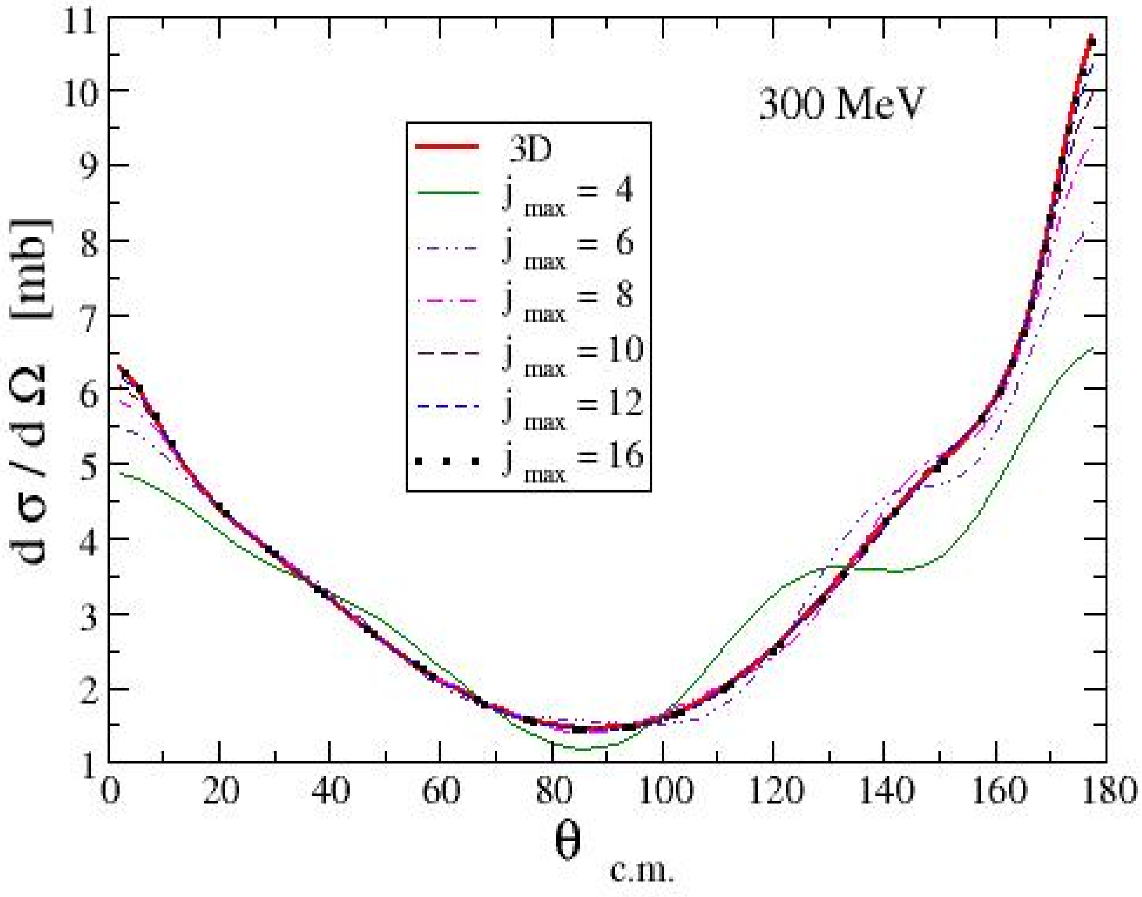
Relativistic Effects at higher energies: What is necessary?

3N and 4N systems:

- standard treatment based on pw projected momentum space successful (3N scattering up to ≈250 MeV) but rather tedious
- 2N: j_{max}=5,   3N: J_{max}=25/2 → 200 `channels'
- Computational maximum today:
- 2N: j_{max}=7,   3N: J_{max}=31/2

⇒ Solution: NO partial waves

Example: NN scattering



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Elab= 300 MeV:

J=16 needed for convergence in cross section

Smarter: **NO** partial waves

Roadmap for 3N problem without PW

Scalar NN model | Realistic NN Model

Scalar NN model

- NN scattering + bound state
- 3N bound state
- 3N bound state + 3NF
- 3N scattering:
- Full Faddeev Calculation
  - Elastic scattering
  - Below and above break-up
  - Break-up

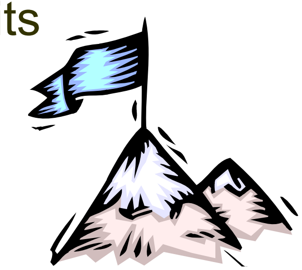
- **Relativistic Calculation**
  - **First Order in t**

Realistic NN Model

- NN scattering + deuteron
  - Potentials AV18 and Bonn-B
- Break-up in first order:
  - (p,n) charge exchange
  - Max. Energy 500 MeV
  - Relativistic kinematics

- Full Faddeev Calculation
  - NN interactions
  - High energy limits

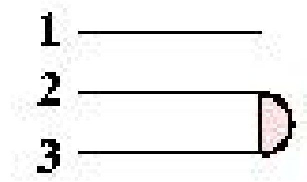
⋮



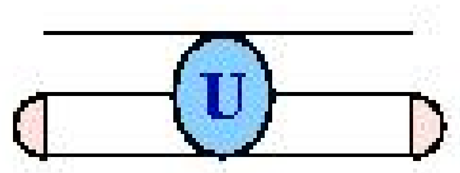
Three-Body Scattering - General

- Initial channel state

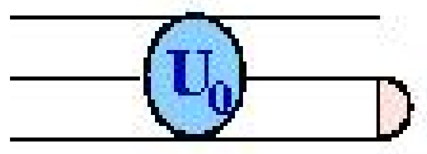
$$\left|\vec{q}_0\varphi_d\right\rangle \equiv$$



- Transition operators
  - elastic scattering



  - breakup



Three-Body Scattering - General

- Transition operator for elastic scattering

$$U = PG_0^{-1} + P\underbrace{tG_0U}$$

- Transition amplitude

$$\boxed{T = tP + tG_0PT}$$

Faddeev Eq.

- Break-up operator

$$U_0 = (1+P)\, tG_0U$$
$$= (1+P)\, T$$

- Here: Consider Spinless Interaction

Faddeev Equation for 3N Scattering

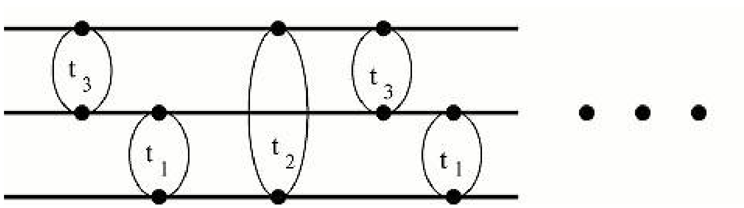
$$T = tP + tG_0 PT$$

NN t-matrix

Free 3N Propagator
Nasty Singularity Structure:
"Moving Singularities"

- Multiple Scattering Series:

$$T = tP + tG_0 PtP + \cdots$$



1^{st} Order in tP

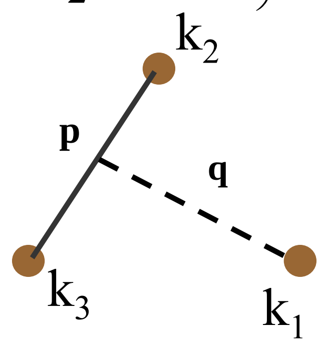
$P = P_{12} P_{23} + P_{13} P_{23} \equiv$ Permutation Operator

3-Body Transition Amplitude (NR)

$$T\left|\mathrm{q}_0\varphi_d\right\rangle = tP\left|\mathrm{q}_0\varphi_d\right\rangle + tG_0PT\left|\mathrm{q}_0\varphi_d\right\rangle$$

$$\mathbf{p} = \frac{1}{2}(\mathbf{k}_2 - \mathbf{k}_3)$$

$$\mathbf{q} = \frac{2}{3}\left(\mathbf{k}_1 - \frac{1}{2}(\mathbf{k}_2 + \mathbf{k}_3)\right)$$



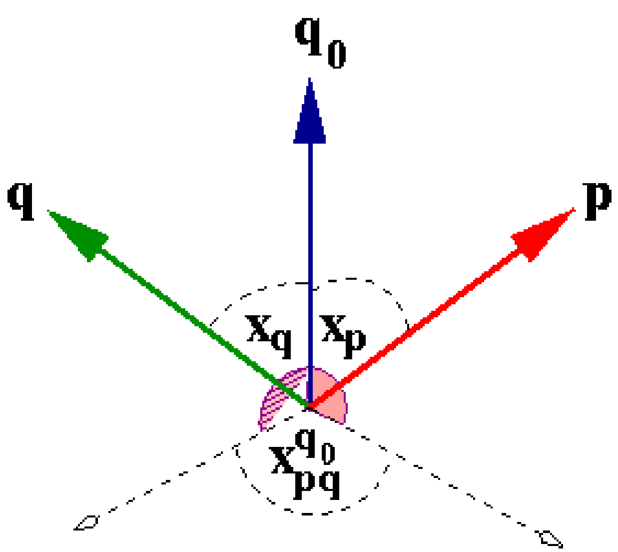
The Faddeev Equation in momentum space by using Jacobi Variables

$$\left\langle \mathrm{pq}\left|\hat{T}\right|\mathrm{q}_0\varphi_d\right\rangle = \varphi_d\left(\mathrm{q}+\tfrac{1}{2}\mathrm{q}_0\right)\hat{t}_s\left(\mathrm{p},\tfrac{1}{2}\mathrm{q}+\mathrm{q}_0, E-\tfrac{3}{4m}q^2\right)$$
$$+\int d^3\mathrm{q}''\frac{\hat{t}_s\left(\mathrm{p},\tfrac{1}{2}\mathrm{q}+\mathrm{q}'', E-\tfrac{3}{4m}q^2\right)}{E-\tfrac{1}{m}\left(q^2+q''^2+\mathrm{q}\cdot\mathrm{q}''\right)+i\varepsilon}\frac{\left\langle \mathrm{q}+\tfrac{1}{2}\mathrm{q}'',\mathrm{q}''\left|\hat{T}\right|\mathrm{q}_0\varphi_d\right\rangle}{E-\tfrac{3}{4m}q''^2-E_d+i\varepsilon}$$

$\hat{t}_s \equiv$ symmetrized 2-body t-matrix

Variables for 3D Calculation

3 distinct vectors in the problem: $\mathbf{q}_0$ $\mathbf{q}$ $\mathbf{p}$



5 independent variables:

$$p = |\mathrm{p}| \ , \ q = |\mathrm{q}|$$

$$x_p = \hat{\mathrm{p}} \cdot \hat{\mathrm{q}}_0 \ , x_q = \hat{\mathrm{q}} \cdot \hat{\mathrm{q}}_0$$

$$x_{pq}^{q_0} = (\mathrm{q}_0 \times \mathrm{q}) \cdot (\mathrm{q}_0 \times \mathrm{p})$$

q system : **z || q**

q_0 system : **z || $\mathbf{q}_0$**

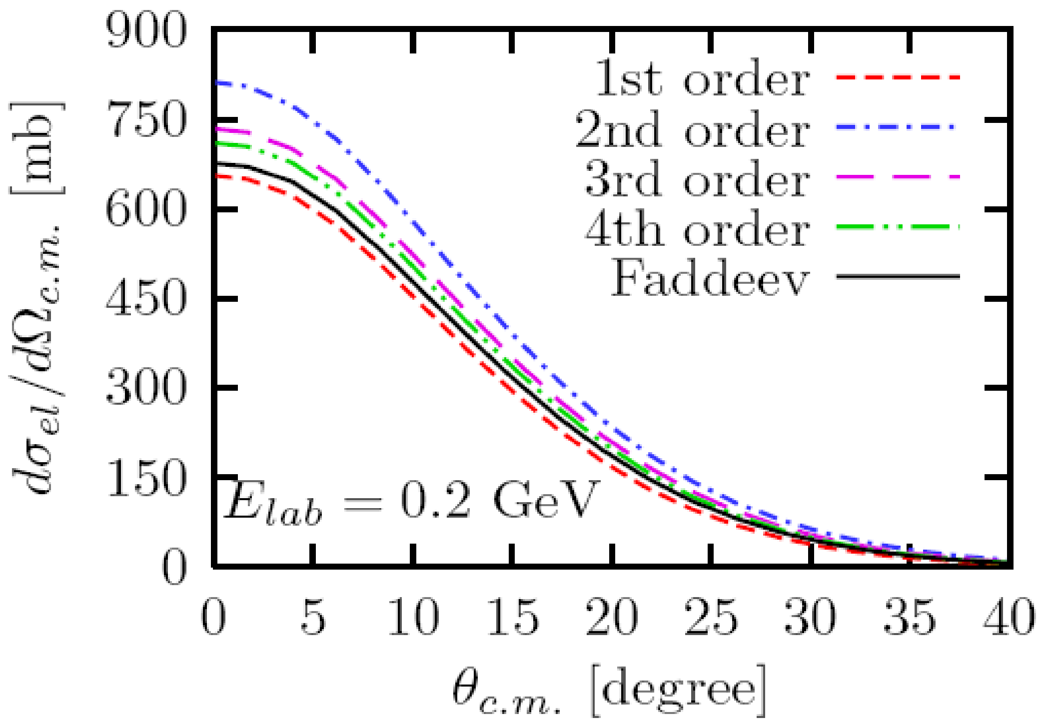
Variables invariant under rotation:

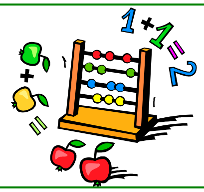
freedom to choose coordinate system for numerical calculation

Elastic Scattering:

(nonrelativistic)

$$T|\phi\rangle = \underbrace{tP|\phi\rangle}_{\text{1st order}} + \underbrace{tG_0PT|\phi\rangle}_{\text{rescattering}}$$

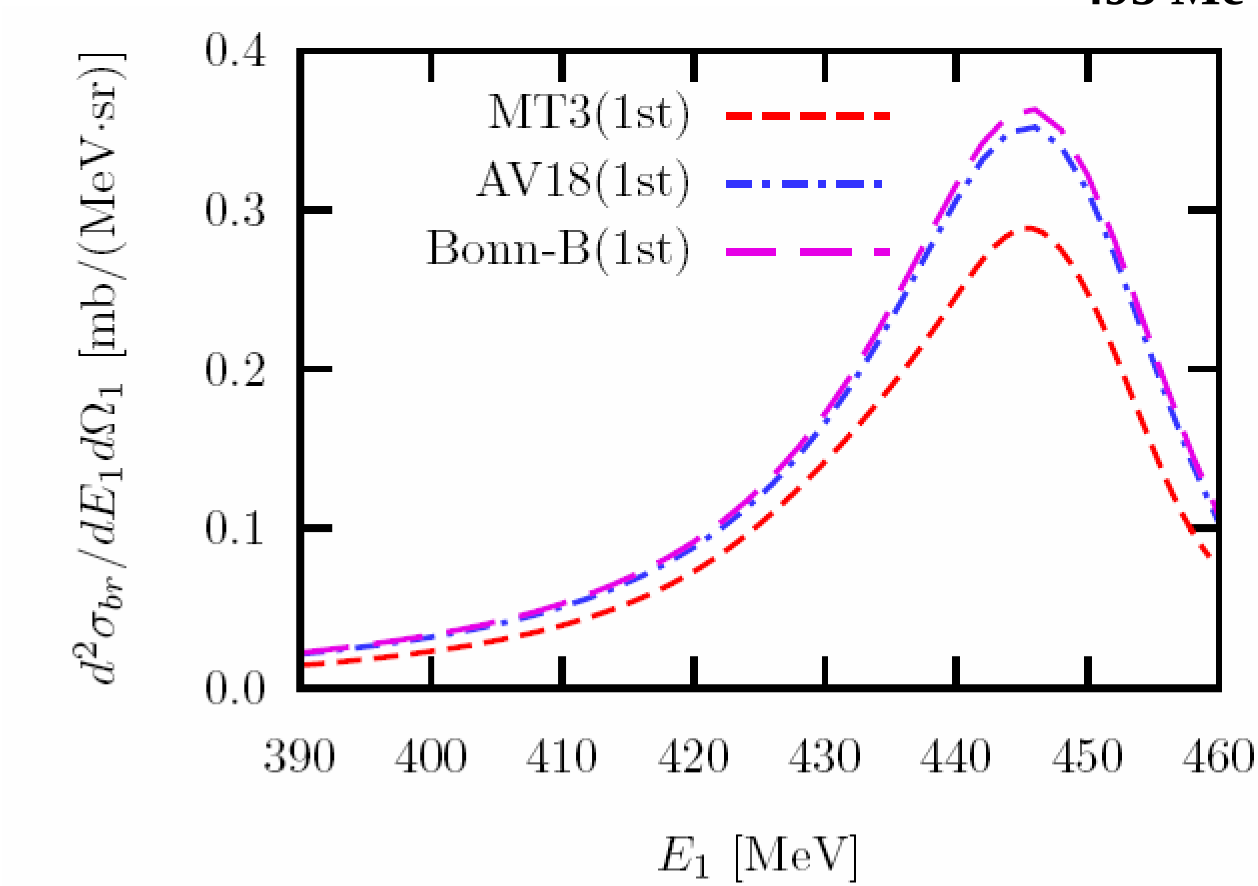




All calculations use a Malfliet-Tjon type potential

Comparison with Realistic NN Potential in first order (NR)

495 MeV, θ=18º



Relativistic Faddeev Calculations

- Context: Poincarė Invariant Quantum Mechanics
  - Poincarė invariance is exact symmetry, realized by a unitary representation of the Poincarė group on a few-particle Hilbert space
  - Instant form
  - Faddeev equations same operator form but different ingredients
- **Kinematics**
  - Lorentz transformations between frames
- **Dynamics**
  - Bakamjian-Thomas Scheme: Mass Operator $M=M_0+V$
  - Interaction embedded in 3-body space

$$V \equiv \sqrt{M^2+q^2} - \sqrt{M_0^2+q^2}$$

Relativistic Kinematics: Phase Space Factors

$$\sigma_{el} = (2\pi)^4 \int d\Omega \frac{E_n^2(q_0) E_d^2(q_0)}{W} \left| \langle \varphi_d \hat{q} q_0 | U | \varphi_d q_0 \rangle \right|^2$$

NR: $(2m/3)^2$

$$W = \sqrt{4(m^2 + p^2) + q^2} + \sqrt{m^2 + q^2} \quad \equiv \text{Invariant Mass}$$

$$\sigma_{br} = \frac{(2\pi)^4}{3} \frac{E_n(q_0) E_d(q_0)}{q_0 W} \int d\Omega_p d\Omega_q dq \frac{p_u q^2}{4} \sqrt{4(m^2 + p_u^2) + q^2} \left| \langle \phi_0 | U_0 | \varphi_d q_0 \rangle \right|^2$$

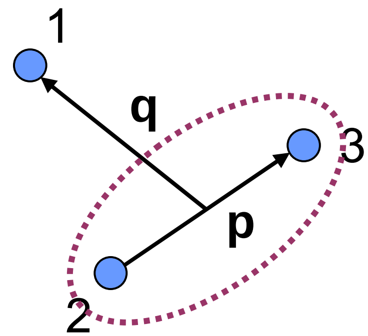
$$|p_u| = 1/2 \sqrt{W^2 - 3m^2 - 2W\sqrt{m^2 + q^2}}$$

$$\sigma_{br}^{NR} = \frac{(2\pi)^4}{3} \frac{m^2}{3q_0} \int d\Omega_p d\Omega_q dq q^2 \sqrt{mE_{cm} - \tfrac{3}{4} q^2} \left| \langle \phi_0 | U_0 | \varphi_d q_0 \rangle \right|^2$$

Kinematics: Poincaré-Jacobi momenta

- **Nonrelativistic (Galilei)**

$$\mathrm{p} = \frac{1}{2}(\mathrm{k}_2 - \mathrm{k}_3)$$
$$\mathrm{q} = \frac{2}{3}(\mathrm{k}_1 - \frac{1}{2}(\mathrm{k}_2 + \mathrm{k}_3))$$



- Relativistic (Lorentz)

$$\mathrm{p} = \frac{1}{2}(\mathrm{k}_2 - \mathrm{k}_3) + \frac{\mathrm{k}_2 + \mathrm{k}_3}{2m_{23}}\left(\frac{(\mathrm{k}_2 - \mathrm{k}_3)\cdot(\mathrm{k}_2 + \mathrm{k}_3)}{(E_2 + E_3) + m_{23}} - (E_2 - E_3)\right)$$
$$\mathrm{q} = \mathrm{k}_1 + \frac{\mathrm{K}}{M}\left(\frac{\mathrm{k}_1 \cdot \mathrm{K}}{E + M} - E_1\right)$$

$$E = E_1 + E_2 + E_3$$
$$\mathrm{K} = \mathrm{k}_1 + \mathrm{k}_2 + \mathrm{k}_3$$
$$M = \sqrt{E^2 - \mathrm{K}^2}$$
$$m_{23} = \sqrt{(E_2 + E_3)^2 - (\mathrm{k}_2 + \mathrm{k}_3)^2}$$

Kinematics: Poincaré-Jacobi Coordinates

3N c.m. frame: k_1, k_2, k_3 with $k_1 + k_2 + k_3 = K = 0$

Poincarė-Jacobi Coordinates:

$$q = k_1$$

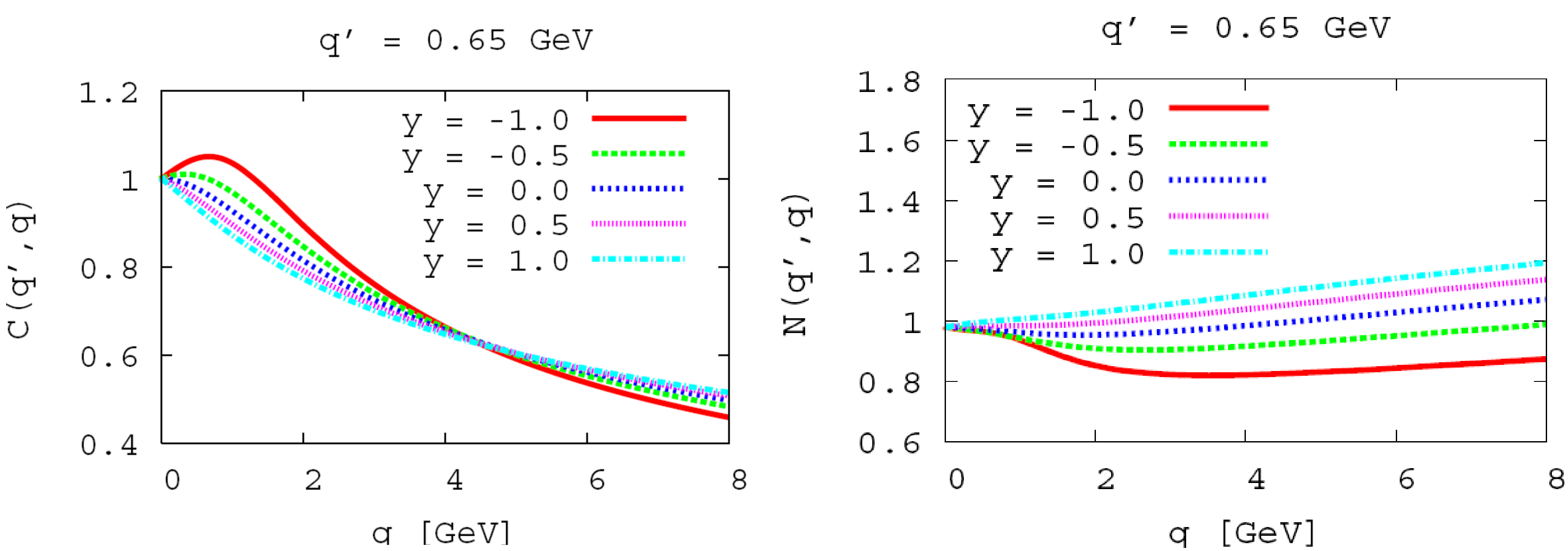
$$p = \frac{1}{2}(k_2 - k_3) - \frac{1}{2}(k_2 + k_3)\left(\frac{E_2 - E_3}{E_2 + E_3 + \sqrt{(E_2 + E_3)^2 - (k_2 + k_3)^2}}\right)$$

$$|k_1 k_2 k_3\rangle = \left|\frac{\partial(Kpq)}{\partial(k_2 k_3)}\right|^{1/2} |Kpq\rangle = \frac{E(p)[E(k_2) + E(k_3)]}{2E(k_2)E(k_3)} |Kpq\rangle$$

- All expressions related to permutations much more complicated
- Depend on vector variables => angle dependent

Permutation Operator: $P=P_{12}P_{23}+P_{13}P_{23}$

$$
\begin{aligned}
{}_1\langle \mathbf{p}'\mathbf{q}'|P|\mathbf{p}''\mathbf{q}''\rangle_1 &= {}_1\langle \mathbf{p}'\mathbf{q}'|\mathbf{p}''\mathbf{q}''\rangle_2 + {}_1\langle \mathbf{p}'\mathbf{q}'|\mathbf{p}''\mathbf{q}''\rangle_3 \\
&= \hat{N}(\mathbf{q}',\mathbf{q}'')\left[\delta\left(\mathbf{p}'-\mathbf{q}''-\frac{1}{2}\mathbf{q}'C(\mathbf{q}'',\mathbf{q}')\right)\delta\left(\mathbf{p}''+\mathbf{q}'+\frac{1}{2}\mathbf{q}''C(\mathbf{q}',\mathbf{q}'')\right)\right. \\
&\quad \left. +\delta\left(\mathbf{p}'+\mathbf{q}''+\frac{1}{2}\mathbf{q}'C(\mathbf{q}'',\mathbf{q}')\right)\delta\left(\mathbf{p}''-\mathbf{q}'-\frac{1}{2}\mathbf{q}''C(\mathbf{q}',\mathbf{q}'')\right)\right]
\end{aligned}
$$



Permutation Operator: $P = P_{12}P_{23} + P_{13}P_{23}$
Explicit expressions:

$$\hat{N}(\mathbf{q}', \mathbf{q}'') = \frac{(E(\mathbf{q}') + E(\mathbf{q}' + \mathbf{q}''))(E(\mathbf{q}'') + E(\mathbf{q}' + \mathbf{q}''))}{4E^2(\mathbf{q}' + \mathbf{q}'')} \times \frac{\sqrt{(E(\mathbf{q}') + E(\mathbf{q}' + \mathbf{q}''))^2 - \mathbf{q}''^2}\sqrt{(E(\mathbf{q}'') + E(\mathbf{q}' + \mathbf{q}''))^2 - \mathbf{q}'^2}}{4E(\mathbf{q}')E(\mathbf{q}'')}$$

$$C(\mathbf{q}', \mathbf{q}'') = 1 + \frac{E(\mathbf{q}') - E(\mathbf{q}' + \mathbf{q}'')}{E(\mathbf{q}'') + E(\mathbf{q}' + \mathbf{q}'') + \sqrt{(E(\mathbf{q}') + E(\mathbf{q}' + \mathbf{q}''))2 - \mathbf{q}''^2}}.$$

2

Kinematic Relativistic Effects:

- Lorentz transformation Lab $\rightarrow$ c.m. frame (3-body)
- Phase space factors in cross sections
- Poincarė-Jacobi momenta
- Permutations

Quantum Mechanics

Galilei Invariant: $$H = \frac{\mathrm{K}^2}{2M_g} + h \quad ; \quad h = h_0 + V_{NR}$$

Poincaré Invariant: $$H = \sqrt{\mathrm{K}^2 + M^2} \quad ; \quad M = M_0 + V_{12} + V_{23} + V_{31}$$

$$V_{ij} = M_{ij} - M_0 = \sqrt{\left(m_{0,ij} + v_{ij}\right)^2 + q_k^2} - \sqrt{m_{0,ij}^2 + q_k^2}$$

Two-body interaction embedded in the 3-particle Hilbert space

$$m_{0,ij} = \sqrt{m_i^2 + p_{ij}^2} + \sqrt{m_j^2 + p_{ij}^2}$$

$$M_0 = \sqrt{m_{0,ij}^2 + q_k^2} + \sqrt{m_k^2 + q_k^2}$$

Two-Body Input: T_1-operator embedded in 3-body system

$$T_1(\mathrm{p}',\mathrm{p};\mathrm{q}) = V(\mathrm{p}',\mathrm{p};\mathrm{q}) + \int d^3k'' \frac{V(\mathrm{p}',\mathrm{k}'';\mathrm{q})\, T_1(\mathrm{k}'',\mathrm{p};\mathrm{q})}{\sqrt{(2E(p'))^2+q^2} - \sqrt{(2E(k''))^2+q^2} + i\varepsilon}$$

Potential:

$$V = \sqrt{(2\sqrt{m^2+\mathbf{p}^2}+v)^2+\mathbf{q}^2} - \sqrt{4(m^2+\mathbf{p}^2)+\mathbf{q}^2}$$

→ 2-body potential in c.m. frame

Hard to compute !

Attempt via spectral expansion in

Kamada, Glöckle, Golak, Elster, PRC66 044010 (2002).

Comment: *works – BUT not well enough*

New Suggestions Kamada-Glöckle nucl-th/0703010:

Solve for V numerically via iteration --- not tested in 3N calculation

Instead:

- Obtain fully off-shell matrix elements $T_1(k,k',W)$ from half shell transition matrix elements by
- Solving a 1st resolvent type equation

$$T_1(W) = T_1(W') + T_1(W)\,[g_0(W) - g_0(W')]\,T_1(W')$$

- For every single off-shell momentum point
- Proposed in
  - Keister & Polyzou, PRC 73, 014005 (2006)
- Carried out for the first time now

Obtain embedded 2N t-matrix T_1(k,k',W):

$$\begin{aligned}\langle \mathbf{k}|T_1(\mathbf{q};z')|\mathbf{k}'\rangle &= \langle \mathbf{k}|V(\mathbf{q})|\mathbf{p}'^{(-)}\rangle \\ &= \frac{2(E_{k'}+E_k)}{\sqrt{4E_{k'}^2+\mathbf{q}^2}+\sqrt{4E_k^2+\mathbf{q}^2}}\,t(\mathbf{k},\mathbf{k}';2E_{k'})\end{aligned}$$

$$t(\mathbf{k},\mathbf{k}';2E_{k'}) = v(\mathbf{k},\mathbf{k}') + \int d\mathbf{k}''\,\frac{v(\mathbf{k},\mathbf{k}'')t(\mathbf{k}'',\mathbf{k}';2E_{k'})}{E_{k'}-2\sqrt{m^2+k''^2}+i\epsilon}$$

Solution of the relativistic 2N LS equation with 2-body potential

→ Exact Boost

Explicit Equation for T_1

$$\langle \mathbf{k}|T_1(\mathbf{q};z)|\mathbf{k}'\rangle = \langle \mathbf{k}|T_1(\mathbf{q};z'|\mathbf{k}'\rangle - \int d\mathbf{k}'' \, \langle \mathbf{k}|T_1(\mathbf{q};z)|\mathbf{k}''\rangle\Big(\frac{1}{z-\sqrt{4(m^2+\mathbf{k}''^2)+\mathbf{q}^2}} - \frac{1}{z'-\sqrt{4(m^2+\mathbf{k}''^2)+\mathbf{q}^2}}\Big)\langle \mathbf{k}''|T_1(\mathbf{q};z')|\mathbf{k}'\rangle$$

where $T_1(z')$ is taken to be right half-shell with $z' = \sqrt{4(m^2+\mathbf{k}'^2)+\mathbf{q}^2} + i\epsilon$

Approximations to the "boosted" potential

$$V = \sqrt{(2\sqrt{m^2 + \mathbf{p}^2} + v)^2 + \mathbf{q}^2} - \sqrt{4(m^2 + \mathbf{p}^2) + \mathbf{q}^2}$$

relativistic interaction in the c.m. frame

$$\begin{aligned} V_0(\mathbf{p}, \mathbf{p}', \mathbf{q}) &= v(\mathbf{p}, \mathbf{p}') \\ V_1(\mathbf{p}, \mathbf{p}', \mathbf{q}) &= v(\mathbf{p}, \mathbf{p}') \left[1 - \frac{\mathbf{q}^2}{8m^2}\right] \\ V_2(\mathbf{p}, \mathbf{p}', \mathbf{q}) &= v(\mathbf{p}, \mathbf{p}') \left[1 - \frac{\mathbf{q}^2}{8E(\mathbf{p})E(\mathbf{p}')}\right] \end{aligned}$$

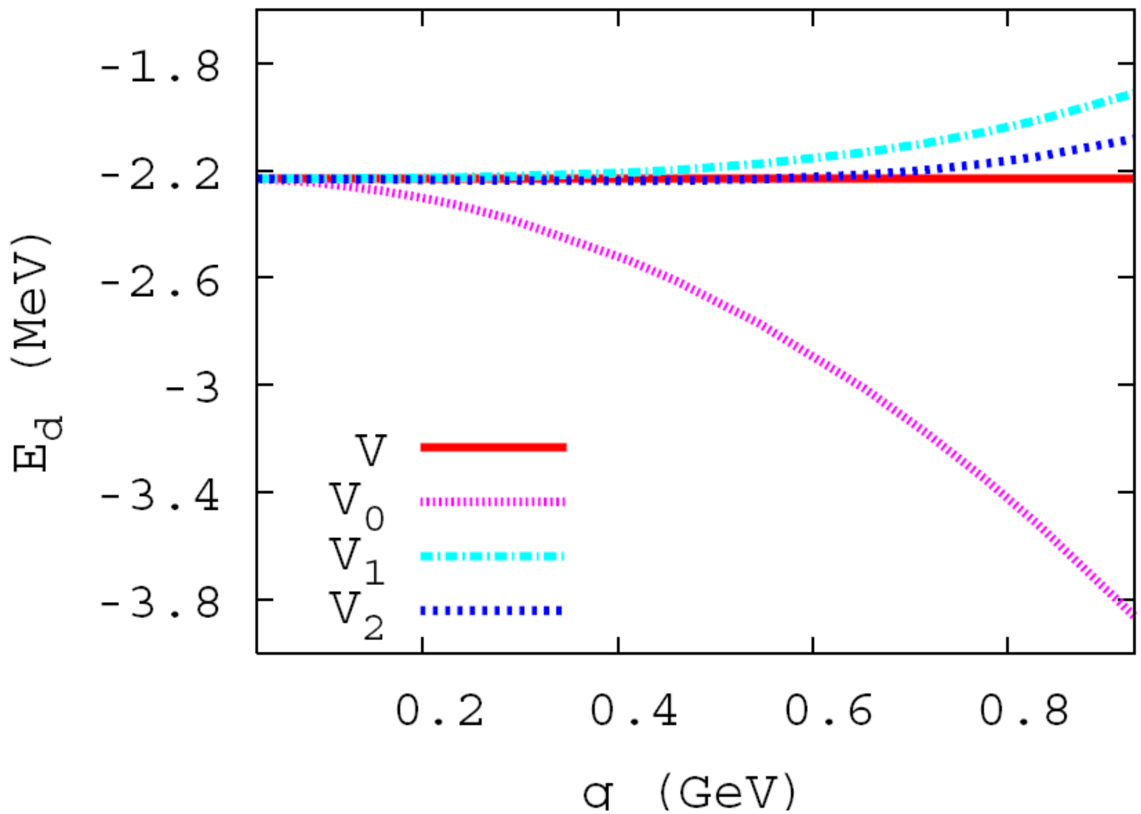
Remark to calculations:

The relativistic potential v(p,p') is phase-shift equivalent to the nonrelativistic potential

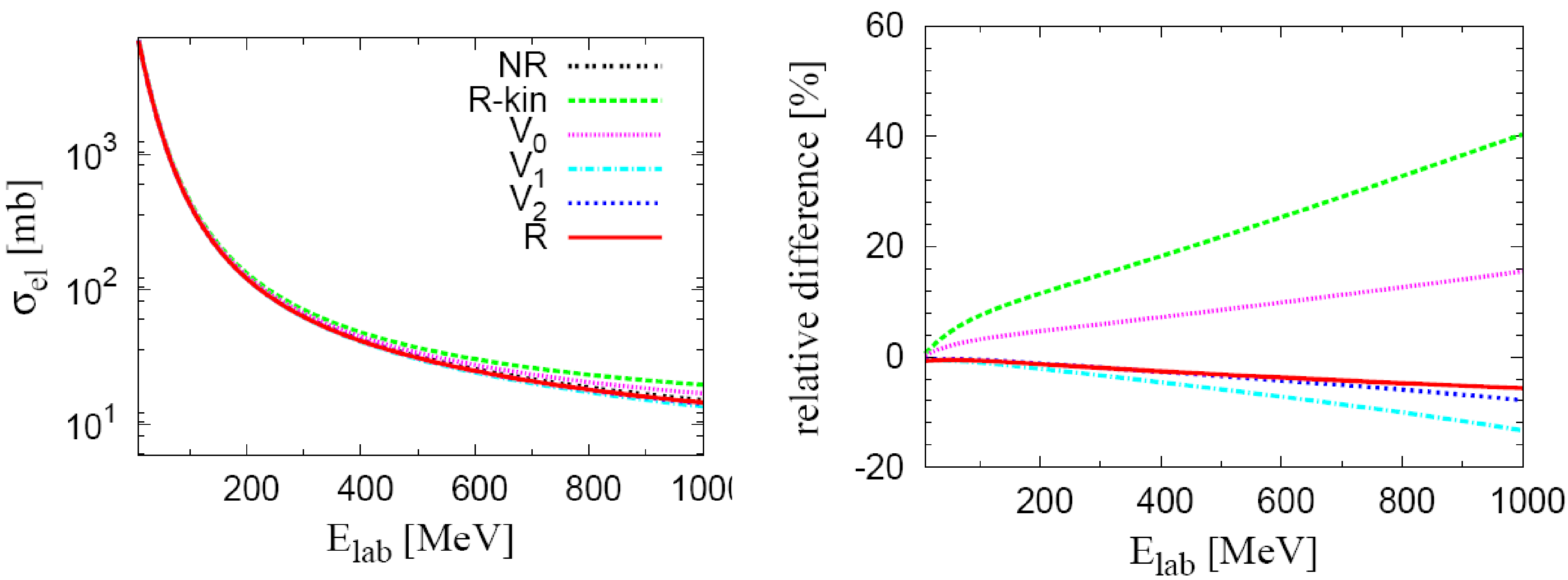
Details on this issue later!!

Deuteron Binding Energy

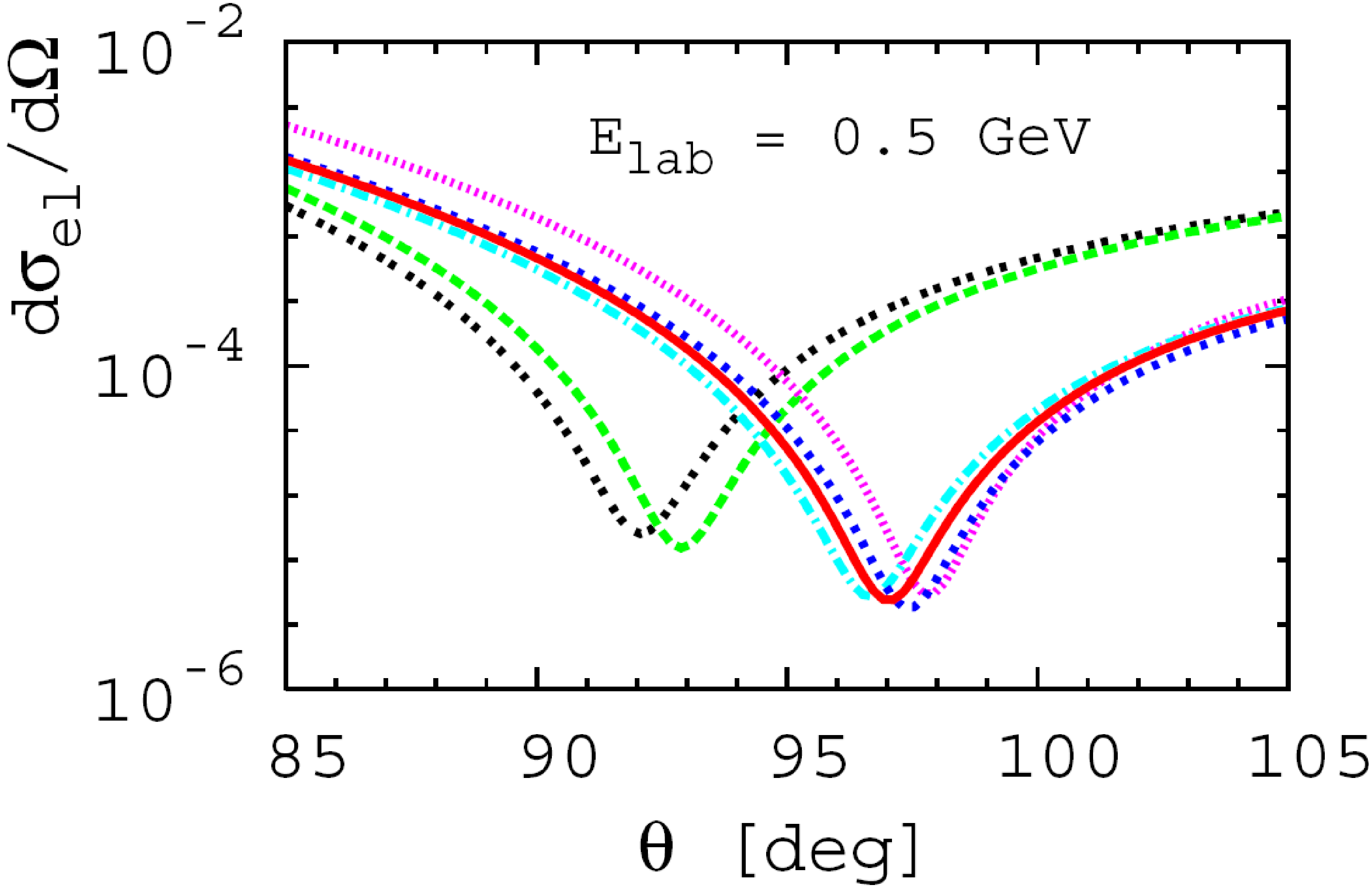
$$\Phi_d(\mathbf{k}) = \frac{1}{\sqrt{M_d^2 + \mathbf{q}^2} - \sqrt{2E_{k_m}^2 + \mathbf{q}^2}} \int d\mathbf{k}' \, V(\mathbf{k}, \mathbf{k}'; \mathbf{q}) \Phi_d(\mathbf{k}').$$



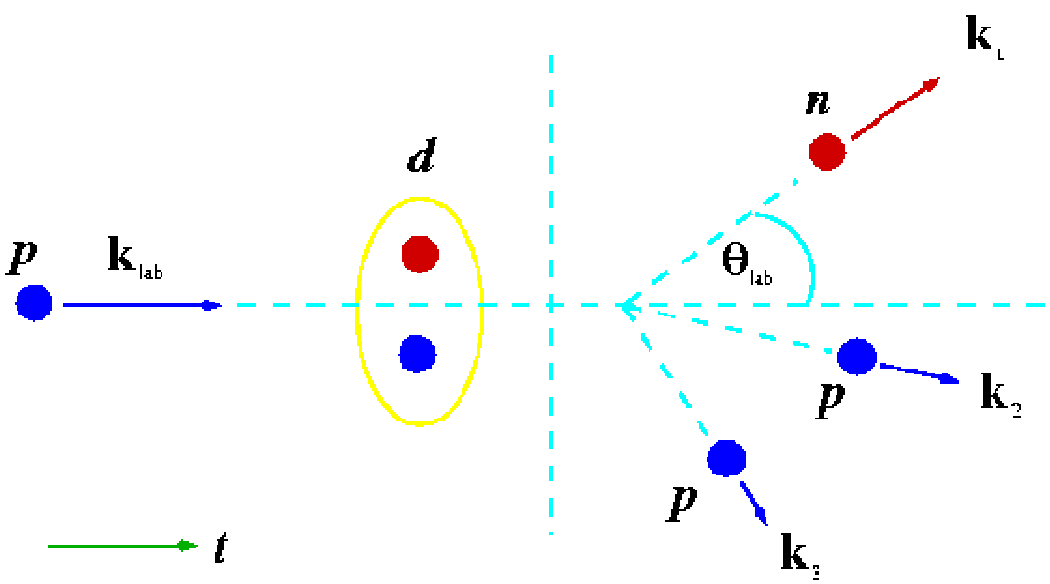
Total Cross Section for Elastic Scattering



Elastic Scattering: Differential Cross Section

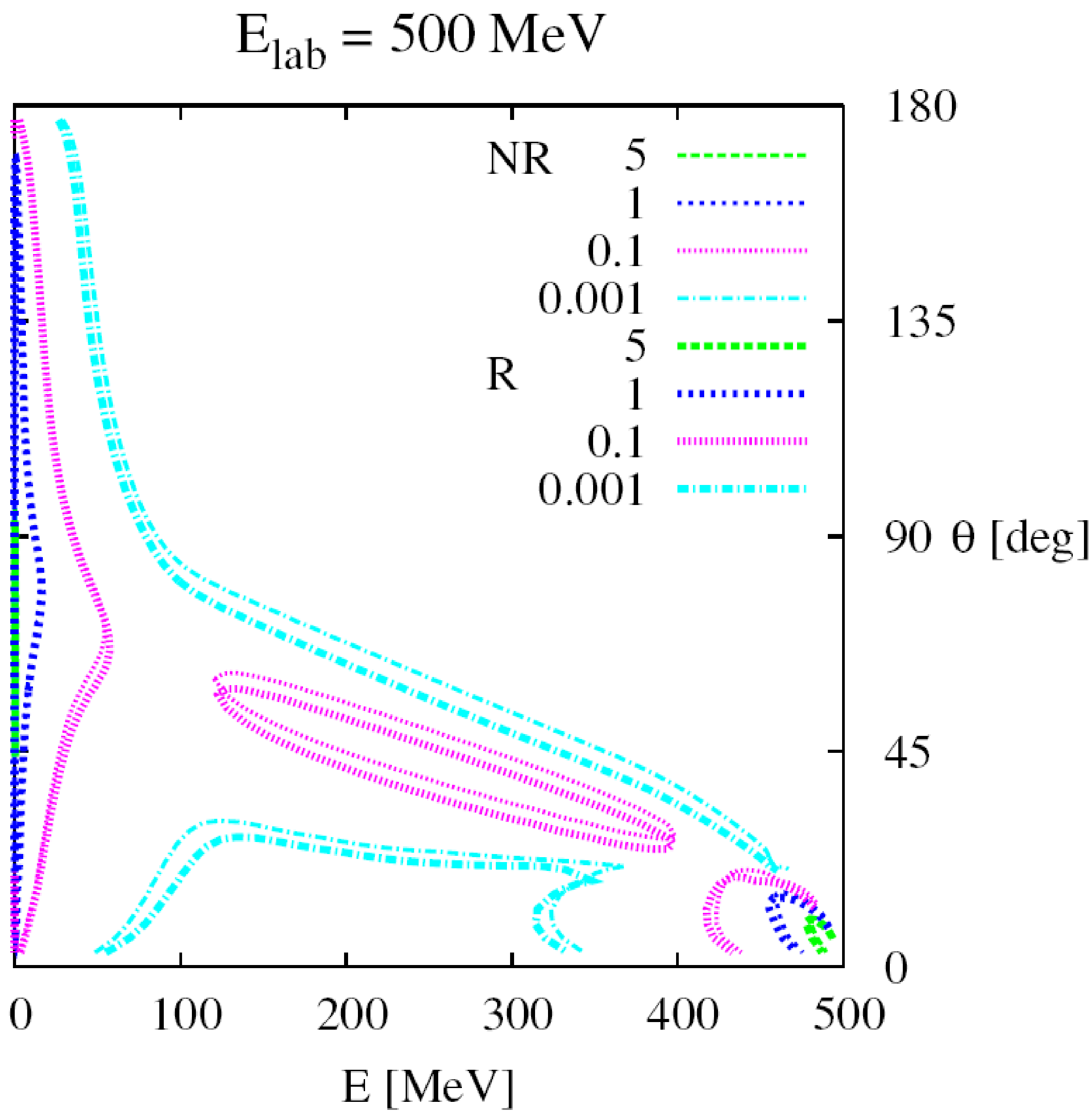


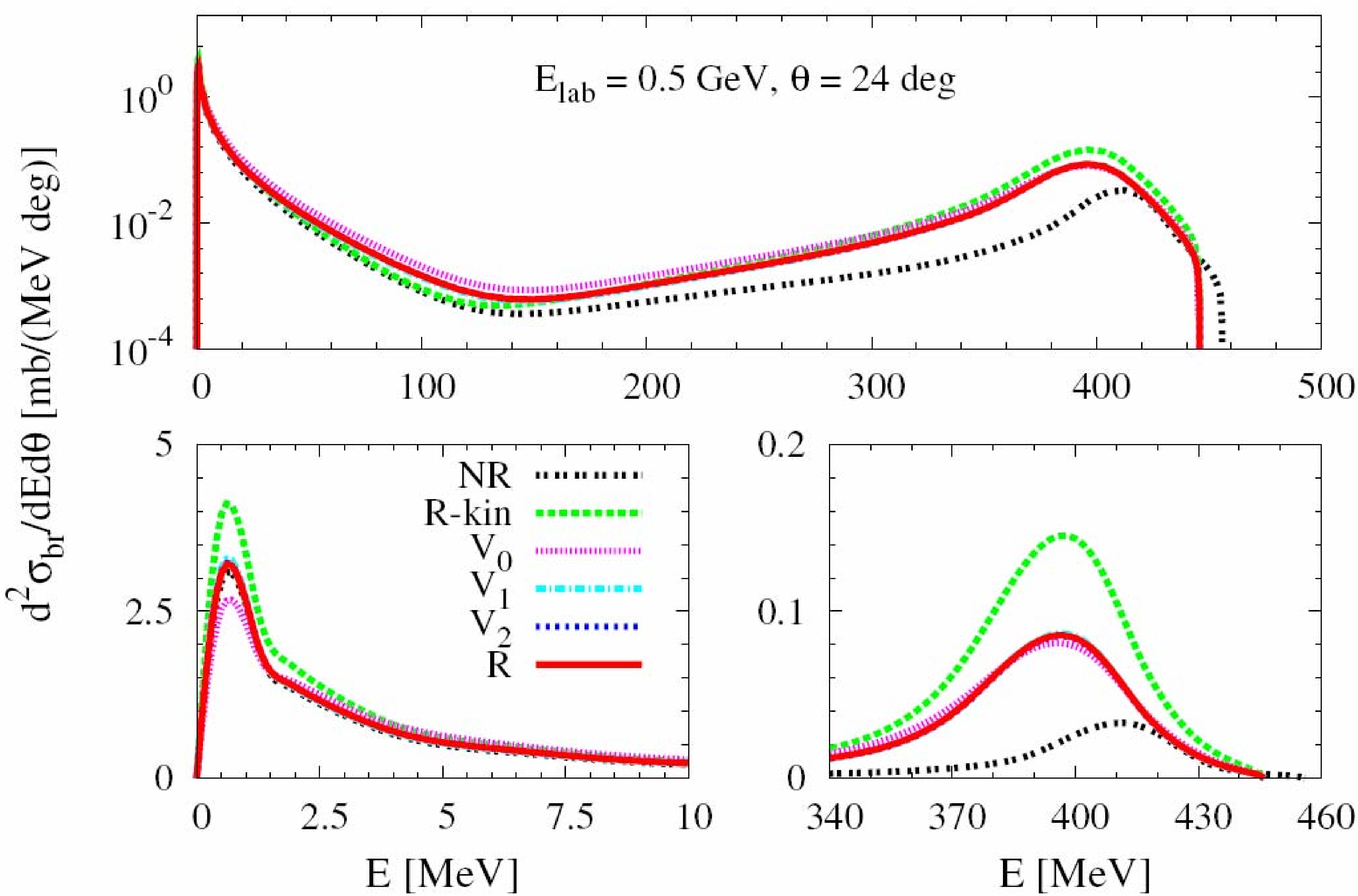
Inclusive Scattering



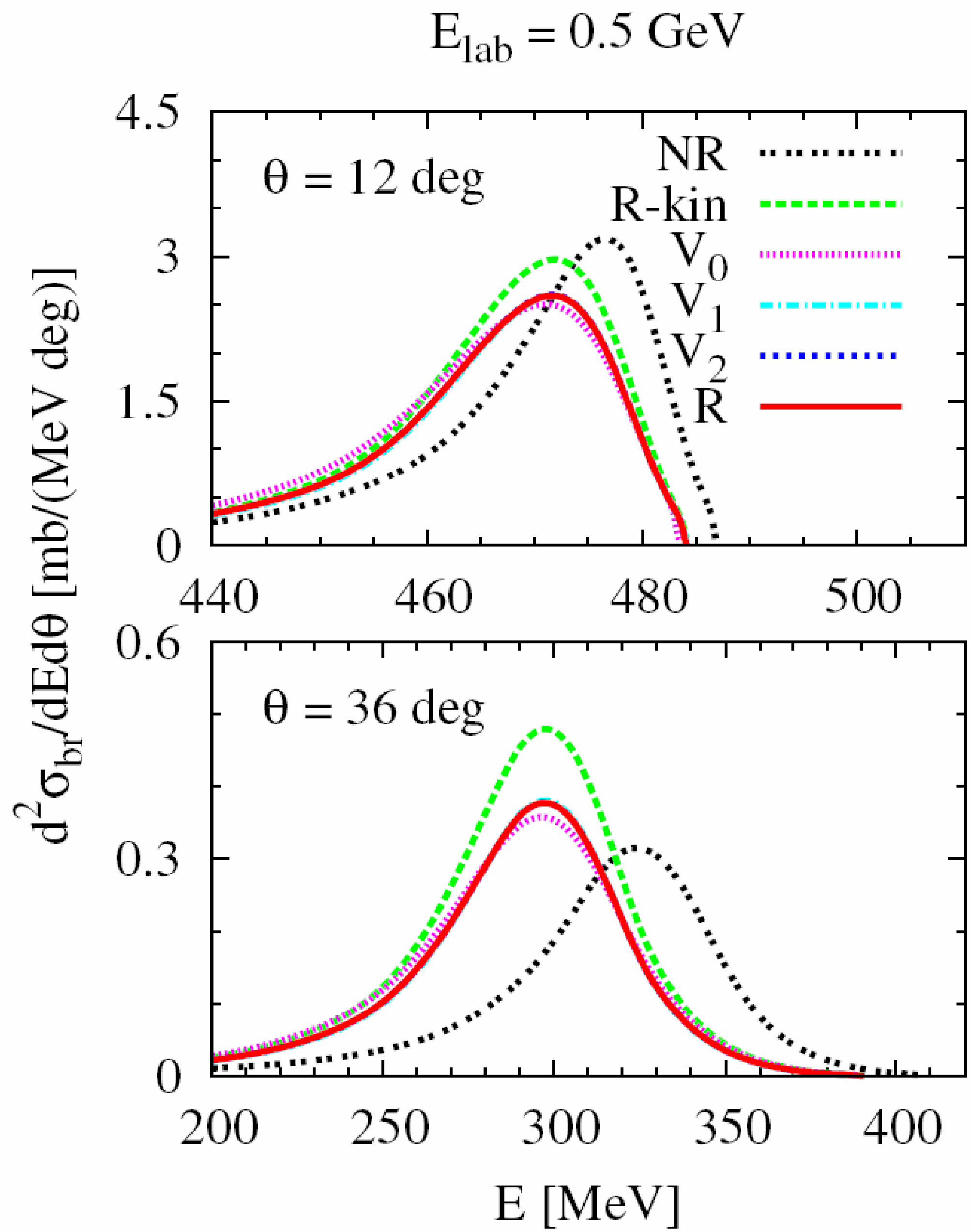
Measured: Energy of one ejected particle as function of angle

Inclusive Breakup Scattering

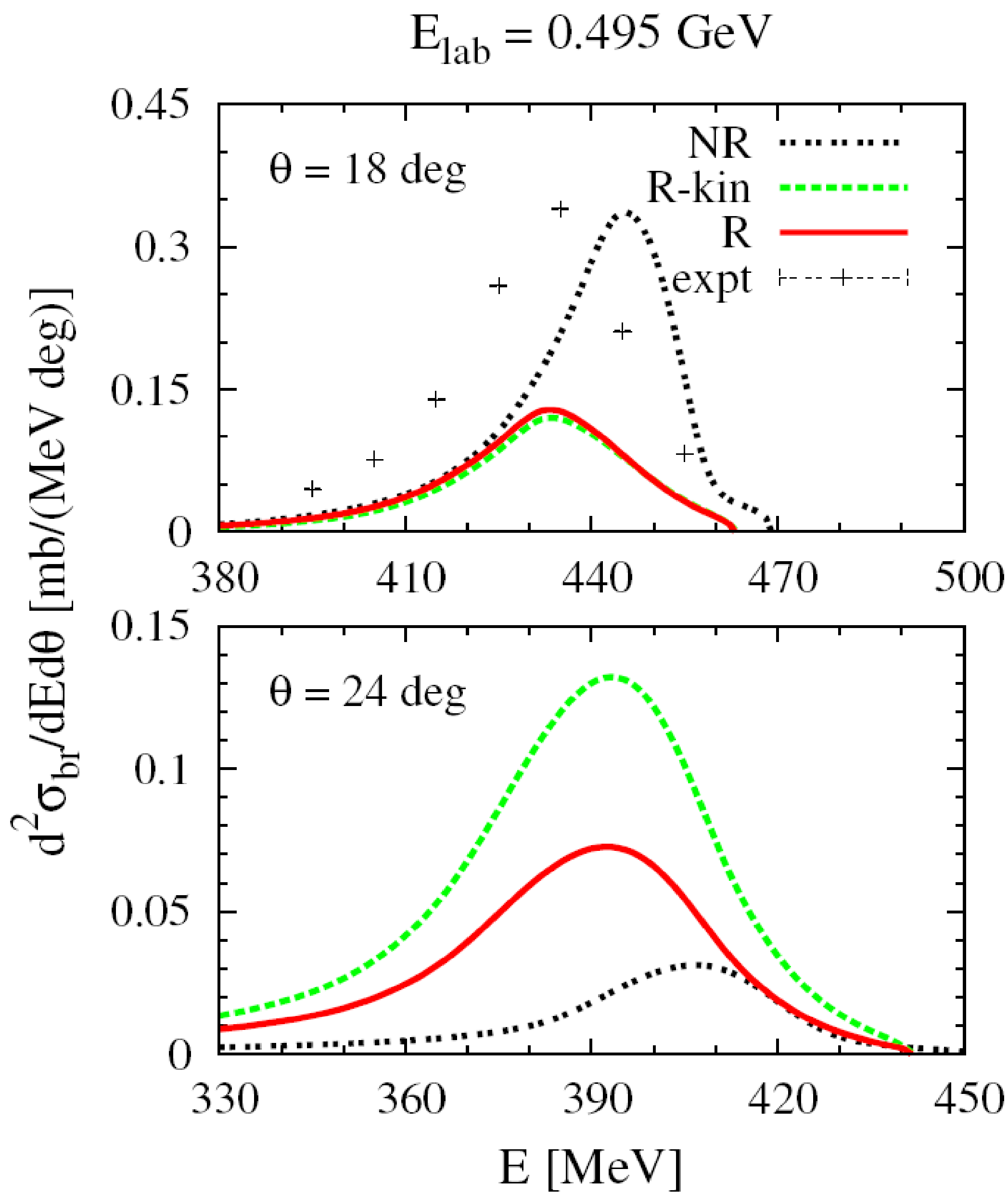




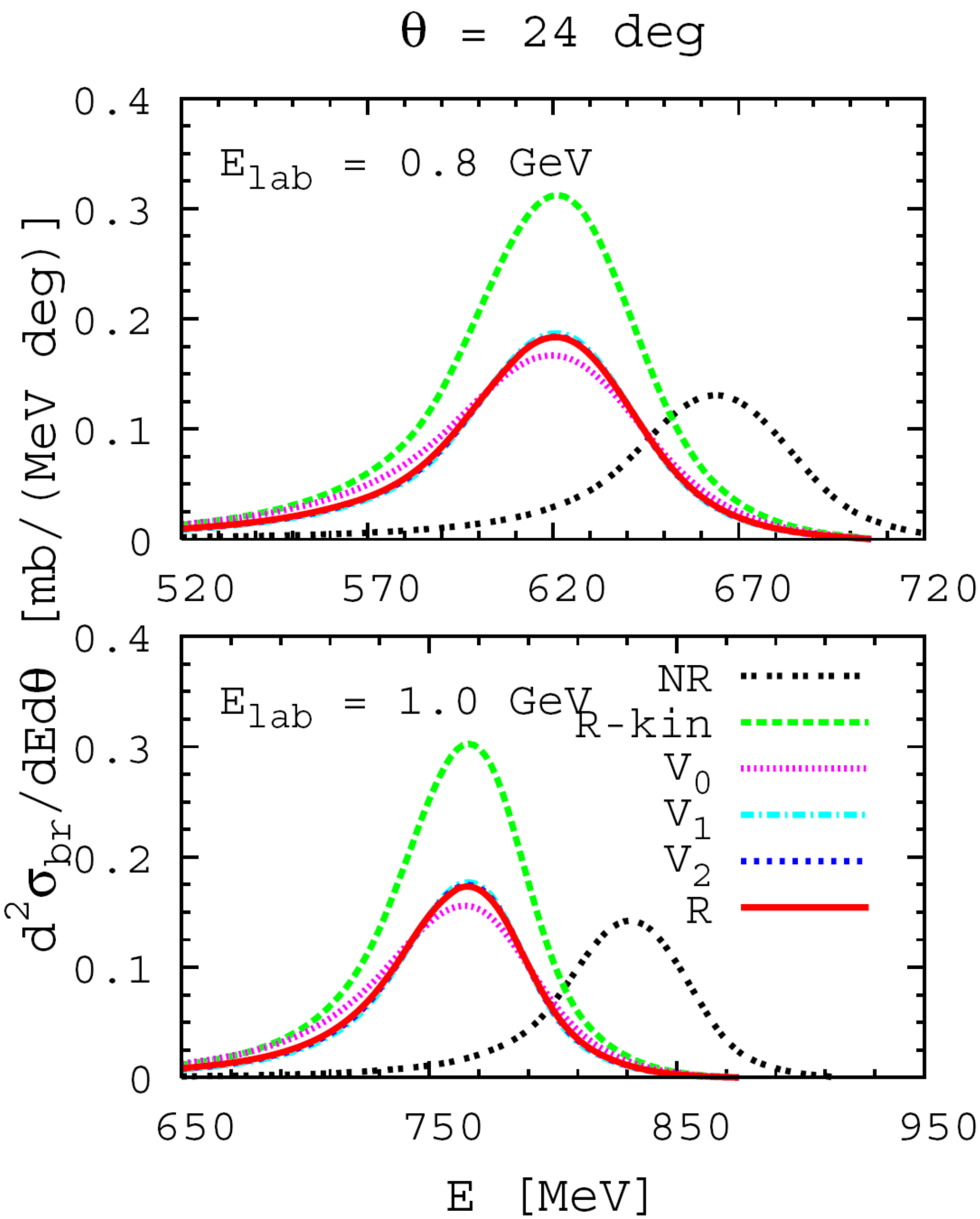
E_{lab} = 0.5 GeV, θ = 24 deg
$d^2\sigma_{br}/dEd\theta$ [mb/(MeV deg)]
NR
R-kin
V_0
V_1
V_2
R
E [MeV]
E [MeV]



Inclusive Breakup Scattering @ Elab=500 MeV

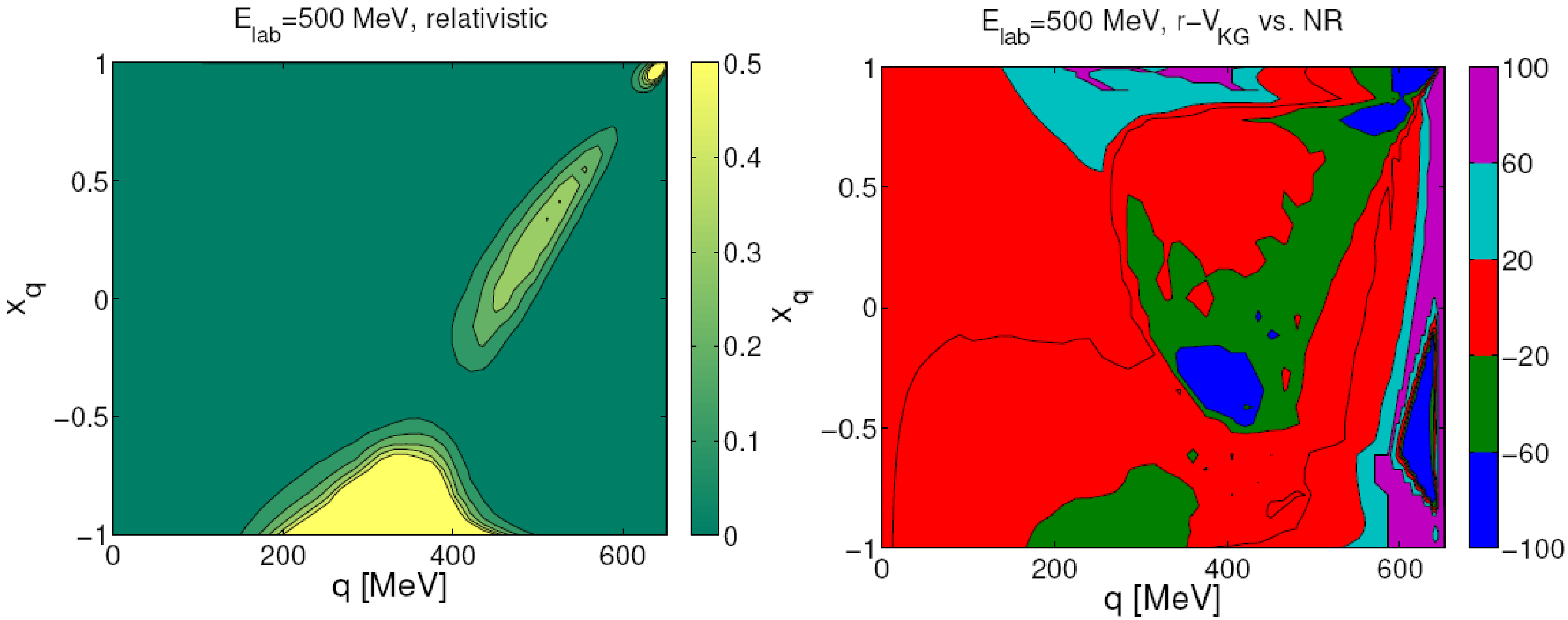


Inclusive Breakup Scattering @ Elab=495 MeV

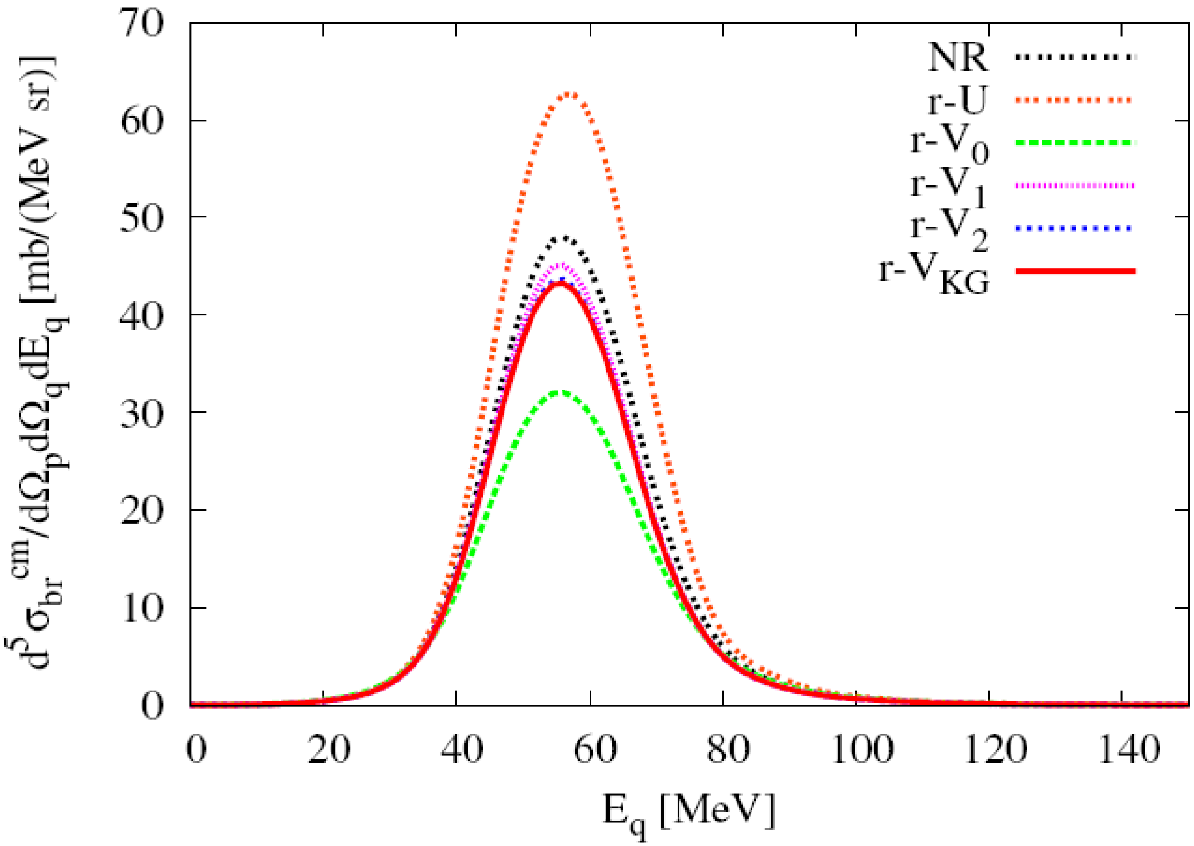


Inclusive Breakup Scattering

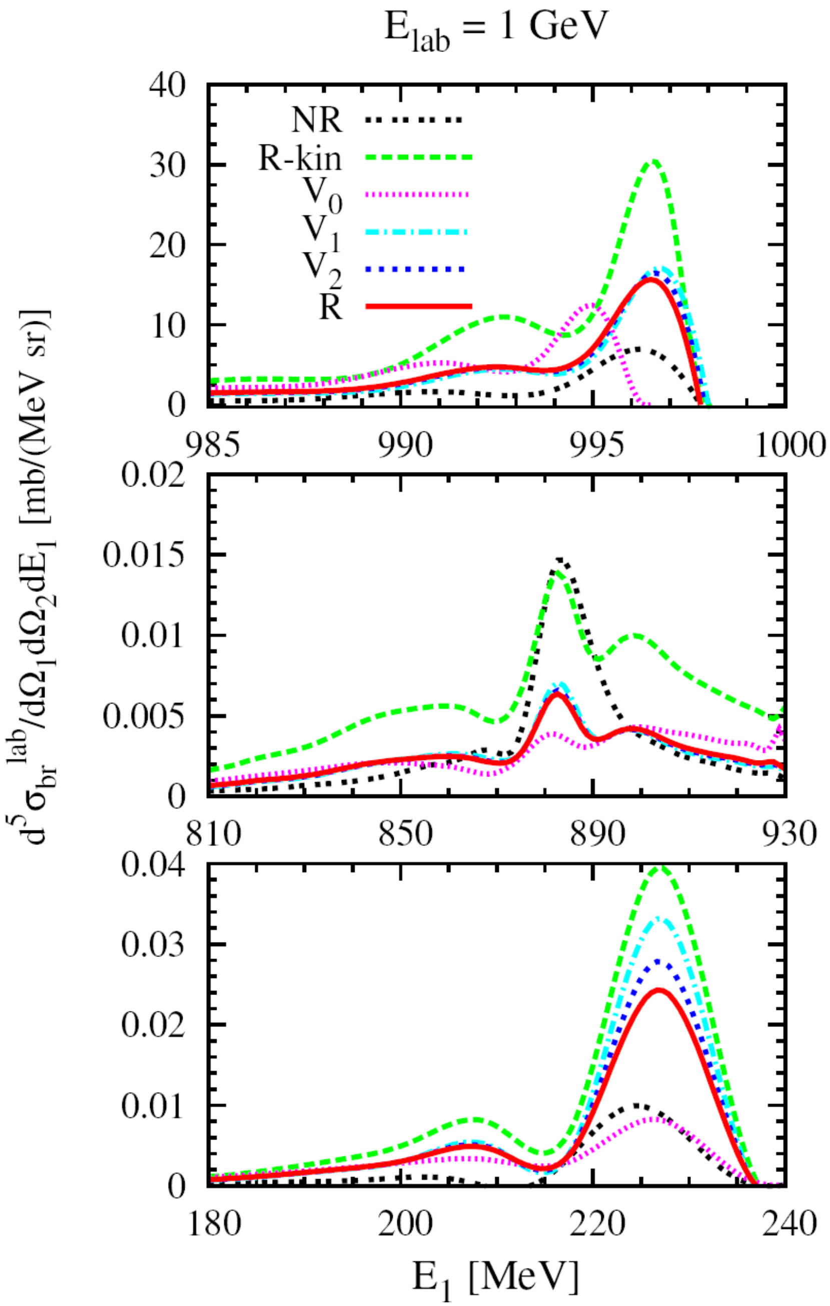
Exclusive Breakup Scattering



Exclusive Breakup Cross Section - QFS



Elab = 500 MeV: x_q = -1, x_p = 1, ϕ_{pq} = 0



$x_q = 1$

$x_p = 0$

$\phi_{pq} = 0$

$x_q = \sqrt{3}/2$

$x_p = -0.5$

$\phi_{pq} = 0$

$x_p = -0.25$

$x_q = -0.9$

$\phi_{pq} = 0$

Consideration for two-body t-matrix

- Relativistic and non-relativistic t-matrix should give identical observables for determining relativistic effects
- Or two-body t-matrices should be phase-shift equivalent
- Four options:
  - Start from relativistic LS equation (natural option)
  - If non-relativistic LS equation is used:
  - Refit of parameters (maybe time consuming in practice)
  - Transformation of Kamada-Glöckle PRL 80, 2547 (1998)
  - Transformation of Coester-Piper-Serduke as given in Polyzou PRC 58, 91 (1998)

Kamada-Glöckle (KG)

- Unitary rescaling of momentum variables to change the nonrelativistic kinetic energy into the relativisitic kinetic energy:

$$2m + \frac{q^2}{m} = 2\sqrt{m^2 + k^2}$$

$$h^2(q) \equiv \frac{q^2}{k^2}\frac{dq}{dk} \Rightarrow \langle k'|v_R|k\rangle = h(k')\langle q(k')|v_{NR}|q(k)\rangle h(k)$$

- Relativistic and nonrelativistic phase shifts are functions of invariant energy E
- Relativistic and nonrelativistic bound states have identical binding energy.

Coester-Piper-Serduke (CPS)

(PRC11, 1 (1975))

- Add interaction to square of non-interacting mass operator

$$M^2 = M_0^2 + u = 4mh \quad \text{with} \;\; h \equiv \frac{k^2}{m} + \frac{u}{4m} + m$$

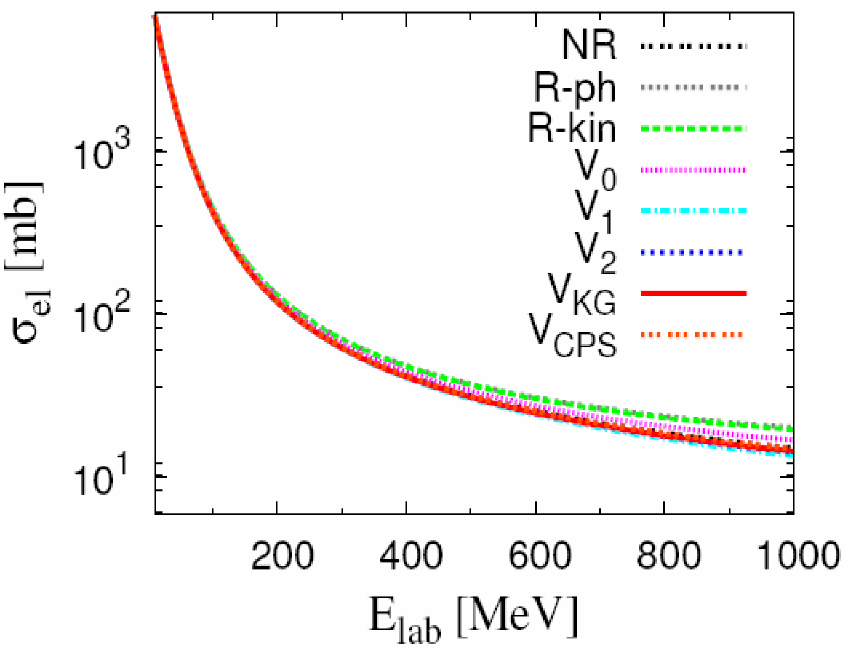
$$u = v^2 + \left\{ M_0^2, v \right\}$$

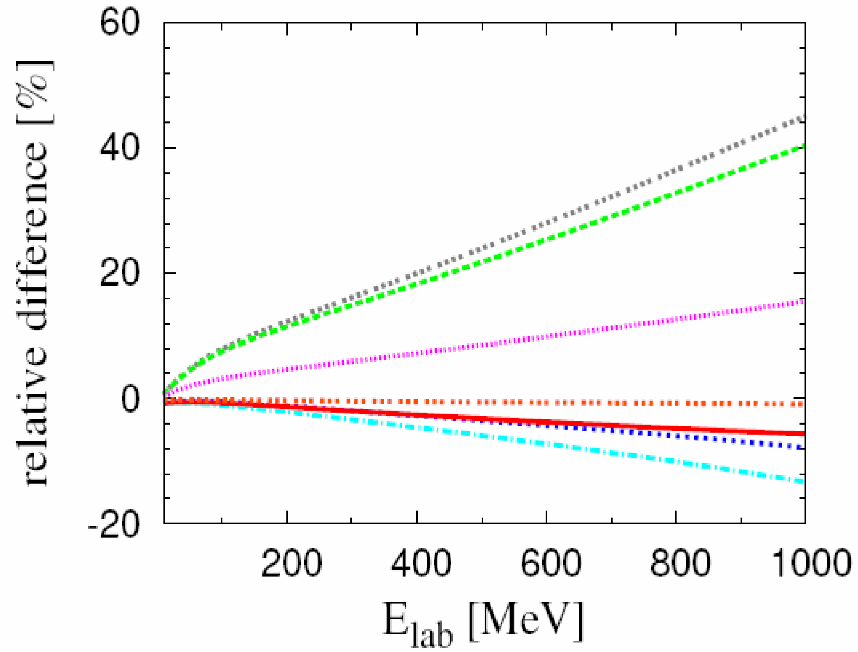
- NO need to evaluate v directly, since M, M^2, h have the same eigenstates
- Relation between half-shell t-matrices

$$\langle k' | \, t_R(e(k)) \, | k \rangle = \frac{4m}{e(k) + e(k')} \langle k' | \, t_{NR}(k^2/m) \, | k \rangle$$

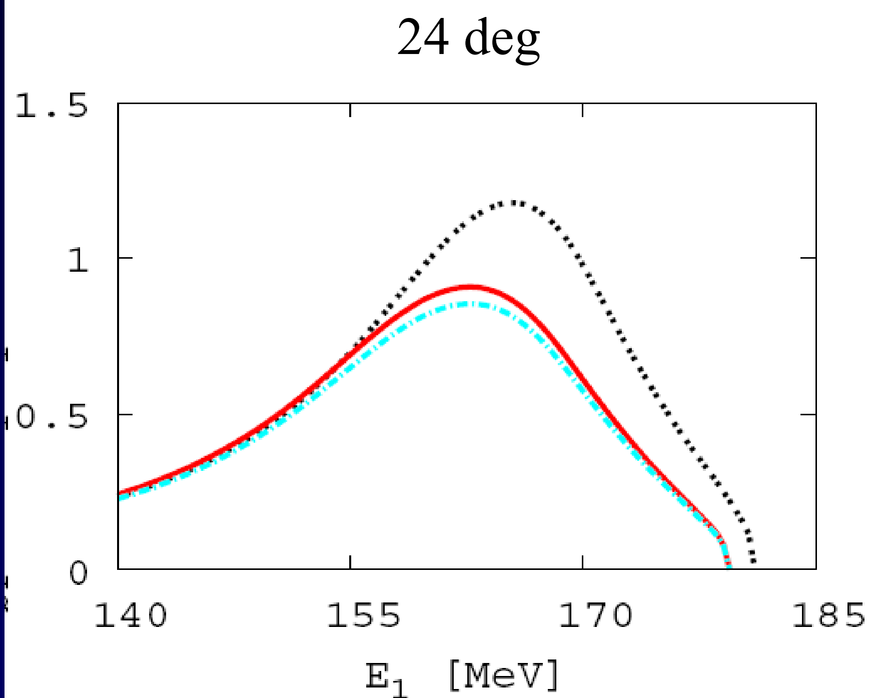
- Relativistic and nonrelativistic cross sections are identical functions of the invariant momentum k

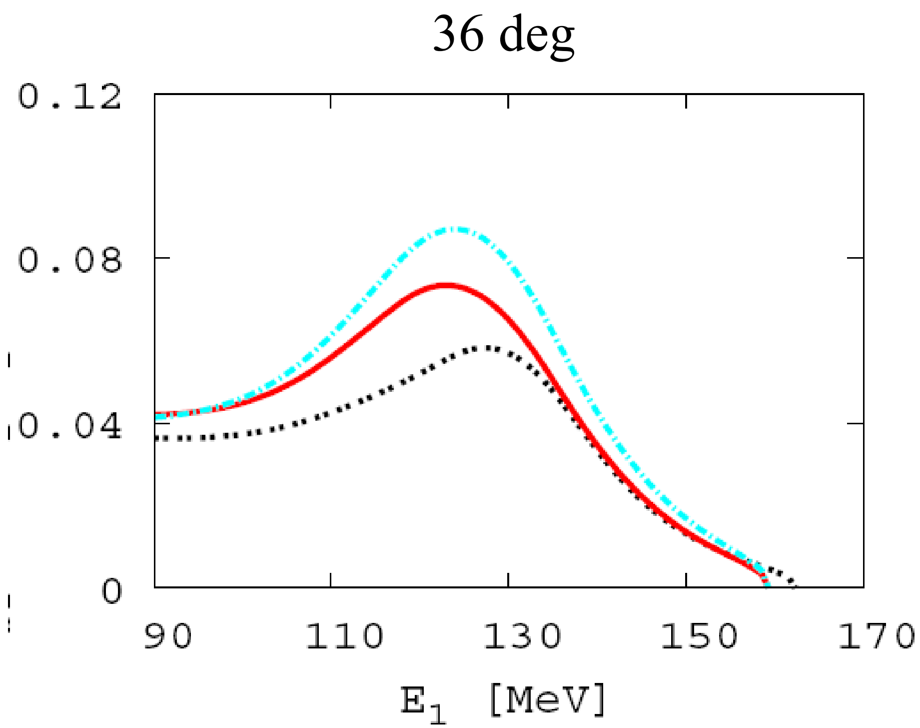
Total Cross Section Elastic Scattering





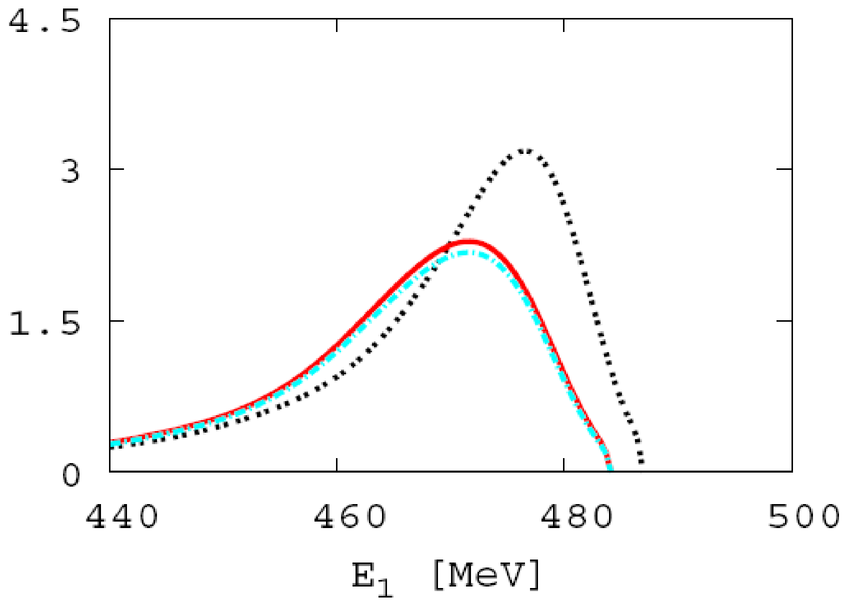
Inclusive Breakup @ 200 MeV



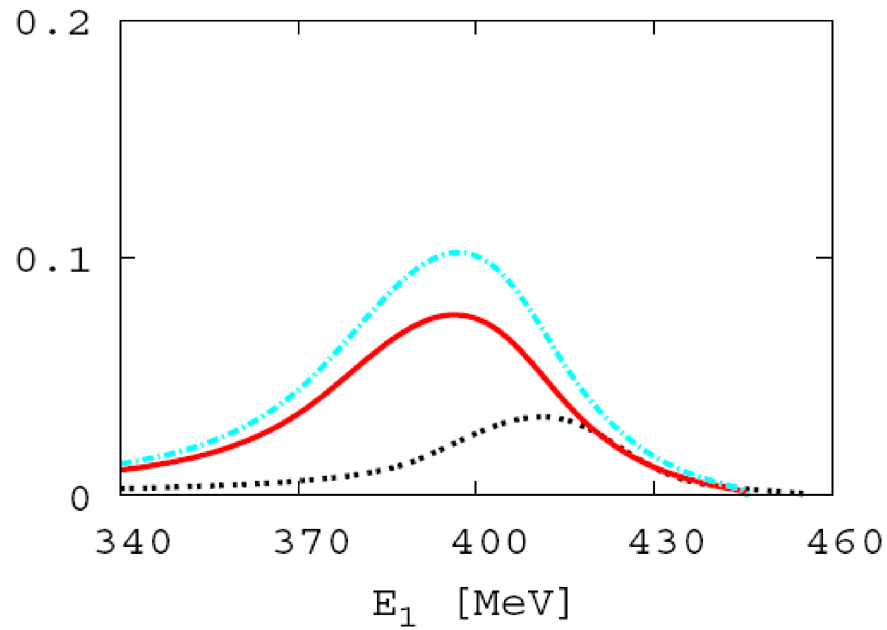


Inclusive Breakup @ 500 MeV

12 deg



24 deg



Relativistic 1st Order Faddeev Calculations

- Kinematics
  - Phase space factors
  - Lorentz Transformation from Lab to c.m. frame
    - Above determines shifts of peaks
  - Lorentz Transformation of Jacobi Coordinates
    - Always reduces effects of phase space factors
- Dynamics
  - Exact calculation of the two-body interaction embedded in the three-particle Hilbert space
  - Approximation V_2 quite good up to ~ 500 MeV
  - Observables show some sensitivity to the construction of the phase-shift equivalent relativistic interaction.

Roadmap - Summary

For higher energies : NO partial waves

- Solve Faddeev equation for 3-particle scattering in vector variables
- Investigate convergence of multiple scattering series as function of energy for different observables and configuarations

1st Order Relativistic Faddeev in tP

- Calculations at intermediate energy show that relativistic effects are quite visible.
- 1st Order = Born term determines kernel of Faddeev Eq.

To Do: Full Solution - Computer Power