Hyper-Spherical Approach to the 3-Body problem

On the Path to 3-Body Potentials

Outline

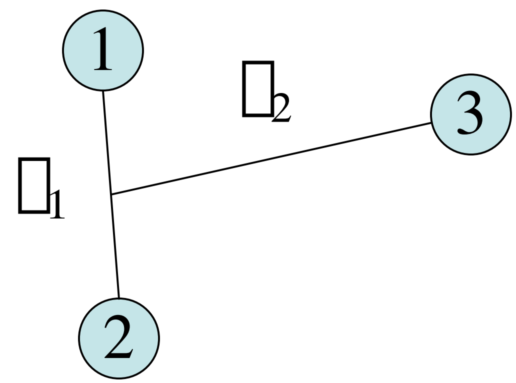
- Introduction to 3-body problem
  - Jacobi coordinates
  - Hyper-Spherical coordinates
- Separation of variables
  - Bound state calculation
- Adiabatic representation
  - Bound state calculation
- Eigenchannel R-matrix approach for scattering
  - Separation of variables
  - Adiabatic representation
- Review of results and conclusions

Introduction

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + V(|x_1 - x_2|) + V(|x_1 - x_3|) + V(|x_2 - x_3|).$$

Jacobi coordinates:

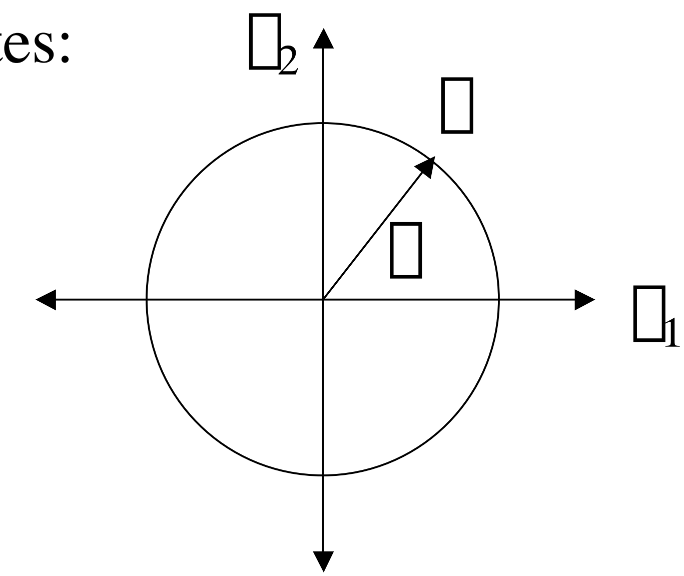
$$X = \frac{1}{\sqrt{3}}(x_1 + x_2 + x_3);\ \xi_1 = \frac{1}{\sqrt{2}}(x_1 - x_2);\ \xi_2 = \frac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3).$$



Hyper-spherical coordinates:

$$\xi_1 = \rho\cos\theta;\ \xi_2 = \rho\sin\theta.$$

$\rho^2 = \xi_1^2 + \xi_2^2$



Schrodinger Equation

$$x_1 - x_2 = \sqrt{2}\xi_1 \qquad x_2 - x_3 = \frac{3}{\sqrt{6}}\xi_2 - \frac{\xi_1}{\sqrt{2}} \qquad x_1 - x_3 = \frac{3}{\sqrt{6}}\xi_2 + \frac{\xi_1}{\sqrt{2}}.$$

$$U(\rho,\theta) = 2m(V(\sqrt{2}\rho|\cos(\theta)|) + V(\sqrt{2}\rho|\cos(\theta+\pi/3)|) + V(\sqrt{2}\rho|\cos(\theta-\pi/3)|)).$$

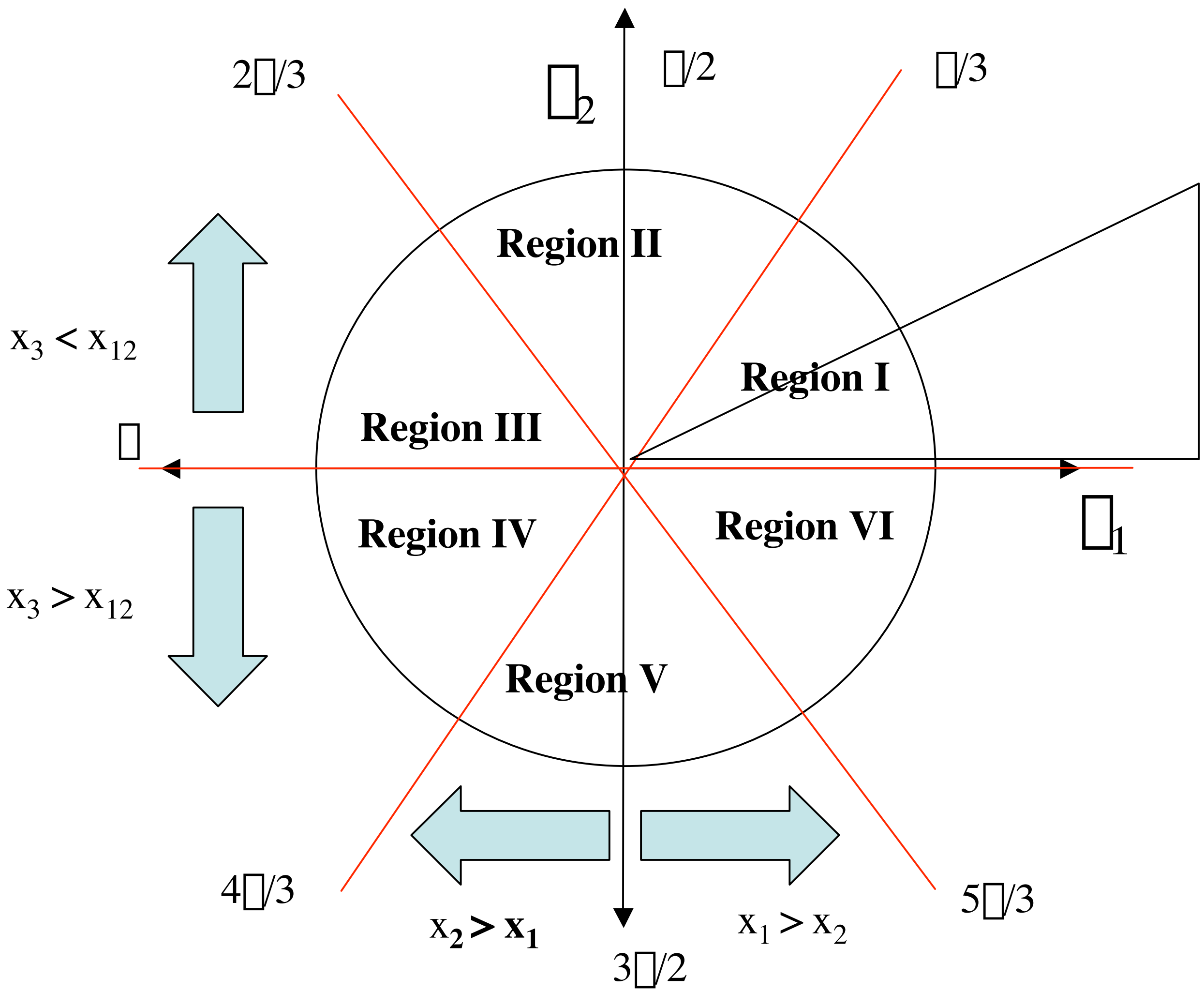
Polar Coordinates!

$$\{\frac{1}{\rho}\partial_\rho(\rho\partial_\rho) + \frac{1}{\rho^2}\partial_\theta^2 + k^2 - U(\rho,\theta)\}\psi(\rho,\theta) = 0$$

1. This potential is EVEN in theta.
2. This potential has 6-FOLD PERIODICITY in theta.

Back to optimized moments Back to cosine moments

Configuration Space



[Back to potential]

Separation of Variables

$$\psi(\rho,\theta) = F(\theta)G(\rho) \qquad G(\rho) = \sum_n g_n(\rho)$$

$$F(\theta) = \sum_n a_n \exp in\theta.$$

U=U(□,□)

The SE will not fully separate!

$$\int_0^{2\pi} \frac{d\theta}{2\pi} \cos(n\theta)\cos(m\theta) = \frac{1}{2}(\delta_{n,m} + \delta_{n,0}\delta_{m,0})$$

$$U_0(\rho) = \int_0^{2\pi} \frac{d\theta}{2\pi} U(\rho,\theta) \qquad U_{n\neq 0}(\rho) = 2\int_0^{2\pi} \frac{d\theta}{2\pi} U(\rho,\theta)\cos(6n\theta)$$

$$\sum_{n=0}^{\infty} \frac{1}{2}(\delta_{l,n} + \delta_{l,0}\delta_{n,0})(g_n^{''} + \frac{1}{\rho}g_n^{'} - \frac{n^2}{\rho^2}g_n + k^2 g_n)$$
$$- \sum_{n=0,m=0}^{\infty} \frac{1}{4} U_m(\delta_{m+n,6l} + \delta_{|m-n|,6l} + \delta_{m+n,0}\delta_{l,0} + \delta_{|m-n|,0}\delta_{l,0})g_n = 0$$

Boundary condition at origin?

Cont.

$$\sum_{n=0}^{N_c} [\hat{T}_{l,n} + \hat{U}_{l,n}] f_n(\rho) = k^2 \sum_{n=0}^{N_c} \frac{1}{2}(\delta_{l,n} + \delta_{l,0}\delta_{n,0}) f_n(\rho)$$

$$\hat{T}_{l,n} = -\frac{1}{2}(\delta_{l,n} + \delta_{l,0}\delta_{n,0})\partial_\rho^2$$

$$\hat{U}_{l,n} = \frac{1}{2}(\delta_{l,n} + \delta_{l,0}\delta_{n,0})(\frac{36n^2 - 1/4}{\rho^2}) + \sum_{m=0}^{2N_c} \frac{1}{4} U_m(\delta_{m+n,6l} + \delta_{|m-n|,6l} + \delta_{m+n,0}\delta_{l,0} + \delta_{|m-n|,0}\delta_{l,0})$$

Now let's apply this to some models!

The Bare 2-Body Potential

$$V(x) = V_s \, e^{-m_s |x|} + V_v \, e^{-m_v |x|} \ .$$

$m_s = 400$ MeV

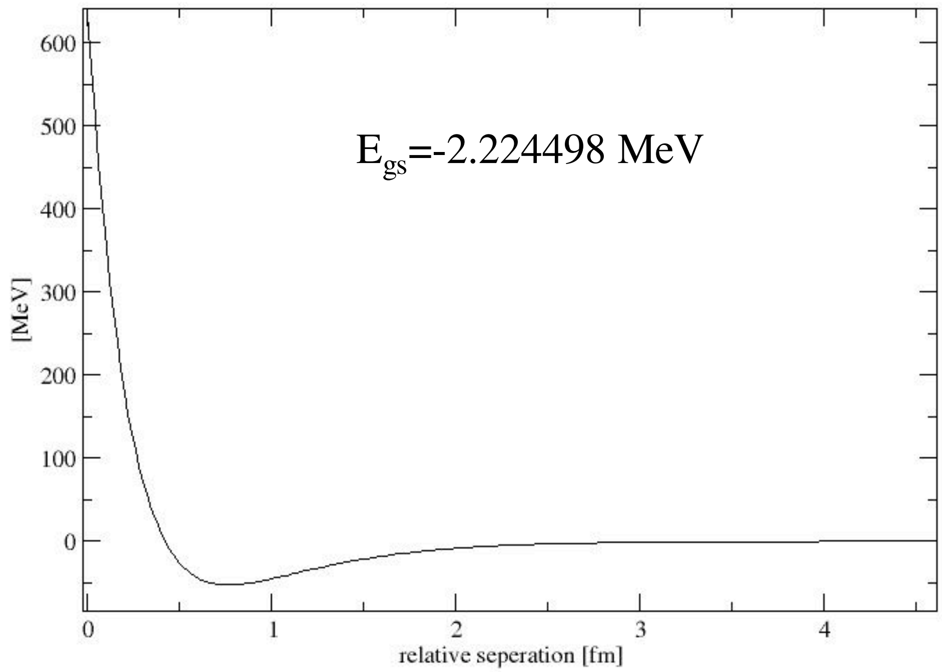
$V_s = -506$ MeV

$m_v = 783$ MeV

$V_v = +1142.49$ MeV

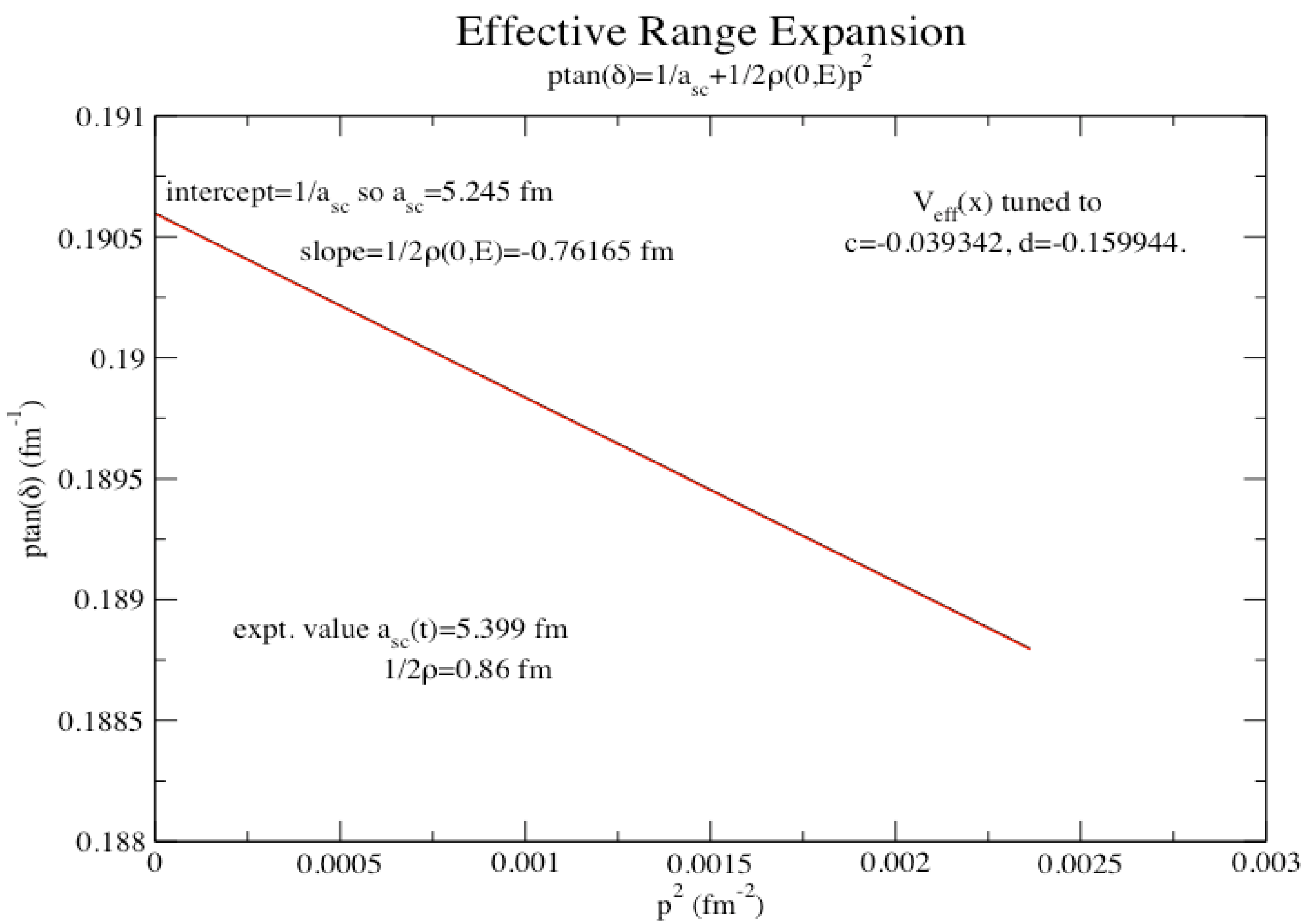
$E_b = -2.2245$ MeV

$r_{rms} = 1.875$ fm.



E_{gs}=-2.224498 MeV

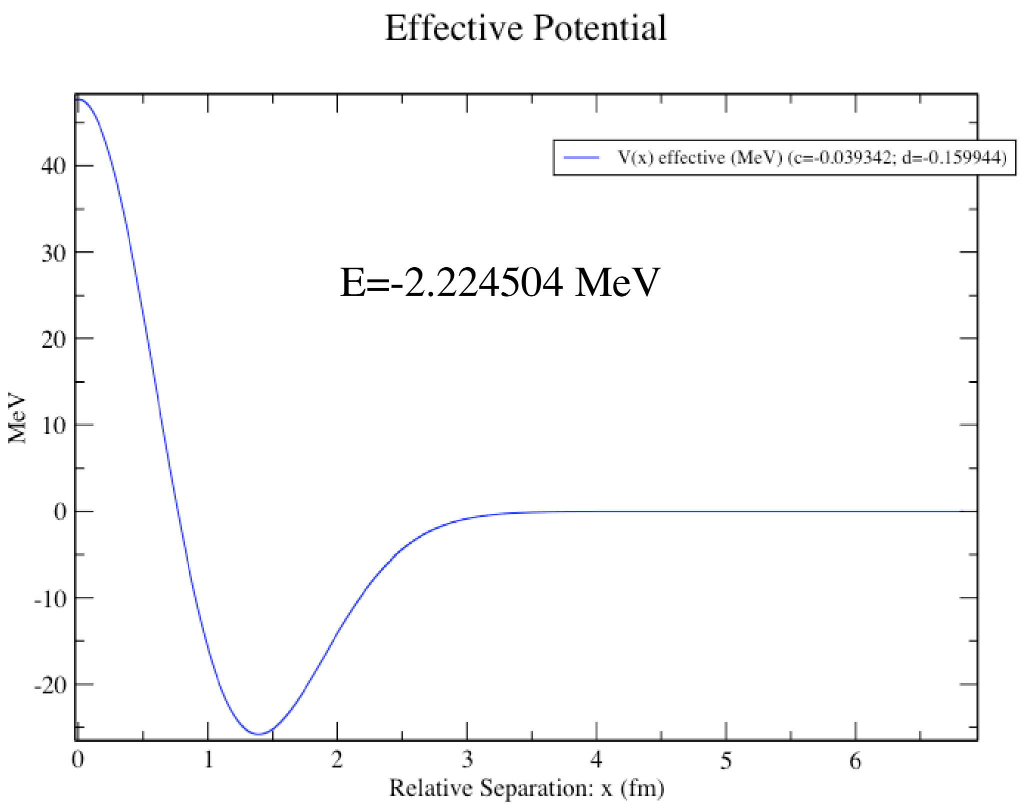
Extracting Low Energy Physics



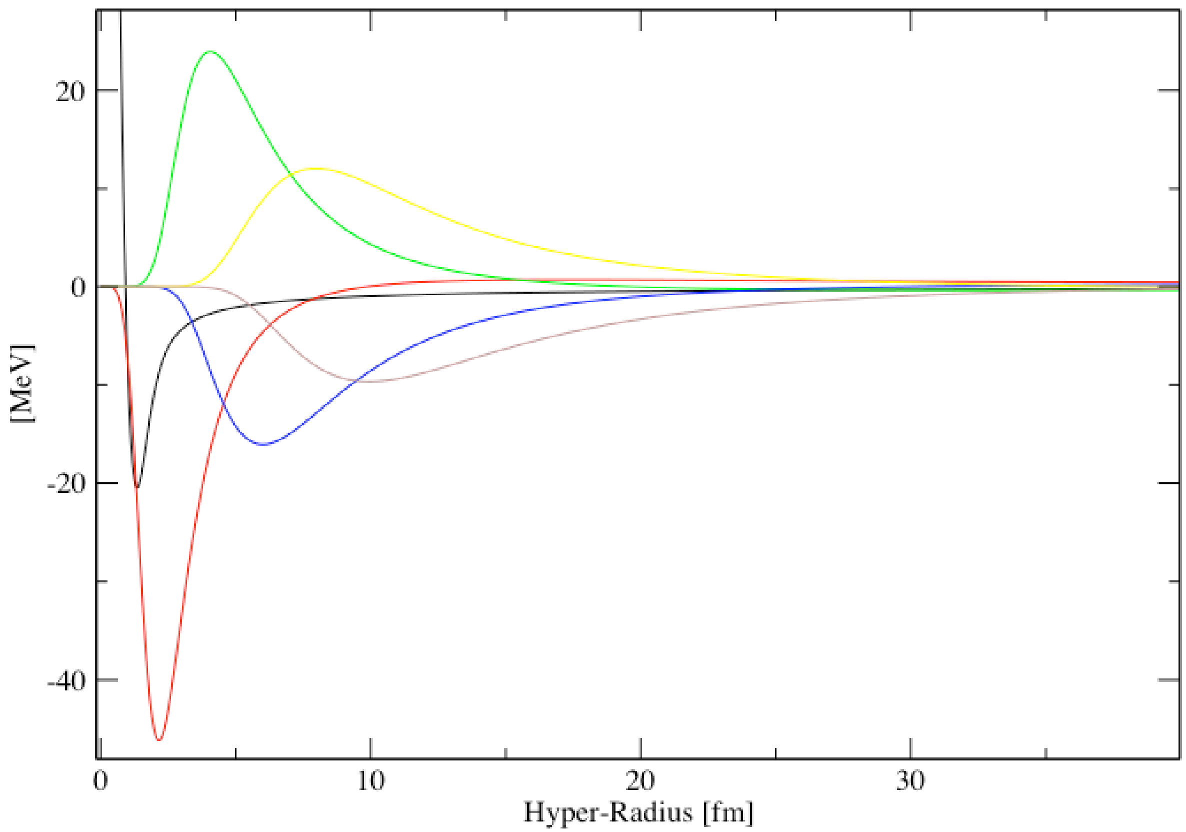
Back to R-Matrix

Effective 2-Body Potential

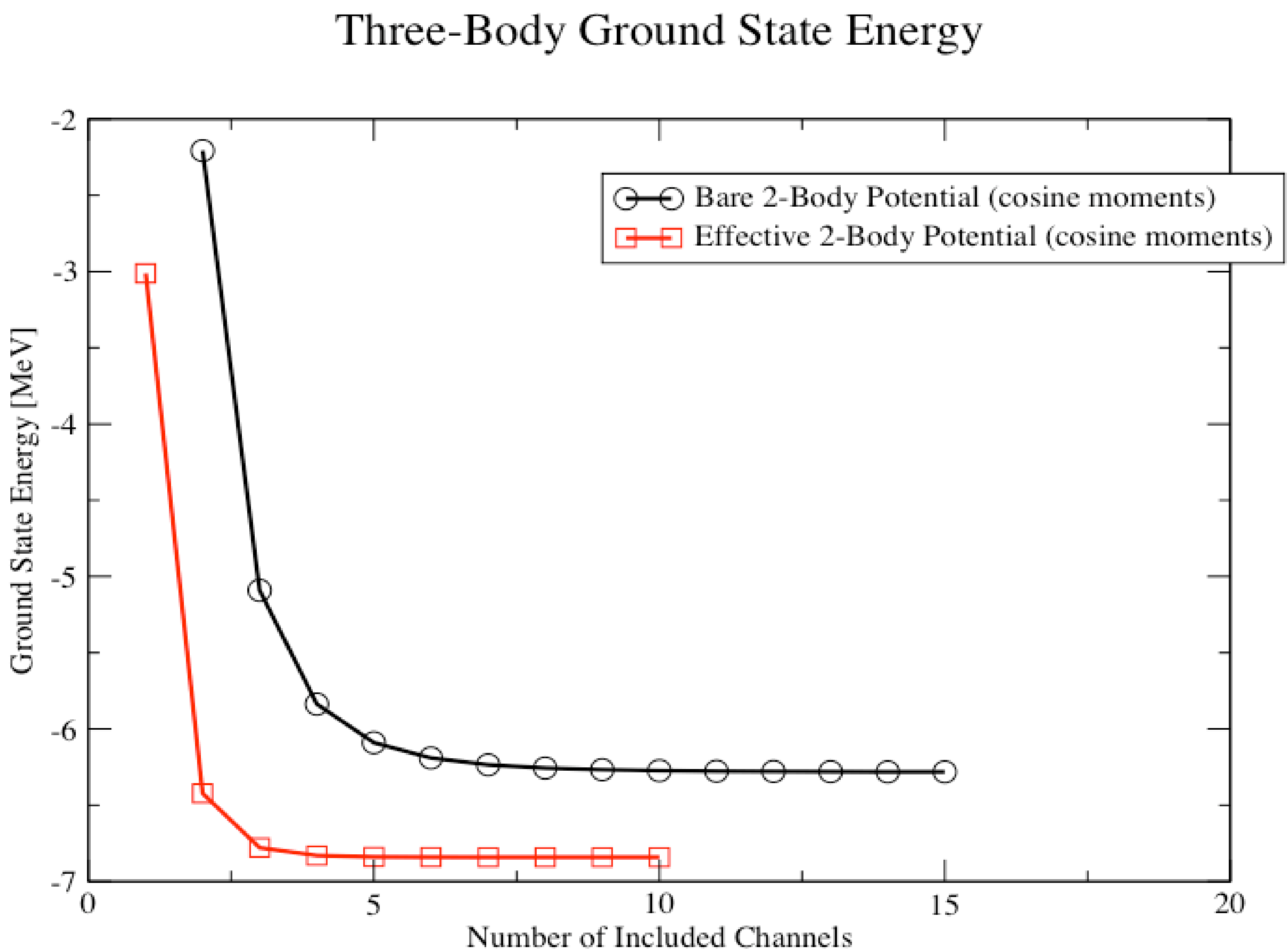
$$V_{\text{eff}}(x) = \frac{1}{a}\left(c + d\frac{\partial^2}{\partial\xi^2} + e\frac{\partial^4}{\partial\xi^4} + \ldots\right)\exp(-\xi^2)\ ; \quad \xi \equiv x/a \qquad a = 1.16\ \text{fm.}$$



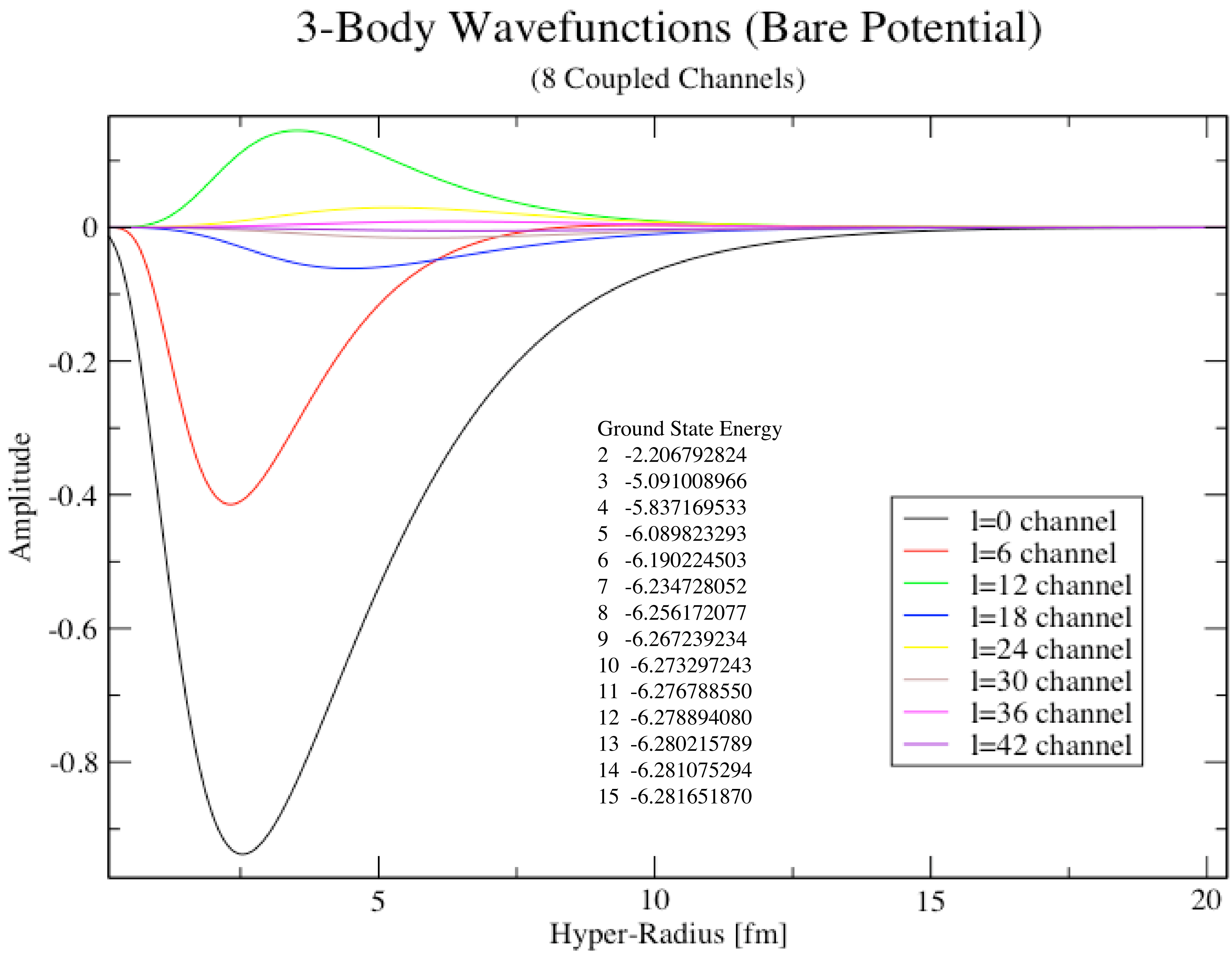
Cosine Moments of the Effective Potential

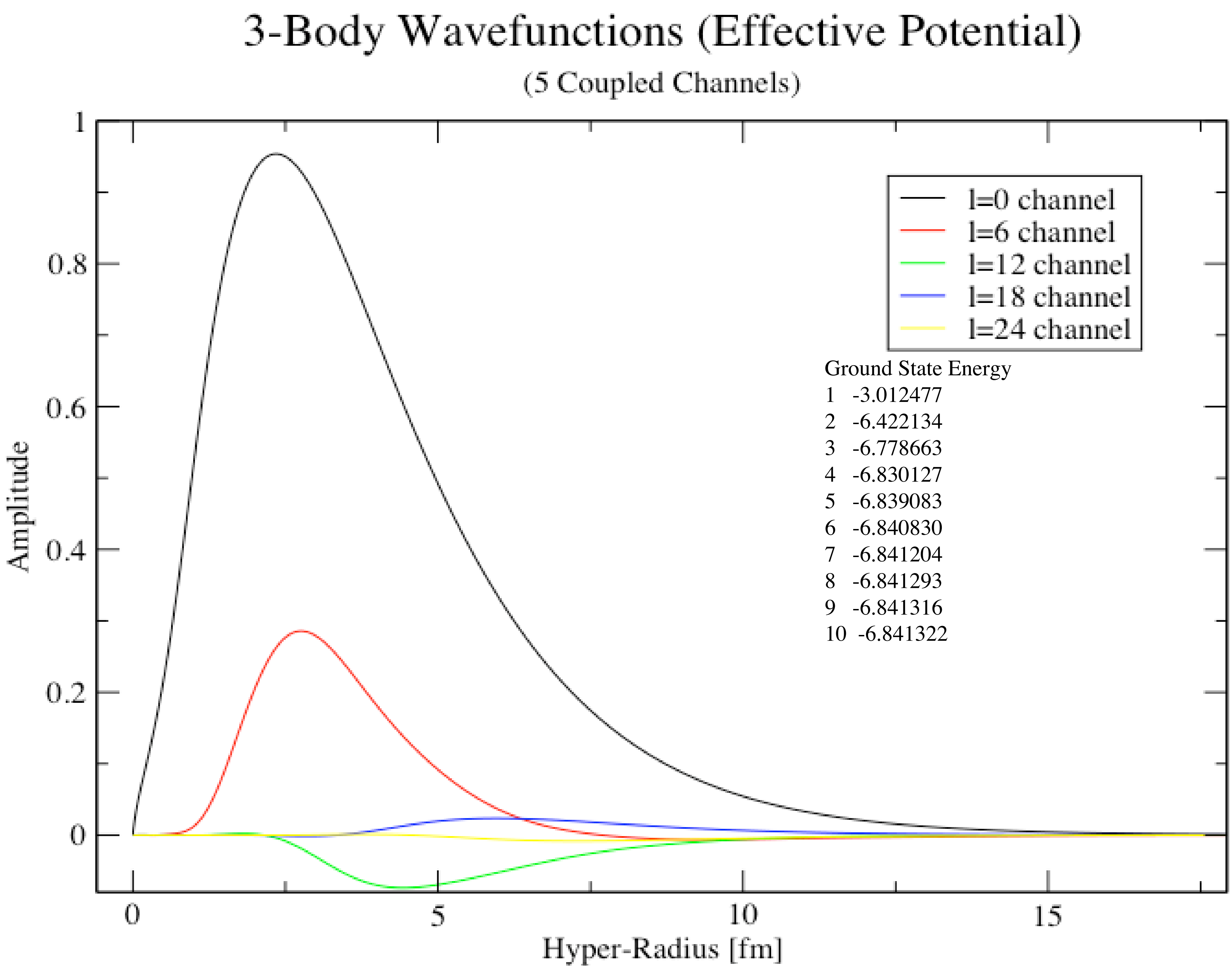


SE



Three-Body Ground State Energy
Bare 2-Body Potential (cosine moments)
Effective 2-Body Potential (cosine moments)
Ground State Energy [MeV]
Number of Included Channels
-2
-3
-4
-5
-6
-7
0
5
10
15
20

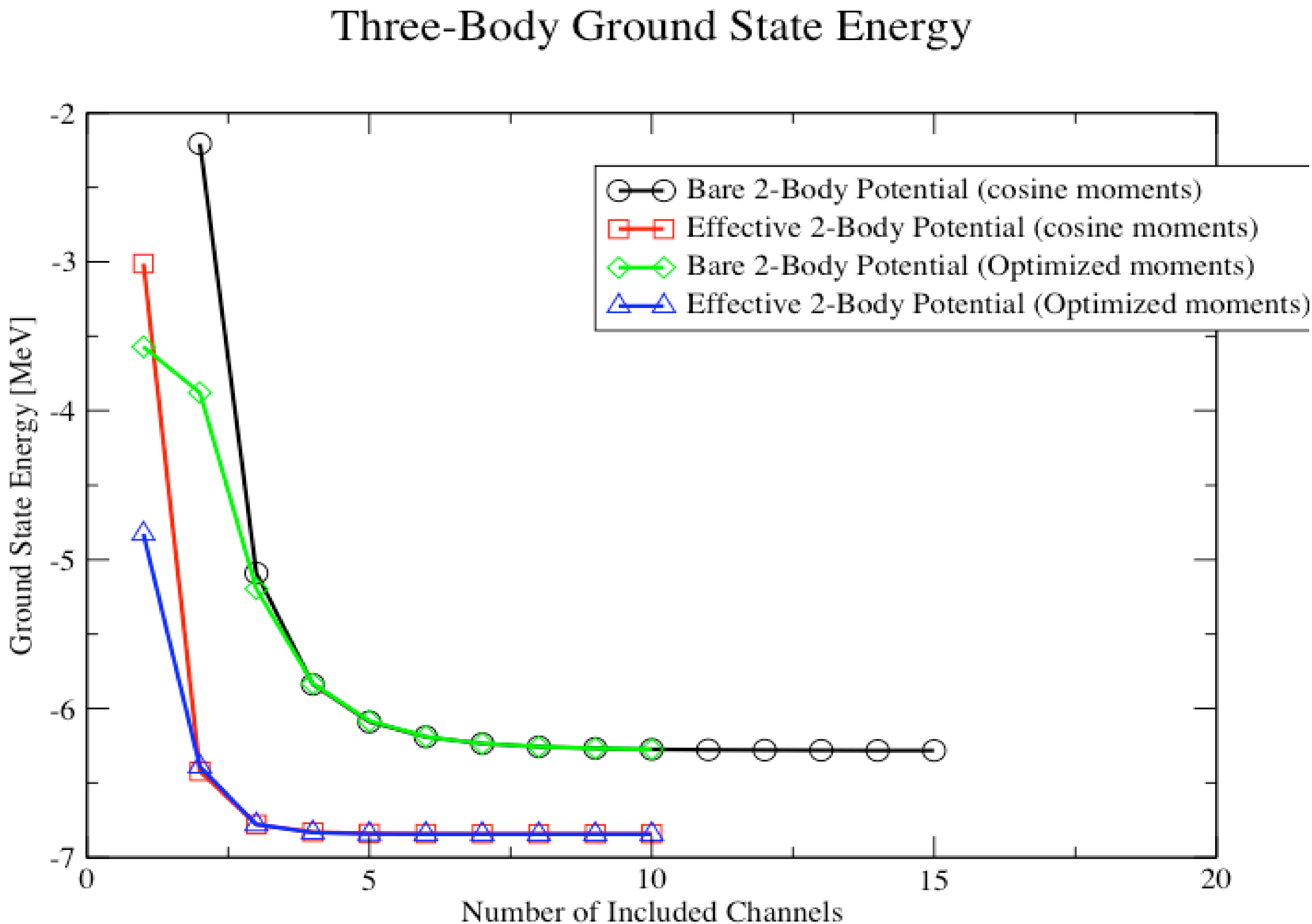

3-Body Wavefunctions (Bare Potential)
(8 Coupled Channels)
Amplitude
Hyper-Radius [fm]
0
-0.2
-0.4
-0.6
-0.8
5
10
15
20
Ground State Energy
2 -2.206792824
3 -5.091008966
4 -5.837169533
5 -6.089823293
6 -6.190224503
7 -6.234728052
8 -6.256172077
9 -6.267239234
10 -6.273297243
11 -6.276788550
12 -6.278894080
13 -6.280215789
14 -6.281075294
15 -6.281651870
l=0 channel
l=6 channel
l=12 channel
l=18 channel
l=24 channel
l=30 channel
l=36 channel
l=42 channel


3-Body Wavefunctions (Effective Potential)
(5 Coupled Channels)
Amplitude
Hyper-Radius [fm]
0
0.2
0.4
0.6
0.8
1
0
5
10
15
l=0 channel
l=6 channel
l=12 channel
l=18 channel
l=24 channel
Ground State Energy
1 -3.012477
2 -6.422134
3 -6.778663
4 -6.830127
5 -6.839083
6 -6.840830
7 -6.841204
8 -6.841293
9 -6.841316
10 -6.841322

Bound State Spectrum

Bare potential spectrum 10 coupled moments [MeV]		Effective potential spectrum 10 coupled moments [MeV]		
1	-6.27330	1	-6.84463	(9.09 %)
2	-1.31846	2	-1.99181	(51.06 %)
3	0.079382	3	-1.14805	
4	0.279674	4	-0.59962	
5	0.660249	5	-0.22500	
6	0.897652	6	0.28711	

Back to conclusions


Three-Body Ground State Energy
Ground State Energy [MeV]
-2
-3
-4
-5
-6
-7
0
5
10
15
20
Number of Included Channels
Bare 2-Body Potential (cosine moments)
Effective 2-Body Potential (cosine moments)
Bare 2-Body Potential (Optimized moments)
Effective 2-Body Potential (Optimized moments)

SE

Adiabatic Approach: Bound States

$$\psi(\rho,\theta) = \sum_n f_n(\rho,\theta) g_n(\rho). \quad \{\frac{1}{\rho}\partial_\rho(\rho\partial_\rho) + \frac{1}{\rho^2}\partial_\theta^2 + k^2 - U(\rho,\theta)\}\psi(\rho,\theta) = 0$$

$$(\frac{1}{\rho^2}\partial_\theta^2 + k_n^2(\rho) - U(\rho,\theta))f_n(\rho,\theta) = 0 = (k_n^2(\rho) - H^{ad}(\rho,\theta))f_n(\rho,\theta)$$

$$(\mathbf{T} + \mathbf{U})\mathrm{g} = k^2\mathrm{g}.$$

$$T_{m,n} = -\delta_{m,n}(\partial_\rho^2 + \frac{2}{\rho}\partial\rho + \frac{1}{4\rho^2}) - P_{m,n}(2\partial_\rho + \frac{2}{\rho}) - W_{m,n}$$

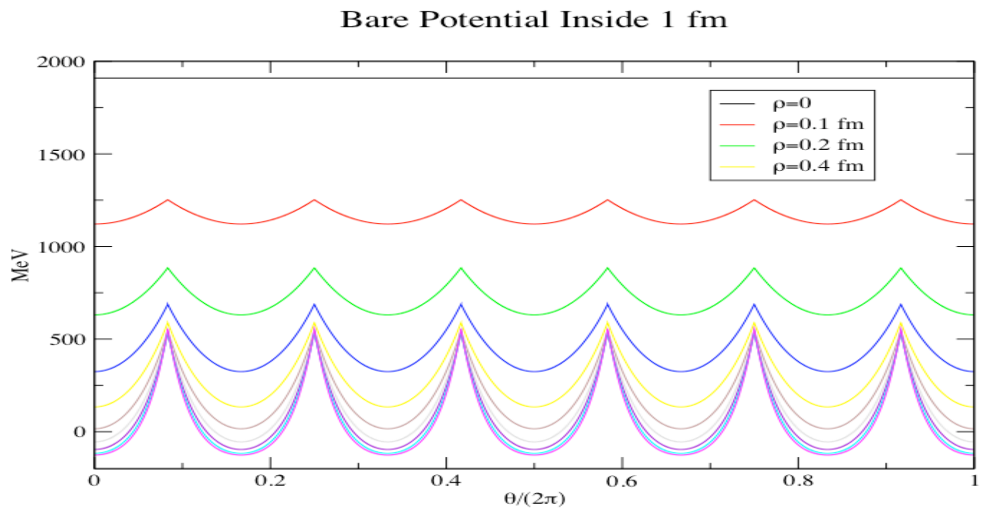
$$U_{m,n} = \delta_{m,n}(k_m^2(\rho) - \frac{1}{4\rho^2})$$

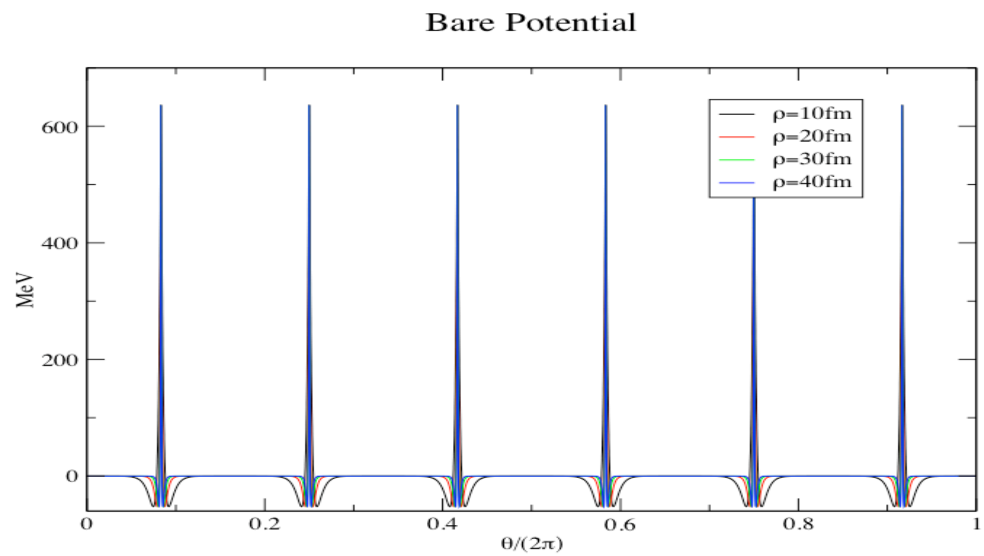
$$P_{m,n} =< f_m|\partial_\rho f_n >, \qquad < f_m|\partial_\rho f_n > = \frac{< f_m|\frac{\partial H}{\partial\rho}|f_n >}{E_n(\rho) - E_m(\rho)}$$

$$W_{m,n} =< f_m|\partial_\rho^2 f_n >$$

Single channel results

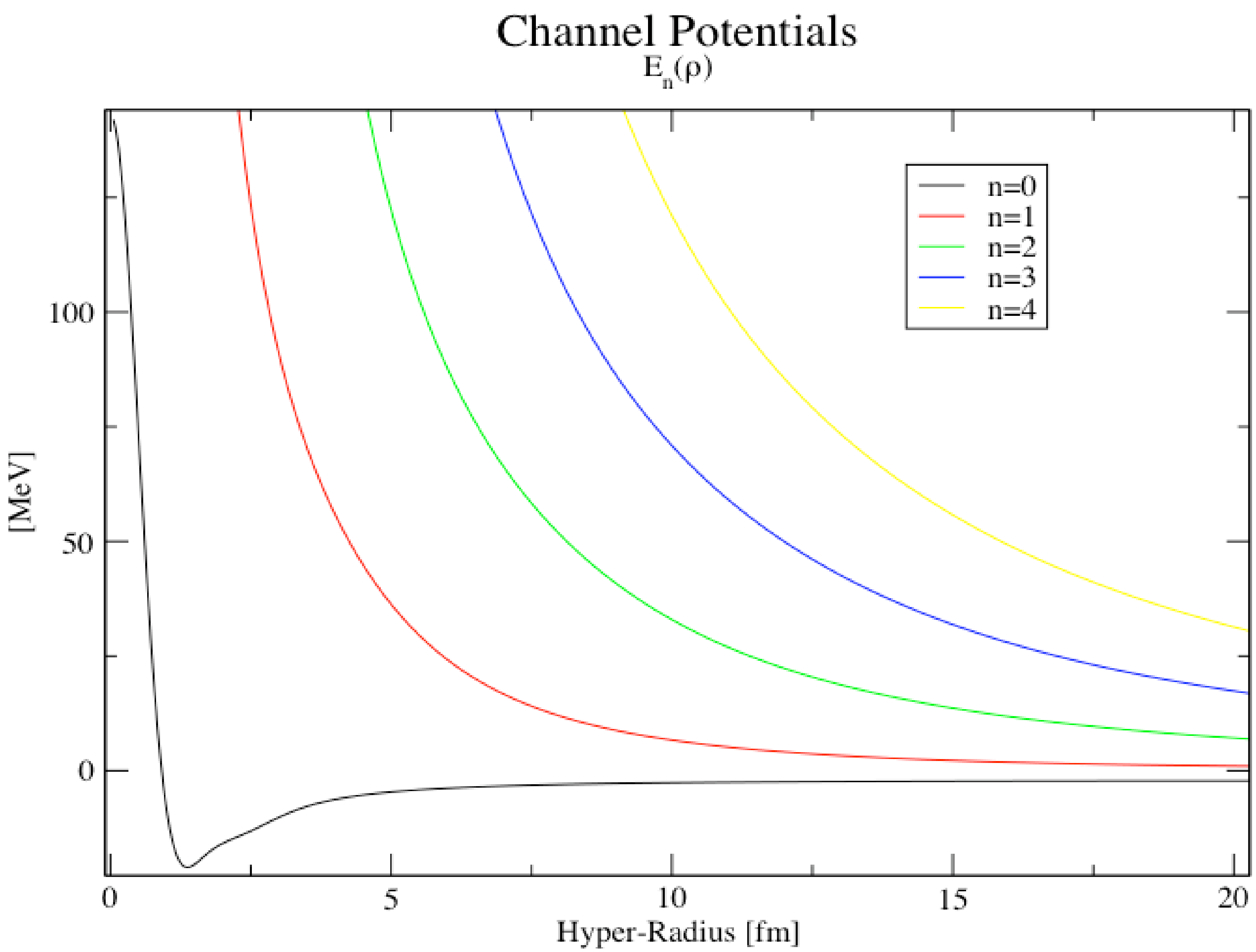
Symmetries





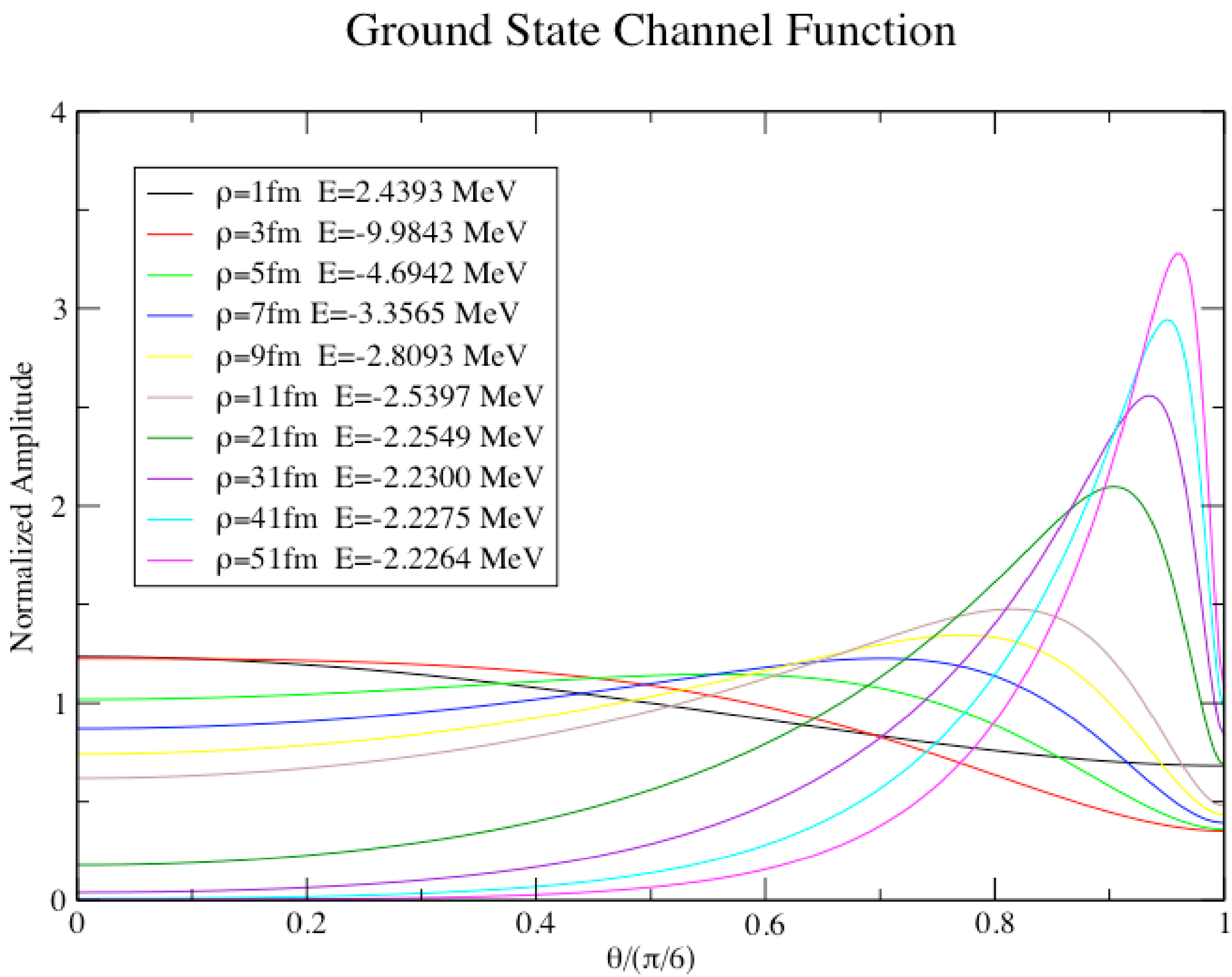
Configuration space

Eigenenergies of H^ad



$$\left(\frac{1}{\rho^2}\partial_\theta^2 + k_n^2(\rho) - U(\rho,\theta)\right)f_n(\rho,\theta) = 0 = \left(k_n^2(\rho) - H^{ad}(\rho,\theta)\right)f_n(\rho,\theta)$$

The Deuteron Channel



Single Channel Results

$$(\mathbf{T}+\mathbf{U})\mathrm{g} = k^2\mathrm{g}.$$

$$U_{m,n} = \delta_{m,n}(k_m^2(\rho) - \frac{1}{4\rho^2})$$

$$T_{m,n} = -\delta_{m,n}(\partial_\rho^2 + \frac{2}{\rho}\partial\rho + \frac{1}{4\rho^2}) - P_{m,n}(2\partial_\rho + \frac{2}{\rho}) - W_{m,n}$$

Ignoring W_{00}	Bare potential spectrum [MeV]	Ignoring W_{00}	Effective potential spectrum [MeV]
1 -6.437203	1 -6.260069	1 -7.024662	1 -6.800459
2 -2.211188	2 -2.184120	2 -2.220957	2 -2.190993
3 -2.046865	3 -2.020407	3 -2.058565	3 -2.032456
4 -1.739340	4 -1.712684	4 -1.755386	4 -1.728973
5 -1.295678	5 -1.267052	5 -1.316991	5 -1.288351
6 -0.721156	6 -0.689651	6 -0.748449	6 -0.716758
7 -0.019101	7 0.015689	7 -0.052924	7 -0.017884
8 0.808193	8 0.846597	8 0.767348	8 0.805966
9 1.759033	9 1.801393	9 1.710658	9 1.753101
10 2.832133	10 2.878232	10 2.775663	10 2.821656

Adiabatic rep.   Back to conclusions

Eigenchannel R-Matrix

$$k^2 = \frac{\int_V \psi^*(-\nabla^2 + U)\psi dV}{\int_V \psi^* \psi dV}$$

$$b_n = -\frac{\partial ln(\psi)}{\partial \hat{n}}$$

$$b = \frac{\int_V dV[-\nabla\vec{\psi}^* \cdot \nabla\vec{\psi}] + \psi^*(k^2 - U)\psi}{\int_\Sigma \psi^* \psi dS}$$

$$\psi_n = \sum_\alpha c_\alpha B_\alpha$$

$$\frac{\partial b}{\partial c_\alpha} = 0.$$

$$\Lambda_{m,n} = \int_\Sigma dS B_m B_n$$

$$b\mathbf{\Lambda c} = \mathbf{\Gamma c}$$

$$\Gamma_{m,n} = \int_V dV[-\nabla\vec{B}_m \cdot \nabla\vec{B}_n + B_m(k^2 - U)B_n]$$

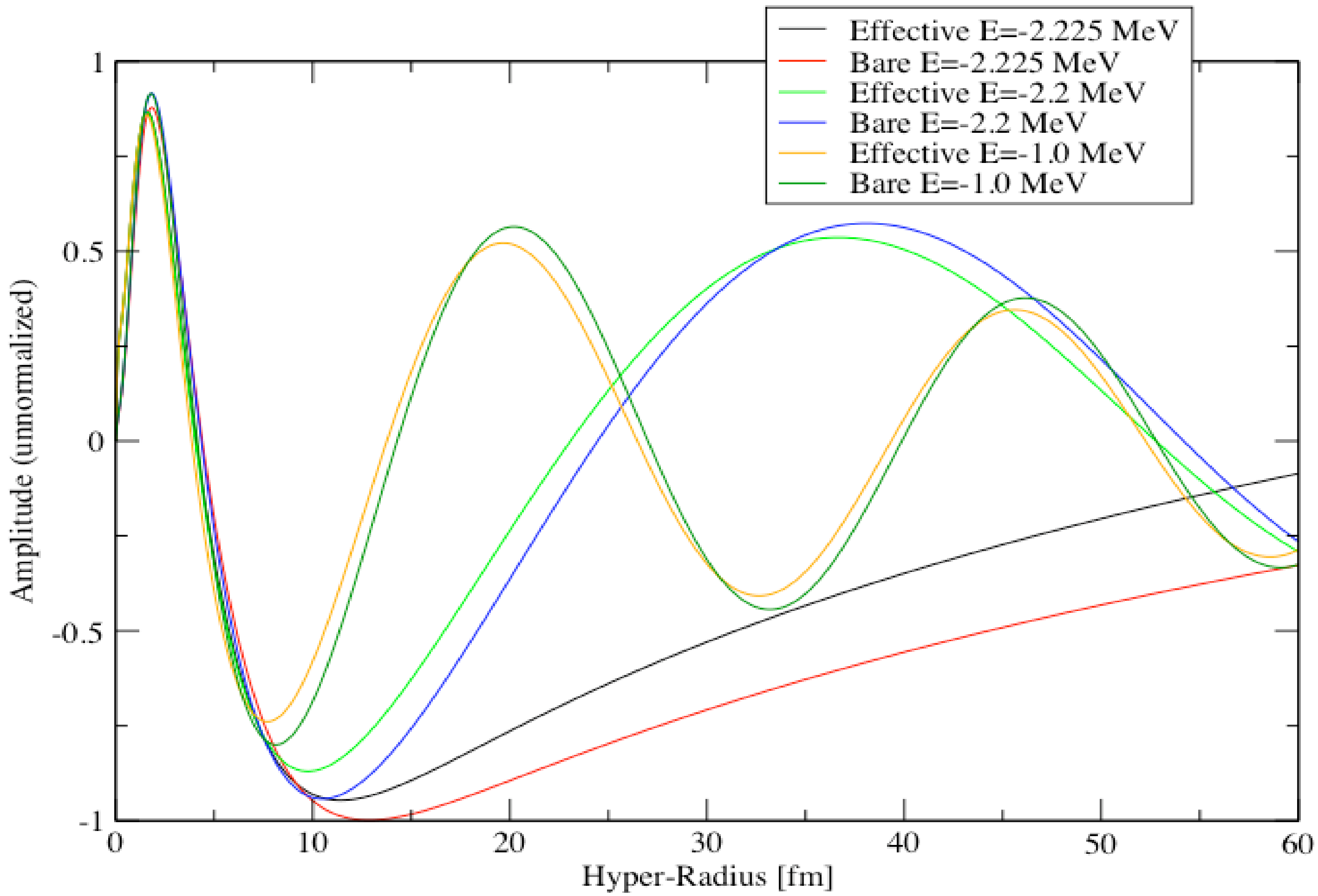
ERT for 2-Body System

$k^2[fm^{-2}]$	$k\tan(\delta)$ R-matrix	$k\tan(\delta)$ Numerov
0.0000000000	0.1905975587	0.1906002044
0.0241149081	0.1722587089	0.1722620450
0.0482298162	0.1540006399	0.1540047025
0.0723447243	0.1358380297	0.1358428543
0.0964596323	0.1177858558	0.1177914786
0.1205745404	0.0998593571	0.0998658142
0.1446894485	0.0820739920	0.0820813204
0.1688043566	0.0644453944	0.0644536291
0.1929192647	0.0469893260	0.0469985028
0.2170341728	0.0297216267	0.0297317808

Tuning V_{eff}

Back to conclusions

Single Channel Scattering Solutions



Back to conclusions

Conclusions

Bound state calculations seem to show more consistency in the adiabatic representation.

see bound state spectrum (SOV) (ad)

R-Matrix theory can be used to tune effective 3-Body potentials (2B) (1CH)