Systematic Low-Energy Expansion for Three Nucleons to All Orders

H. W. Grießhammer

Technische Universität München, Germany

1. Few-Nucleon Physics at Very Low Energies
2. The Three-Body System to All Orders
3. Summary and Rewards

Bedaque/hg/Hammer/Rupak: *Nucl. Phys.* **A714** (2003), 589 [nucl-th/0207034]

Bedaque/hg: *Nucl. Phys.* **A671** (2000), 357 [nucl-th/9907077]

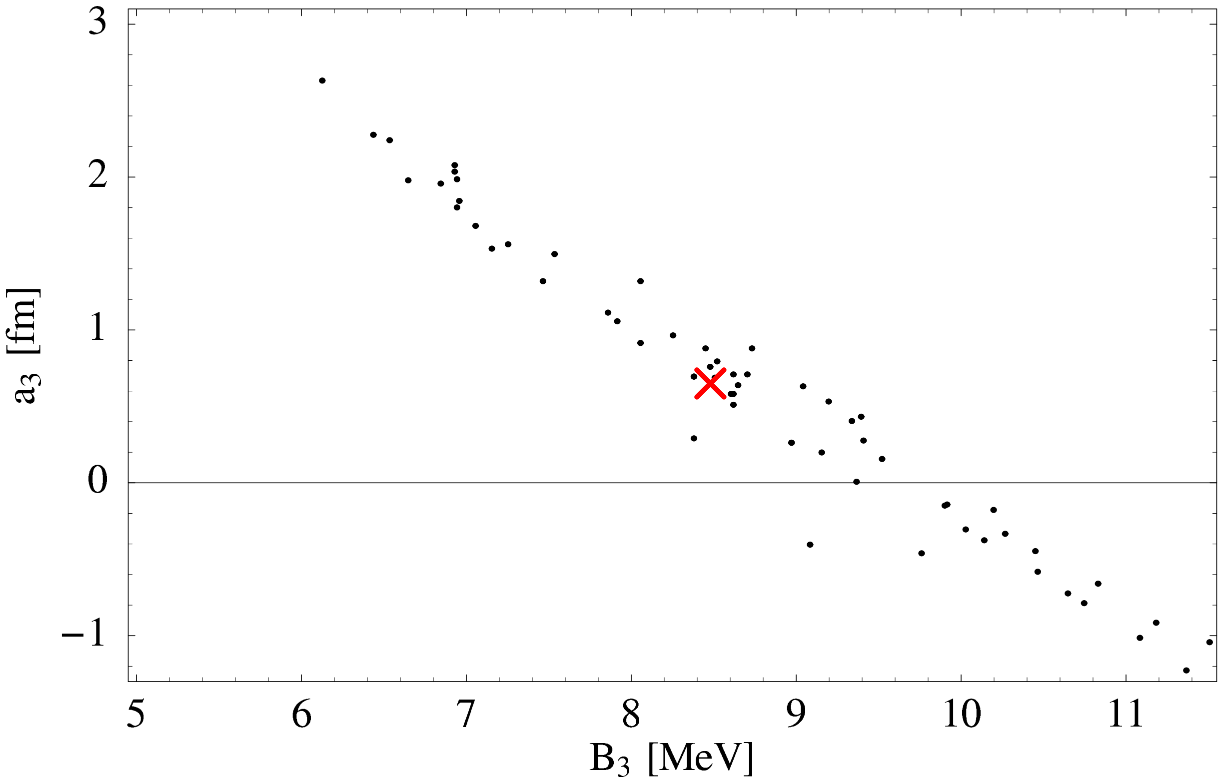
Bedaque/Gabbiani/hg: *Nucl. Phys.* **A675** (2000), 601 [nucl-th/9911034]

Mathematica code: `http://www-nsdth.lbl.gov` or
`http://www.ph.tum.de/~hgrie`

1. Why Few-Nucleon Physics at Very Low Energies?

(a) When do Three-Body Forces Enter?

NN potentials predict vastly differing triton binding energy and scattering length:



Phillips line (1969):

nd scattering length in $^2\mathrm{S}_{\frac{1}{2}}$ channel vs. triton binding energy

×: exp.

Ad-hoc three-body forces added to make up for difference.

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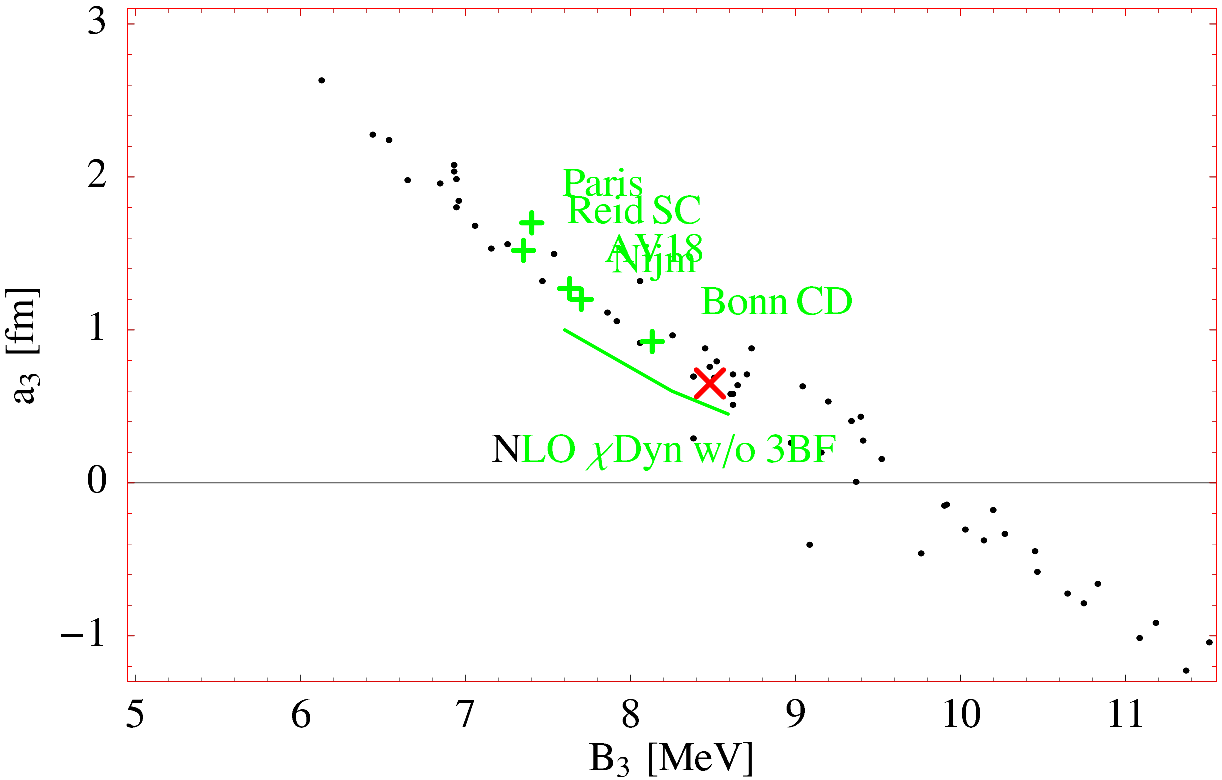

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$nd \to t\gamma$ at thermal energies:

AV 18 $+$ Urbana IX: $\sigma = 0.6$ mb

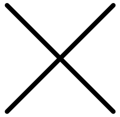
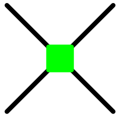
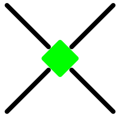
experiment: $\sigma = [0.508 \pm 0.015]$ mb

(b) Interactions and Q Power Counting in EFT($\not{\pi}$)

(Bethe 1949, Kaplan/Savage/Wise & van Kolck 1997)

Most general Lagrangean out of local interactions between low-energy degrees of freedom respecting all symmetries of underlying theory (Galilei/Lorentz order by order, particle conservation, flavour, gauge, . . .):

Two-Body Sector: Systematisation and Extension of Effective Range Theory of NN scattering.

$-\mathrm{i}C_0$  , $\mathrm{i}p^2C_2$  , $\mathrm{i}p^4C_4$  , $\ldots$

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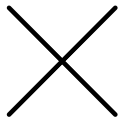
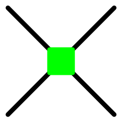
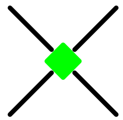
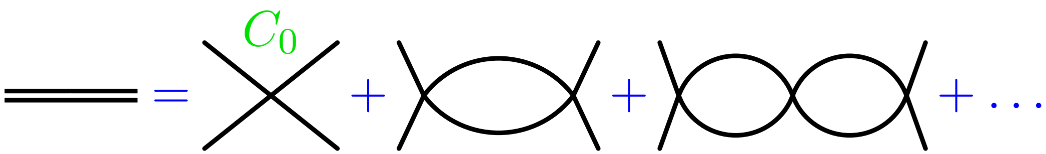
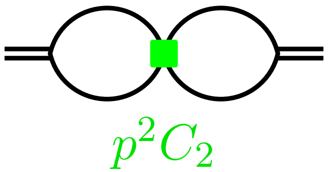
$-\mathrm{i}C_0$  , $\mathrm{i}p^2C_2$  , $\mathrm{i}p^4C_4$  , ...

Coefficients C_{2n} from simple observables:

LO (30%) : 

C_0: correct binding energies of anomalously shallow bound states

NLO (10%) : 

C_2: correct effective ranges ρ_0

Power counting: Typical momentum $\gamma = \sqrt{MB_{\text{deut}}} \approx 45\ \text{MeV}$

$$\Longrightarrow \text{Expansion parameter } Q = \frac{\gamma}{\Lambda \approx m_\pi} \simeq \frac{1}{3}$$

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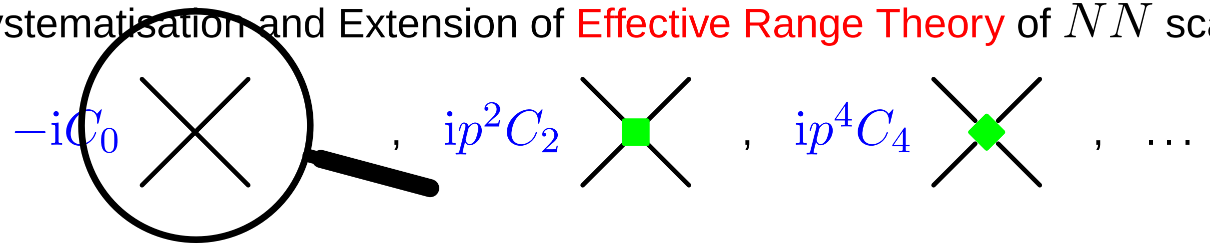
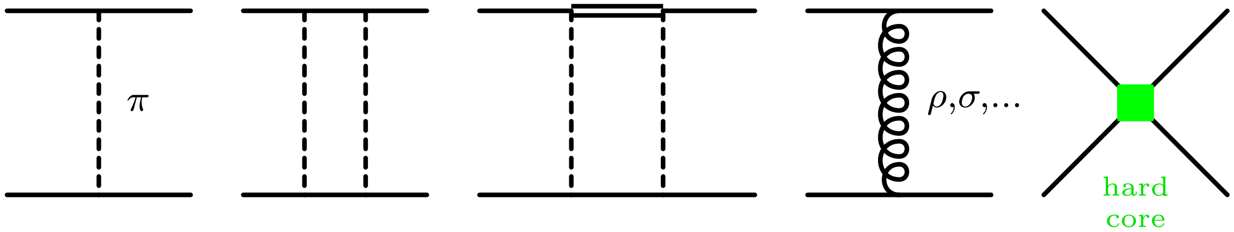


Coefficients C_{2n} encode UV physics:



...: pion integrated out

"We can do without that Pion crap." cf. Rob Timmermans

(c) Comments and Answers on EFT($\not\pi$) at Very Low Energies

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EFT($\not\pi$) does not exist in the Chiral Limit.

Yes. How close is the world to it?

(c) Comments and Answers on EFT($\not\pi$) at Very Low Energies

EFT($\not\pi$) is just Bethe's Effective Range Theory.

– True in the simplest system: NN scattering.

– No self-consistent Effective Range Theory in 3-Nucleon system. see later

– Beyond Effective Range:
- relativistic effects;
- manifestly gauge invariant & conserving symmetries;
- external and exchange currents,
- inelastic reactions,
- finite temperature/chemical potential,
- . . .

(c) Comments and Answers on EFT($\not{\pi}$) at Very Low Energies

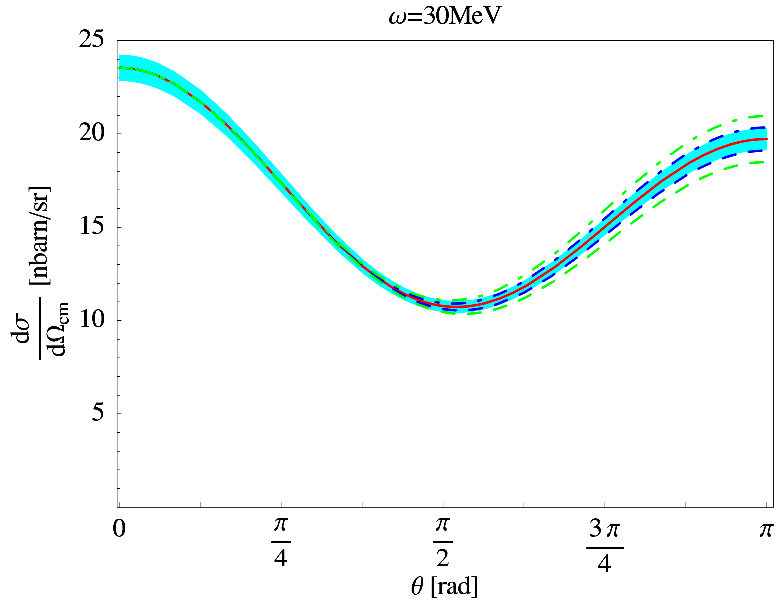
EFT($\not{\pi}$) is just a toy model. ("White-Cockroach-Argument")

Need for model-independent, systematic predictions and extractions of fundamental nucleon properties at $E < 10$ MeV:

– Fundamental neutron properties from light nuclei:
  How strong are nuclear binding effects?

  e.g. neutron-polarisabilities from $\gamma d \to \gamma d$ at 30 MeV

  (hg/Rupak 1999 for TUNL-proposal)



– Plethora of pivotal physical processes hard to access experimentally (rates, targets, . . .):
  - Big Bang Nuclear Synthesis,
  - neutrino-nucleus interactions, e.g. νd to calibrate SNO

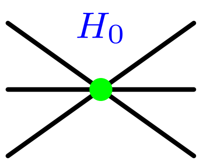
– Universality: Applications in e.g.
  - Λ-hypernuclei
  - atomic trimers (e.g. three ^{4}He atoms)
  - neutron-rich nuclei
  - loss-rates in Bose-Einstein Condensates

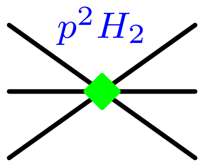
– *systematic* understanding of long-standing Nuclear Physics puzzles:

(d) Three-Body Forces in EFT($\not{\pi}$)

How good is Effective Range Expansion in 3-body system?

EFT: All interactions permitted by the symmetries of QCD $\Longrightarrow$ 3-body interactions already present:

$H_0\,(N^\dagger N)^3$: 

, $p^2\;H_2\,(N^\dagger N)^3$: 

, etc.

$\Longrightarrow$ Look for channels and observables most sensitive to these new forces.

How important are they? $\Longleftrightarrow$ **At which order in Q do they start to contribute?**

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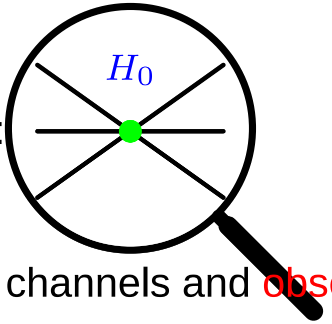
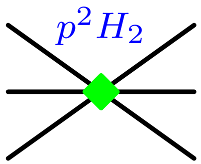
$H_0\,(N^\dagger N)^3$: 

, $p^2\ H_2\,(N^\dagger N)^3$: 

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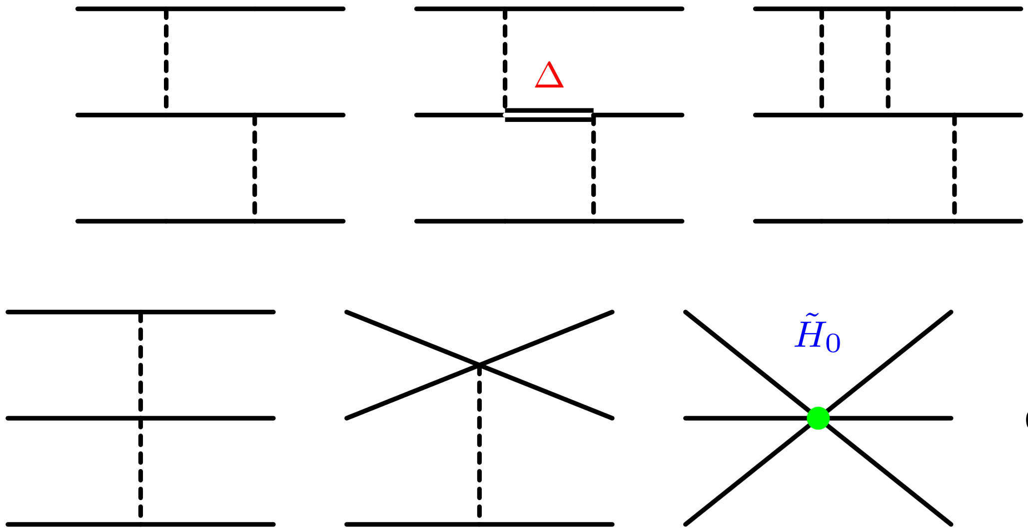


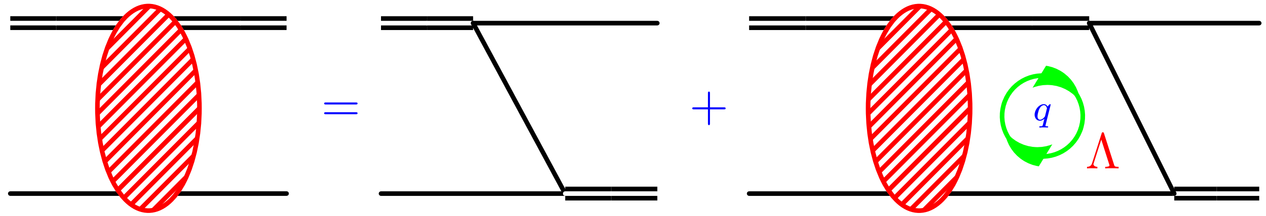
etc.

2. The Three-Body System: Neutron-Deuteron Scattering

(a) 3-Body System in EFT: Nd scattering, Quartet Channels

Sum deuteron and use iteration to obtain Faddeev integral equation for half off-shell amplitude $\mathcal{A}(k, q)$:
(Skorniakov/Ter-Martirosian 1957)

LO:

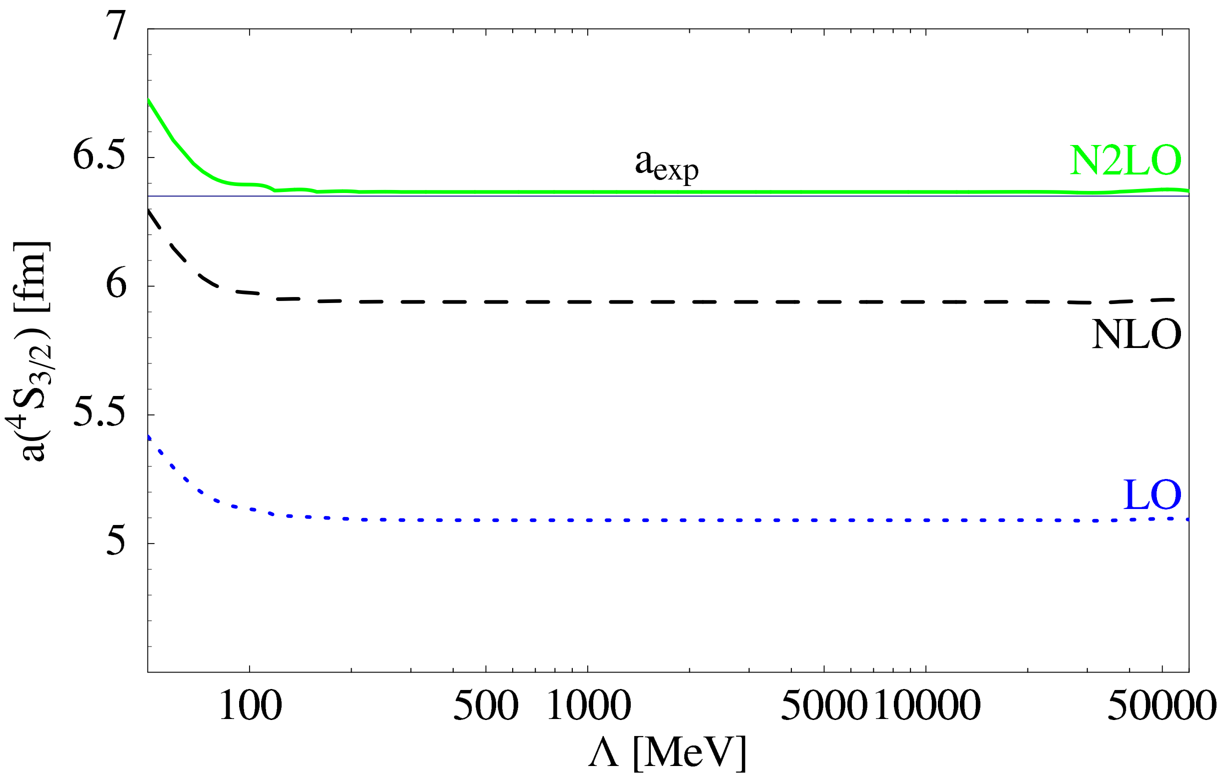


Observable cut-off independent,
good convergence.

parameter-free

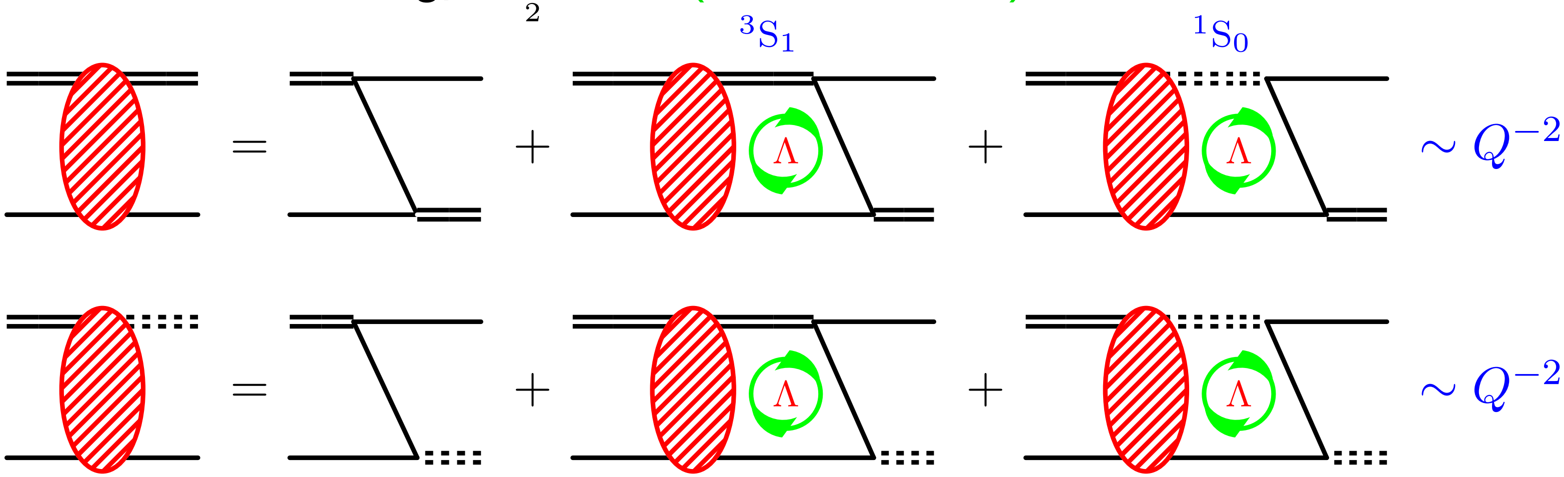
Quartet partial waves
and Doublet higher waves:

Pauli exclusion or centrifugal barrier
forbids momentum-independent 3BF.

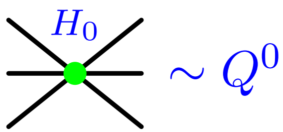


(LO: Skorniakov/Ter-Martirosian 1957, NLO: Efi mov 1991, N2LO: Bedaque/van Kolck 1998)

(b) The Problem: nd-Scattering, ${}^2\mathrm{S}_{\frac{1}{2}}$ Wave ("triton channel")



No Pauli principle, no centrifugal barrier $\Longrightarrow$ 3-body forces at N2LO?



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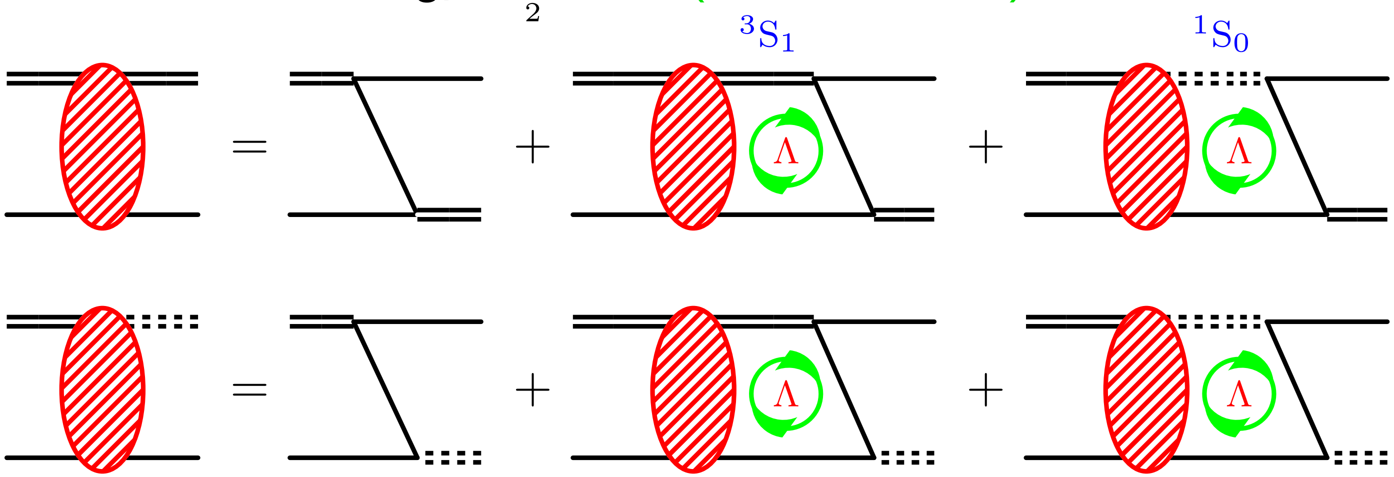

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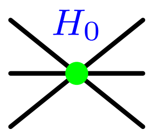
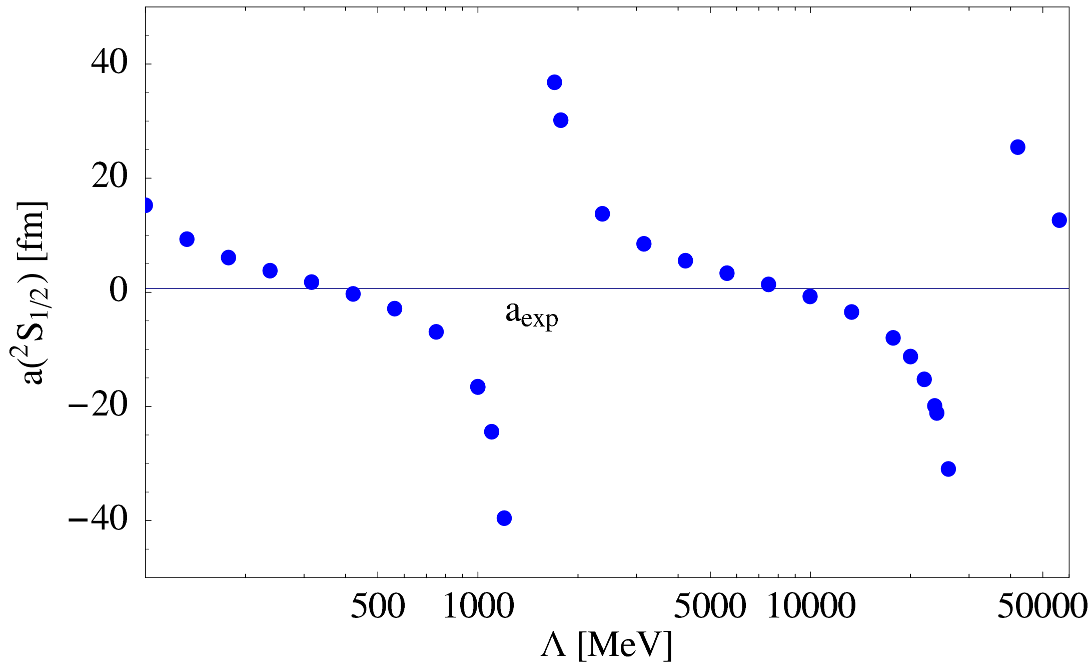


$\sim Q^0$



Slight cut-off variation has dramatic effect on scattering length $a(^2\mathrm{S}_{\frac{1}{2}})$.

Danilov, Minlos/Faddeev 1961

$\Longrightarrow$ No self-consistent Effective Range Expansion in 3-body system!

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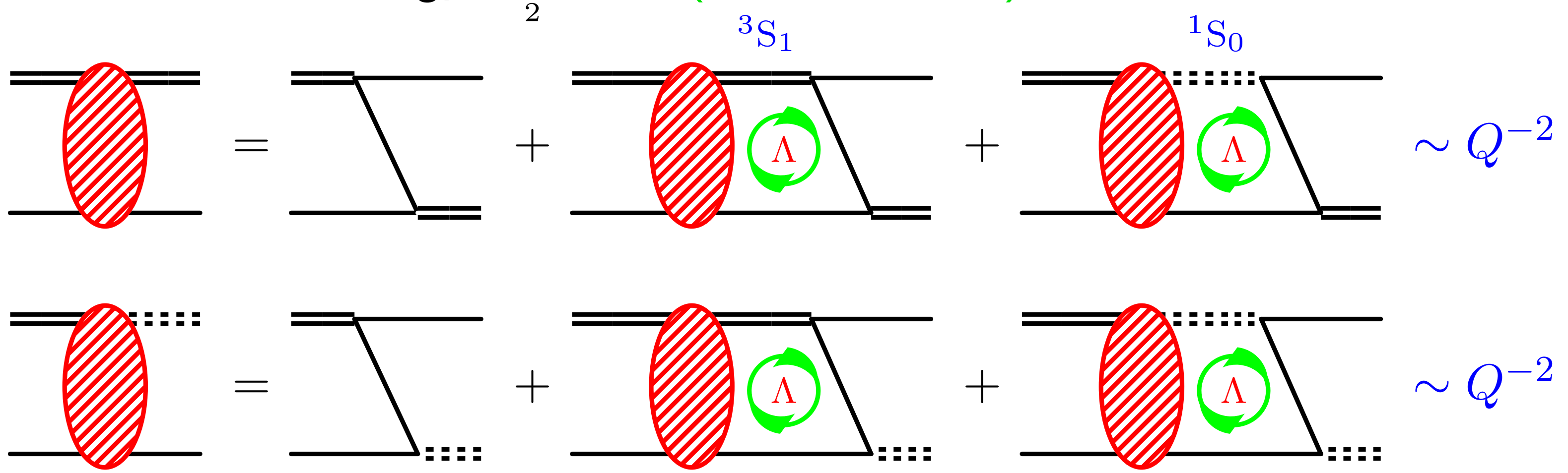

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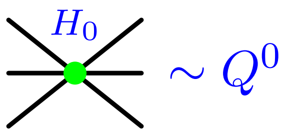

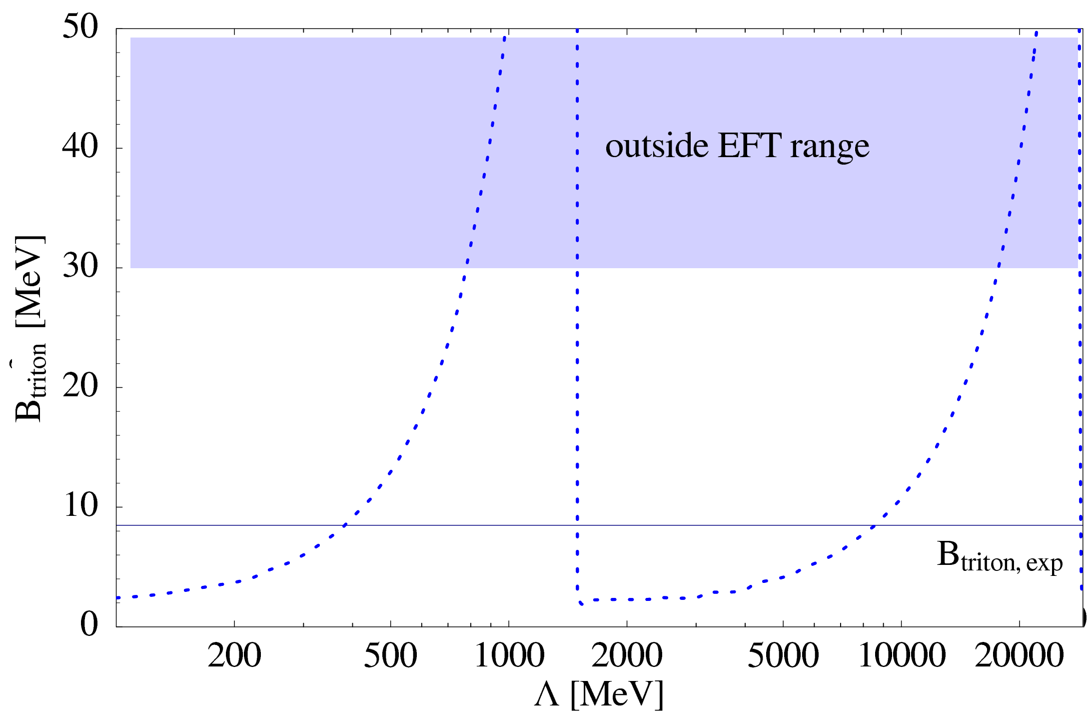

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Thomas Effect (1935):

$$\left[-\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R}+\frac{s_0{}^2}{R^2}-ME\right]F_0(R)=0.$$

$s_0^2 < 0$: attractive $\frac{1}{r^2}$-pot. has infinitely many, deeply bound states.

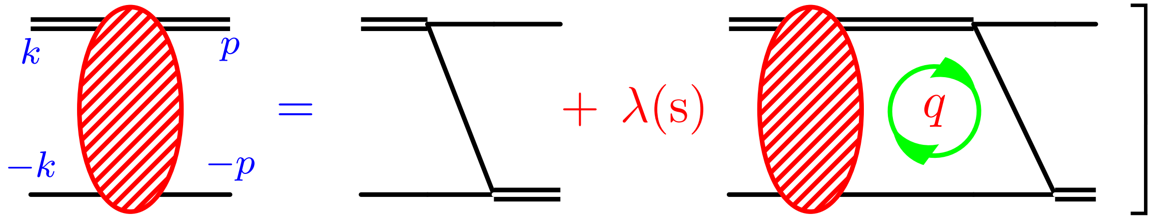
(c) Some Math

(Danilov 1961; Bedaque/Hammer/van Kolck 1998; hg 2000/03)

IR/long-range physics must be insensitive to UV/short-range.

$\Longrightarrow$ Look at UV, droping low-energy scales: $\Lambda \gg q \gg k \sim \gamma, \ldots$

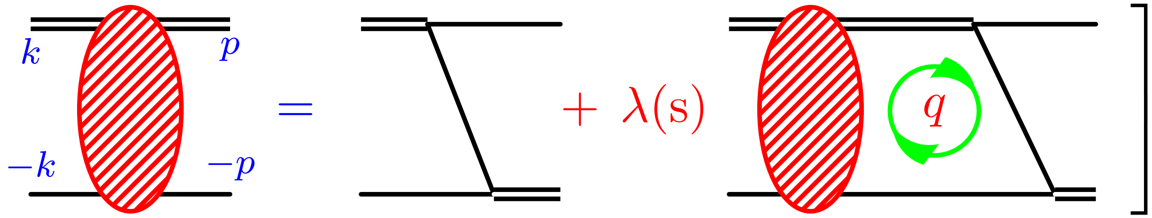
$\Longrightarrow$ Wigner's $\mathrm{SU}(4)$–symmetry of combined spin and iso-spin rotations



lth partial wave

Decoupled Faddeev eq. in UV dominated by zero mode, depending of partial wave l, spin s:

$$a_{(l,s)}(0,p) \quad = \quad (-)^l \, \frac{4\,\lambda(\mathrm{s})}{\sqrt{3}\,\pi} \int\limits_0^\infty \frac{\mathrm{d}q}{p} \; a_{(l,s)}(0,q) \; Q_l \left[\frac{p}{q} + \frac{q}{p}\right]$$



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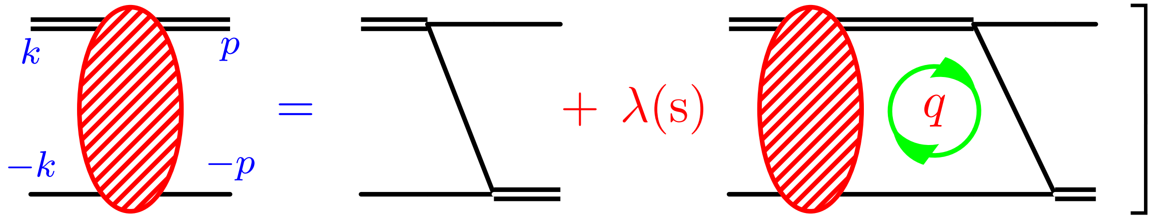
Ansatz: $a_{(l,s)}(0,p) \propto \dfrac{p^{-s_0}}{p}$ (Mellin)



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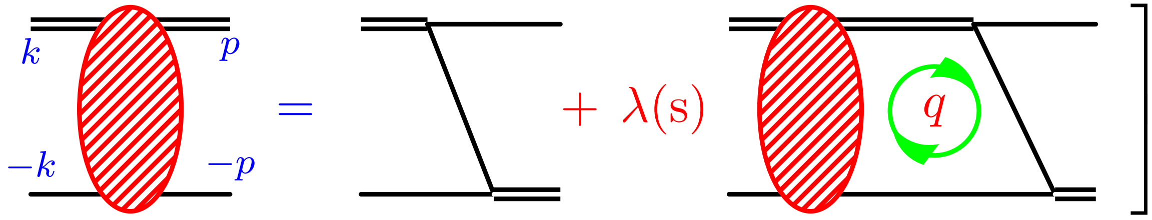
$$\Longrightarrow \quad 1 = (-)^l \frac{2^{1-l}\,\lambda}{\sqrt{3\pi}} \frac{\Gamma[\frac{l+1+s_0}{2}]\,\Gamma[\frac{l+1-s_0}{2}]}{\Gamma[l+\frac{3}{2}]}\; {}_2F_1[\frac{l+1+s_0}{2}, \frac{l+1-s_0}{2}; l+\frac{3}{2}; \frac{1}{4}]$$



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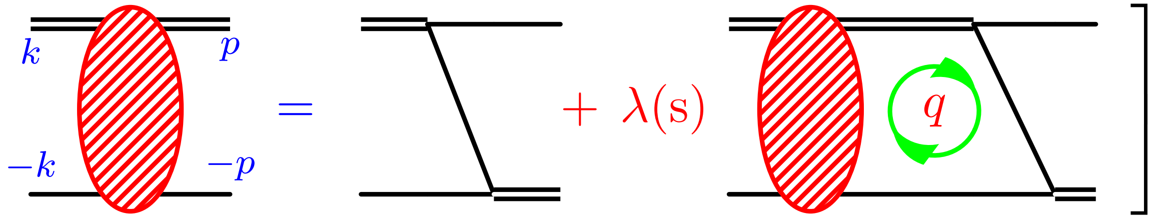
$s_0(l,s)$	$l=0$	$l=1$	$l=2$	$l=3$
Quartet ($s=\frac{3}{2}$): $\lambda=-\frac{1}{2}$	2.16	1.77	3.10	4.04
Doublet ($s=\frac{1}{2}$): $\lambda=1$	$\pm 1.0062\,\mathrm{i}$	2.86	2.82	3.92

Naïvely: $\frac{1}{p^2}$: $s_0=1$

UV-scaling not as guessed.



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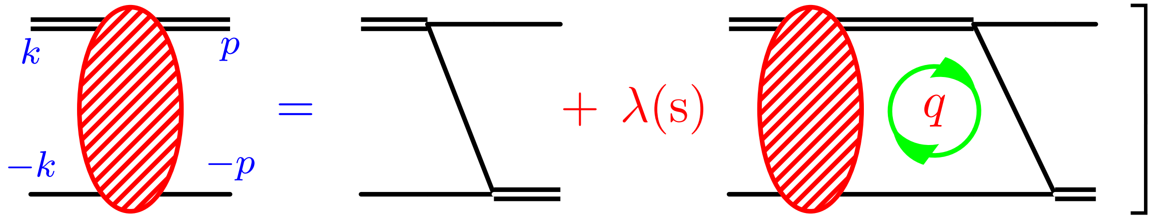
$$\mathcal{A}_{(l=0,s=\frac{1}{2})}(k\to 0,p) \propto \frac{\cos[1.0062\ln p+\delta]}{p}$$



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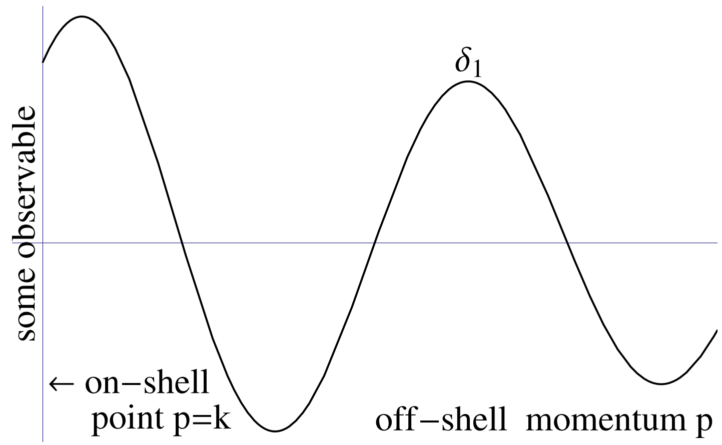
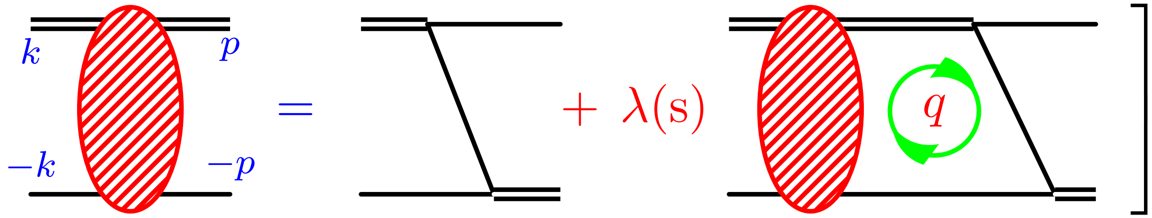




lth partial wave

UV dominated by zero mode $$a_{(l,s)}(0,p) \;=\; (-)^l\,\frac{4\,\lambda(\mathrm{s})}{\sqrt{3}\,\pi}\int\limits_0^\infty \frac{\mathrm{d}q}{p}\,a_{(l,s)}(0,q)\,Q_l\left[\frac{p}{q}+\frac{q}{p}\right]$$

Ansatz: $a_{(l,s)}(0,p) \propto \dfrac{p^{-s_0}}{p}$ (Mellin)

$$\Longrightarrow \quad 1 \;=\; (-)^l\,\frac{2^{1-l}\,\lambda}{\sqrt{3\pi}}\,\frac{\Gamma[\frac{l+1+s_0}{2}]\,\Gamma[\frac{l+1-s_0}{2}]}{\Gamma[l+\frac{3}{2}]}\;{}_2F_1[\frac{l+1+s_0}{2},\frac{l+1-s_0}{2};l+\frac{3}{2};\frac{1}{4}]$$

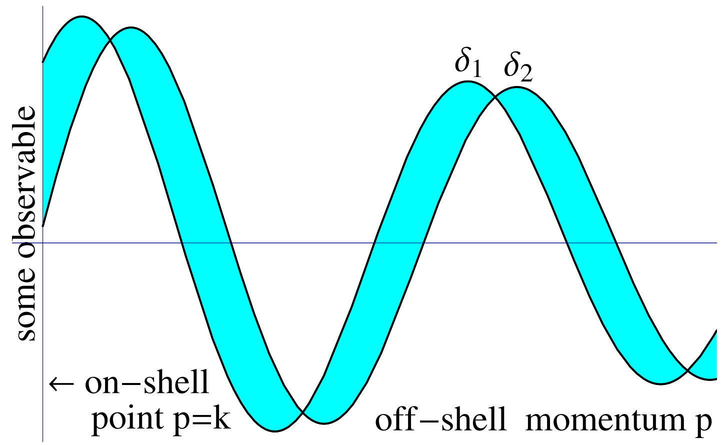
$s_0(l,s)$	$l=0$	$l=1$	$l=2$	$l=3$
Quartet ($s=\frac{3}{2}$): $\lambda=-\frac{1}{2}$	2.16	1.77	3.10	4.04
Doublet ($s=\frac{1}{2}$): $\lambda=1$	$\pm 1.0062\,\mathrm{i}$	2.86	2.82	3.92

Naïvely: $\dfrac{1}{p^2}$: $s_0=1$

UV-scaling not as guessed.

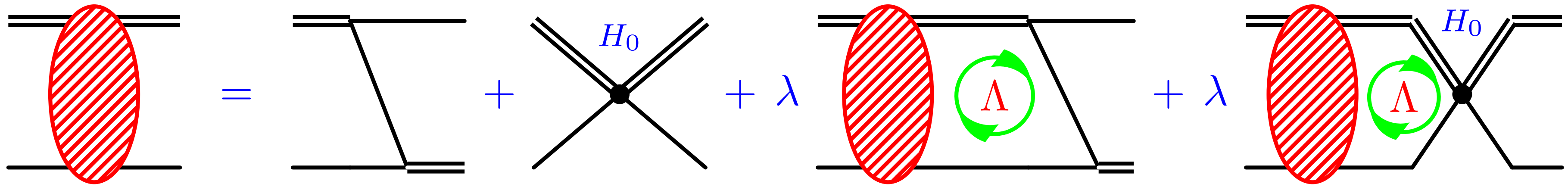
$\Longrightarrow$ On-shell depends on UV-phase δ only in triton ($^2\mathrm{S}_{\frac{1}{2}}$) channel:

$$\mathcal{A}_{(l=0,s=\frac{1}{2})}(k\to 0,p) \propto \frac{\cos[1.0062\ln p+\delta]}{p}$$



Tenet: Include specific 3BF if and only if needed to cancel off-shell dependence of observables.

Spin-flavour-symmetric three-body force with strength $H_0(\Lambda) \sim Q^{-2}$ to absorb cut-off dependence.



Tune running coupling $H_0(\Lambda)$ such that $\mathcal{A}$ cut-off independent in UV

$\mathcal{A}$ known **analytically** in UV $\Longrightarrow$ $H_0(\Lambda) = \dfrac{\sin[s_0 \ln \frac{\Lambda}{\Lambda_0} + \arctan s_0]}{\sin[s_0 \ln \frac{\Lambda}{\Lambda_0} - \arctan s_0]}$

Numerically:

Fix $\mathcal{A}$ to one observable, e.g. scatt. length.

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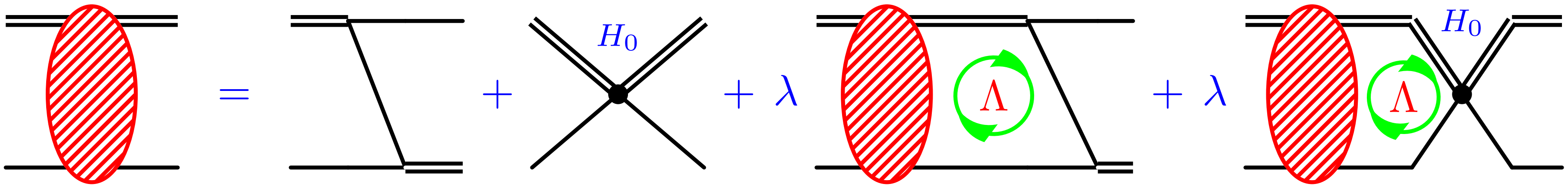

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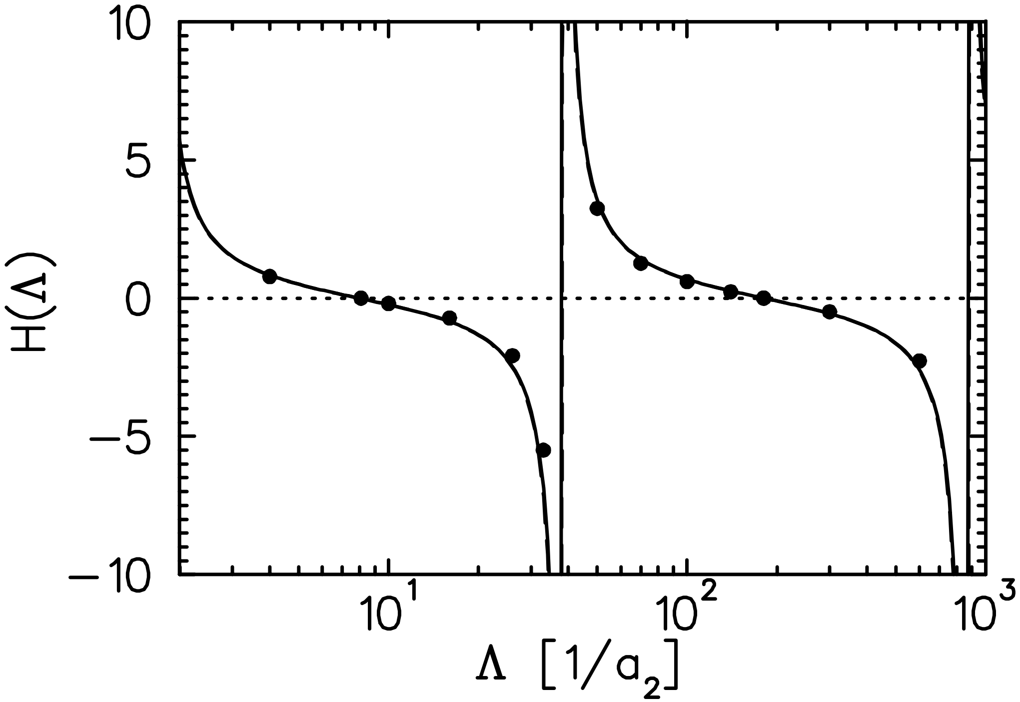

Numerically:

Fix $\mathcal{A}$ to one observable, e.g. scatt. length.

Limit Cycle of $H_0(\Lambda)$: new RG phenomenon.

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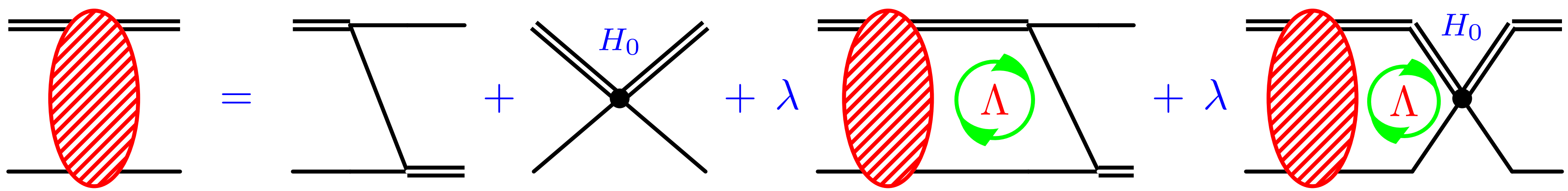

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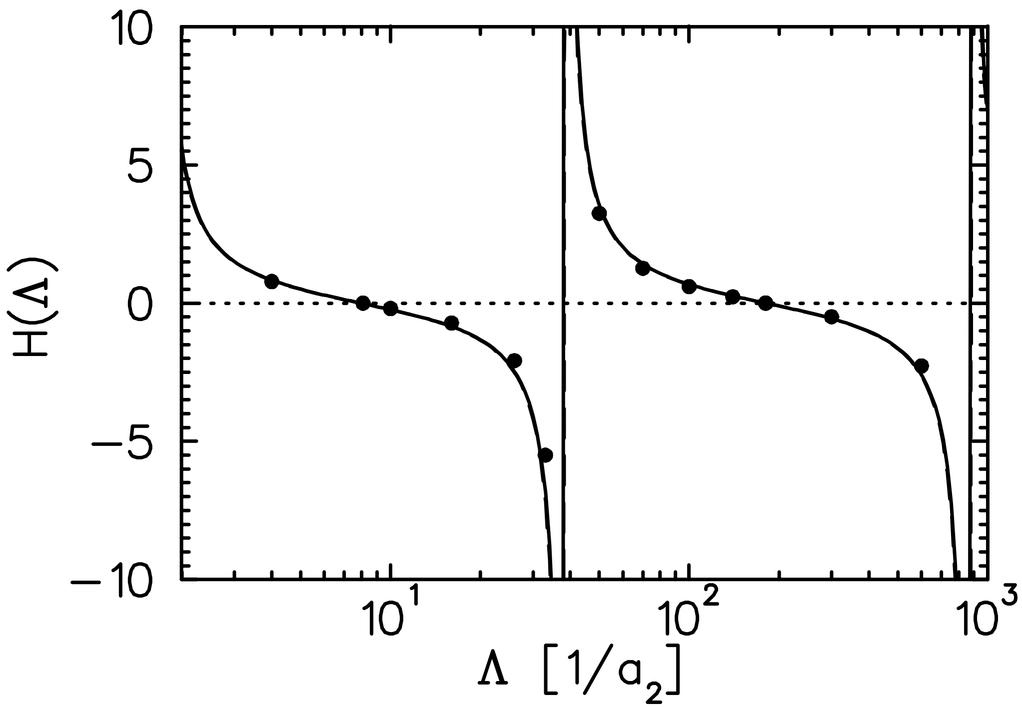


Numerically:

Fix $\mathcal{A}$ to one observable, e.g. scatt. length.

Limit Cycle of $H_0(\Lambda)$: new RG phenomenon.

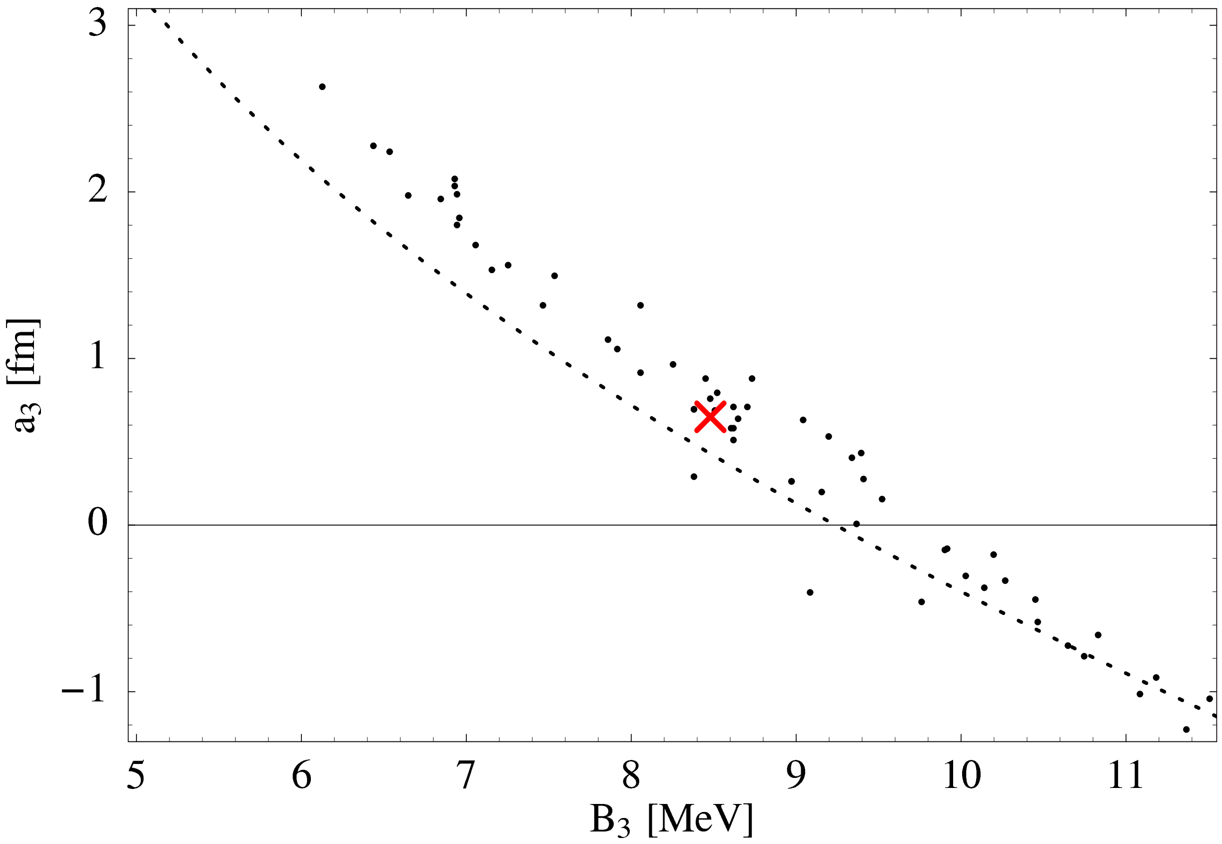
Different 2-body off-shell behaviour leads to different 3-body force.

Not three-body force is "large", but its effect!

Naïve dimensional analysis was too naïve!

Bedaque/Hammer/van Kolck 1998 cf. Efi mov

The one new, free parameter Λ_0 explains Phillips line of Nuclear Physics.



×: exp. $B_3 = 8.48\ \mathrm{MeV}$

LO EFT: $8.0\ \mathrm{MeV}$

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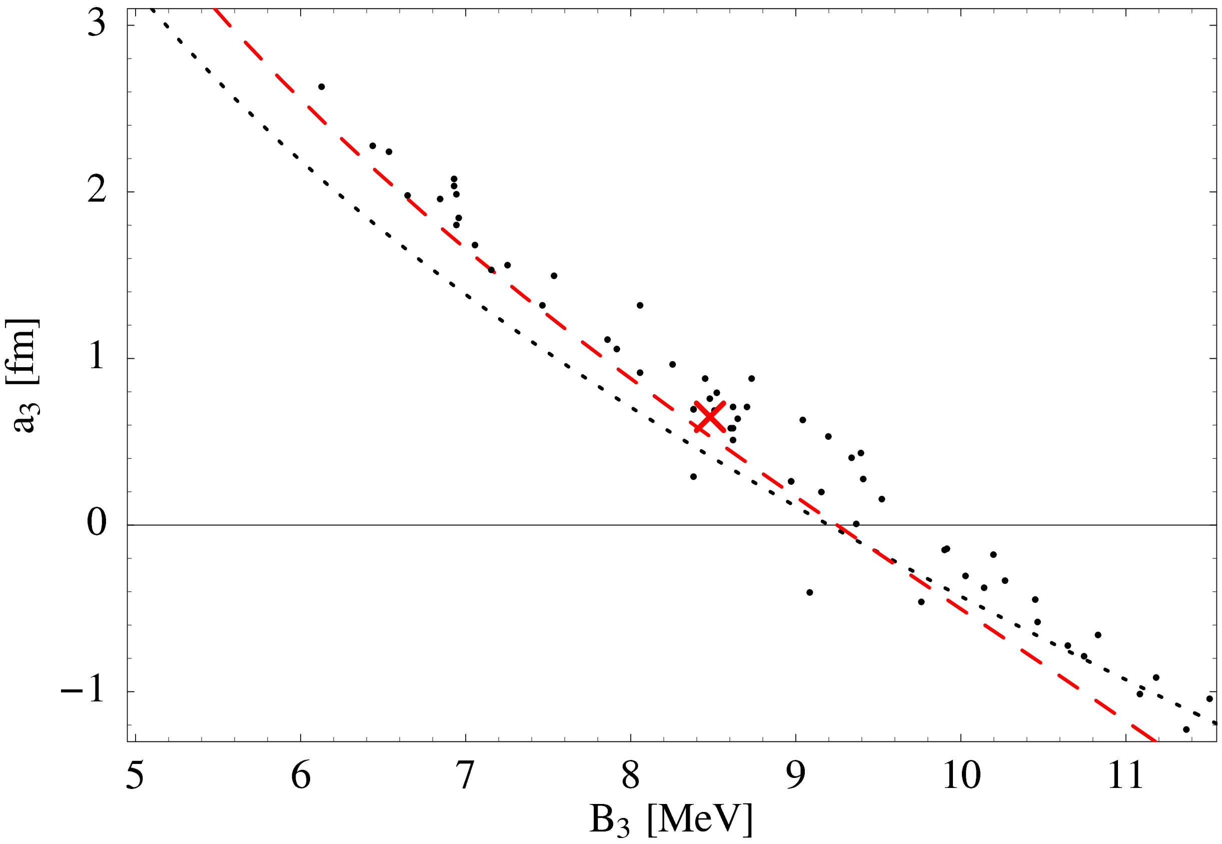

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LO EFT: 8.0 MeV

NLO EFT: 8.35 MeV

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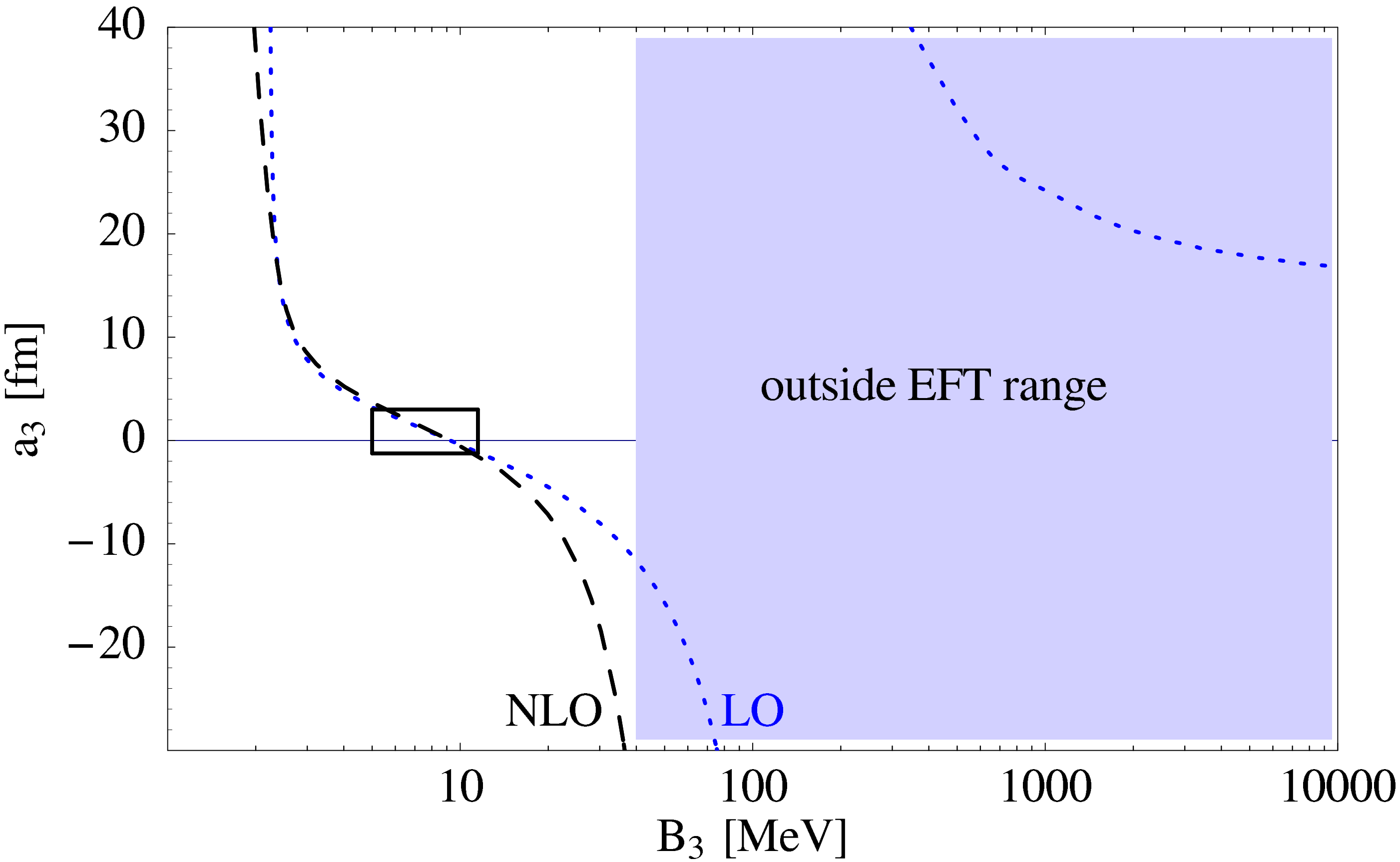


×: exp. $B_3 = 8.48$ MeV

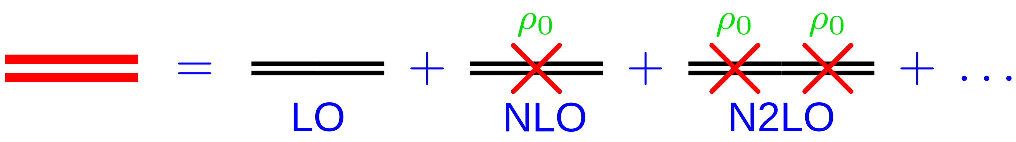
LO EFT: 8.0 MeV

NLO EFT: 8.35 MeV

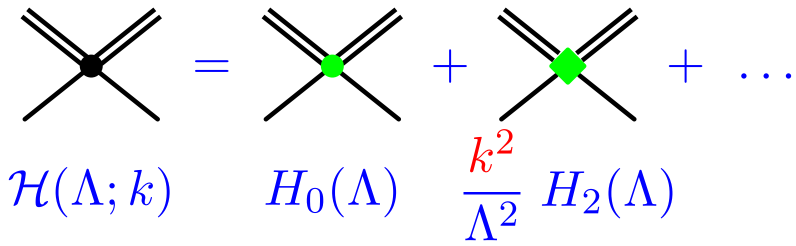
(d) Systematisation: Higher Order Corrections

(Bedaque/hg/Hammer/Rupak 2002)

- Improve deuteron by effective range ρ_0 etc.:



- Expand generic 3BF in on-shell momentum k:



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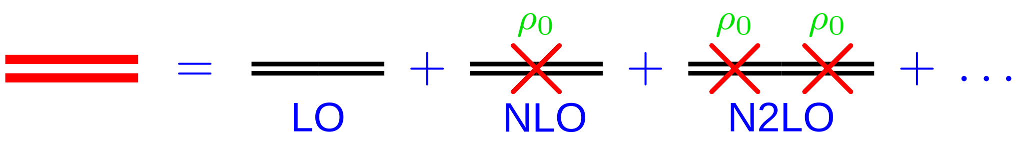

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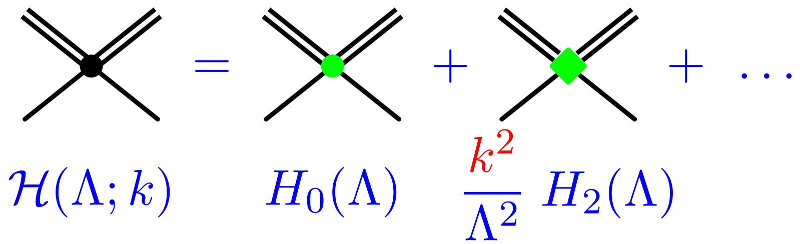

Wishlist:

– analytic at least in UV to establish power counting;

– numerically fast and simple.

(d) Systematisation: Higher Order Corrections

(Bedaque/hg/Hammer/Rupak 2002)

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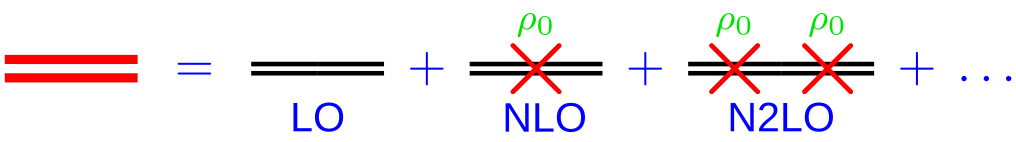

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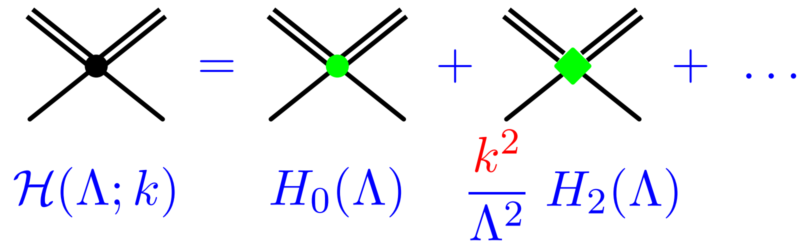
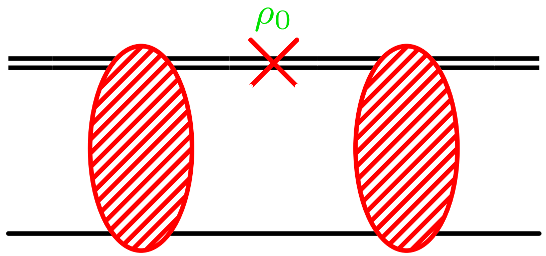


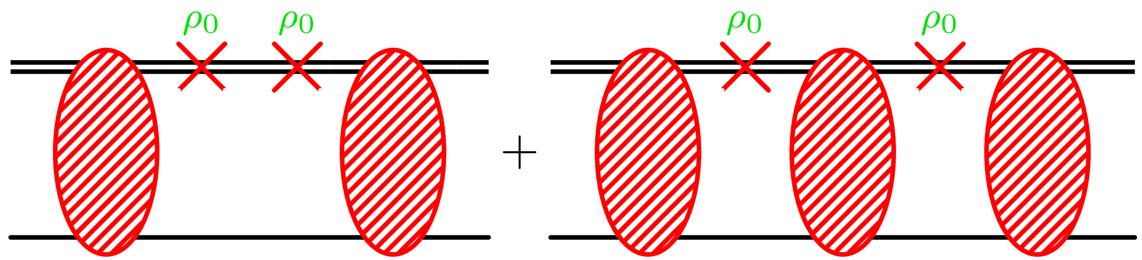
Strict Perturbation Theory

(NLO: Hammer/Mehen 2001)

NLO:



N2LO:



– Mix analytical and numerical renormalisation to cancel Λ^n divergences.

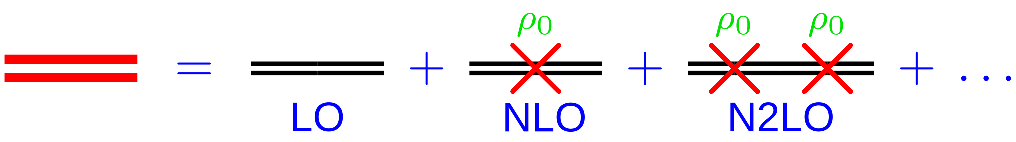
– Need full off-shell amplitude: numerically costly.

– Analytic running of $H_0(\Lambda),\ H_2(\Lambda)$?

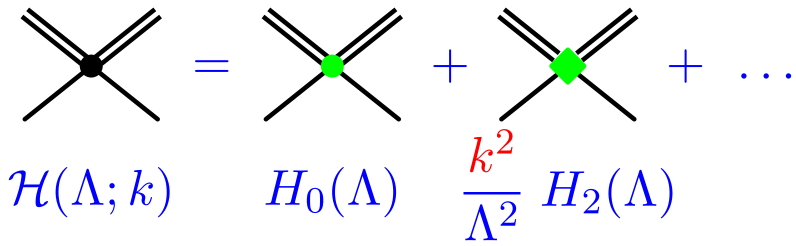
(d) Systematisation: Higher Order Corrections

(Bedaque/hg/Hammer/Rupak 2002)

- Improve deuteron by effective range ρ_0 etc.:



- Expand generic 3BF in on-shell momentum k:



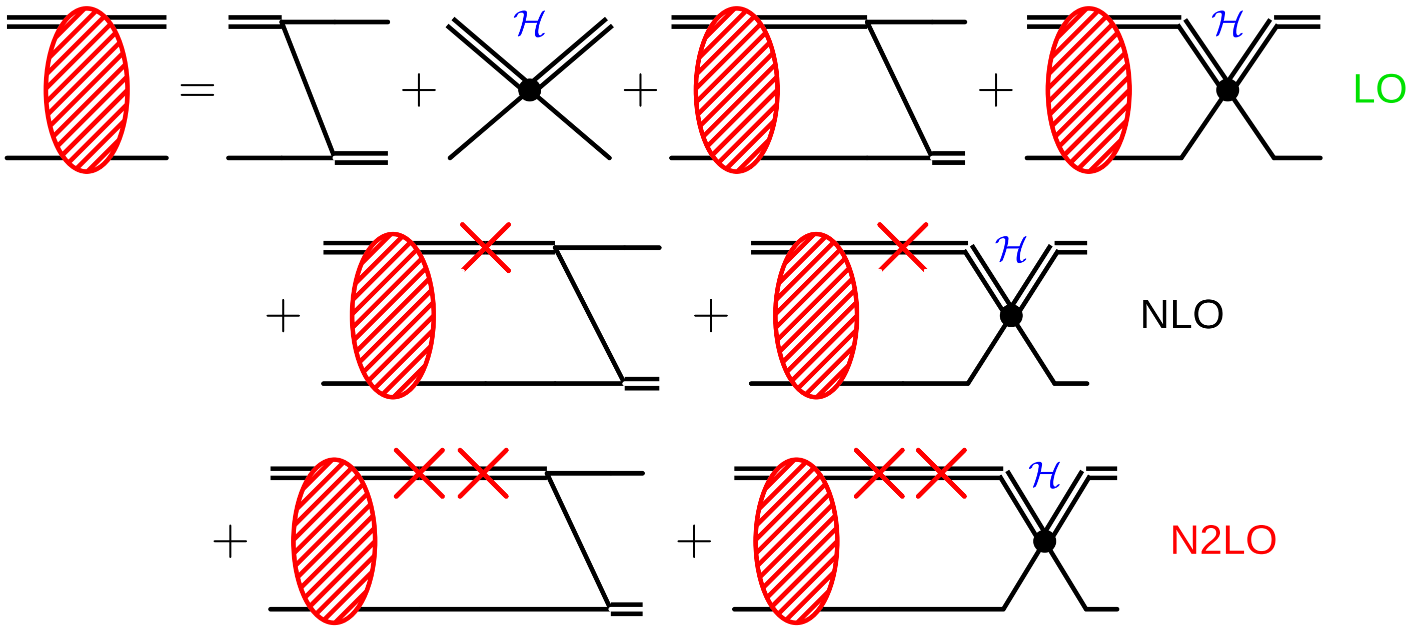
Perturbative in Kernel, Iterated

(Bedaque/hg/Hammer/Rupak 2002)

(i) Expand *potential* (kernel) in powers of Q and regularise.

(ii) Iterate kernel by inserting into integral equation.

$\Longrightarrow$ Include some (not all) higher-order graphs for convenience: not necessary, no increased accuracy.

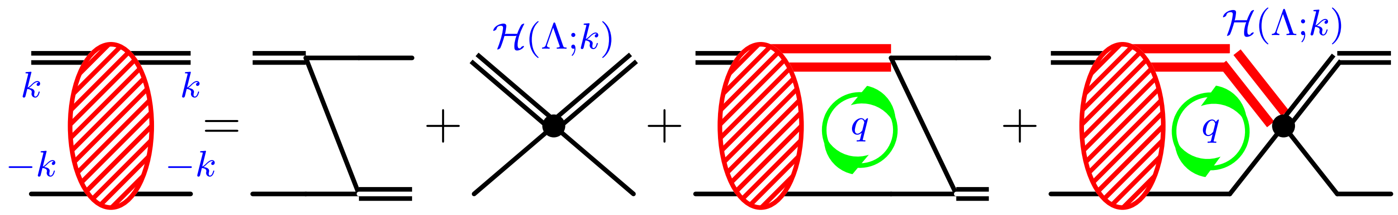


Classifying the Three-Body Forces with the EFT Tenet:

Include specific 3-body force $H_{2n}(\Lambda)$ if and only if needed as counter term to cancel off-shell dependence of low-energy observables which are stronger that that of neglected terms.

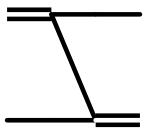
Cut-off dependence of on-shell $\mathcal{A}(k,k)$ from analytical, perturbative solution of Faddeev equation in UV limit:

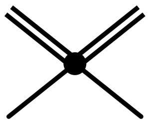
$k \sim \gamma \ll \dfrac{1}{\rho_0} \ll \Lambda,\ q$ (off-shell mom.) $\Longrightarrow$ Lowest 3-body forces Wigner-$\mathrm{SU}(4)$-symmetric.



LO

 $\longrightarrow \sqrt{\dfrac{4}{3}}\,\dfrac{1}{q}$

 $\longrightarrow \dfrac{1}{q^2}$

$\Longrightarrow$  $= H_0^{\mathrm{LO}}(\Lambda)$

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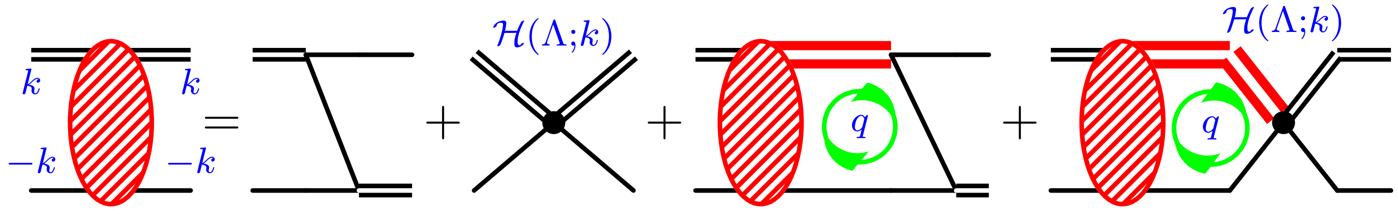
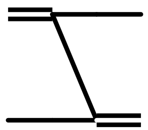
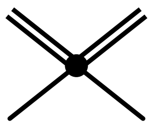


LO NLO

 $\rightarrow \sqrt{\dfrac{4}{3}}\,\dfrac{1}{q} \qquad + \dfrac{4\gamma}{3\,q^2} + \dfrac{\rho_0}{2}$

 $\rightarrow \dfrac{1}{q^2}$

$\Longrightarrow$  $= H_0^{\mathrm{LO}}(\Lambda) \qquad + H_0^{\mathrm{NLO}}(\Lambda)$

LO and NLO ($< 10\%$ accuracy):

One free parameter H_0,

fixed e.g. by triton binding energy.

Classifying the Three-Body Forces with the EFT Tenet:

Include specific 3-body force $H_{2n}(\Lambda)$ if and only if needed as counter term to cancel off-shell dependence of low-energy observables which are stronger that that of neglected terms.

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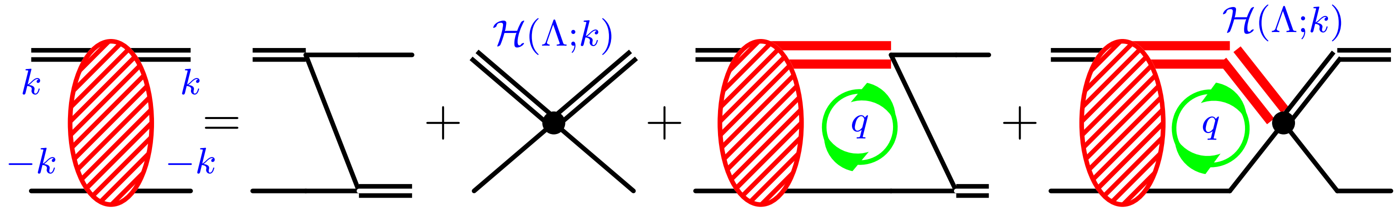
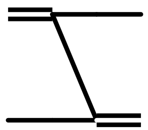
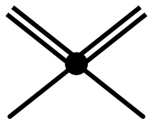


LO NLO N2LO

 $\rightarrow \sqrt{\frac{4}{3}}\,\frac{1}{q} \qquad + \frac{4\gamma}{3\,q^2} + \frac{\rho_0}{2} \qquad + \frac{3k^2+4\gamma^2}{3\sqrt{3}\,q^3} + \frac{2\gamma\rho_0}{\sqrt{3}\,q} + \frac{\sqrt{3}}{8}\,q\rho_0^2$

 $\rightarrow \frac{1}{q^2} \qquad + \frac{k^2 - 12\,\gamma^2}{12 q^2}$

$\Longrightarrow$  $= H_0^{\mathrm{LO}}(\Lambda) \qquad + H_0^{\mathrm{NLO}}(\Lambda) \qquad + \frac{k^2}{\Lambda^2}\,H_2(\Lambda) + H_0^{\mathrm{N2LO}}(\Lambda)$

LO and NLO ($<10\%$ accuracy):
One free parameter H_0,
fixed e.g. by triton binding energy.

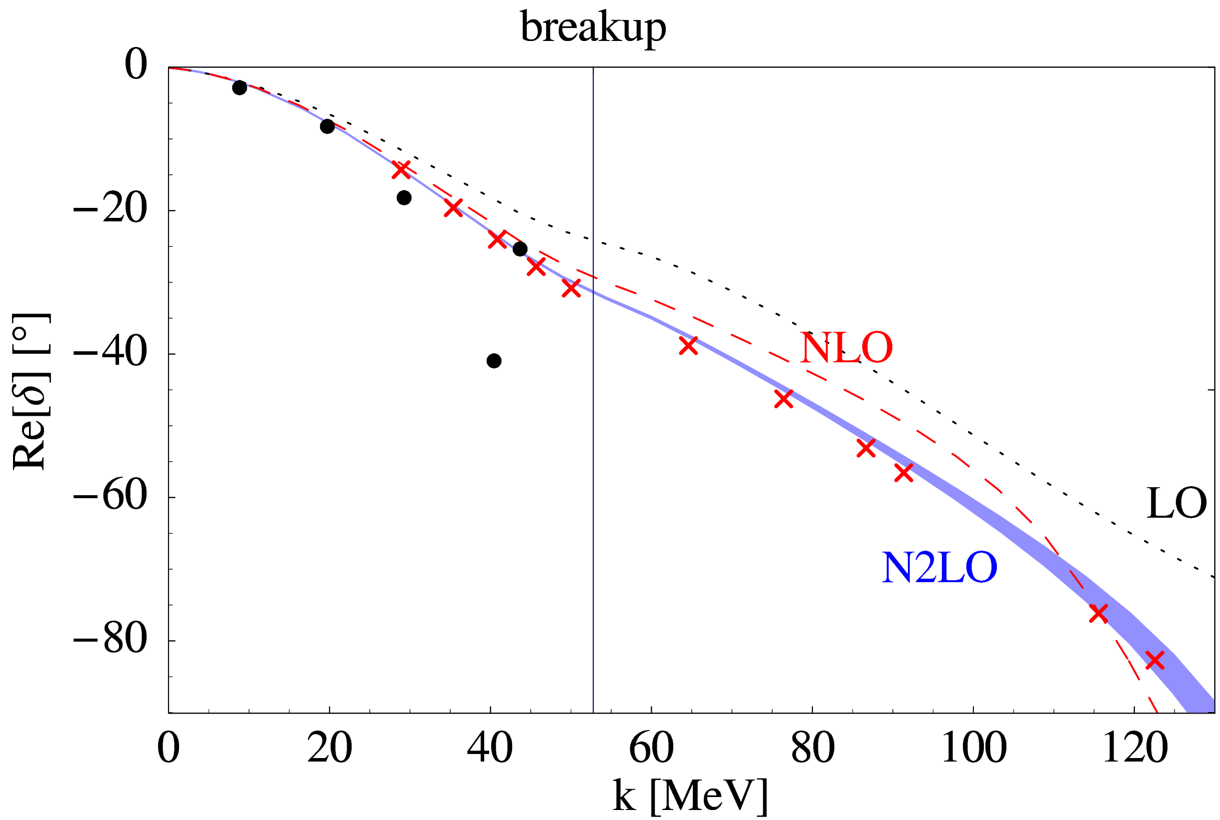
N2LO and N3LO ($<1\%$ accuracy):
One more free parameter H_2,
fixed best by scattering length.

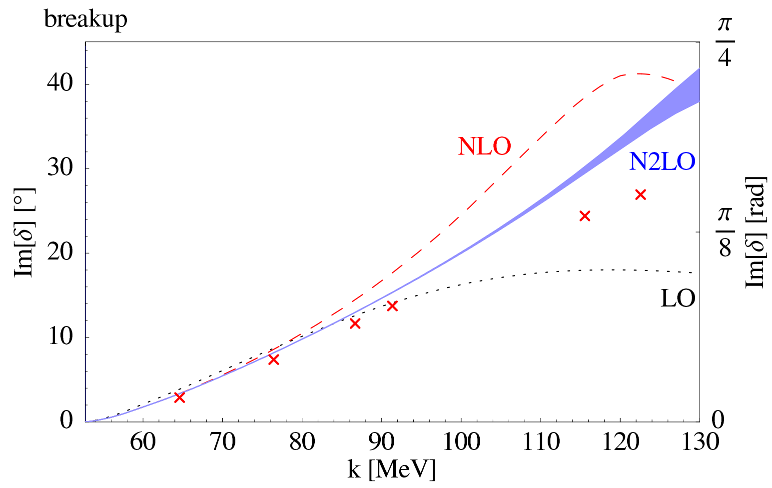
(e) Doublet-S Wave nd Phase Shift

(Bedaque/hg/Hammer/Rupak 2002)

Numerically: Fix H_0 to one observable (a_3).

N2LO: H_0&H_2 to $(B_3,\ a_3)$





×: AV18+U IX (Kievski 2002) ●: PWA 1967 (Seagrave/van Oers)

blue corridor: N2LO with $\Lambda \in [200;\infty]$ MeV: estimates higher order effects ↔ variation of resolution

$\Longrightarrow \mathcal{A}(k, p = k)$ on-shell cut-off independent

Agrees well with sophisticated, modern potential model calculations,

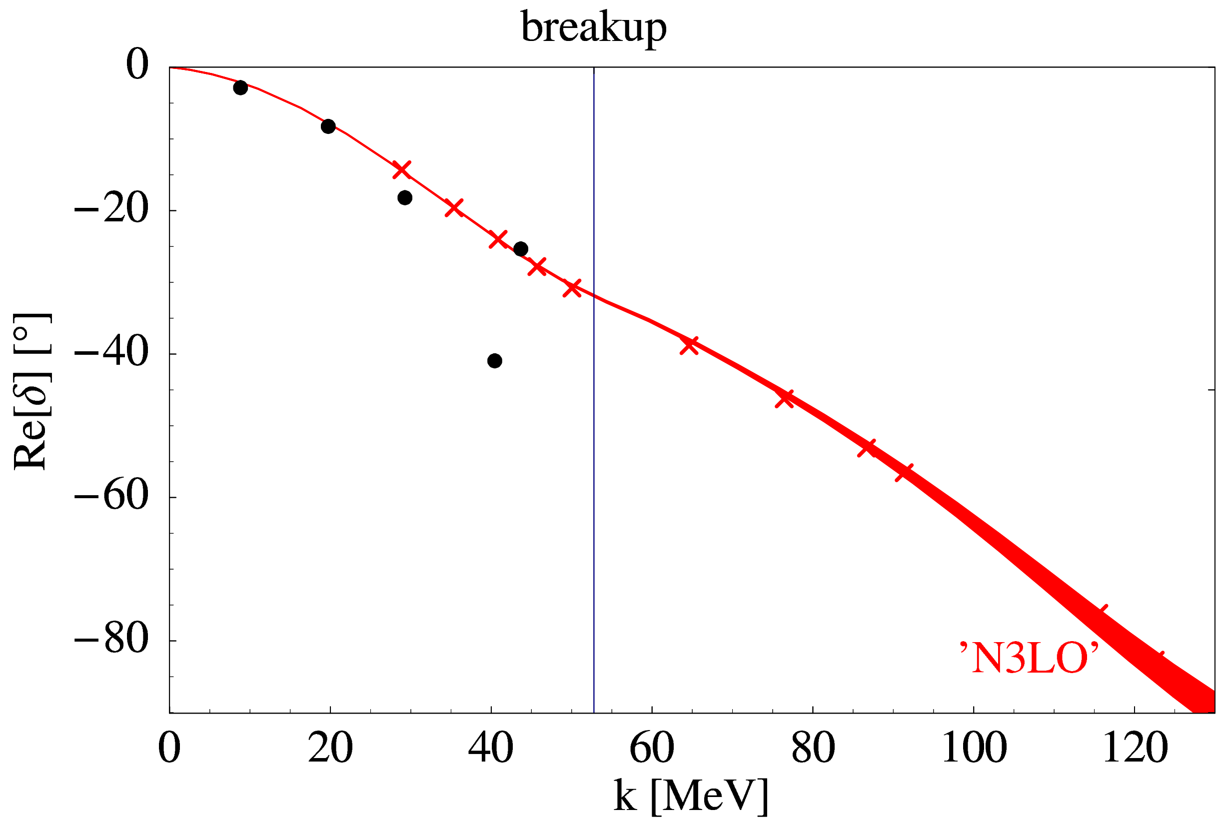
no free parameters after B_{triton} fixed, plus a_3 at N2LO.

(e) Doublet-S Wave nd Phase Shift

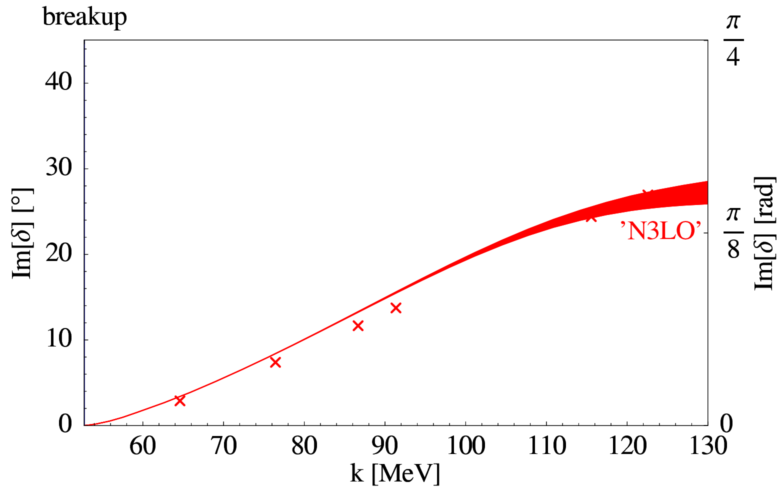
(Bedaque/hg/Hammer/Rupak 2002)

Numerically: Fix H_0 to one observable (a_3).

N2LO: $H_0 \& H_2$ to ($B_3,\ a_3$)



'N3LO': only eff. range effects, no partial-wave mixing.



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Agrees well with sophisticated, modern potential model calculations, no free parameters after B_{triton} fixed, plus a_3 at N2LO.

(f) A Problem with Experiment

Theory error by neglecting "higher order" interactions at N2LO: $1-3\%$

Experimentally induced error by uncertainty in nd scattering length: $6-10\%$

$$a_3(\text{exp}) = [0.64 \pm 0.04]\ \text{fm}$$ (Dilg/Koester/Nistler 1971)

Measure for triton wave function at low momenta: $\mathcal{A} \propto \dfrac{1}{k\cot\delta - ik}$ with $k\cot\delta\,|_{k=0} = -\dfrac{1}{a_3}$

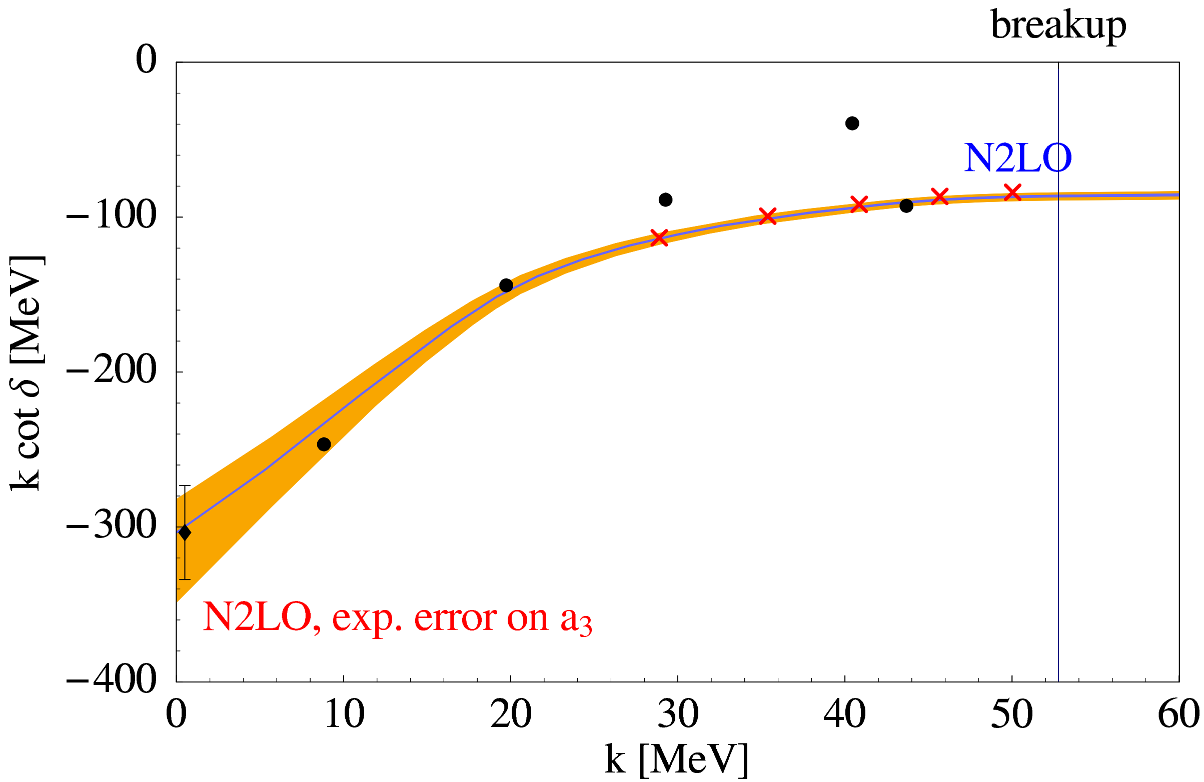
blue corridor: N2LO, $\Lambda \in [200; \infty]$ MeV

red corridor: EFT prediction when a_3 varied within exp. error

● PWA 1967 (Seagrave/van Oers)

× AV18+Urbana IX (Kievski 2002)

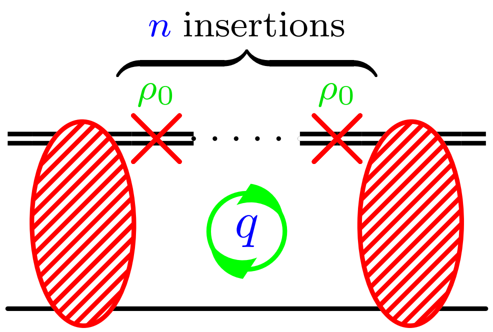
Most sensitive to change in scattering length at low momenta.



$\Longrightarrow$ Re-measure Doublet-S Wave scattering length: Zimmer/Glättli/hg et al. at PSI

(g) Leading Three-Body Forces in the 3-Nucleon Channels: Wigner-$\mathrm{SU}(4)$-symmetric (hg 2000/03)

Most divergent in perturbation theory with m external momenta:



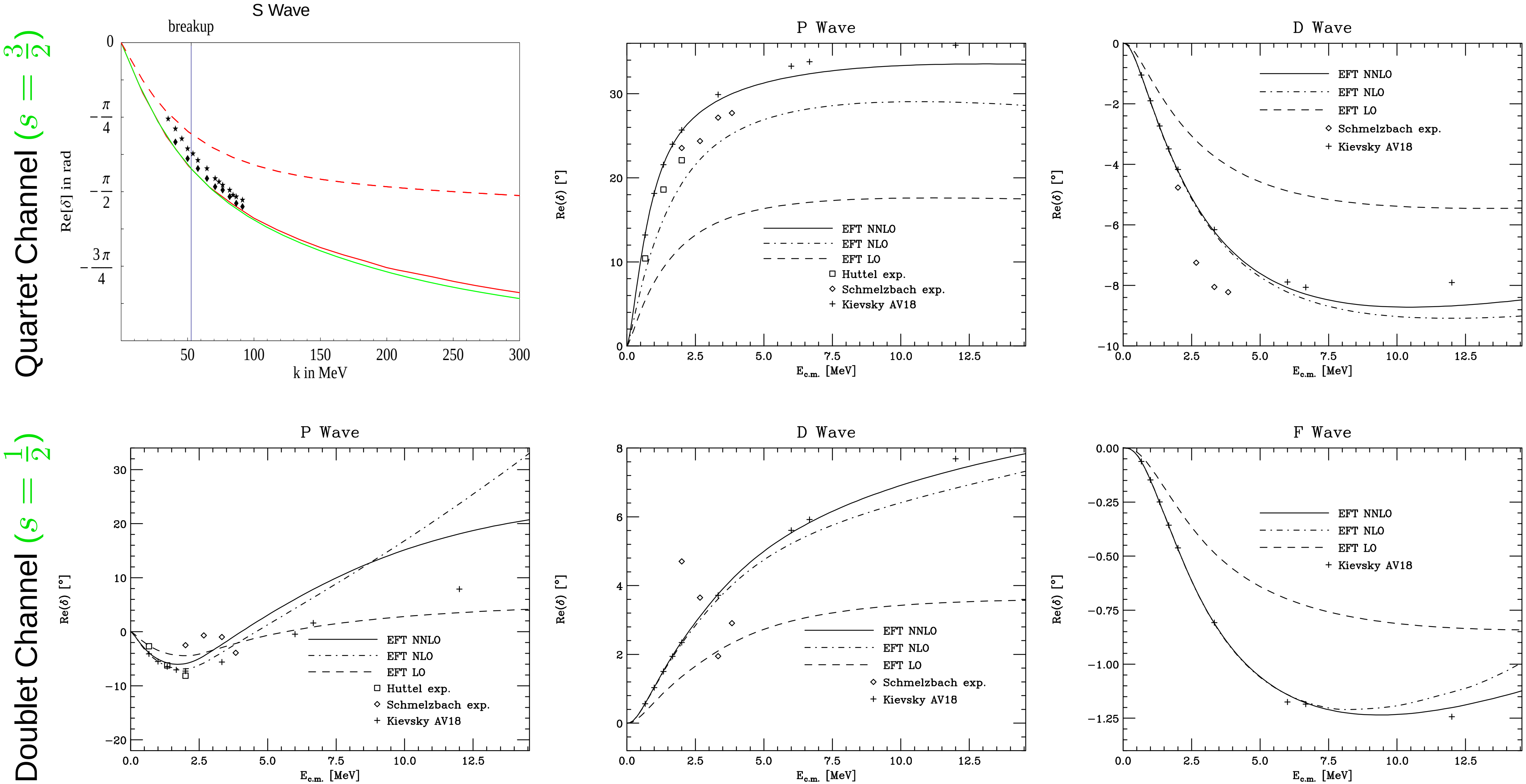
$$\xrightarrow{\text{UV}} \frac{1}{q^{2s_0+2}}\,\frac{q^3}{q}\,(\rho_0 q)^n\,\frac{k^m}{q^m} = k^m\,q^{n-m-2s_0}$$

3-body force with m derivatives as counter term needed when UV-divergent, i.e. $n \geq \mathrm{Re}[m+2s_0]$.

UV limit of LO amplitude $\mathcal{A}_{(l,s)}(0,q) \propto \dfrac{q^{-s_0(l,s)}}{q}$ from analytic solution.

partial wave	s_0	first 3-body force at m	order	naïve dim. an. $(s_0 = 1)$
Doublet-S	$\pm 1.0062\ \mathrm{i}$	0: H_0	LO	N^{2}LO: Promoted
		2: H_2	N^{2}LO	N^{4}LO: Promoted
Doublet-P	2.86	2	N$^{7.72}$LO	N^{3}LO: Demoted
Doublet-D	2.82	4	N$^{9.64}$LO	N^{4}LO: Demoted
Quartet-S	2.16	2	N$^{6.32}$LO	N^{4}LO: Demoted
Quartet-P	1.77	2	N$^{5.54}$LO	N^{3}LO: Demoted
Quartet-D	3.10	4	N$^{10.2}$LO	N^{4}LO: Demoted

Bedaque/hg: NP**A671**(2000), 357; Bedaque/Gabbiani/hg: NP**A675**(2000), 601.



Numerically simple: N2LO code runs within a minute on PC

Agrees well with sophisticated, modern potential model calculations.

N3LO (3-body force!): Splitting/mixing of partial waves $\Longrightarrow A_y$

(h) Universality of EFT: Lessons Learned Applied Elsewhere

In systems with anomalously large 2-body scattering length/shallow 2-body bound state.

Hypertriton Λnp: $B_3 = [0.13 \pm 0.05]\ \mathrm{MeV}$ (exp.) **(Hammer 2001)**

$$\Longrightarrow a_{\Lambda d} = 16.8^{+4.4}_{-2.4}\ \mathrm{fm},\ r_{\Lambda d} = [2.3 \pm 0.3]\ \mathrm{fm}$$

Atomic Physics **(Bedaque/Hammer/van Kolck 1999)**

Bound state of two He-atoms ("He-dimer"): $B_3 = [62 \pm 10]$ Å

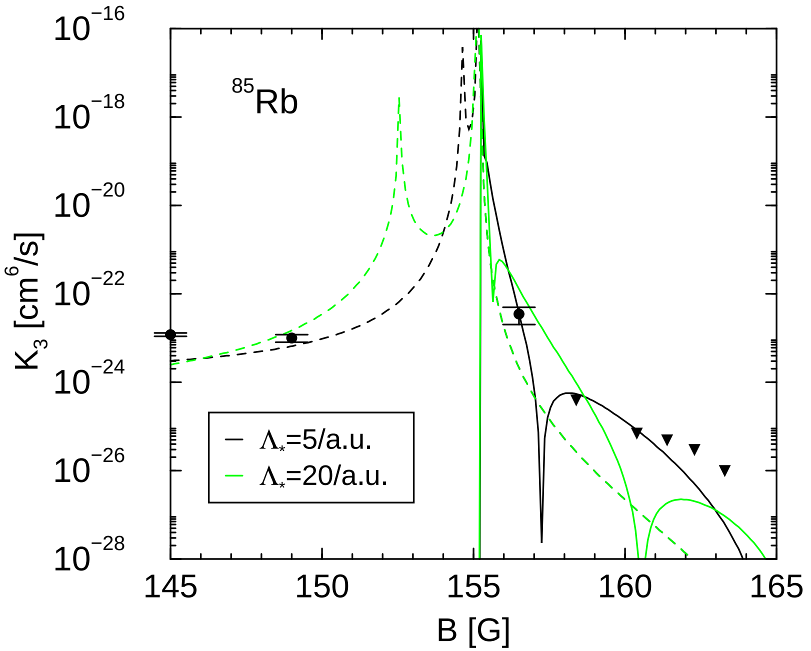
Prediction for $^4\mathrm{He}$ trimer: Bound state energy not calculable from dimer properties only.
If $a_3 = 195$ Å, predict $B_3 = [1.2 \pm 0.1]\ \mathrm{mK}$.

Bose-Einstein Condensates **(Braaten/Hammer 2001)**

Feshbach resonances: two-body scattering length tunable at will by magnetic field.

Strong contribution to loss rate from three-particle recombination into shallow three-particle bound state.

Compares well to experiment.



3. Summary and Rewards

(a) Summary

– Few-nucleon system at very low energies: 3-body force puzzle $\Longrightarrow$ Systematic, model-independent approach: **Effective Field Theory** with local interactions between nucleons only.

– Systematic classification of all three-body forces achieved.

– In most channels, 3-body force only for very high precision.

– Doublet-S wave (triton as bound state): 3-body observables independent of high-energy/off-shell behaviour only if spin-iso-spin symmetric 3-body forces included. Naïve power counting is naïve.

– Limit Cycle: Only combination of 2-body and 3-body force physically meaningful.

– Need 2 empirical three-body data for 1% accuracy: triton binding energy, nd scattering length in triton channel (to be determined better!).

– Wide range of applications: $nd \rightarrow t\gamma$, calibrating SNO, BBN, A_y, atomic trimers, BEC, etc.

– Successful test of EFT methods.

Bedaque/hg/Hammer/Rupak: *Nucl. Phys.* **A714** (2003), 589 [nucl-th/0207034]
Bedaque/hg: *Nucl. Phys.* **A671** (2000), 357 [nucl-th/9907077]
Bedaque/Gabbiani/hg: *Nucl. Phys.* **A675** (2000), 601 [nucl-th/9911034]

Mathematica code: `http://www-nsdth.lbl.gov` or `http://www.ph.tum.de/~hgrie`

(b) Challenges and Rewards Once H_0 and H_2 are Fixed

- model-independent understanding of 3BF at low p to $\lesssim 1\%$ accuracy;
- accurate triton wave function;
- Can H_0, H_2 be saturated by one-pion exchange?
- fundamental three-body processes like $nd \to t\gamma$ at thermal energies:

  AV 18 + Urbana IX: $\sigma = 0.6$ mb   experiment: $\sigma = [0.508 \pm 0.015]$ mb
- triton form factors, Compton scattering (nucleon polarisabilities);
- comparing ^{3}H and ^{3}He: iso-spin breaking, charge dependence;
- triton β-decay: calibrating SNO
- stellar & primordial nucleosynthesis,

  e.g. deuteron abundance sensitive to primordial baryon density.

  theory side uncertainty $\approx$ observational error

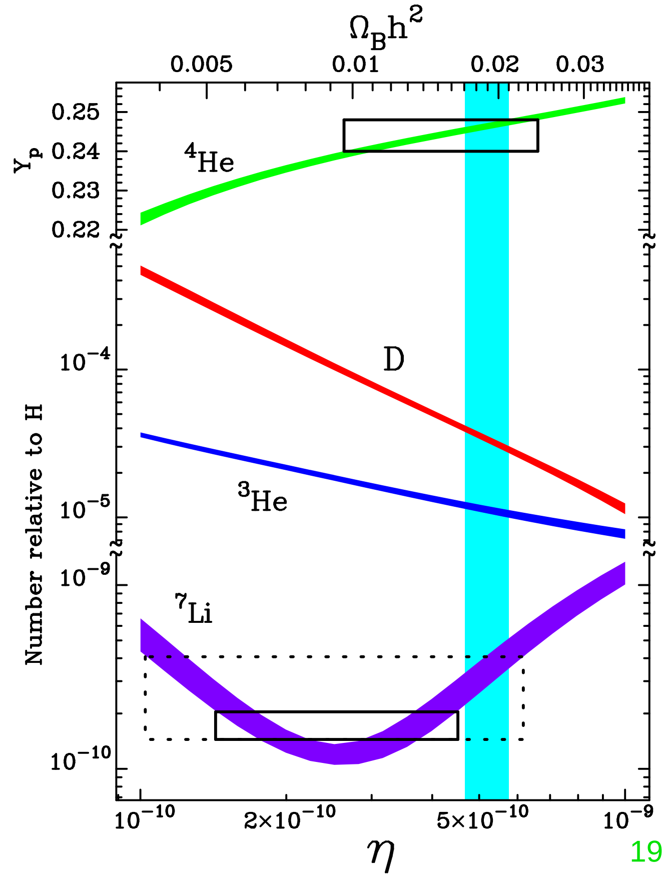
  $dd \to pt$@100 keV:

  Need $2-3\%$ accurate cross-sections at $30-300$ keV.

  ^{3}H and ^{3}He wave functions biggest source of uncertainty.

  30% of error bar in d abundance

  also $dt \to n\alpha$, $n^3\text{He} \to pt$



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Comparing Deuteron Wave Functions: Unnatural scales in NN

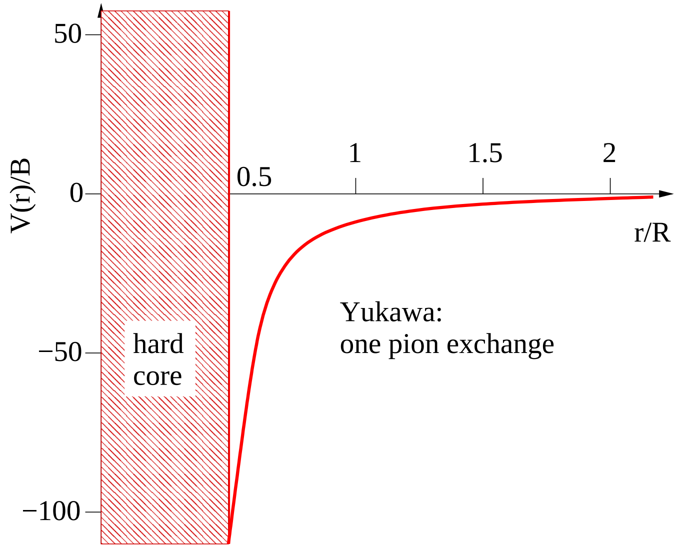
$$-1/a(^1\mathrm{S}_0) = 8\ \mathrm{MeV},$$

$$\gamma_{\mathrm{deut.}} = 45\ \mathrm{MeV} \ll m_\pi,\ \Lambda_{\mathrm{QCD}}$$

Potential models use (unphysical) hard core

at short distances to regulate One Pion Exchange:

Fine-tuning.



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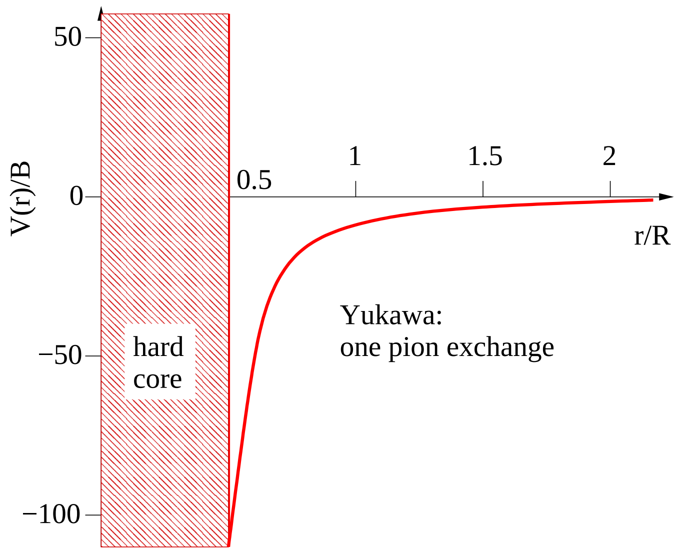

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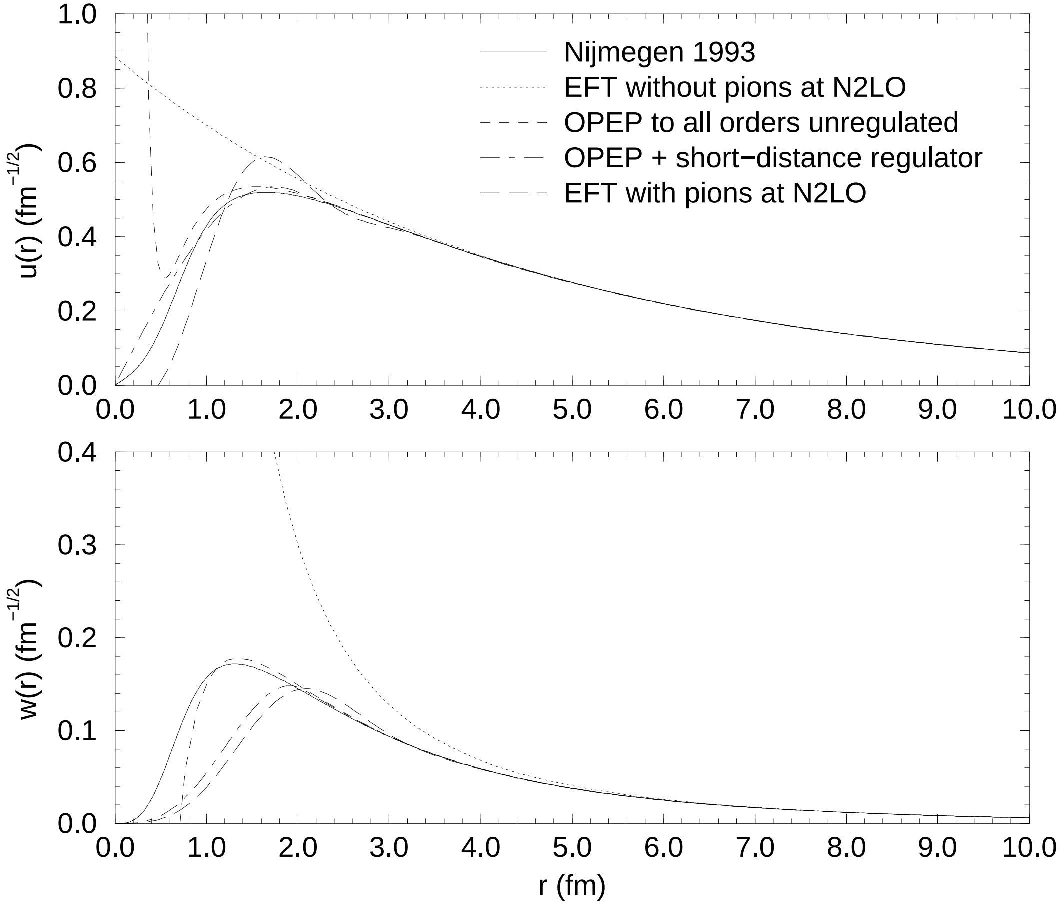
$^3\mathrm{S}_1$

$^3\mathrm{D}_1$



Short distance very different

but unimportant,

long distance asymptotics identical:

$\propto \frac{\mathrm{e}^{-\gamma r}}{r}$,

mid distance feels pion effects.

Systematic expansion in

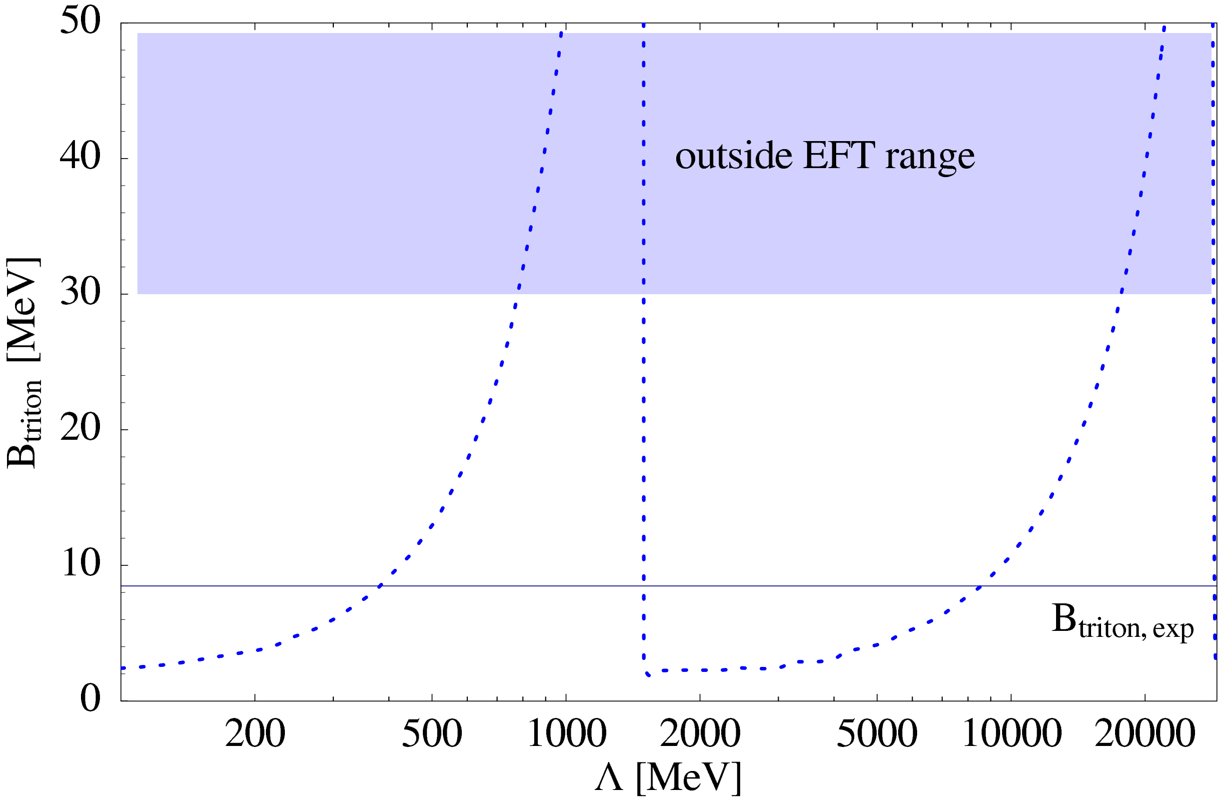
$$Q = \frac{\text{typ. momentum}}{\text{breakdown scale}} = \frac{\text{target size}}{\text{resolution}}$$

Are the other observables cut-off independent, too?

numerically: H_0 such that scattering length $a_3 = 0.64$ fm,
i.e. on-shell amplitude cut-off independent at $\mathcal{A}(k = 0, p = 0)$.

Is on-shell amplitude $\mathcal{A}(k \neq 0, p = k)$ cut-off independent everywhere?

$\Longrightarrow$ Calculate position of pole in $\mathcal{A}$: Triton binding energy B_3.



Cut-off sensitive without three-body force

Once a_3 given, B_3 fixed. $\Longrightarrow$ Explains Phillips line.

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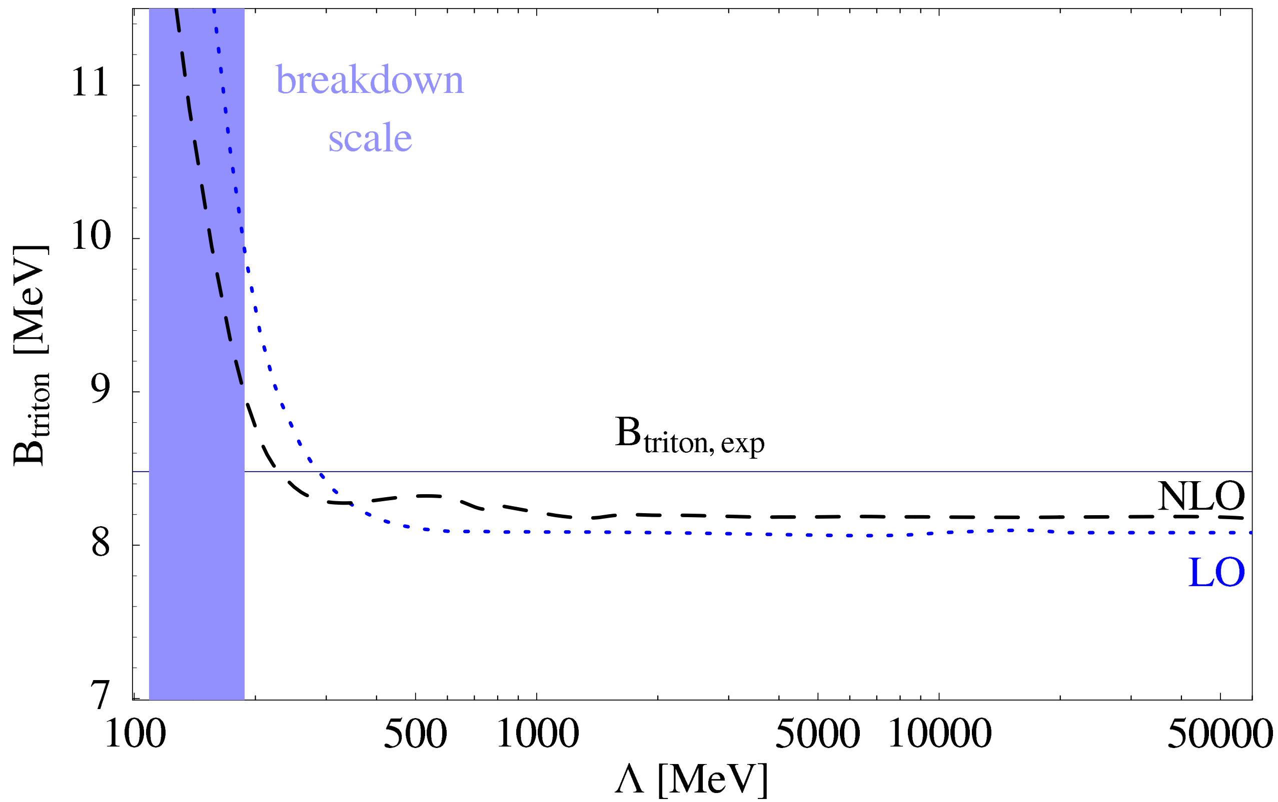

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