INT Mini-Workshop:
Dilute Strongly Interacting Fermions
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Superfluids with Mismatched Fermi Surfaces

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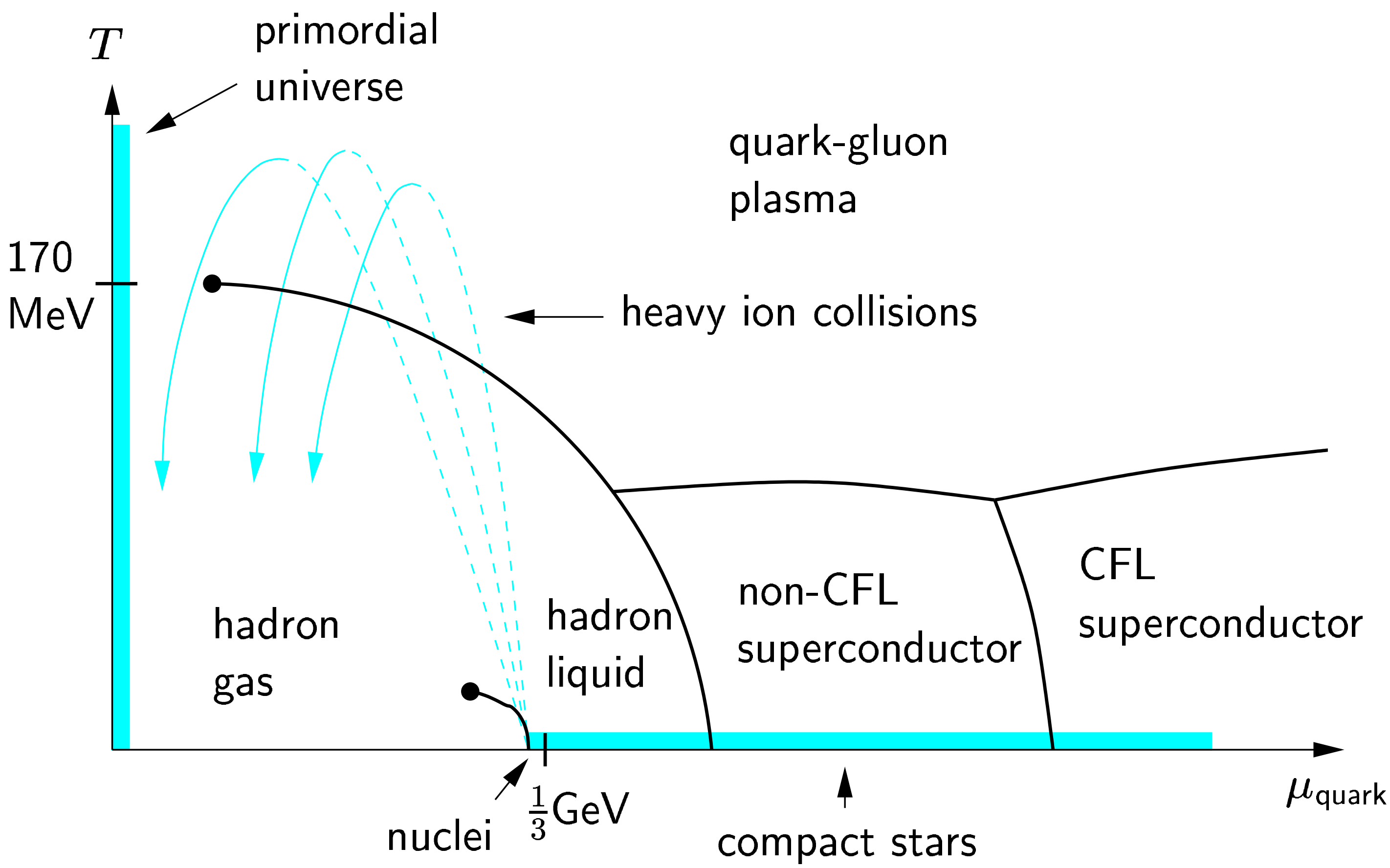
Phase structure of QCD



- Cold dense quark matter is a color superconductor

- High density: color-flavor-locked (CFL) phase, invariant under simultaneous $SU(3)$ rotations in color and flavor

- Intermediate density: flavor asymmetry $(m_s \neq 0)$ can disrupt the CFL phase. Possible phases of non-CFL color superconductivity include:
  - single-flavor phase
  - breached-pair phase
  - crystalline phase

High density

- Can neglect the strange quark mass

- Color-flavor-locking (CFL) phase: a BCS condensate of quark Cooper pairs. Condensate structure:

$$\langle \psi_{i\alpha a}(\mathbf{x}) \psi_{j\beta b}(\mathbf{x}) \rangle \propto \Delta_0 \epsilon_{\alpha\beta A} \epsilon_{ijA} (C\gamma_5)_{ab}$$

  ($i, j =$ flavor, $\alpha, \beta =$ color, $a, b =$ spin)

  – $\epsilon_{\alpha\beta A} \Rightarrow$ color $\bar{\mathbf{3}}$ (gives attractive color force)
  – $C\gamma_5 \Rightarrow$ spin 0 (Lorentz scalar)
  – $\epsilon_{ijA} \Rightarrow$ flavor $\bar{\mathbf{3}}$ ($\langle ud \rangle$, $\langle us \rangle$, and $\langle ds \rangle$ pairing)
  – $\Delta_0 \sim T_c \sim$ 10-100 MeV at $\mu \sim 400$ MeV

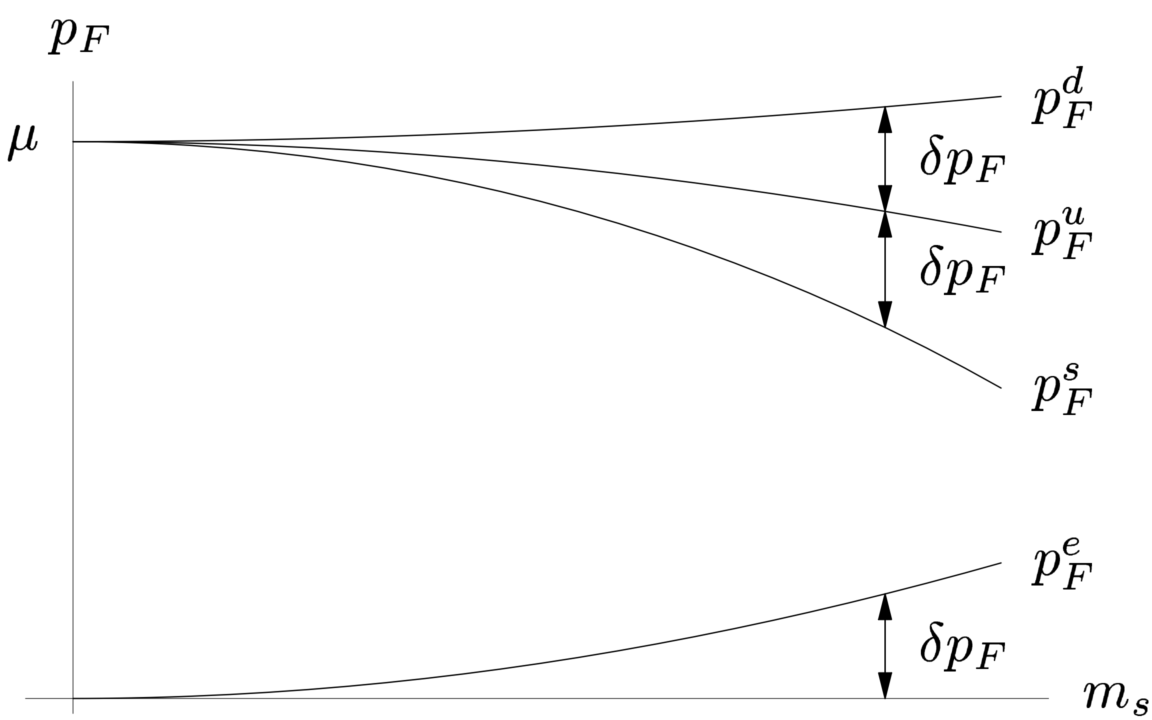
- QCD interaction favors Cooper pairs that are antisymmetric in color and in flavor

- Quark-quark pairing $(\mathbf{p}, -\mathbf{p})$ with $|\mathbf{p}| \approx p_F$
  $(p_F^u = p_F^d = p_F^s = \mu)$

- Charge neutral with $n_u = n_d = n_s$, no electrons

Intermediate density

- Strange quark mass becomes important
  - Consider noninteracting quarks: $m_{u,d} = 0$, $m_s \neq 0$
  - Impose electric neutrality and weak decay equilibrium
  - Fermi surfaces separate: $\delta p_F \approx m_s^2/4\mu$.



- CFL "unlocking" at $m_s^2/4\mu \approx \Delta_0$ (i.e. $\mu \approx m_s^2/4\Delta_0$)

- What happens after CFL? A few possibilities:
  - Hadronic matter
  - Non-CFL color superconductivity:
    - ∗ Single-flavor pairing
    - ∗ Breached pairing
    - ∗ Crystalline phase

Single-flavor pairing

Schaefer, PR **D62**, 094007 (2000);
Schmitt, Wang, Rischke, PR **D66**, 114010 (2002);
Alford, JAB, Cheyne, Cowan, PR **D67**, 054018 (2003)

- New diquark condensate:

$$\langle \psi_{i\alpha a}(\mathbf{x}) \psi_{j\beta b}(\mathbf{x}) \rangle \propto \Delta \epsilon_{\alpha\beta 3} \delta_{i1} \delta_{j1} (C\gamma_3)_{ab}$$

  ($i, j =$ flavor, $\alpha, \beta =$ color, $a, b =$ spin)

  – $\epsilon_{\alpha\beta 3} \Rightarrow$ color $\bar{\mathbf{3}}$ (gives attractive color force)
  – $C\gamma_3 \Rightarrow j = 1$ (vector) condensate
  – $\delta_{i1}\delta_{j1} \Rightarrow \langle uu \rangle$, $\langle dd \rangle$, or $\langle ss \rangle$ pairing

- Small gaps: $\Delta \lesssim 1$ MeV at $\mu \sim 400$ MeV
  (numbers are very model-dependent)

- Gapless modes at poles of the Fermi surface
  (may affect transport properties and neutrino emissivity)

- Spin-wave Goldstone bosons
  (spontaneously-broken rotational symmetry)

- "Color-spin locking" (Schaefer): $\epsilon_{\alpha\beta 3} C\gamma_3 \to \epsilon_{\alpha\beta A} C\gamma_A$

  – fully-gapped Fermi surface
  – restored rotational symmetry
  – analogous to ^{3}He B phase

NJL survey

Alford, JAB, Cheyne, Cowan, PR **D67**, 054018 (2003)

- Mean-field survey of attractive diquark channels for various NJL models:

electric gluons:	$\bar{\psi}\gamma_0 T_A \psi\ \bar{\psi}\gamma_0 T_A \psi$
magnetic gluons:	$\bar{\psi}\gamma_k T_A \psi\ \bar{\psi}\gamma_k T_A \psi$
instanton vertex:	$\bar{\psi}_{Li} T_A \psi_{Rj}\ \bar{\psi}_{Lk} T_A \psi_{Rl}\ \epsilon_{ik}\epsilon_{jl}$

- Exhaustive survey of condensates of the form

$$\langle \psi_{i\alpha a}\psi_{j\beta b}\rangle \propto \Delta \mathfrak{C}_{\alpha\beta}\mathfrak{F}_{ij}\Gamma_{ab}$$

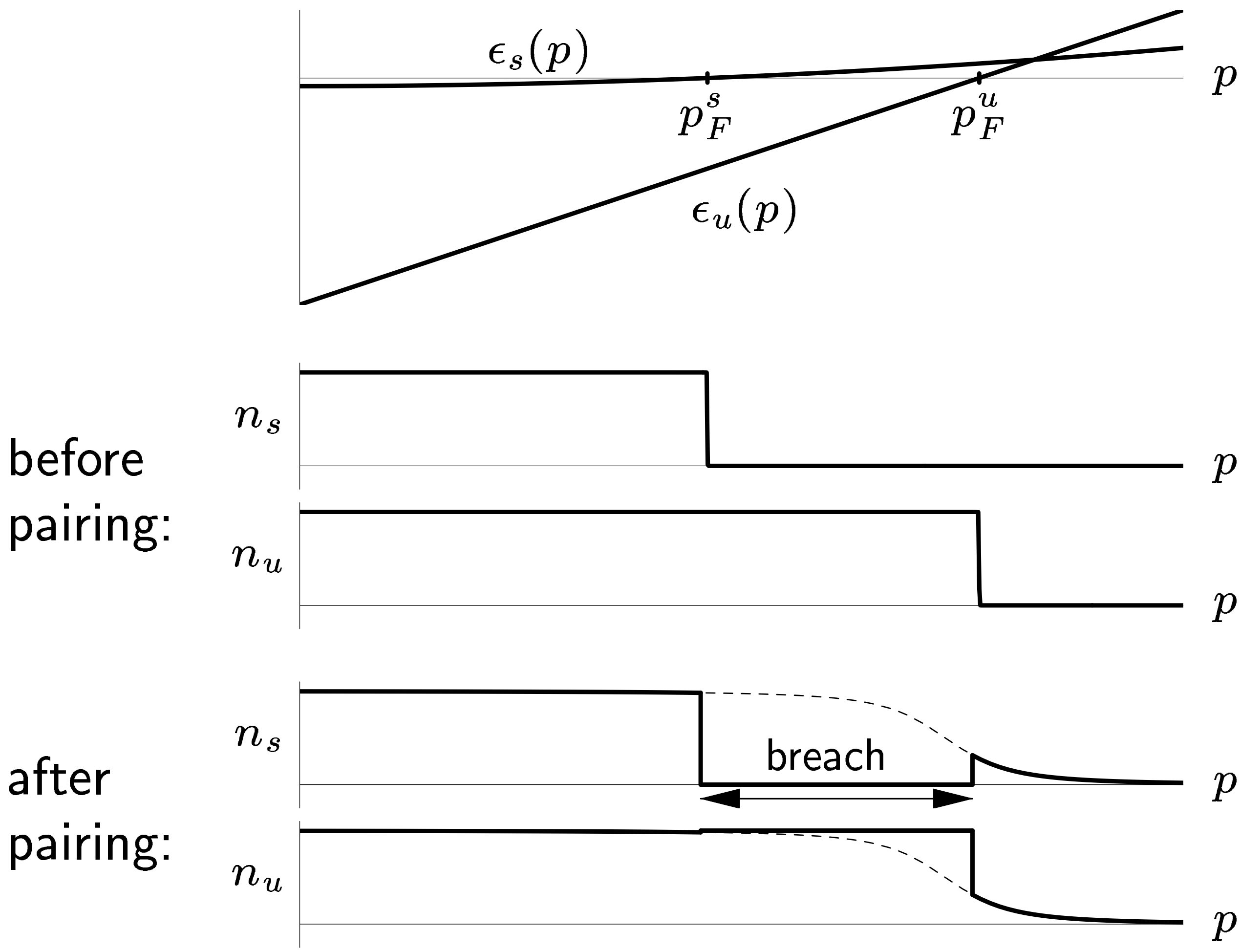
- Summary of attractive channels:

N_c	N_f	$\mathfrak{C}$	$\mathfrak{F}$	Γ	j	Δ
2	2	$\epsilon_{\alpha\beta 3}$	ϵ_{ij3}	$C\gamma_5$	0	10-100 MeV
2	2	$\epsilon_{\alpha\beta 3}$	ϵ_{ij3}	$C\gamma_3\gamma_5$	1	$\lesssim$ 1 MeV
1	2	$\delta_{\alpha 1}\delta_{\beta 1}$	ϵ_{ij3}	$C\sigma_{03}$	1	$\lesssim$ 1 MeV
2	1	$\epsilon_{\alpha\beta 3}$	$\delta_{i1}\delta_{j1}$	$C\gamma_3$	1	$\lesssim$ 1 MeV
1	1	$\delta_{\alpha 1}\delta_{\beta 1}$	$\delta_{i1}\delta_{j1}$	$C\gamma_0\gamma_5$	0	$\lesssim$ 0.01 MeV

Breached pairing

Gubankova, Liu, and Wilczek, hep-ph/0304016

- Originally proposed by Sarma (1963) for an electron superconductor with a Zeeman splitting
- Might occur in quark matter for pairing between a heavy and a light quark (i.e. us or ds). A shell of heavy quarks is promoted from one Fermi surface to the other to allow pairing:



- Pairing is blocked in a "breach" region, with gapless quasiquark excitations at edges of breach

Breached pairing

- Stability (relative to unpaired state) is sensitive to whether particle number or chemical potential is fixed:

Fix:	Breached pairing favored?
μ_s and μ_u	no, never
n_s and n_u	yes, always
n_{tot} and $\delta\mu$	yes, if $v_s/v_u \lesssim 0.17$

- In quark matter with no strange quarks ($m_s > \mu_s$), breached ud pairing can occur when electric neutrality and weak equilibrium are imposed, i.e. fix:

$$\begin{aligned} \frac{2}{3}n_u - \frac{1}{3}n_d - n_e &= 0 \\ \mu_d - \mu_s &= 0 \\ \mu_d - \mu_u - \mu_e &= 0 \\ \mu_u + \mu_d + \mu_s &= \mu_{\text{baryon}} \end{aligned}$$

(Shovkovy and Huang, hep-ph/0302142)

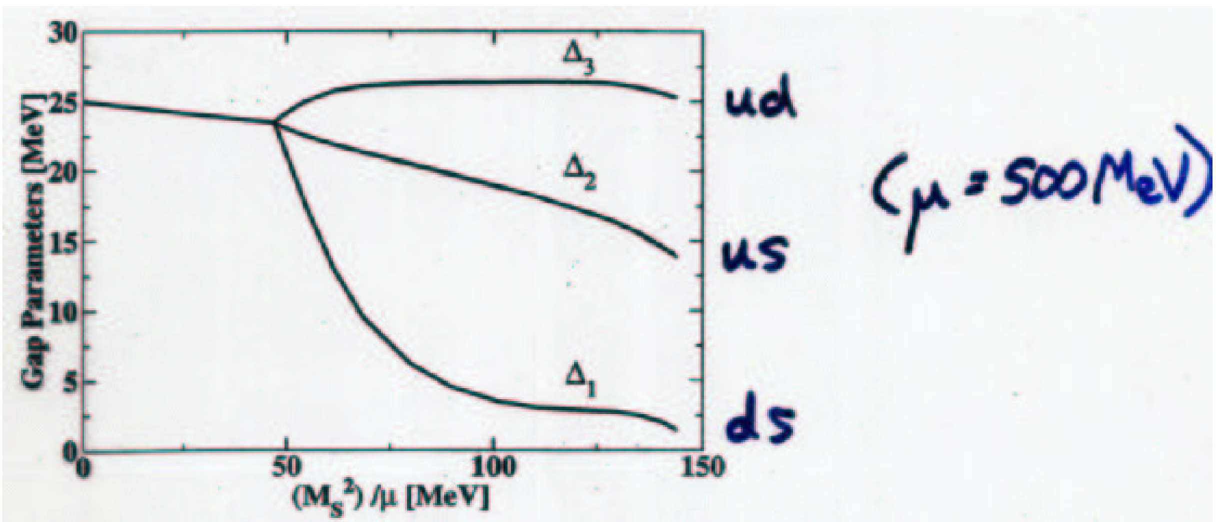
Three-flavor pairing: gapless CFL

Alford, Kouvaris, Rajagopal, /hep-ph/0311286

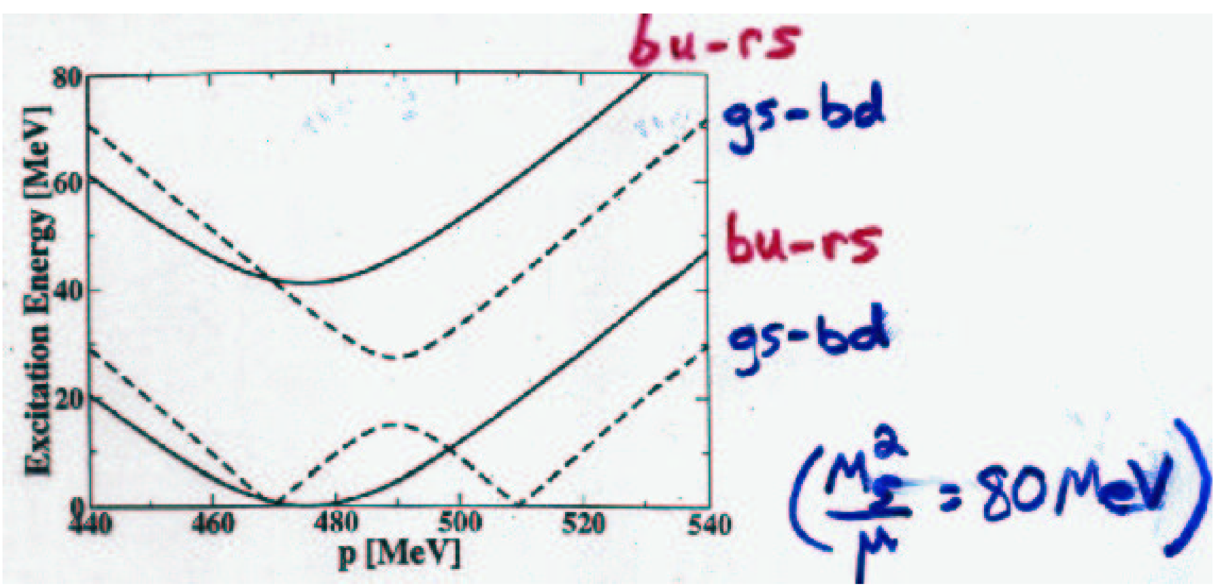
- More general pairing ansatz:

$$\langle \psi_{i\alpha}\psi_{j\beta} \rangle \sim \Delta_1 \epsilon_{\alpha\beta 1}\epsilon_{ij1} + \Delta_2 \epsilon_{\alpha\beta 2}\epsilon_{ij2} + \Delta_3 \epsilon_{\alpha\beta 3}\epsilon_{ij3}$$

- Impose color and electric neutrality and vary m_s



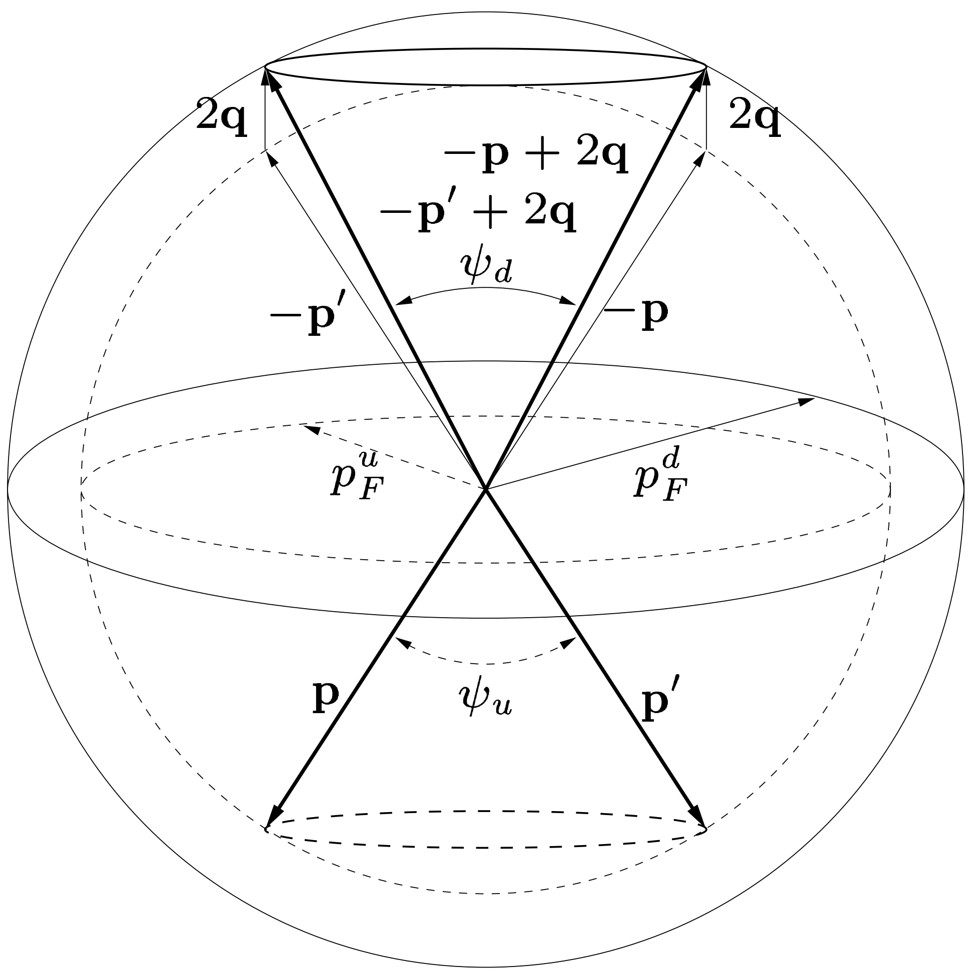
- Gapless quasiquark excitations



Crystalline superconductor

Originally proposed by Larkin, Ovchinnikov, Fulde, Ferrell (1964) for an electron superconductor with a Zeeman splitting. LOFF idea: try Cooper pairs $(\mathbf{p}, -\mathbf{p}+2\mathbf{q})$

- total momentum $2\mathbf{q}$ for each and every pair
- each quark at its Fermi surface, even with $p_F^u \neq p_F^d$
- $\hat{q}$ chosen spontaneously, $|\mathbf{q}|$ determined from Fermi surface separation (result is $|\mathbf{q}| = q_0 \approx 0.6\delta p_F$; result from modelling QCD interaction as pointlike & s-wave)
- condensate forms a ring on each Fermi surface, with opening angle $\psi_u \approx \psi_d \approx 2\cos^{-1}(\delta p_F/2q_0) \approx 67^\circ$



Crystalline superconductor

- If each and every Cooper pair has the same total momentum $2\mathbf{q}$, condensate varies like a single plane wave in position space:

  $$\langle\psi(\mathbf{x})\psi(\mathbf{x})\rangle \sim \Delta e^{i2\mathbf{q}\cdot\mathbf{x}}$$

  System becomes unstable to the formation of a plane wave condensate at a second-order critical point δp_F^{crit}

- If the system is unstable to the formation of a single plane wave condensate, we expect that a condensate of *multiple* plane waves is favored:

  $$\langle\psi(\mathbf{x})\psi(\mathbf{x})\rangle \sim \sum_{\mathbf{q},\ |\mathbf{q}|=q_0} \Delta_{\mathbf{q}} e^{i2\mathbf{q}\cdot\mathbf{x}}$$

  – each $\mathbf{q}$ corresponds to a "ring" of radius angle 67.1° on each Fermi surface
  – rings can "interact" with each other
  – ground state obtained by exploring space of crystal order parameters $\{\Delta_{\mathbf{q}}\}$ to find the crystal structure that is a global minimum of the free energy functional $\Omega[\Delta(\mathbf{x})] = \Omega(\{\Delta_{\mathbf{q}}\})$

Ginzburg-Landau method

- Free energy approximated by a Ginzburg-Landau potential:

$$\Omega(\{\Delta_{\mathbf{q}}\}) \propto \sum_{\mathbf{q},\ |\mathbf{q}|=q_0} \alpha(q_0)\Delta^*_{\mathbf{q}}\Delta_{\mathbf{q}}$$
$$+\frac{1}{2}\sum_{\mathbf{q}_1\cdots\mathbf{q}_4,\ |\mathbf{q}_i|=q_0} J(\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3\mathbf{q}_4)\Delta^*_{\mathbf{q}_1}\Delta_{\mathbf{q}_2}\Delta^*_{\mathbf{q}_3}\Delta_{\mathbf{q}_4}\delta_{\mathbf{q}_1-\mathbf{q}_2+\mathbf{q}_3-\mathbf{q}_4}$$
$$+\cdots$$

  α changes sign to indicate the onset of the plane wave instability: $\alpha \simeq (\delta p_F - \delta p_F^{\text{crit}})/\delta p_F^{\text{crit}}$. J and higher coefficients characterize interactions between the different modes.

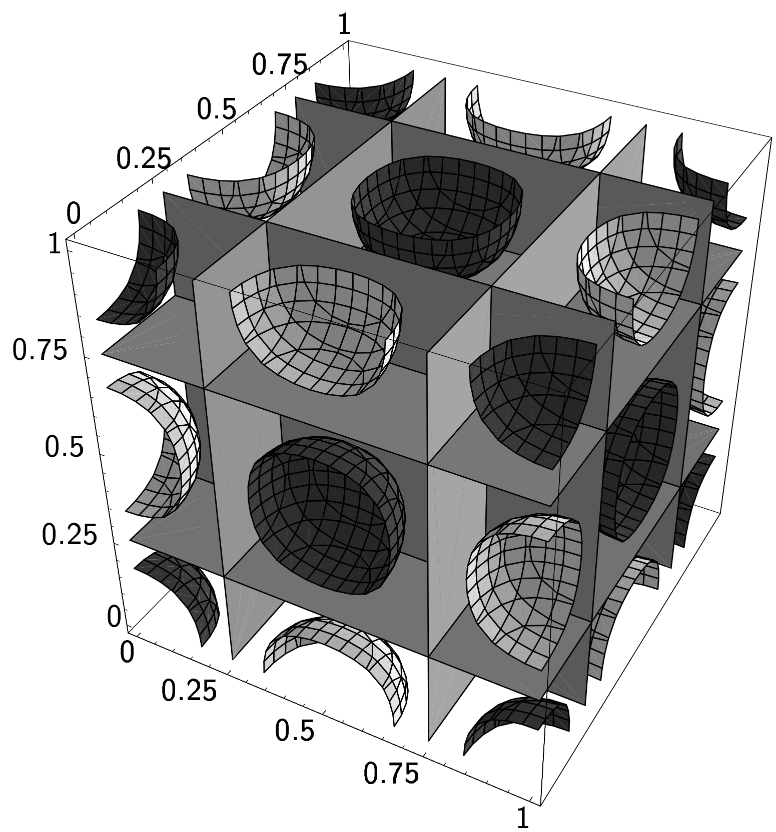
- We investigate crystal structures, each a set of condensate wave vectors $\{\mathbf{q}_a, \mathbf{q}_b, \ldots\}$ with equal lengths $|\mathbf{q}| = q_0$ and equal gaps $\Delta_a = \Delta_b = \cdots = \Delta$. Qualitative lessons of Ginzburg-Landau analysis:
  - Structures with intersecting pairing rings are disfavored. At most nine rings with opening angle 67° can be "packed" on a sphere without intersections
  - "Regular" structures are favored, i.e. sets of $\mathbf{q}$'s for which many combinations of $\mathbf{q}$'s sum to zero
  - There is no regular nine-wave structure, but there is a very regular eight-wave structure: eight $\mathbf{q}$'s pointing towards the eight corners of a cube

FCC Crystal

- The cube structure is the favored ground state: eight wave vectors pointing towards the corners of a cube, forming the eight shortest vectors in the reciprocal lattice of a face-centered-cubic crystal. The gap function is

$$\Delta(\mathbf{x}) = 2\Delta\left[\cos\frac{2\pi}{a}(x+y+z)+\cos\frac{2\pi}{a}(x-y+z) + \cos\frac{2\pi}{a}(x+y-z)+\cos\frac{2\pi}{a}(-x+y+z)\right]$$

A unit cell:



with contours $\Delta(\mathbf{x}) = +4\Delta$ (black), 0 (gray), -4Δ (white). Lattice constant is $a = \sqrt{3}\pi/|\mathbf{q}| \simeq 6.012/\Delta_0$.

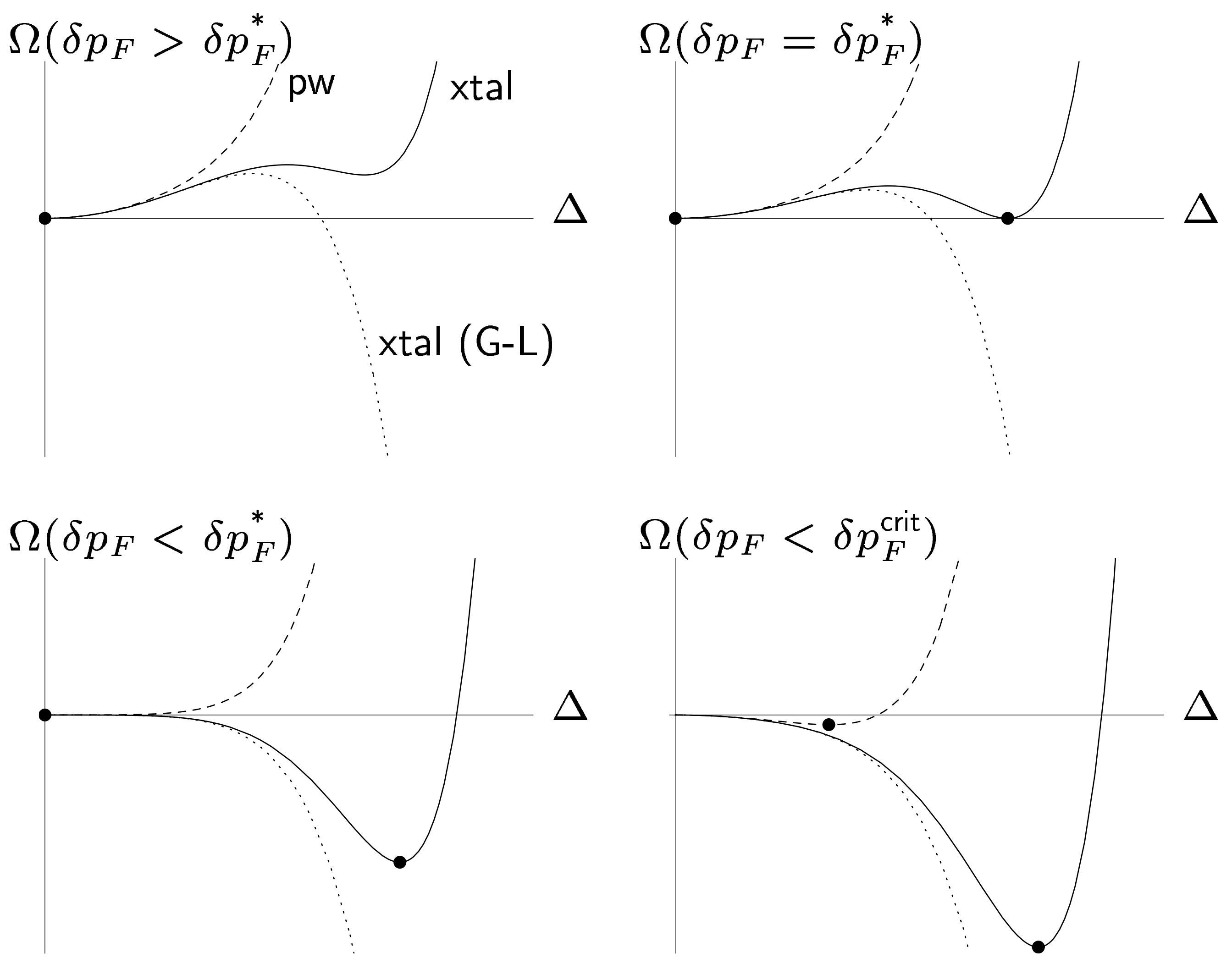
Unstable structure?

- G-L potential for cube has negative 4$^{\text{th}}$, 6$^{\text{th}}$ order coeffs.

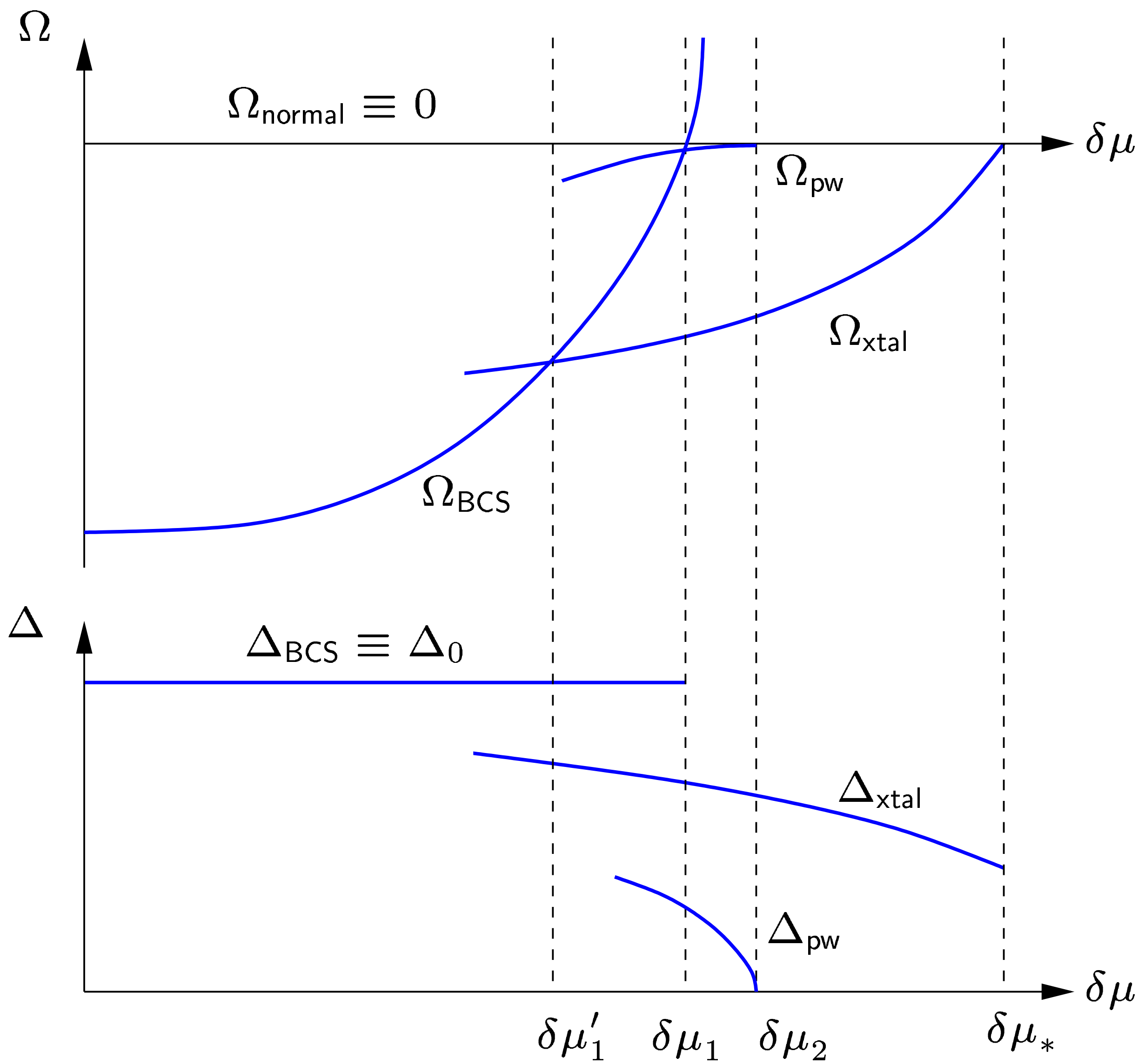
- Ginzburg-Landau instability guarantees a strong first-order transition at some $\delta p_F = \delta p_F^* > \delta p_F^{\text{crit}}$

- Δ, Ω_{min} are large, but cannot be predicted by the Ginzburg-Landau method

- Larger instability $\Rightarrow$ more robust ground state (cube has the most unstable Ginzburg-Landau free energy)



Comparison of phases



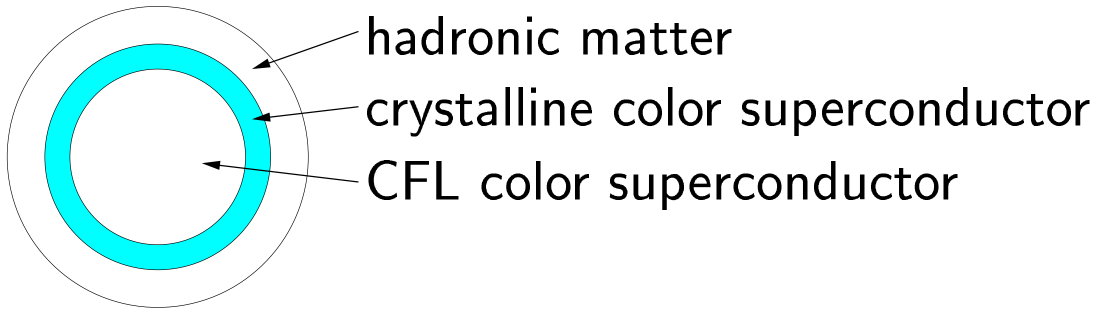
- Single plane wave state ("pw"): LOFF interval is $[\delta\mu_1, \delta\mu_2] \approx [0.707\Delta_0, 0.754\Delta_0]$, transitions to BCS and normal states are first and second order, respectively

- Multiple plane wave state ("xtal"): LOFF interval is $[\delta\mu_1', \delta\mu_*]$, both transitions are first order

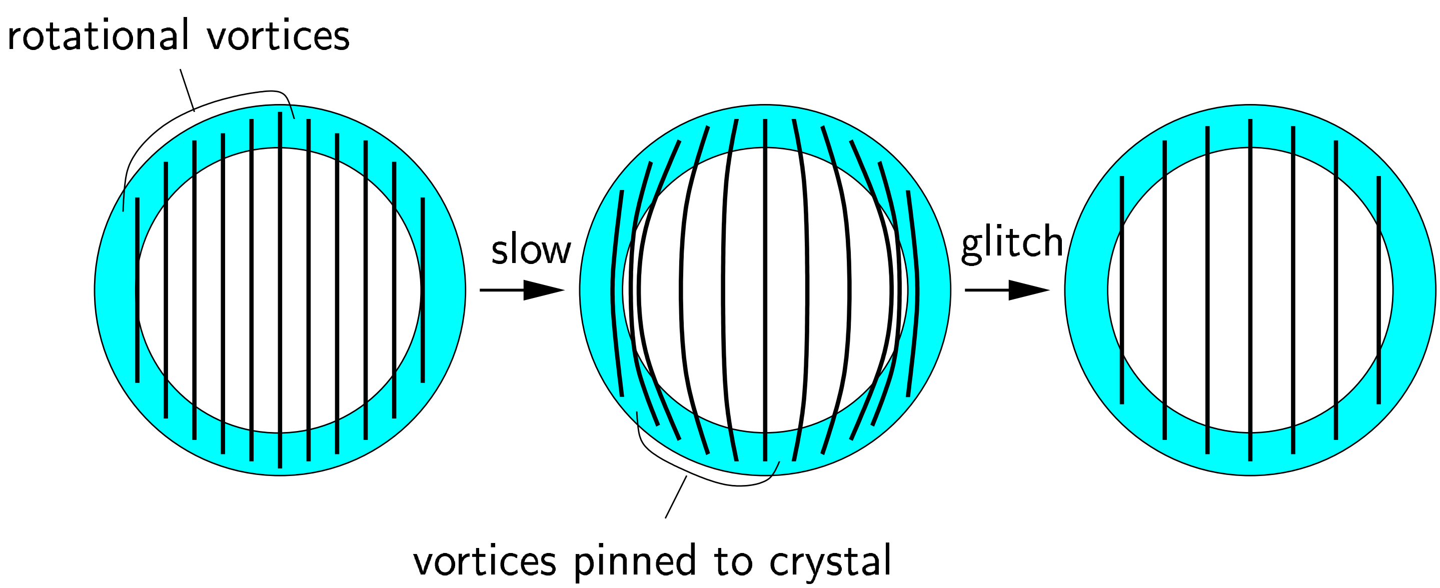
- Crystal is much more robust: $(\delta\mu_* - \delta\mu_1') \gg (\delta\mu_2 - \delta\mu_1)$

Crystalline phase and pulsar glitches

- Crystalline phase may occur within a shell in the interior of a compact star:



- Rotational glitches can occur when vortices are pinned to the crystal lattice:



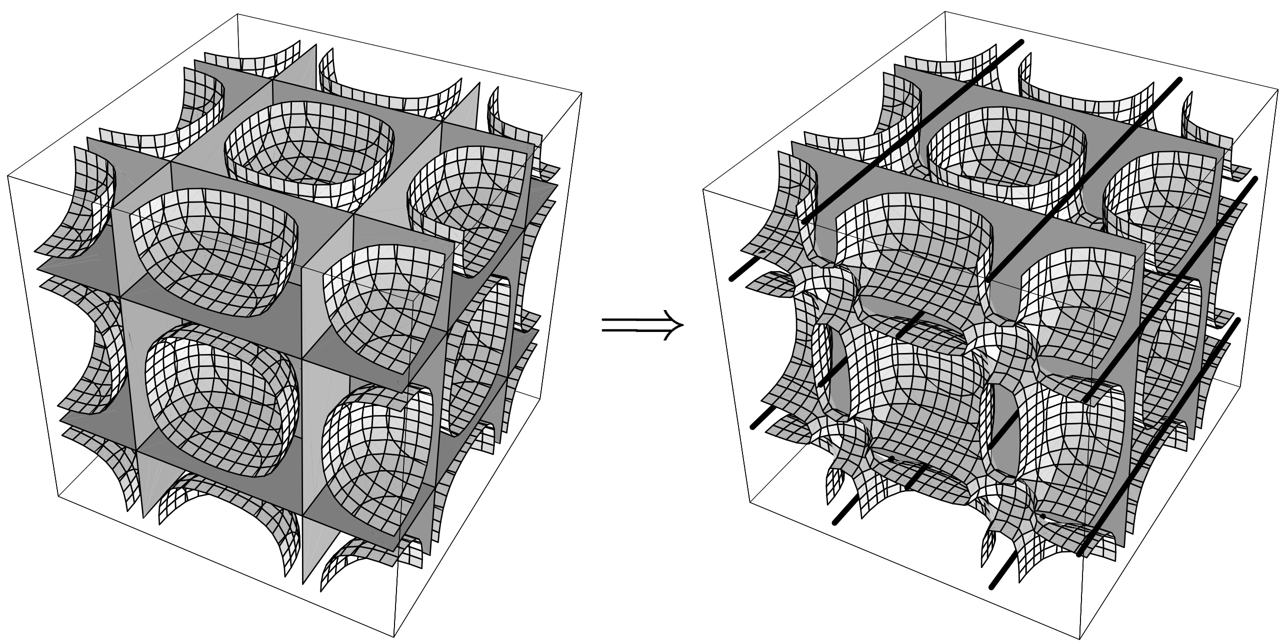
- Calculations are proceeding to construct a vortex in the crystalline phase and calculate the pinning force and crystal shear modulus

Vortices

- Supercurrent vanishes on nodal planes:

$$\Delta(\mathbf{x}) \to \Delta(\mathbf{x})e^{i\theta(\mathbf{x})} \implies \mathbf{J} = (\boldsymbol{\nabla}\theta(\mathbf{x}))|\Delta(\mathbf{x})|^2$$

- Crystal structure must change to accommodate vorticity. Eliminate nodal planes by applying small phase differences for the different plane wave modes, i.e. $\Delta_{\mathbf{q}} \to e^{i\phi_{\mathbf{q}}}\Delta_{\mathbf{q}}$:



Supercurrents can circulate in the layers between remaining nodal planes. Vortices may pin at the new nodal lines.

- Pinned vortices can still move by crystal dislocation if critical shear stress is exceeded (elastic moduli can be extracted from Casalbuoni *et al*'s phonon dispersion relation; shear modulus is $\kappa \sim (q^2 - \delta\mu^2) f_\phi^2/6$)

Crystalline phase in atomic physics

- Experimenters (Thomas *et al*, Ketterle *et al*, others) are trapping and cooling gases of fermionic ^{6}Li to high degeneracy ($T \sim 0.2 T_F$). Fermi temperature $T_F \simeq 10\mu$K corresponds to atomic density $n \sim 10^{14}$ cm^{-3}.

- Magnetically-tunable Feshbach resonance: attractive s-wave interaction between two lowest hyperfine states (range $R \sim 50$ Å, scattering length $a \sim 0.5\ \mu$m, atomic spacing $l \sim 0.2\ \mu$m $\Rightarrow$ dilute, strongly interacting gas)

- Tune hyperfine population difference $N_B - N_A$
  - $N_B - N_A = 0$: BCS superfluid
  - $N_B - N_A \neq 0$: LOFF crystalline superfluid

- Might literally "see" the crystal structure: spatial modulation of atomic density $\delta n(\mathbf{x})/n \sim |\Delta(\mathbf{x})|^2/E_F^2$. Nodal spacing $b \sim 1 - 6\ \mu$m; cigar-shaped cloud has dimensions $r_z \sim 700\ \mu$m, $r_\perp \sim 10\ \mu$m.

- Detecting the crystalline phase: shut off trapping potential and let the cloud expand
  - Fluid-like expansion: spatial pattern may dilate until it can be visualized (as occurs for BEC vortex lattice)
  - Ballistic expansion: momentum anisotropy due to arrangement of pairing rings on Fermi surface

Outlook

- Ginzburg-Landau analysis indicates that the FCC crystal is the likely ground state, but does not predict numbers for the gap and free energy. Going beyond Ginzburg-Landau, use eight-wave ansatz for the crystalline condensate, with non-small Δ. Obtain *bounded* free energy to determine Δ and Ω for the FCC crystal
- Three-flavor analysis: $\langle ud \rangle$, $\langle us \rangle$, and $\langle ds \rangle$ condensates with a realistic m_s. Crystalline CFL? Or breached us and ds pairing?
- Vortex pinning in the crystalline phase: What is the structure of a vortex? Are vortices pinned? What is the pinning force? What is the crystal shear modulus? Is it possible to unpin vortices by crystal dislocation?
- Gapless modes: crystalline, single-flavor, and breached-pair phases all have gapless quasiquark modes. Could these modes contribute significantly to neutrino cooling (e.g. direct URCA) in compact stars?
- Observing the crystal in an atomic trap: how does the cloud expand when the trapping potential is shut off?
- Crytstal at asymptotically-high densities: $|\mathbf{q}|/\delta p_F$ decreases and therefore the opening angle of the pairing rings decreases. Crystal structure may be qualitatively different from that which we have found (but asymptotic analysis may not be relevant for compact stars).