The Uses of Conformal Symmetry in QCD

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- Preliminaries
- Algebraic aspects: conformal operators
- Conformal symmetry and its breaking in QCD
- Reconstruction of hard scattering amplitudes and evolution kernels
- Example: $\gamma^*\gamma^*$ -to- π transition form factor
- Conclusions

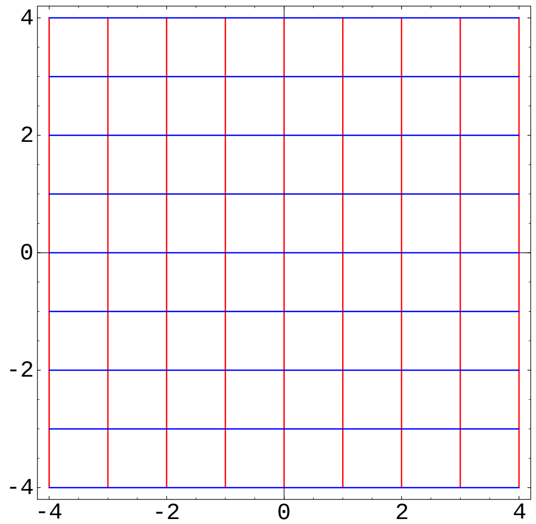
Review on this subject is given in
V.M. Braun, G.P. Korchemsky, and D.M. hep-ph/0306057
Prog. Part. Nucl. Phys. 51 (2003) 2 (in print)

Preliminaries

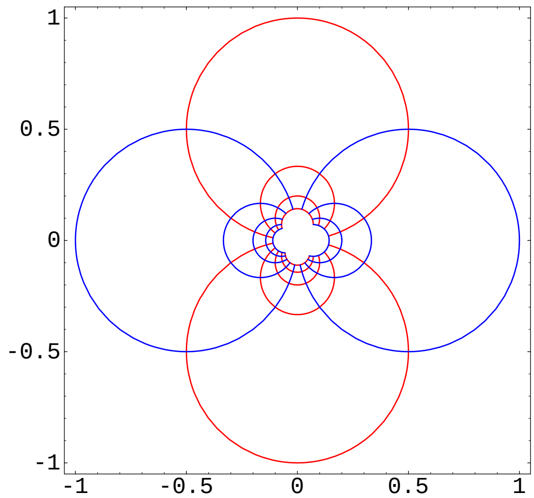
Definition of conformal transformations

Among the general coordinate transformations exist transformations that only change the *normalization* of the metric

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \omega(x) g_{\mu\nu}(x) \quad \Longrightarrow \quad x^2 = 0 = x'^2$$



$\Rightarrow$



History

- Electrodynamics is invariant under conformal transformations (applications in electrostatic)
- massless Dirac equation is conformally invariant, too
- inspired by both scaling behaviour in high energy experiments and second order phase transition phenomena conformal symmetry has been intensively studied in field theory at the end of 60th and 70th
- in QCD conformal symmetry has been 'discovered' at short, light-like distances, and at high energy (BFKL) around 80th (or before), renewed interest in the 90th by different authors
- conformal field theories in two-dimensions

Conformal transformations in $D = 4$ Minkowski space

Global transformations

The *maximal extension* of the Poincaré group SO(3,1)

$$x^\mu \to x'^\mu = \Lambda_{\mu\nu} x^\nu \quad \text{with} \quad x'^2 = x^2 \qquad (6 \text{ parameter})$$

$$x^\mu \to x'^\mu = x^\mu + a^\mu \qquad (4 \text{ parameter})$$

that *leaves the light-cone invariant* is given by

$$x^\mu \to x'^\mu = \lambda x^\mu \qquad \textit{dilatation}$$

$$x^\mu \to x'^\mu = \frac{x^\mu + a^\mu}{1 + 2a \cdot x + a^2 x^2} \qquad 4 \textit{ special conformal transformations}$$

($\mathbf{I T}_a \mathbf{I}\, x^\mu$ with $\mathbf{I}\, x^\mu = x^\mu / x^2$), the conformal group $SO(4,2)$.

Lie–algebra

Global transformations are generated by infinitesimal ones.
The generators of Poincaré transformations satisfy

$$i[\mathbf{P}_\mu, \mathbf{P}_\nu] = 0\,, \quad i[\mathbf{M}_{\alpha\beta}, \mathbf{P}_\mu] = g_{\alpha\mu}\mathbf{P}_\beta - g_{\beta\mu}\mathbf{P}_\alpha\,,$$

$$i[\mathbf{M}_{\alpha\beta}, \mathbf{M}_{\mu\nu}] = g_{\alpha\mu}\mathbf{M}_{\beta\nu} - g_{\beta\mu}\mathbf{M}_{\alpha\nu} - g_{\alpha\nu}\mathbf{M}_{\beta\mu} + g_{\beta\nu}\mathbf{M}_{\alpha\mu}$$

and for *dilatation* $\mathbf{D}$ and *special conformal transformations* $\mathbf{K}_\mu$ one has

$$i[\mathbf{D}, \mathbf{P}_\mu] = \mathbf{P}_\mu\,, \quad i[\mathbf{D}, \mathbf{K}_\mu] = -\mathbf{K}_\mu\,,$$

$$i[\mathbf{M}_{\alpha\beta}, \mathbf{K}_\mu] = g_{\alpha\mu}\mathbf{K}_\beta - g_{\beta\mu}\mathbf{K}_\alpha\,, \quad i[\mathbf{P}_\mu, \mathbf{K}_\nu] = -2g_{\mu\nu}\mathbf{D} + 2\mathbf{M}_{\mu\nu}\,,$$

$$i[\mathbf{D}, \mathbf{M}_{\mu\nu}] = i[\mathbf{K}_\mu, \mathbf{K}_\nu] = 0$$

Conformal symmetry *can only hold* for *massless* theories

$$\exp\{i\lambda\mathbf{D}\}\,\mathbf{P}^2 \exp\{-i\lambda\mathbf{D}\} = \exp\{2\lambda\}\,\mathbf{P}^2.$$

Field theoretical representations

The conformal transformations of a generic field $\Phi(x)$ are induced by

$$\delta_M^{\mu\nu}\Phi(0) = -\Sigma^{\mu\nu}\Phi(0)\,, \quad \delta_D\Phi(0) = \ell\;\Phi(0)\,, \quad \delta_K^{\mu}\Phi(0) = 0\,.$$

with ℓ is the *scaling dimension*. From the 'little algebra'

$$\begin{aligned} \Sigma^{\mu\nu}\phi &= 0, & \Sigma^{\mu\nu}\psi &= \frac{i}{2}\sigma^{\mu\nu}\psi, & \Sigma^{\mu\nu}A^{\alpha} &= g^{\nu\alpha}A^{\mu} - g^{\mu\alpha}A^{\nu}\,, \\ \ell_\phi &= 1, & \ell_\psi &= 3/2, & \ell_A &= 1, \end{aligned}$$

and by translations

$$\delta_P^{\mu}\Phi(x) \;\equiv\; i[\mathbf{P}^{\mu}, \Phi(x)] \;=\; \partial^{\mu}\Phi(x)\,,$$

one finds

$$\begin{aligned} \delta_M^{\mu\nu}\Phi(x) &\equiv i[\mathbf{M}^{\mu\nu}, \Phi(x)] = (x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu} - \Sigma^{\mu\nu})\,\Phi(x)\,, \\ \delta_D\Phi(x) &\equiv i[\mathbf{D}, \Phi(x)] = (x\cdot\partial + \ell)\,\Phi(x)\,, \\ \delta_K^{\mu}\Phi(x) &\equiv i[\mathbf{K}^{\mu}, \Phi(x)] = \left(2x^{\mu}x\cdot\partial - x^2\partial^{\mu} + 2\ell x^{\mu} - 2x_{\nu}\Sigma^{\mu\nu}\right)\Phi(x) \end{aligned}$$

Casimir operators

- Poincaré group with $\mathbf{W}_{\mu} = (1/2)\epsilon_{\mu\alpha\beta\gamma}\mathbf{P}^{\alpha}\mathbf{M}^{\beta\gamma}$:

$$\mathbf{P}^2\Phi(x) = M^2\Phi(x) \qquad \mathbf{W}^2\Phi(x) = s(s+1)M^2\Phi(x),$$

- Conformal group, eigenvalues of Casimir operators are expressed by spin s and scaling dimension ℓ :

$$\mathbf{C}_2 = \frac{1}{2}\mathbf{M}_{\mu\nu}\mathbf{M}^{\mu\nu} - \mathbf{K}_{\mu}\mathbf{P}^{\mu} - 4i\mathbf{D} - \mathbf{D}^2, \quad \mathbf{C}_3, \quad \mathbf{C}_4$$

Collinear conformal group

An ultra-relativistic particle propagates close to the light-cone, thus

$$\Phi(x) \to \Phi(\alpha n) \equiv \Phi(\alpha), \qquad x^\mu = x_- n^\mu + x_+ \overline{n}^\mu + x_\perp^\mu, \quad n^2 = \overline{n}^2 = 0, \; n \cdot \overline{n} = 1.$$

Choosing the special conformal parameter $a_\mu = a\overline{n}_\mu$ light-like, one finds the following *collinear conformal* subgroup [projection on a line]

$$x_- \to x'_- = \frac{x_-}{1 + 2a\, x_-}, \qquad x_- \to x_- + c, \quad x_- = \lambda x_-$$

The fields are living on the light cone are classified with respect to their spin projection s on the "+" direction:

$$\Sigma_{+-}\Phi(\alpha) = s\,\Phi(\alpha)$$

Then the global transformations are

$$\Phi(\alpha) \to \Phi'(\alpha) \quad = \quad (c\alpha + d)^{-2j}\Phi\left(\frac{a\alpha + b}{c\alpha + d}\right), \quad ad - bc = 1\,,$$

where a, b, c, d are real numbers and with the *conformal spin*

$$\boxed{j = (\ell + s)/2}.$$

The conformal generators are

$$\mathbf{L}_+ = \mathbf{L}_1 + i\mathbf{L}_2 = -i\mathbf{P}_+\,, \qquad \mathbf{L}_- = \mathbf{L}_1 - i\mathbf{L}_2 = (i/2)\mathbf{K}_-\,,$$

$$\mathbf{L}_0 = (i/2)(\mathbf{D} + \mathbf{M}_{-+})\,, \qquad \mathbf{E} = (i/2)(\mathbf{D} - \mathbf{M}_{-+})\,.$$

satisfying the $SL(2,\mathbb{R}) \sim O(2,1)$ algebra

$$[\mathbf{L}_0, \mathbf{L}_\mp] = \mp\mathbf{L}_\mp\,, \quad [\mathbf{L}_-, \mathbf{L}_+] = -2\mathbf{L}_0\,, \quad [\mathbf{E}, \mathbf{L}_{0,\mp}] = 0.$$

Algebraic aspects

Eigenvalues of $\mathbf{E}$ counts the *twist* t

$$[\mathbf{E}, \Phi(\alpha)] = \frac{1}{2}(\ell - s)\Phi(\alpha) \quad t = \ell - s$$

Simplest representations

Multiplets are classified by the eigenvalues of $\sum_{i=0,1,2}[\mathbf{L}_i, [\mathbf{L}_i, \Phi(\alpha)]]$

$$\boxed{[\mathbf{L}_0, [\mathbf{L}_0, \Phi(\alpha)]] - [\mathbf{L}_0, \Phi(\alpha)] + [\mathbf{L}_+, [\mathbf{L}_-, \Phi(\alpha)]] = j(j-1)\Phi(\alpha)}$$

The lowest member of the multiplet is characterized by

$$[\mathbf{L}_-, \Phi(0)] = \frac{i[\mathbf{K}_-, \Phi(0)]}{2} = 0\,, \quad [\mathbf{L}_0, \Phi(0)] = j\,\Phi(0)\,.$$

Applying the step-up operator

$$\mathcal{O}_{k+1} = [\mathbf{L}_+, \mathcal{O}_k] = (-n \cdot \partial)\mathcal{O}_k\,, \qquad \mathcal{O}_0 = \Phi(0)\,.$$

provides a *conformal tower* of states wich have conformal spin projection

$$\boxed{[\mathbf{L}_0, \mathcal{O}_k] = (k + j)\mathcal{O}_k \quad \text{with} \quad k = 0, 1, 2, \ldots, \infty}.$$

Action of the step-down operator is

$$[\mathbf{L}_-, \mathcal{O}_k] = -k(k + 2j - 1)\mathcal{O}_{k-1}\,.$$

Decomposition of the primary field operator $\Phi(\alpha)$ in the states $\mathcal{O}_k$

$$\Phi(\alpha) = \sum_{k=0}^{\infty} \frac{(-\alpha)^k}{k!}\mathcal{O}_k\,.$$

Classification of two-(multi) particle operators

Conformal multiparticle operators $\mathbb{O}_n$ are constructed by reduction

$$[j_1] \otimes [j_2] = \bigoplus_{n \geq 0} [j_1 + j_2 + n] \, .$$

$$\mathbf{L}_i = \mathbf{L}_{1,i} + \mathbf{L}_{2,i} \, , \qquad \mathbf{L}^2 \mathbb{O}^{j_1, j_2}_{n,l} \equiv \sum_{i=0,1,2} [\mathbf{L}_i, [\mathbf{L}_i, \mathbb{O}_n]] = j_n (j_n - 1) \mathbb{O}^{j_1, j_2}_{n,l} ,$$

where $j_n = j_1 + j_2 + n$. For the lowest state it is required that

$$[\mathbf{L}_-, \mathbb{O}^{j_1, j_2}_n] = 0 \quad \text{and} \quad [\mathbf{L}_0, \mathbb{O}^{j_1, j_2}_n] = j_n \, \mathbb{O}^{j_1, j_2}_n$$

from which the CG coefficients, given by Legendre polynomials, follows

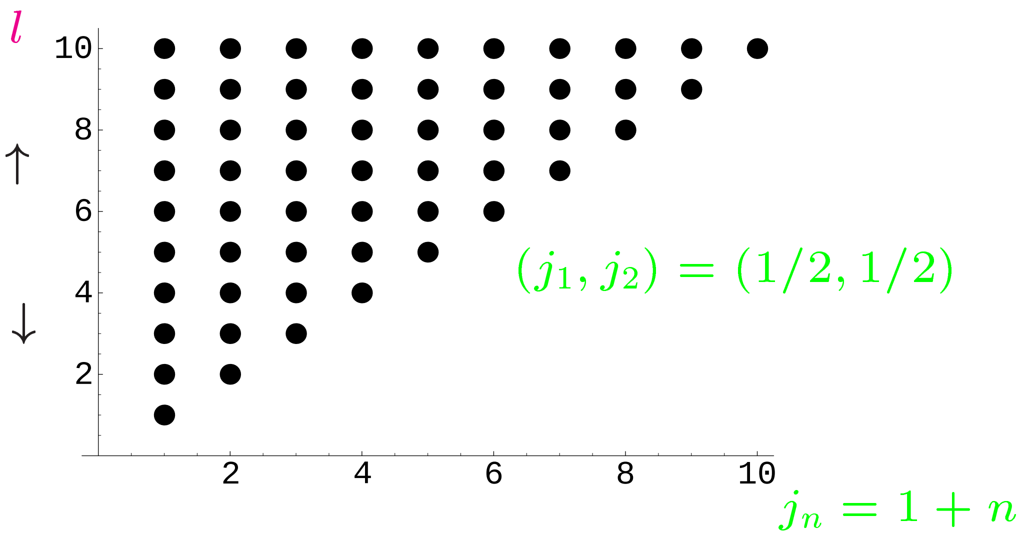
$$\mathbb{O}^{j_1, j_2}_n = \partial_+^n \left[\Phi_{j_1} P_n^{(2j_1 - 1, 2j_2 - 1)} \left(\frac{\overrightarrow{\partial}_+ - \overleftarrow{\partial}_+}{\overrightarrow{\partial}_+ + \overleftarrow{\partial}_+} \right) \Phi_{j_2} \right]$$

States with higher conformal spin projections are given by

$$\mathbb{O}^{j_1, j_2}_{n,l} = i^{l-n} (\partial_{+1} + \partial_{+2})^{l-n} \, \mathbb{O}^{j_1, j_2}_{n,n} \, , \quad l \geq n , \quad \mathbb{O}^{j_1, j_2}_{n,n} \equiv \mathbb{O}^{j_1, j_2}_n .$$

$\mathbf{L}_+ = (1/i) \mathbf{P}_+ \ \uparrow$

$\mathbf{L}_- = (i/2) \mathbf{K}_- \ \downarrow$



QCD example

Definite spin projection of fermionic field $\psi = \psi_+ + \psi_-$:

$$\psi_+ = \frac{1}{2}\gamma_-\gamma_+\psi \quad \left(s = +\frac{1}{2}\right), \qquad \psi_- = \frac{1}{2}\gamma_+\gamma_-\psi \quad \left(s = -\frac{1}{2}\right).$$

The components of a bilocal vector operator are classified by the *twist*:

$$\begin{aligned}
\text{twist-2:} \quad \mathcal{Q}_+(\alpha,\beta) &= \bar{\psi}_+(\beta)\gamma_+\psi_+(\alpha) \equiv \mathcal{Q}^{1,1}(\alpha,\beta), \\
\text{twist-3:} \quad \mathcal{Q}_\perp(\alpha,\beta) &= \bar{\psi}_+\gamma_\perp\psi_- + \bar{\psi}_-\gamma_\perp\psi_+ \equiv \mathcal{Q}^{1,1/2} + \mathcal{Q}^{1/2,1} \\
\text{twist-4:} \quad \mathcal{Q}_-(\alpha,\beta) &= \bar{\psi}_-\gamma_-\psi_- \equiv \mathcal{Q}^{1/2,1/2},
\end{aligned}$$

and can be decomposed in conformal operators

$$\mathbb{Q}^{1,1}_n(x) = (i\partial_+)^n \left[\bar{\psi}(x)\,\gamma_+\,C^{3/2}_n\left(\overleftrightarrow{D}_+/\partial_+\right)\psi(x)\right],$$

Analogous treatment for gluonic operators gives at leading twist:

$$\mathbb{G}^{3/2,3/2}_n(x) = (i\partial_+)^n \left[G^a_{+\perp}(x)\,C^{5/2}_n\left(\overleftrightarrow{D}_+/\partial_+\right)G^a_{+\perp}(x)\right].$$

Remarks

- Higher twist two-particle operators are reducible to multiple particle operators + lower twist operators by equation of motion.
- The spectrum of 3, 4, . . . -particle operators is degenerated.
- Such an algebraic classification can be also employed in the case that conformal symmetry is broken, e.g., $m \neq 0$.

Conformal partial wave expansion

The conformal expansion on the operator level implies the partial wave expansion for (generalized) distribution amplitudes, e.g., pion distribution amplitude

$$\langle 0|[\bar{d}(0)[0,\alpha]\gamma_+\gamma_5 u(\alpha)]_{\rm ren}|\pi^+(p)\rangle = i f_\pi p_+ \int_0^1 du\, e^{-iu\alpha p_+}\phi_\pi(u,\mu)$$

$$\phi_\pi(u,\mu) = \sum_{n=0}^{\infty} \frac{6u(1-u)}{N_n} C_n^{3/2}(2u-1)\langle 0|\mathbb{Q}_n^{1,1}(\mu)|\pi^+(p)\rangle$$

At LO the conformal operators are multiplicative renormalizable:

$$\langle 0|\mathbb{Q}_n^{1,1}(\mu)|\pi^+(p)\rangle = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\gamma_n^{(0)}/\beta_0} \langle 0|\mathbb{Q}_n^{1,1}(\mu_0)|\pi^+(p)\rangle,$$

where $\beta_0 = 11 - 2n_f/3$ and $\gamma_n^{(0)}$ are know from DIS

$$\gamma_n^{(0)} = C_F\left(1 - \frac{2}{(n+1)(n+2)} + 4\sum_{m=2}^{n+1}\frac{1}{m}\right) .$$

Alltogether, we have the solution of the ER-BL equation to LO

$$\mu^2\frac{d}{d\mu^2}\phi_\pi(u,\mu) = \frac{\alpha_s(\mu)}{2\pi}\int_0^1 dv\, V^{(0)}(u,v)\,\phi_\pi(v,\mu) .$$

and the kernel has the following representation

$$V_0(u,v) = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{6u(1-u)}{N_n}\gamma_n^{(0)}\, C_n^{3/2}(2u-1)C_n^{3/2}(2v-1) .$$

This expansion *does not hold true* in NLO for MS scheme.

Conformal operator product expansion

The OPE of two currents (fields) with definite *spin projections* s_A, s_B and *scale dimensions* ℓ_A, ℓ_B for $x_+, x_\perp \to 0$, x_--fixed read

$$A(x)B(0) \simeq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} C_{n,k}^{j_A \cdots j_2} \left(\frac{1}{x^2}\right)^{\frac{t_A+t_B-t_n}{2}} x_-^{n+k+s_1+s_2-s_A-s_B}\, \mathbb{O}_{n,n+k}^{j_1,j_2}(0)$$

Additional conditions arise from conformal symmetry

$$[\mathbf{L}_-, A(x)B(0)] \simeq \left\{ x_- \left(2j_A + x \cdot \partial_x\right) A(x) - \frac{x^2}{2}\, \bar{n} \cdot \partial_x A(x) \right\} B(0)$$

$$\left[\mathbf{L}_-, \mathbb{O}_{n,n+k}^{j_1,j_2}\right] = -k(k + 2j_n - 1)\mathbb{O}_{n,n+k-1}^{j_1,j_2}\,, j_n = j_1 + j_2 + n\,.$$

They provide a recurrence relation for the Wilson coefficients

$$\boxed{C_{n,k+1}^{j_A \cdots j_2} = -\frac{j_A - j_B + j_n + k}{(k+1)(k+2j_n)} C_{n,k}^{j_A \cdots j_2}} \Longrightarrow C_{n,k}^{j_A \cdots j_2} (C_{n,0}^{j_A \cdots j_2} \equiv C_n)$$

that allow to resum the total derivatives

$$A(x)B(0) \simeq \sum_{n=0}^{\infty} C_n \left(\frac{1}{x^2}\right)^{\frac{t_A+t_B-t_n}{2}} \frac{x_-^{n+s_1+s_2-s_A-s_B}}{\mathrm{B}(j_A - j_B + j_n, j_B - j_A + j_n)}$$

$$\times \int_0^1 du\, u^{j_A - j_B + j_n - 1} (1-u)^{j_B - j_A + j_n - 1} \mathbb{O}_n^{j_1,j_2}(u x_-)\,,$$

Wilson-coefficients C_n can be often borrowed from known results:

$$\langle P|A(x)B(0)|P\rangle \simeq \sum_{n=0}^{\infty} C_n \left(\frac{1}{x^2}\right)^{\frac{t_A+t_B-t_n}{2}} x_-^{n+s_1+s_2-s_A-s_B} \langle P|\mathbb{O}_n^{j_1,j_2}|P\rangle\,.$$

Conformal symmetry and its breaking in QCD

Energy-momentum tensor

Conformal symmetry for $G = \{P_\mu, M_{\mu\nu}, D, K_\mu\}$

$$\delta_G S = \int d^4x \delta_G \mathcal{L} = 0, \quad \mathcal{L}_{\rm QCD} = \bar\psi i \not{D} \psi - \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \dots$$

implies the existens of *15 conserved* currents

$$J^\mu_{P,\alpha} = \Theta^\mu{}_\alpha\,, \quad J^\mu_{M,\alpha\beta} = x_\alpha \Theta^\mu{}_\beta - x_\beta \Theta^\mu{}_\alpha\,,$$
$$J^\mu_D = x^\nu \Theta^\mu{}_\nu\,, \quad J^\mu_{K,\alpha} = \left(2x^\nu x_\alpha - g_\alpha{}^\nu x^2\right) \Theta^\mu{}_\nu,$$

where $\partial^\mu \Theta_{\mu\nu} \overset{\rm EOM}{=} 0$, $\Theta_{\mu\nu} = \Theta_{\nu\mu}$, and

$$\partial_\mu J^\mu_D = 0,\ \partial_\mu J^\mu_{K,\alpha} = 0 \quad \Leftrightarrow \quad \Theta^\mu{}_\mu \overset{\rm EOM}{=} 0.$$

In QCD the *improved energy–momentum tensor* formally reads

$$\Theta^{\rm QCD}_{\mu\nu} = -g_{\mu\nu}\mathcal{L}_{\rm QCD} - G^a_{\mu\lambda} G^a_{\nu\lambda} + \frac{i}{4}\Big[\bar\psi \gamma_\mu \overrightarrow{D}_\nu \psi - \bar\psi \overleftarrow{D}_\nu \gamma_\mu \psi + (\mu \leftrightarrow \nu)\Big] + \dots$$

where the trace is zero on classical level

$$g^{\mu\nu}\Theta^{\rm QCD}_{\mu\nu} \overset{\rm EOM}{=} 0 + \dots.$$

In a quantized theory the trace anomaly appears due to *UV divergencies*:
(for instance in dimensional regularization $d = 4 - 2\epsilon$, $i.e. g^\nu{}_\nu = d$)

$$\boxed{g^{\mu\nu}\Theta^{\rm QCD}_{\mu\nu} \overset{\rm EOM}{=} \frac{d-4}{4}(G^a_{\mu\nu})^2 + \dots = \frac{-\epsilon g + \beta(g)}{2g}[(G^a_{\mu\nu})^2] + \dots}$$

Conformal Ward Identities

Ward identies for Green functions are derived from the reparametrization invariance of the path integral [Sarkar (74); D.M. (90), A.V. Belitsky (98)]

$$\langle 0|\mathrm{T}[\mathcal{O}]\Phi(y)\Phi(z)|0\rangle \equiv \frac{1}{\mathcal{N}}\int \mathcal{D}\Phi\,[\mathcal{O}(\Phi)]\Phi(y)\Phi(z)\,e^{i\int d^dx\mathcal{L}(\Phi)}$$

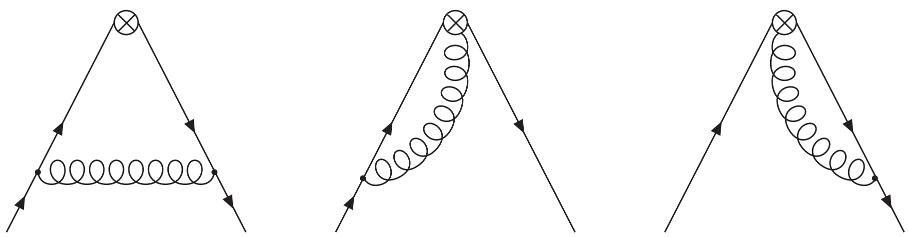
under infinitesimal transformations for $G = \{P_\mu, M_{\mu\nu}, D, K_\mu\}$:

$$\Phi(x) \to \Phi'(x) = \Phi(x) + \varepsilon\delta_G\Phi(x) \quad \text{with} \quad \delta_G\Phi(x) = \mathcal{G}(x,\partial)\Phi(x)\,,$$

and read in the renormalized theory

$$[\mathcal{G}(y,\partial_y) + \mathcal{G}(z,\partial_z)]\,\langle 0|\mathrm{T}[\mathbb{Q}_{nl}]\psi(y)\overline{\psi}(z)|0\rangle = \\ -\langle 0|\mathrm{T}\,(\delta_G[\mathbb{Q}_{nl}])\,\psi(y)\overline{\psi}(z)|0\rangle - \langle 0|\mathrm{T}[\mathbb{Q}_{nl}]\psi(y)\overline{\psi}(z)\,(\delta_G S)\,|0\rangle$$

Renormalization of composite operators in the MS-scheme



(Landau gauge)

Poincaré invariance gives constraints on the form of mixing:

$$[\mathbb{Q}_{nl}] = \sum_{m=0}^{n} Z_{nm}\mathbb{Q}_{ml}\,,\;\; Z_{nm} = \delta_{nm} + \frac{1}{2\epsilon}\left(\frac{\alpha_s}{2\pi}\gamma^{(0)}_{nm} + O(\alpha_s^2)\right) + O(1/\epsilon^2),$$

thus the conformal variations of the renormalized operator insertion are

$$\delta_D[\mathbb{Q}_{nl}] = \ell^{\mathrm{can}}[\mathbb{Q}_{nl}],\quad \delta_K^-[\mathbb{Q}_{nl}] = -i\sum_{m=0}^{n}\left\{\hat{Z}\hat{a}(l)\hat{Z}^{-1}\right\}_{nm}[\mathbb{Q}_{ml-1}],$$

where $\ell^{\mathrm{can}} = l+3$, $a_{nm}(l) = a(n,l)\delta_{nm}$ with $a(n,l) = 2(n-l)(n+l+3)$.

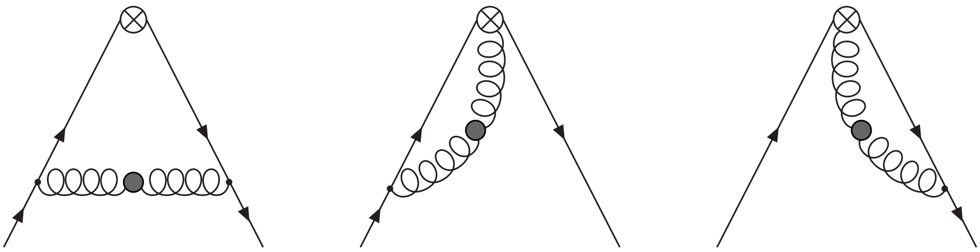
Renormalization of operator products in the MS-scheme

$$\left\{ \begin{matrix} \delta_D \\ \delta_K^- \end{matrix} \right\} [S] = \frac{-\epsilon g + \beta(g)}{2g} \int d^d x \left\{ \begin{matrix} 1 \\ 2x^- \end{matrix} \right\} [(G^a_{\mu\nu})^2](x) + \ldots$$

Thus the operator product needs *renormalization* at $x = 0$

$$i[\mathbb{Q}_{nl}(0)][(G^a_{\mu\nu})^2(x)] = i[\mathbb{Q}_{nl}(0)(G^a_{\mu\nu})^2(x)]$$
$$+\delta^d(x) \sum_{m=0}^{n} Z^A_{nm}[\mathbb{Q}_{ml}(0)] + \frac{i}{2}\partial_+ \delta^d(x) \sum_{m=0}^{n} Z^{A-}_{nm}[\mathbb{Q}_{ml-1}(0)] + \cdots$$

Renormalization constants are calculable from Feynman diagrams



$$Z^A_{nm} : \qquad D_\alpha{}^\mu(k)\, i \left(g_{\mu\nu} k^2 - k_\mu k_\nu \right) D^\nu{}_\beta(k) = D_{\alpha\beta}(k)$$

$$Z^{A-}_{nm} : \qquad 2\bar{n}\cdot \overleftarrow{\partial}_k \, D_{\alpha\beta}(k) - D_{\alpha\beta}(k)\, 2\bar{n}\cdot \overrightarrow{\partial}_k$$

A straigthforward LO calculation provides

$$Z^A_{nm} = \frac{1}{\epsilon}\frac{\alpha_s}{2\pi}\gamma^{(0)}_{nm} + \cdots, \quad Z^{A-}_{nm} = \frac{1}{\epsilon}\frac{\alpha_s}{2\pi}\left(-\left\{ \hat{\gamma}^{(0)} \hat{b} \right\}_{nm} + w_{nm} \right) + \cdots,$$

$$\left[2l - (2u-1)\frac{d}{du} \right] C^{3/2}_n(2u-1) = \sum_{n=0}^{m} b_{nm}(l) C^{3/2}_m(2u-1)$$

$$\int_0^1 du\, C^{3/2}_n(2u-1) \left[-C_F \theta(v-u) \frac{u}{v} \frac{2}{(u-v)^2} + \left\{ \begin{matrix} u \to 1-u \\ v \to 1-v \end{matrix} \right\} \right]_+$$
$$= \sum_{m=0}^{n} w_{nm} C^{3/2}_m(2v-1)\,.$$

Conformal Ward identies in the MS-scheme

$$[\mathcal{D}_y+\mathcal{D}_z]\langle[\mathbb{Q}_{nl}]\psi\overline{\psi}\rangle(y,z) = -\sum_{m=0}^{n}\left[\ell^{\rm can}\delta_{nm} + \hat{\gamma}_{nm}\right]\langle[\mathbb{Q}_{ml}]\psi\overline{\psi}\rangle(y,z) + \frac{\beta}{g}\langle[\mathbb{Q}_{ml}\Delta^{g}]\psi\overline{\psi}\rangle(y,z)+\cdots$$

$$[\mathcal{K}^-_y+\mathcal{K}^-_z]\langle[\mathbb{Q}_{nl}]\psi\overline{\psi}\rangle(y,z) = i\sum_{m=0}^{n}\left[a(n,l)\delta_{nm}+\hat{\gamma}^c_{nm}(l)\right]\langle[\mathbb{Q}_{ml-1}]\psi\overline{\psi}\rangle(y,z) + \frac{\beta}{g}\langle[\mathbb{Q}_{ml}\Delta^{g}_{-}]\psi\overline{\psi}\rangle(y,z)+\cdots$$

with the *conformal anomalies*

$$\gamma_{nm} = \frac{\alpha_s}{2\pi}\gamma^{(0)}_n\delta_{nm} + \frac{\alpha_s^2}{(2\pi)^2}\gamma^{(1)}_{nm}+\cdots,\quad \gamma^{c(0)}_{nm}(l) = -b_{nm}(l)\gamma^{(0)}_m + w_{nm},$$

Constraints for conformal anomalies

The conformal algebra implies *constraints* for the *anomalies* [D.M. 90]:

$$[\mathcal{K}_-,\mathcal{P}_+] = -2(\mathcal{D}+\mathcal{M}_{-+}) \Rightarrow \boxed{\hat{\gamma}^c(l+1)-\hat{\gamma}^c(l) = -2\hat{\gamma}},$$

$$[\mathcal{D},\mathcal{K}_-] = \mathcal{K}_- \quad \Rightarrow \boxed{\left[\hat{a}(l)+\hat{\gamma}^c(l)+2\tfrac{\beta(g)}{g}\hat{b}(l),\hat{\gamma}\right] = 0},$$

The second constraint can be read as

$$2(n-m)(n+m+3)\gamma_{nm}(\alpha_s) = \left[\hat{\gamma}(\alpha_s),\hat{\gamma}^c(l;\alpha_s)+2\frac{\beta(g)}{g}\hat{b}(l)\right]_{nm}, n>m,$$

and thus $\gamma^{(0)}_{nm} = 0$ for $n > m$ and at NLO ($\beta_0 = 11 - 2N_f/3$):

$$\boxed{\gamma^{(1)}_{nm} = \frac{\gamma^{(0)}_n - \gamma^{(0)}_m}{2(n-m)(n+m+3)} \left(-b_{nm}\gamma^{(0)}_m + w_{nm} - \beta_0 b_{nm}\right)}, n > m$$

Restoration of conformal covariance

Can we *restore* conformal covariance?

Yes, for $\beta = 0$. The solution of the conformal constraints provide us that the rotation to the basis of multiplicative renormalizable operators

$$\mathbb{Q}^{\rm CS}_{nl} = \sum_{m=0}^{n} B^{-1}_{nm}[\mathbb{Q}_{ml}]\,, \quad \text{with} \quad \hat{\gamma}^{\rm CS} \equiv \hat{\gamma}^{\rm D} = \hat{B}^{-1}\hat{\gamma}\hat{B},$$

which is only given by the special conformal anomaly

$$\hat{B} = \hat{1} - \mathcal{J}\hat{\gamma}^c + \mathcal{J}(\hat{\gamma}^c\mathcal{J}\hat{\gamma}^c) - \cdots, \quad \mathcal{J}\hat{A} := \left\{\theta(n > m)\frac{A_{nm}}{a(n,m)}\right\}.$$

Rotating the special conformal anomaly in MS-scheme gives

$$\hat{B}^{-1}\left[\hat{a}(l) + \hat{\gamma}^c(l)\right]\hat{B} = \{2(n-l)(n+l+3+\gamma_{nn})\delta_{nm}\}\ .$$

Consequently, in the coformal subtraction (CS) scheme we have

$$[\mathcal{D}_y + \mathcal{D}_z]\langle\mathbb{Q}^{\rm CS}_{nl}\psi\overline{\psi}\rangle(y,z) = -\left[\ell^{\rm can} + \gamma_{nn}\right]\langle\mathbb{Q}^{\rm CS}_{nl}\psi\overline{\psi}\rangle(y,z),$$
$$[\mathcal{K}^-_y + \mathcal{K}^-_z]\langle\mathbb{Q}^{\rm CS}_{nl}\psi\overline{\psi}\rangle(y,z) = i\left[a(n,l) + 2(n-l)\gamma_{nn}\right]\langle\mathbb{Q}^{\rm CS}_{nl-1}\psi\overline{\psi}\rangle(y,z),$$

and so the final expected answer turns out to be *true* in the *CS scheme*

$$\ell^{\rm can}_n \quad\Rightarrow\quad \ell_n(\alpha_s) = \ell^{\rm can}_n + \gamma_n(\alpha_s)\,, \qquad \gamma_n \equiv \gamma_{nn}$$
$$a(n,l) \quad\Rightarrow\quad a(n,l) + 2(n-l)\gamma_n(\alpha_s) = 2(n-l)(\ell_n(\alpha_s) + l)\,.$$

Reconstruction of hard scattering amplitudes and evolution kernels

The found results can be applied in *two different ways*:

- The CS scheme allows a *trivial prediction* for Wilson-coefficients (NNLO) and anomalous dimensions (NLO, partly at NNLO) in terms of the forward quantities known from deep inelastic scattering at NNLO [D.M.]. The β *proportional terms* remain scheme dependent and have either to be calculated (NNLO) or are fixed by a "renormalization group improvement" [B.Melicz, D.M., K.Passek].
- In the MS scheme one can *predict* the remaining anomalous dimensions and Wilson-coefficients at NLO [A.V.Belitsky, D.M.] and reconstruct the hard scattering amplitudes and evolution kernels [A.V. Belitsky,D.M., A.Freund].

Flavour singlet anomalous dimensions

We consider now the operators

$$\mathbf{O}_{nl} = (i\partial_+)^{l-n} \begin{pmatrix} \mathbb{Q}_n^{1,1} \\ \mathbb{G}_{n-1}^{3/2,3/2} \end{pmatrix} \quad \text{with} \quad l \geq n, \quad \text{for} \left\{ \begin{matrix} \text{vector} \\ \text{axial vector} \end{matrix} \right\} \text{case}$$

The constraints have the same form as in the non-singlet case:

$$\boxed{\left[\hat{\boldsymbol{a}}(l) + \hat{\boldsymbol{\gamma}}^c(l) + 2\frac{\beta}{g}\hat{\boldsymbol{b}}(l), \hat{\boldsymbol{\gamma}}\right] = 0 \quad \text{and} \quad \hat{\boldsymbol{\gamma}}^c(l+1) - \hat{\boldsymbol{\gamma}}^c(l) = -2\hat{\boldsymbol{\gamma}}},$$

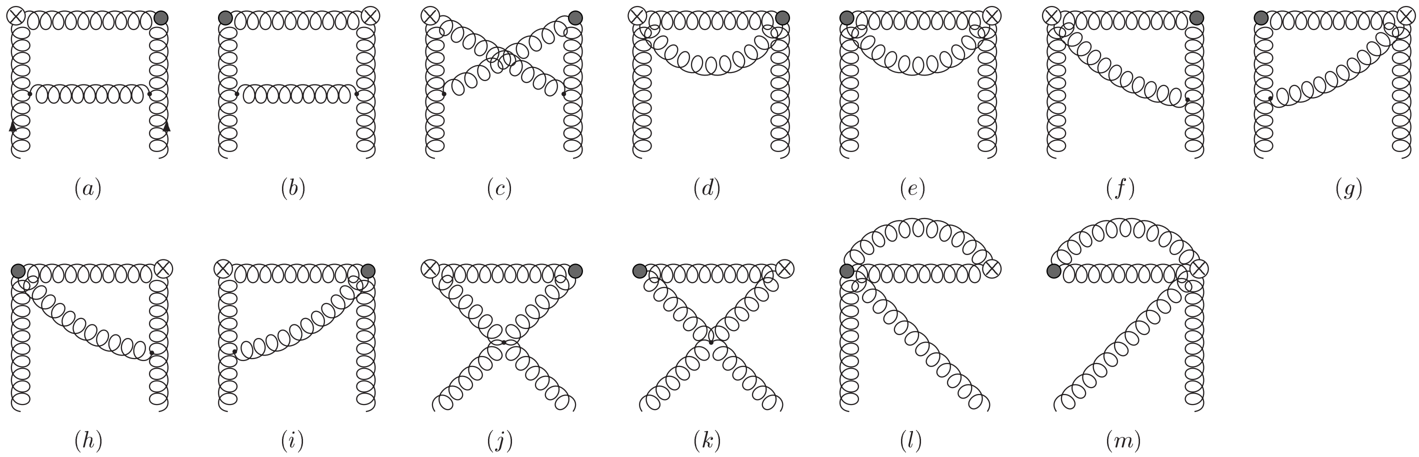
$$\hat{\boldsymbol{a}} = \begin{pmatrix} \hat{a} & 0 \\ 0 & \hat{a} \end{pmatrix}, \; \hat{\boldsymbol{b}} = \begin{pmatrix} \hat{b} & 0 \\ 0 & \hat{b} \end{pmatrix}, \; \hat{\boldsymbol{\gamma}} = \begin{pmatrix} {}^{QQ}\hat{\gamma} & {}^{QG}\hat{\gamma} \\ {}^{GQ}\hat{\gamma} & {}^{GG}\hat{\gamma} \end{pmatrix}, \; \hat{\boldsymbol{\gamma}}^c = \begin{pmatrix} {}^{QQ}\hat{\gamma}^c & {}^{QG}\hat{\gamma}^c \\ {}^{GQ}\hat{\gamma}^c & {}^{GG}\hat{\gamma}^c \end{pmatrix}.$$

NLO conformal predictions

The special conformol anomaly

$$\hat{\boldsymbol{\gamma}}^{c(0)} = -\hat{\boldsymbol{b}}\hat{\boldsymbol{\gamma}}^{(0)} + \hat{\boldsymbol{w}}$$

is known from a LO calculation, e.g,



$\hat{\boldsymbol{\gamma}}^{(1)} = \hat{\boldsymbol{\gamma}}^{\mathrm{D}(1)} + \hat{\boldsymbol{\gamma}}^{\mathrm{ND}(1)}$ with $\hat{\boldsymbol{\gamma}}^{\mathrm{ND}(1)} = -\left[\hat{\boldsymbol{\gamma}}^{(0)}, \hat{\boldsymbol{d}}\right]\left(\beta_0 \hat{\mathbf{1}} + \hat{\boldsymbol{\gamma}}^{(0)}\right) + \left[\hat{\boldsymbol{\gamma}}^{(0)}, \hat{\boldsymbol{g}}\right]$,

where $\boldsymbol{d}_{nm} = \boldsymbol{b}_{nm}/a(n,m)$ and $\boldsymbol{g}_{nm} = \boldsymbol{w}_{nm}/a(n,m)$.

Consistency checks

- In the flavor non-singlet sector the conformal prediction coincides with the explicit NLO calculation.
- In the singlet sector the β_0 proportional terms have been checked by the calculation of Feynman graphs with quark bubble insertions.
- Four supersymmetric constraints, arising from the reduction of QCD to supersymmetric $\mathcal{N} = 1$ Yang-Mills theory, for the off-diagonal part of the anomalous dimensions are satisfied.
- Superconformal symmetry, which is anomalously broken, implies four constraints for the special conformal anomaly. Evaluation of the superconformal anomaly in LO shows that these constraints are consistent with the special conformal anomaly.

Construction of evolution kernels to NLO

The result for the anomalous dimensions imply the following structure:

$$\boldsymbol{V}^{(1)} = \boldsymbol{D}(u,v) + \boldsymbol{G}(u,v) - \left\{ \dot{\boldsymbol{V}} \otimes \boldsymbol{V}^{(0)} - \frac{\beta_0}{2}\dot{\boldsymbol{V}} + \left[\boldsymbol{g} \overset{\otimes}{,} \boldsymbol{V}^{(0)} \right]_- \right\}(u,v)$$

where the separate kernels are obtained by

- $[\hat{\boldsymbol{\gamma}}^{(0)}, \hat{\boldsymbol{d}}]_{\boldsymbol{nm}} \Leftrightarrow \dot{\boldsymbol{V}}(u,v)$ and $\boldsymbol{g}_{\boldsymbol{nm}} \Leftrightarrow \boldsymbol{g}(u,v)$ (up to diagonal entries)
- ${}^{QQ}G(u,v) \;\Rightarrow\; \boldsymbol{G}(u,v)$ arise from crossed ladder diagrams (no subdivergencies and thus one can use six supersymmetric relations)
- $\boldsymbol{D}(u,v)$ has a simple representation as convolution of LO kernels

Extension of the support is unique and so the evolution for generalized parton distributions is [D.M., D.Robaschik, B.Geyer, F.M.Dittes, J.Hořejši (94)]:

$$\mu^2 \frac{d}{d\mu^2} \boldsymbol{q}(x,\eta) = \int_{-1}^{1} \frac{dy}{2|\eta|} \begin{pmatrix} {}^{QQ}V & \eta^{-1}\,{}^{QG}V \\ \eta\,{}^{GQ}V & {}^{GG}V \end{pmatrix} \left(\frac{\eta + x}{2\eta}, \frac{\eta + y}{2\eta} \right) \boldsymbol{q}(y,\eta)$$

Construction of hard scattering amplitudes

Physical motivation

Considering the Compton scattering process (or crossed channels)

$$\gamma^*(q_1) + h(p_1) \to \gamma^{(*)}(q_2) + h(p_2)$$

in the generalized Bjorken kinematics

$$\nu = p\cdot q \to \infty,\, Q^2 = -q^2 \to \infty,\, \xi = \frac{1}{\omega} = \frac{Q^2}{p\cdot q},\, \eta = \frac{\Delta\cdot q}{p\cdot q} \;\; \text{fixed},$$

where $p = p_1 + p_2$, $\Delta = p_2 - p_1$ and $q = (q_1 + q_2)/2$.

Conformal prediction

The QCD dynamics is encoded in the hadronic tensor

$$T_{\mu\nu}(p,\Delta,q) = i\int d^4x\, e^{i\,q\cdot x}\langle h(p_2)|T J_\mu(x/2)J_\nu(-x/2)\,|h(p_1)\rangle,$$

which is predicted to leading twist-two by the conformal OPE ($\beta = 0$):

$$T(\omega,\eta,Q^2) = \sum_{n=0}^{\infty} c_n(\alpha_s)\left(\frac{\mu^2}{Q^2}\right)^{\frac{\gamma_n(\alpha_s)}{2}} \frac{2^{n+1}B(n+1,n+2)}{B(n{+}2{+}\gamma_n/2, n{+}2{+}\gamma_n/2)}$$
$$\times\int_0^1 du\, \frac{\omega^{n+1}[u(1-u)]^{n+1+\gamma_n(\alpha_s)/2}}{[1-\eta\omega(2u-1)]^{n+1+\gamma_n(\alpha_s)/2}}\langle\!\langle \mathbb{Q}_n^{\mathrm{CS}}(0)\rangle\!\rangle(\eta,\Delta^2,\mu^2)\,.$$

The *normalization* can be borrowed from DIS results ($\eta = 0$)

$$c_n(\alpha_s) = c_n^{(0)} + \frac{\alpha_s}{2\pi}c_n^{(1)} + \frac{\alpha_s^2}{(2\pi)^2}c_n^{(2)} + O(\alpha_s^3) \quad \text{with} \quad c_n^{(0)} = 1$$

and the renormalization group equation is diagonal

$$\mu\frac{d}{d\mu}\langle\!\langle \mathbb{Q}_n^{\mathrm{CS}}(0)\rangle\!\rangle = -\gamma_n(\alpha_s(\mu))\langle\!\langle \mathbb{Q}_n^{\mathrm{CS}}(0)\rangle\!\rangle\,,$$

where γ_n are completly (partial) known to NLO (NNLO):

$$\gamma_n(\alpha_s) = \frac{\alpha_s}{2\pi}\gamma_n^{(0)} + \frac{\alpha_s^2}{(2\pi)^2}\gamma_n^{(1)} + \frac{\alpha_s^3}{(2\pi)^3}\gamma_n^{(2)} + O(\alpha_s^3)$$

Note:

conformal prediction for $c_n^{(1)}$ is exact (no β_0 term)

Momentum fraction representation

The local conformal OPE prediction is related to the momentum fraction representation by

$$\int_{-1}^{1} dx\, \eta^n\, C_n^{3/2}\left(\frac{x}{\eta}\right) q^{\text{CS}}(x, \eta, \Delta^2, \mu^2) = \langle\langle \mathbb{Q}_n^{\text{CS}}(0) \rangle\rangle(\eta, \Delta^2, \mu^2) \, .$$

Rotating to the MS scheme

$$\langle\langle [\mathbb{Q}_{nl}] \rangle\rangle = \sum_{m=0}^{n} B_{nm} \langle\langle \mathbb{Q}_{ml}^{\text{CS}} \rangle\rangle \, , \;\; B_{nm} = \delta_{nm} - \theta(n > m) \frac{\alpha_s}{2\pi} \frac{\gamma_{nm}^{c(0)}}{a(n, m)} - \cdots ,$$

allows the resummation of all conformal partial waves

$$\xi T \simeq \int_{-1}^{1} \frac{dx}{|\eta|} \left[\mathcal{T}\left(\frac{\xi}{\eta}, \frac{x}{\eta}, \frac{Q}{\mu}; \alpha_s(\mu) \right) \mp \{\xi \to -\xi\} \right] q(x, \eta, \Delta^2, \mu^2) \, ,$$

which coincedes with the explicit NLO calculation for quarks and gluons
[L. Mankiewicz et al.; X. Ji and J. Osborne (1998)]

$\gamma^*\gamma^*$ -to- π transition form factor

Kinematics

$$\gamma^*(q_1) + \gamma^*(q_2) \to \pi^0(P), \qquad Q^2 = -\frac{(q_1 - q_2)^2}{4} = -\frac{q_1^2 + q_2^2}{2}$$

$$\omega = \frac{P \cdot q}{Q^2} = \frac{q_1^2 - q_2^2}{q_1^2 + q_2^2}, \quad \eta = 1.$$

Due to parity conservation the hadronic tensor must be of the form

$$T_{\mu\nu}(\omega, \eta = 1, Q^2) = ie^2 \epsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta F_{\gamma\pi}(\omega, Q)$$

$$F = \frac{\sqrt{2} f_\pi}{3Q^2} \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} C_n\left(\omega \middle| \alpha_s(\mu), \frac{Q}{\mu}\right) \phi_n(\mu) + \mathcal{O}\left(\frac{1}{Q^4}\right),$$

where in the conformal scheme the Wilson–coefficients read

$$C_n\left(\omega \middle| \alpha_s, \frac{Q}{\mu}\right) = c_n(\alpha_s) \frac{2^{n+1} B(n+1, n+2)}{B(n+2+\gamma_n/2, n+2+\gamma_n/2)} \left(\frac{\mu^2}{Q^2}\right)^{\frac{\gamma_n(\alpha_s)}{2}}$$

$$\times \int_0^1 du \, \frac{\omega^n [u(1-u)]^{n+1+\gamma_n(\alpha_s)/2}}{[1 - \omega(2u-1)]^{n+1+\gamma_n(\alpha_s)/2}},$$

ω–dependence

For $(n+1)\ln(1 + \sqrt{1-\omega^2}) > \sqrt{2}$ it is exponentially suppressed:

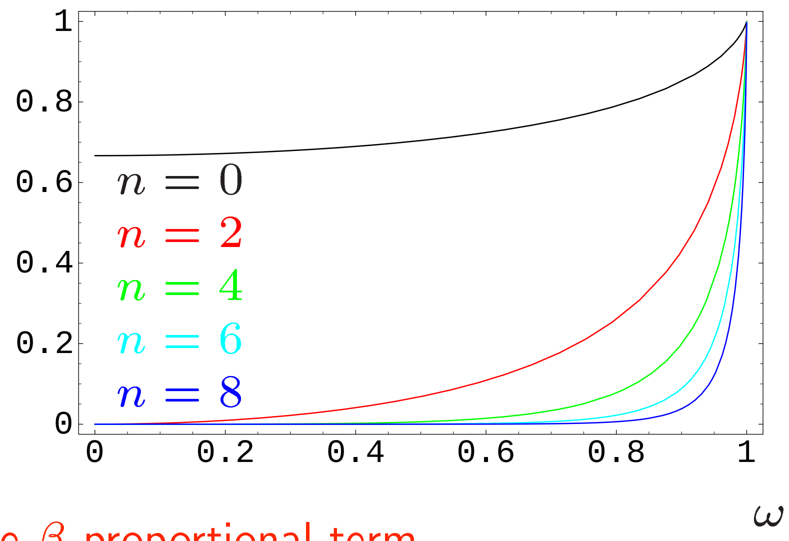
$$C_n\left(\omega \middle| \alpha_s, \frac{Q}{\mu}\right) \propto \omega^n \exp\left\{ -\left(n + 1 + \frac{\gamma_n}{2}\right) \ln\left(1 + \sqrt{1-\omega^2}\right) \right\},$$

i.e., C_n is strongly concentrated in $|\omega| = 1$, taking the value

$$C_n\left(\omega = 1\middle|\alpha_s, \frac{Q}{\mu}\right) = \frac{c_n(\alpha_s)B(n+1,n+2)}{B(n+2+\gamma_n/2, n+3+\gamma_n/2)}\left(\frac{\mu^2}{2Q^2}\right)^{\frac{\gamma_n(\alpha_s)}{2}}$$

$$= \frac{3+2n}{(n+1)(n+2)} + \mathcal{O}(\alpha_s)$$

$$\frac{C_n^{\mathrm{LO}}(\omega)}{C_n^{\mathrm{LO}}(\omega=1)}$$



Restoration of the β proportional term

Either one can take the $\overline{\mathrm{MS}}$ result or one requires that the partial waves do not mix in the full theory:

$$\mu\frac{d}{d\mu}\phi'_n(\mu) = -\gamma_n(\alpha_s(\mu))\phi'_n(\mu) \text{ with } \phi'_n(\mu) = \sum_{m=0}^{n} \mathcal{B}^{-1}_{nm}(\mu)\phi_m(\mu)\,,$$

$$\mu\frac{d}{d\mu}C'_n\left(\omega\middle|\alpha_s(\mu),\frac{Q}{\mu}\right) = \gamma_n(\alpha_s(\mu))C'_n\left(\omega\middle|\alpha_s(\mu),\frac{Q}{\mu}\right), C'_n = \sum_{m=n}^{\infty} C_m\mathcal{B}_{mn}\,.$$

Moreover, we require the initial condition

$$C'_n\left(\omega\middle|\alpha_s(\mu),\frac{Q}{\mu}\right)\Big|_{\mu=Q} = C_n\left(\omega\middle|\alpha_s(Q),1\right).$$

What is and what can be measured?

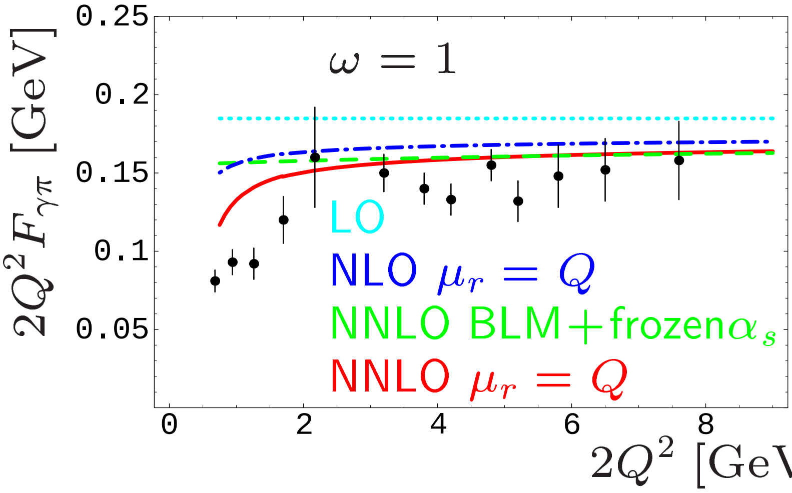
Quasi–real photon limit $\omega = 1$

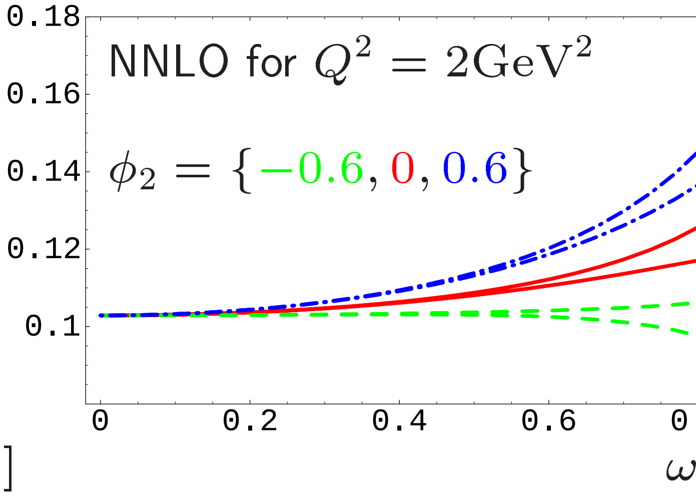
i) In this singular limit all partial waves contribute.
ii) In the CS scheme the prediction for the lowest one is parameter free (independent on the factorization scale μ_f):

$$F_{\gamma\pi}(\omega = 1, Q) = \frac{\sqrt{2} f_\pi}{2Q^2} \left\{ 1 - \frac{\alpha_s(\mu_r)}{\pi} - \frac{\alpha_s^2(\mu_r)}{\pi^2} \times \left[3.583 - 2.25 \ln\left(\frac{Q^2}{\mu_r^2}\right)\right] + O(\alpha_s^3) \right\},$$

Variation of photon virtualities $|\omega| < 1$

i) Variation of $0.4 \lesssim |\omega| \lesssim 0.8$ allows to access the first few moments.
ii) Perturbative corrections essentially generate only a vertical shift.





Approximative equal photon virtualities $|\omega| \lesssim 0.4$

In this case we have a Bjorken like sume rule $\omega^{\rm cut} \lesssim 0.4$:

$$\frac{Q^2}{\omega^{\rm cut}} \int_0^{\omega^{\rm cut}} d\omega F_{\gamma\pi}(\omega, Q) = \frac{\sqrt{2} f_\pi}{3} c_{\rm Bj}(\alpha_s(\mu_r), Q/\mu_r) + O\left(\frac{1}{Q^2}\right),$$

$$c_{\rm Bj}(\alpha_s, 1) = 1 - \frac{\alpha_s}{\pi} - 3.583\frac{\alpha_s^2}{\pi^2} - 20.215\frac{\alpha_s^3}{\pi^3} + O(\alpha_s^4),$$

Conclusions

Conformal symmetry *holds* for $\beta = 0$ in the perturbative QCD sector and it is *rather useful* for several problems:

- Classification and partial wave decomposition of (generalized) distribution amplitudes.
- Solving of evolution equations, for instance, via the Hamiltonian approach (discovering of hidden symmetries).
- Reconstruction of exclusive quantities, e.g., evolution kernels and hard scattering amplitudes, from the corresponding forward results.
- Relating different observables, Crewther relation

$$\boxed{C(\alpha_s(\mu_{\rm BLM}))D(\alpha_s(\mu^*_{\rm BLM})) = 1}$$ *verified in NNNLO*

What else can be done with conformal symmetry?

- Evaluation of NNLO corrections: $\gamma^*\gamma^* \to \eta$, $\gamma^*\gamma^* \to \pi^+\pi^-$, DVCS.
- Conformal symmetry can be used for the evaluation of NLO corrections in the twist-3 sector.
- Hopefully, conformal symmetry has the potential to understand (better) twist-four corrections in exclusive channels.
- Is there something else in pQCD ?