# Hadron Tomography

## *GPDs $\Rightarrow$ 3-d images of the nucleon where $x - y$ plane in position space and $z$ axis in momentum space*

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# Outline

- form factor $\Rightarrow$ charge distribution in position space
- Deeply virtual Compton scattering (DVCS)
- $\hookrightarrow$ Generalized parton distributions (GPDs)
- Probabilistic interpretation of GPDs as Fourier transforms of impact parameter dependent PDFs
  - $H(x, 0, -\mathbf{\Delta}_\perp^2) \longrightarrow q(x, \mathbf{b}_\perp)$
  - $\tilde{H}(x, 0, -\mathbf{\Delta}_\perp^2) \longrightarrow \Delta q(x, \mathbf{b}_\perp)$
  - $E(x, 0, -\mathbf{\Delta}_\perp^2) \longrightarrow \perp$ distortion of PDFs when the target is transversely polarized
- Chromodynamik lensing and $\perp$ single-spin asymmetries (SSA)

$$\left.\begin{array}{c}\text{transverse distortion of PDFs}\\ +\ \text{final state interactions}\end{array}\right\} \Rightarrow \ \perp \text{SSA in } \ \gamma N \longrightarrow \pi + X$$

- Summary

# Form factor vs. charge distribution (non-relativistic)

- plane wave states have uniform charge distribution

$\hookrightarrow$ meaningful definition of $\rho(\vec{r})$ requires that state is localized in position space!

$\hookrightarrow$ define localized state (center of mass frame)

$$\left|\vec{R}=\vec{0}\right\rangle \equiv \mathcal{N}\int d^3\vec{p}\,|\vec{p}\rangle$$

- define charge distribution (for this localized state)

$$\rho(\vec{r}) \equiv \left\langle \vec{R}=\vec{0}\right| j^0(\vec{r}) \left|\vec{R}=\vec{0}\right\rangle$$

# Form factor vs. charge distribution (non-relativistic)

- use translational invariance to relate to same matrix element that appears in def. of form factor

$$\rho(\vec{r}) \equiv \left\langle \vec{R}=\vec{0}\right| j^0(\vec{r}) \left|\vec{R}=\vec{0}\right\rangle$$

$$= |\mathcal{N}|^2 \int d^3\vec{p} \int d^3\vec{p}\,' \left\langle \vec{p}\,'\right| j^0(\vec{r}) \left|\vec{p}\right\rangle$$

$$= |\mathcal{N}|^2 \int d^3\vec{p} \int d^3\vec{p}\,' \left\langle \vec{p}\,'\right| j^0(\vec{0}) \left|\vec{p}\right\rangle e^{i\vec{r}\cdot(\vec{p}-\vec{p}\,')},$$

$$= |\mathcal{N}|^2 \int d^3\vec{p} \int d^3\vec{p}\,' F\left(-\left(\vec{p}\,'-\vec{p}\right)^2\right) e^{i\vec{r}\cdot(\vec{p}-\vec{p}\,')}$$

$\hookrightarrow$

$$\rho(\vec{r}) = \int \frac{d^3\vec{q}}{(2\pi)^3} F(-\vec{q}^{\,2}) e^{i\vec{q}\cdot\vec{r}}$$

back

# Form Factors (relativistic)

- Lorentz invariance, parity, current conservation $\Rightarrow$

$$\langle p' | j^\mu(0) | p\rangle = \begin{cases} \left(p^\mu + p^{\mu\prime}\right) F(q^2) & \text{(spin } 0) \\ \bar{u}(p') \left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2M} F_2(q^2)\right] u(p) & \text{(spin } \frac{1}{2}) \end{cases}$$

with $q^\mu = p^\mu - p^{\mu\prime}$.

- issue: “energy factors” spoil simple interpretation of form factors as FT of charge distributions

# Form Factor vs. Charge Distribution (relativistic)

- wave packet

$$|\Psi\rangle = \int \frac{d^3p \;\; \psi(\vec{p})}{\sqrt{2E_{\vec{p}}(2\pi)^3}} \, |\vec{p}\rangle \, ,$$

- $E_{\vec{p}} = \sqrt{M^2 + \vec{p}^2}$ and covariant normalization $\langle \vec{p}' | \vec{p} \rangle = 2E_{\vec{p}} \delta(\vec{p}' - \vec{p})$
- Fourier transform of charge distribution in the wave packet

$$\begin{aligned}
\tilde{\rho}(\vec{q}) &\equiv \int d^3x e^{-i\vec{q}\cdot\vec{x}} \langle \Psi | \, j^0(\vec{x}) \, | \Psi \rangle \\
&= \int \frac{d^3p}{\sqrt{2E_{\vec{p}} 2E_{\vec{p}'}}} \Psi^*(\vec{p}+\vec{q}) \Psi(\vec{p}) \langle \vec{p}' | \, j^0(\vec{0}) \, | \vec{p} \rangle \\
&= \frac{1}{2} \int d^3p \frac{E_{\vec{p}} + E_{\vec{p}'}}{\sqrt{E_{\vec{p}} E_{\vec{p}'}}} \Psi^*(\vec{p}+\vec{q}) \Psi(\vec{p}) F(q^2).
\end{aligned}$$

# Form Factor vs. Charge Distribution (relativistic)

- Nonrelativistic case:
  - $$\frac{E_{\vec{p}} + E_{\vec{p}'}}{2\sqrt{E_{\vec{p}} E_{\vec{p}'}}} = 1 \qquad \text{and} \qquad q^2 = -\vec{q}^2$$

$\hookrightarrow$ Forier transform of charge distribution in the wave packet

$$\tilde{\rho}(\vec{q}) = \int d^3p \Psi^*(\vec{p}+\vec{q})\Psi(\vec{p})F(\vec{q}^2)$$

  - choose $\Psi(\vec{p})$ very localized in position space
    $\Psi^*(\vec{p}+\vec{q}) \approx \Psi^*(\vec{p})$

$$\hookrightarrow \qquad \tilde{\rho}(\vec{q}) = F(\vec{q}^2)$$

# Form Factor vs. Charge Distribution (relativistic)

- Relativistic corrections (example rms radius):

$$\tilde{\rho}(\vec{q}^2) = 1 - \frac{R^2}{6}\vec{q}^2 - \frac{R^2}{6}\int d^3p\,|\Psi(\vec{p})|^2 \frac{(\vec{q}\cdot\vec{p})^2}{E_{\vec{p}}^2} + \int d^3p\left|\vec{q}\cdot\vec{\nabla}\Psi(\vec{p})\right|^2 - \frac{1}{8}\int d^3p\,|\Psi(\vec{p})|^2\frac{(\vec{q}\cdot\vec{p})^2}{E_{\vec{p}}^4},$$

[$R^2$ defined as usual: $F(q^2) = 1 + \frac{R^2}{6}q^2 + \mathcal{O}(q^4)$]

- If one completely localizes the wave packet, i.e. $\int d^3p\left|\vec{q}\cdot\vec{\nabla}\Psi(\vec{p})\right|^2 \rightarrow 0$, then relativistic corrections diverge ($\Delta x \Delta p \sim 1$)

$$\frac{R^2}{6}\int d^3p\,|\Psi(\vec{p})|^2\frac{(\vec{q}\cdot\vec{p})^2}{E_{\vec{p}}^2} \rightarrow \infty, \qquad \frac{1}{8}\int d^3p\,|\Psi(\vec{p})|^2\frac{(\vec{q}\cdot\vec{p})^2}{E_{\vec{p}}^4} \rightarrow \infty$$

# Form Factor vs. Charge Distribution (relativistic)

- in rest frame, rel. corrections contribute $\Delta R^2 \sim \lambda_C^2 = \frac{1}{M^2}$ identification of charge distribution in rest frame with Fourier transformed form factor only unique down to scale $\lambda_C$
- standard remedy: interpret $F(\vec{q})$ as Fourier transform of charge distribution in Breit "frame" $\vec{p}' = -\vec{p}$ (note: Breit "frame" is actually a different frame for each $\vec{q}$!)

# Form Factor vs. Charge Distribution (relativistic)

- infinite momentum frame: rel. corrections governed by $\frac{\vec{p}\cdot\vec{q}}{E_{\vec{p}}^2}$ and $\frac{\vec{q}^2}{E_{\vec{p}}^2}$

  consider wave packet $\Psi(\vec{p}_\perp)$ in transverse direction, with
  - sharp longitudinal momentum $P_z \to \infty$
  - transverse size of wave packet $r_\perp$, with $R \gg r_\perp \gg \frac{1}{P_z}$
  - take purely transverse momentum transfer

  $\hookrightarrow$ $\tilde{\rho}(\vec{q}_\perp) = F(\vec{q}_\perp^2)$

  $\hookrightarrow$ form factor can be interpreted as Fourier transform of charge distribution w.r.t. impact parameter in $\infty$ momentum frame (without $\lambda_C$ uncertainties!)
  - impact parameter measured w.r.t. $\perp$ center of momentum $\mathbf{R}_\perp = \sum_{i\in q,g} x_i \mathbf{r}_\perp^i$

# Same Derivation in LF-Coordinates

- light-front (LF) coordinates

$$p^+ = \frac{1}{\sqrt{2}}\left(p^0 + p^3\right) \qquad\qquad p^- = \frac{1}{\sqrt{2}}\left(p^0 - p^3\right)$$

- form factor for spin $\frac{1}{2}$ target (Lorentz invariance, parity, charge conservation)

$$\langle p' \left| j^\mu(0) \right| p \rangle = \bar{u}(p') \left[ \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2M} F_2(q^2) \right] u(p)$$

with $q^\mu = p^\mu - p^{\mu\prime}$.

- If $q^+ = 0$ (Drell-Yan-West frame) then

$$\left\langle p', \uparrow \left| j^+(0) \right| p, \uparrow \right\rangle = 2p^+ F_1(-\mathbf{q}_\perp^2)$$

# $F(\mathbf{q}_\perp^2) \to \rho(\mathbf{r}_\perp)$ in LF-Coordinates

- define state that is localized in $\perp$ position:

$$\left|p^+, \mathbf{R}_\perp = \mathbf{0}_\perp, \lambda\right\rangle \equiv \mathcal{N} \int d^2\mathbf{p}_\perp \left|p^+, \mathbf{p}_\perp, \lambda\right\rangle$$

  Note: $\perp$ boosts in IMF form Galilean subgroup $\Rightarrow$ this state has
  $\mathbf{R}_\perp \equiv \frac{1}{P^+} \int dx^- d^2\mathbf{x}_\perp\, \mathbf{x}_\perp T^{++}(x) = \mathbf{0}_\perp$
  (cf.: working in CM frame in nonrel. physics)

- define charge distribution in impact parameter space

$$2p^+ \rho(\mathbf{b}_\perp) \equiv \frac{1}{2p^+} \left\langle p^+, \mathbf{R}_\perp = \mathbf{0}_\perp \right| j^+(0^-, \mathbf{b}_\perp) \left| p^+, \mathbf{R}_\perp = \mathbf{0}_\perp \right\rangle$$

# $F(\mathbf{q}_\perp^2) \to \rho(\mathbf{r}_\perp)$ in LF-Coordinates

- use translational invariance to relate to same matrix element that appears in def. of form factor

$$
\begin{aligned}
\rho(\mathbf{b}_\perp) &\equiv \frac{1}{2p^+}\left\langle p^+, \mathbf{R}_\perp = \mathbf{0}_\perp \right| j^+(0^-, \mathbf{b}_\perp) \left| p^+, \mathbf{R}_\perp = \mathbf{0}_\perp \right\rangle \\
&= \frac{|\mathcal{N}|^2}{2p^+} \int d^2\mathbf{p}_\perp \int d^2\mathbf{p}'_\perp \left\langle p^+, \mathbf{p}'_\perp \right| j^+(0^-, \mathbf{b}_\perp) \left| p^+, \mathbf{p}_\perp \right\rangle \\
&= \frac{|\mathcal{N}|^2}{2p^+} \int d^2\mathbf{p}_\perp \int d^2\mathbf{p}'_\perp \left\langle p^+, \mathbf{p}'_\perp \right| j^+(0^-, \mathbf{0}_\perp) \left| p^+, \mathbf{p}_\perp \right\rangle e^{i\mathbf{q}_\perp \cdot \mathbf{b}_\perp} \\
&= |\mathcal{N}|^2 \int d^2\mathbf{p}_\perp \int d^2\mathbf{p}'_\perp F_1(-\mathbf{q}_\perp^2) e^{i\mathbf{q}_\perp \cdot \mathbf{b}_\perp}
\end{aligned}
$$

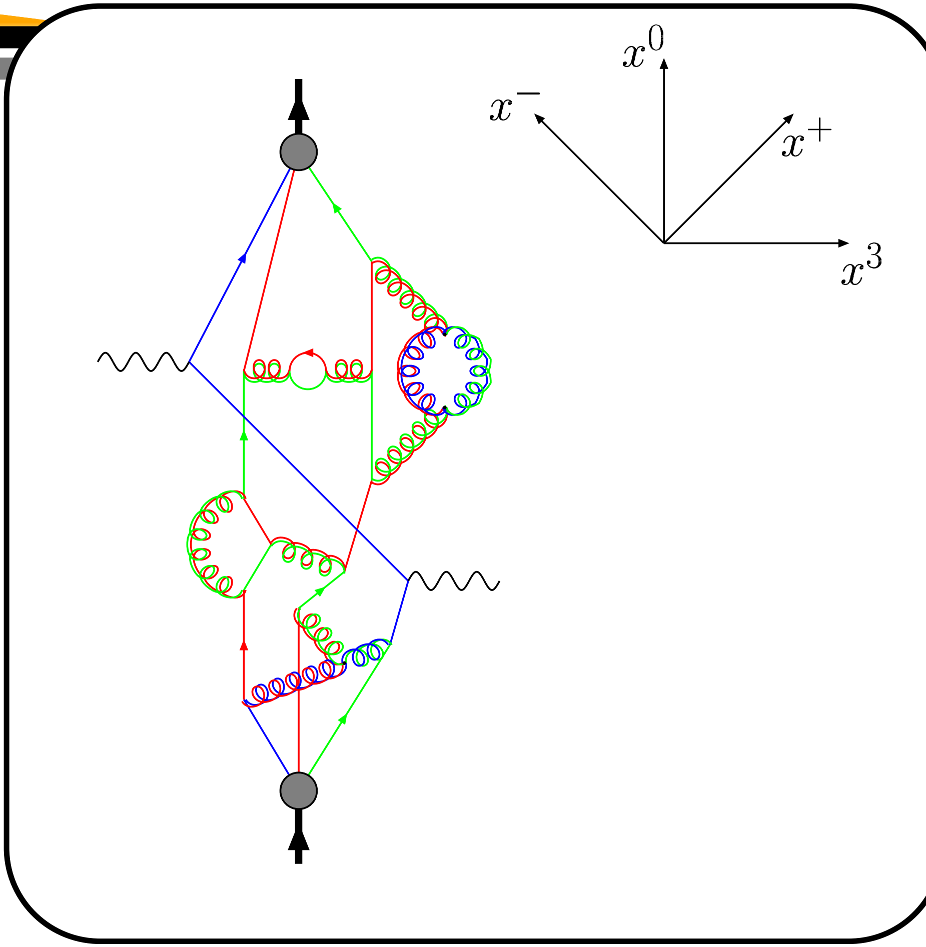
$\hookrightarrow$

$$
\boxed{\rho(\mathbf{b}_\perp) = \frac{\int d^2\mathbf{q}_\perp}{(2\pi)^2} F_1(-\mathbf{q}_\perp^2) e^{i\mathbf{q}_\perp \cdot \mathbf{b}_\perp}}
$$

# Summary: Form Factor vs. Charge Distribution

- fixed target: FT of form factor = charge distribution in position space
- “moving” target:
  - nonrelativistically: FT of form factor = charge distribution in position space, where position is measured relative to **center of mass**
  - relativistic corrections usually make idendification $F(q^2) \overset{FT}{\leftrightarrow} \rho(\vec{r})$ ambigous at scale $\Delta R \sim \lambda_C = \frac{1}{M}$
  - exceptions:
    - Breit “frame”
    - $\infty$ momentum frame ($\rightarrow$ Galilean subgroup of $\perp$ boosts)
      Reference point: **transverse center of longitudinal momentum**

# DIS $\longrightarrow$ light-cone correlations




light-cone coordinates:

$$x^+ = \left(x^0 + x^3\right)/\sqrt{2}$$

$$x^- = \left(x^0 - x^3\right)/\sqrt{2}$$

DIS related to correlations along light–cone

$$q(x_{Bj}) = \int \frac{dx^-}{2\pi} \langle P|\overline{q}(0^-, \mathbf{0}_\perp)\gamma^+ q(x^-, \mathbf{0}_\perp)|P\rangle \, e^{ix^- x_{Bj} P^+}$$

Probability interpretation!

No information about transverse position of partons!

# Generalized Parton Distributions (GPDs)

$$\int \frac{dx^-}{2\pi} e^{ix^-\bar{p}^+x} \left\langle p' \left| \bar{q}\left(-\frac{x^-}{2}\right)\gamma^+ q\left(\frac{x^-}{2}\right)\right| p \right\rangle = H(x,\xi,\Delta^2)\bar{u}(p')\gamma^+ u(p) + E(x,\xi,\Delta^2)\bar{u}(p')\frac{i\sigma^{+\nu}\Delta_\nu}{2M}u(p)$$

$$\int \frac{dx^-}{2\pi} e^{ix^-\bar{p}^+x} \left\langle p' \left| \bar{q}\left(-\frac{x^-}{2}\right)\gamma^+\gamma_5 q\left(\frac{x^-}{2}\right)\right| p \right\rangle = \tilde{H}(x,\xi,\Delta^2)\bar{u}(p')\gamma^+\gamma_5 u(p) + \tilde{E}(x,\xi,\Delta^2)\bar{u}(p')\frac{\gamma_5\Delta^+}{2M}u(p)$$

where $\Delta = p' - p$ is the momentum transfer and $\xi$ measures the longitudinal momentum transfer on the target $\Delta^+ = \xi(p^+ + p^{+\prime})$.

# Parton Interpretation

$$\int \frac{dx^-}{2\pi} e^{ix^-\bar{p}^+x} \left\langle p' \left| \bar{q}\left(-\frac{x^-}{2}\right)\gamma^+ q\left(\frac{x^-}{2}\right)\right| p\right\rangle = H(x,\xi,\Delta^2)\bar{u}(p')\gamma^+ u(p)$$
$$+E(x,\xi,\Delta^2)\bar{u}(p')\frac{i\sigma^{+\nu}\Delta_\nu}{2M}u(p)$$

- $x$ is mean long. momentum fraction carried by active quark
- In general no probabilistic interpretation since initial and final state not the same
- Instead: interpretation as transition amplitude
- $\int dx H(x,\xi,\Delta^2) = F_1(\Delta^2)$ and $\int dx E(x,\xi,\Delta^2) = F_2(\Delta^2)$

$\hookrightarrow$ GPDs provide a decomposition of form factor w.r.t. the momentum fraction (in IMF) carried by the active quark

- Actually $H = H(x,\xi,\Delta^2,q^2)$, but will not discuss $q^2$ dependence of GPDs today!

# What is Physics of GPDs ?

- Definition of GPDs resembles that of form factors

$$\left\langle p' \left| \hat{O} \right| p \right\rangle = H(x,\xi,\Delta^2)\bar{u}(p')\gamma^+u(p) + E(x,\xi,\Delta^2)\bar{u}(p')\frac{i\sigma^{+\nu}\Delta_\nu}{2M}u(p)$$

with $\hat{O} \equiv \int \frac{dx^-}{2\pi} e^{ix^-\bar{p}^+x}\bar{q}\left(-\frac{x^-}{2}\right)\gamma^+ q\left(\frac{x^-}{2}\right)$

$\hookrightarrow$ relation between PDFs and GPDs similar to relation between a charge and a form factor

$\hookrightarrow$ If form factors can be interpreted as Fourier transforms of charge distributions in position space, what is the analogous physical interpretation for GPDs ?

# Form Factors vs. GPDs

| operator | forward matrix elem. | off-forward matrix elem. | position space |
|---|---|---|---|
| $\bar{q}\gamma^+ q$ | $Q$ | $F(t)$ | $\rho(\vec{r})$ |
| $\int \frac{dx^- e^{ixp^+x^-}}{4\pi} \bar{q}\left(\frac{-x^-}{2}\right) \gamma^+ q\left(\frac{x^-}{2}\right)$ | $q(x)$ | $H(x,\xi,t)$ | ? |

# Form Factors vs. GPDs

| operator | forward matrix elem. | off-forward matrix elem. | position space |
|---|---|---|---|
| $\bar{q}\gamma^+ q$ | $Q$ | $F(t)$ | $\rho(\vec{r})$ |
| $\int \frac{dx^- e^{ixp^+x^-}}{4\pi} \bar{q}\left(\frac{-x^-}{2}\right)\gamma^+ q\left(\frac{x^-}{2}\right)$ | $q(x)$ | $H(x,0,t)$ | $q(x,\mathbf{b}_\perp)$ |

$q(x,\mathbf{b}_\perp)$ = impact parameter dependent PDF

# Impact parameter dependent PDFs

- define state that is localized in $\perp$ position:

$$\left|p^+, \mathbf{R}_\perp = \mathbf{0}_\perp, \lambda\right\rangle \equiv \mathcal{N} \int d^2\mathbf{p}_\perp \left|p^+, \mathbf{p}_\perp, \lambda\right\rangle$$

Note: $\perp$ boosts in IMF form Galilean subgroup $\Rightarrow$ this state has
$\mathbf{R}_\perp \equiv \frac{1}{P^+} \int dx^- d^2\mathbf{x}_\perp \, \mathbf{x}_\perp T^{++}(x) = \mathbf{0}_\perp$
(cf.: working in CM frame in nonrel. physics)

- define impact parameter dependent PDF

$$q(x, \mathbf{b}_\perp) \equiv \int \frac{dx^-}{4\pi} \left\langle p^+, \mathbf{R}_\perp = \mathbf{0}_\perp \right| \bar{q}(-\frac{x^-}{2}, \mathbf{b}_\perp)\gamma^+ q(\frac{x^-}{2}, \mathbf{b}_\perp) \left|p^+, \mathbf{R}_\perp = \mathbf{0}_\perp\right\rangle e^{ixp^+x^-}$$

# Impact parameter dependent PDFs

- use translational invariance to relate to same matrix element that appears in def. of GPDs

$$q(x,\mathbf{b}_\perp) \equiv \int dx^- \langle p^+,\mathbf{R}_\perp = \mathbf{0}_\perp | \bar{q}(-\frac{x^-}{2},\mathbf{b}_\perp)\gamma^+ q(\frac{x^-}{2},\mathbf{b}_\perp) | p^+,\mathbf{R}_\perp = \mathbf{0}_\perp \rangle e^{ixp^+x^-}$$

$$= |\mathcal{N}|^2 \int d^2\mathbf{p}_\perp \int d^2\mathbf{p}'_\perp \int dx^- \langle p^+,\mathbf{p}'_\perp | \bar{q}(-\frac{x^-}{2},\mathbf{b}_\perp)\gamma^+ q(\frac{x^-}{2},\mathbf{b}_\perp) | p^+,\mathbf{p}_\perp \rangle e^{ixp^+x^-}$$

# Impact parameter dependent PDFs

- use translational invariance to relate to same matrix element that appears in def. of GPDs

$$q(x,\mathbf{b}_\perp) \equiv \int dx^- \left\langle p^+,\mathbf{R}_\perp=\mathbf{0}_\perp\right| \bar{q}(-\frac{x^-}{2},\mathbf{b}_\perp)\gamma^+ q(\frac{x^-}{2},\mathbf{b}_\perp)\left|p^+,\mathbf{R}_\perp=\mathbf{0}_\perp\right\rangle e^{ixp^+x^-}$$

$$= |\mathcal{N}|^2\int d^2\mathbf{p}_\perp\int d^2\mathbf{p}'_\perp\int dx^- \left\langle p^+,\mathbf{p}'_\perp\right| \bar{q}(-\frac{x^-}{2},\mathbf{b}_\perp)\gamma^+ q(\frac{x^-}{2},\mathbf{b}_\perp)\left|p^+,\mathbf{p}_\perp\right\rangle e^{ixp^+x^-}$$

$$= |\mathcal{N}|^2\int d^2\mathbf{p}_\perp\int d^2\mathbf{p}'_\perp\int dx^- \left\langle p^+,\mathbf{p}'_\perp\right| \bar{q}(-\frac{x^-}{2},\mathbf{0}_\perp)\gamma^+ q(\frac{x^-}{2},\mathbf{0}_\perp)\left|p^+,\mathbf{p}_\perp\right\rangle e^{ixp^+x^-}$$

$$\times e^{i\mathbf{b}_\perp\cdot(\mathbf{p}_\perp-\mathbf{p}'_\perp)}$$

# Impact parameter dependent PDFs

- use translational invariance to relate to same matrix element that appears in def. of GPDs

$$q(x,\mathbf{b}_\perp) \equiv \int dx^- \left\langle p^+, \mathbf{R}_\perp = \mathbf{0}_\perp \right| \bar{q}(-\frac{x^-}{2}, \mathbf{b}_\perp)\gamma^+ q(\frac{x^-}{2}, \mathbf{b}_\perp) \left| p^+, \mathbf{R}_\perp = \mathbf{0}_\perp \right\rangle e^{ixp^+x^-}$$

$$= \left|\mathcal{N}\right|^2 \int d^2\mathbf{p}_\perp \int d^2\mathbf{p}'_\perp \int dx^- \left\langle p^+, \mathbf{p}'_\perp \right| \bar{q}(-\frac{x^-}{2}, \mathbf{b}_\perp)\gamma^+ q(\frac{x^-}{2}, \mathbf{b}_\perp) \left| p^+, \mathbf{p}_\perp \right\rangle e^{ixp^+x^-}$$

$$= \left|\mathcal{N}\right|^2 \int d^2\mathbf{p}_\perp \int d^2\mathbf{p}'_\perp \int dx^- \left\langle p^+, \mathbf{p}'_\perp \right| \bar{q}(-\frac{x^-}{2}, \mathbf{0}_\perp)\gamma^+ q(\frac{x^-}{2}, \mathbf{0}_\perp) \left| p^+, \mathbf{p}_\perp \right\rangle e^{ixp^+x^-} \times e^{i\mathbf{b}_\perp\cdot(\mathbf{p}_\perp - \mathbf{p}'_\perp)}$$

$$= \left|\mathcal{N}\right|^2 \int d^2\mathbf{p}_\perp \int d^2\mathbf{p}'_\perp H\left(x, 0, -\left(\mathbf{p}'_\perp - \mathbf{p}_\perp\right)^2\right) e^{i\mathbf{b}_\perp\cdot(\mathbf{p}_\perp - \mathbf{p}'_\perp)}$$

$$\hookrightarrow \quad \boxed{q(x,\mathbf{b}_\perp) = \int \frac{d^2\mathbf{\Delta}_\perp}{(2\pi)^2} H(x, -\mathbf{\Delta}_\perp^2) e^{-i\mathbf{b}_\perp\cdot\mathbf{\Delta}_\perp}}$$

# Impact parameter dependent PDFs

- $$q(x,\mathbf{b}_\perp) = \int \frac{d^2\mathbf{\Delta}_\perp}{(2\pi)^2} H(x, -\mathbf{\Delta}_\perp^2) e^{-i\mathbf{b}_\perp\cdot\mathbf{\Delta}_\perp}$$

  **($\mathbf{\Delta}_\perp = \mathbf{p}'_\perp - \mathbf{p}_\perp$, $\xi = 0$)**
- $q(x,\mathbf{b}_\perp)$ has physical interpretation of a density

$$q(x,\mathbf{b}_\perp) \geq 0 \quad \text{for} \quad x > 0$$

$$q(x,\mathbf{b}_\perp) \leq 0 \quad \text{for} \quad x < 0$$

# Discussion: $GPD \leftrightarrow q(x, \mathbf{b}_\perp)$

- GPDs allow simultaneous determination of longitudinal momentum and transverse position of partons

$$q(x, \mathbf{b}_\perp) = \int \frac{d^2 \mathbf{\Delta}_\perp}{(2\pi)^2} H(x, 0, -\mathbf{\Delta}_\perp^2) e^{-i \mathbf{b}_\perp \cdot \mathbf{\Delta}_\perp}$$

- $q(x, \mathbf{b}_\perp)$ has interpretation as density (positivity constraints!)

$$\begin{aligned} q(x, \mathbf{b}_\perp) &\sim \left\langle p^+, \mathbf{0}_\perp \left| b^\dagger(xp^+, \mathbf{b}_\perp) b(xp^+, \mathbf{b}_\perp) \right| p^+, \mathbf{0}_\perp \right\rangle \\ &= \left| b(xp^+, \mathbf{b}_\perp) | p^+, \mathbf{0}_\perp \rangle \right|^2 \geq 0 \end{aligned}$$

$\hookrightarrow$ positivity constraint on models

# Discussion: $GPD \leftrightarrow q(x, \mathbf{b}_\perp)$

- Nonrelativistically such a result not surprising!
  Absence of relativistic corrections to identification
  $H(x, 0, -\mathbf{\Delta}_\perp^2) \xrightarrow{FT} q(x, \mathbf{b}_\perp)$ due to Galilean subgroup in IMF
- $\mathbf{b}_\perp$ distribution measured w.r.t. $\mathbf{R}_\perp^{CM} \equiv \sum_i x_i \mathbf{r}_{i,\perp}$
  $\hookrightarrow$ width of the $\mathbf{b}_\perp$ distribution should go to zero as $x \to 1$, since the active quark becomes the $\perp$ center of momentum in that limit!
  $\hookrightarrow$ $H(x, t)$ must become $t$-indep. as $x \to 1$.
- very similar results for impact parameter dependent polarized quark distributions (nucleon longitudinally polarized)

$$\Delta q(x, \mathbf{b}_\perp) = \int \frac{d^2 \mathbf{\Delta}_\perp}{(2\pi)^2} \tilde{H}(x, 0, -\mathbf{\Delta}_\perp^2) e^{-i \mathbf{b}_\perp \cdot \mathbf{\Delta}_\perp}$$

# Discussion: $GPD \leftrightarrow q(x, \mathbf{b}_\perp)$

- Use intuition about nucleon structure in position space to make predictions for GPDs:
  large $x$: quarks from localized valence 'core',
  small $x$: contributions from larger ' meson cloud'
  $\hookrightarrow$ expect a gradual increase of the $t$-dependence ($\perp$ size) of $H(x,0,t)$ as $x$ decreases
- small $x$, expect transverse size to increase
- very simple model: $H_q(x,0,-\mathbf{\Delta}_\perp^2) = q(x)e^{-a\mathbf{\Delta}_\perp^2(1-x)\ln\frac{1}{x}}$.

# Other topics

- QCD evolution
- extrapolating to $\xi = 0$

# The physics of $E(x,0,-\boldsymbol{\Delta}_\perp^2)$

- So far: only unpolarized (or long. polarized) nucleon

  In general, use ( $\Delta^+ = 0$)

$$\int \frac{dx^-}{4\pi} e^{ip^+x^-x} \left\langle P{+}\Delta,\uparrow \left| \bar{q}(0)\, \gamma^+ q\left(x^-\right) \right| P,\uparrow \right\rangle = H(x,0,-\boldsymbol{\Delta}_\perp^2)$$

$$\int \frac{dx^-}{4\pi} e^{ip^+x^-x} \left\langle P{+}\Delta,\uparrow \left| \bar{q}(0)\, \gamma^+ q\left(x^-\right) \right| P,\downarrow \right\rangle = -\frac{\Delta_x - i\Delta_y}{2M} E(x,0,-\boldsymbol{\Delta}_\perp^2).$$

- Consider nucleon polarized in $x$ direction (in IMF)
  $|X\rangle \equiv |p^+, \mathbf{R}_\perp = \mathbf{0}_\perp, \uparrow\rangle + |p^+, \mathbf{R}_\perp = \mathbf{0}_\perp, \downarrow\rangle$.

$\hookrightarrow$ unpolarized quark distribution for this state:

$$q_X(x,\mathbf{b}_\perp) = q(x,\mathbf{b}_\perp) - \frac{1}{2M}\frac{\partial}{\partial b_y}\int \frac{d^2\boldsymbol{\Delta}_\perp}{(2\pi)^2} E(x,-\boldsymbol{\Delta}_\perp^2) e^{-i\mathbf{b}_\perp\cdot\boldsymbol{\Delta}_\perp}$$

# The physics of $E(x,0,-\Delta_\perp^2)$

$q_X(x,\mathbf{b}_\perp)$ in transversely polarized nucleon is transversely distorted compared to longitudinally polarized nucleons !

- mean displacement of flavor $q$ ($\perp$ flavor dipole moment)

$$d_y^q \equiv \int dx \int d^2\mathbf{b}_\perp q_X(x,\mathbf{b}_\perp) b_y = \frac{1}{2M}\int dx E_q(x,0,0) = \kappa_q^p$$

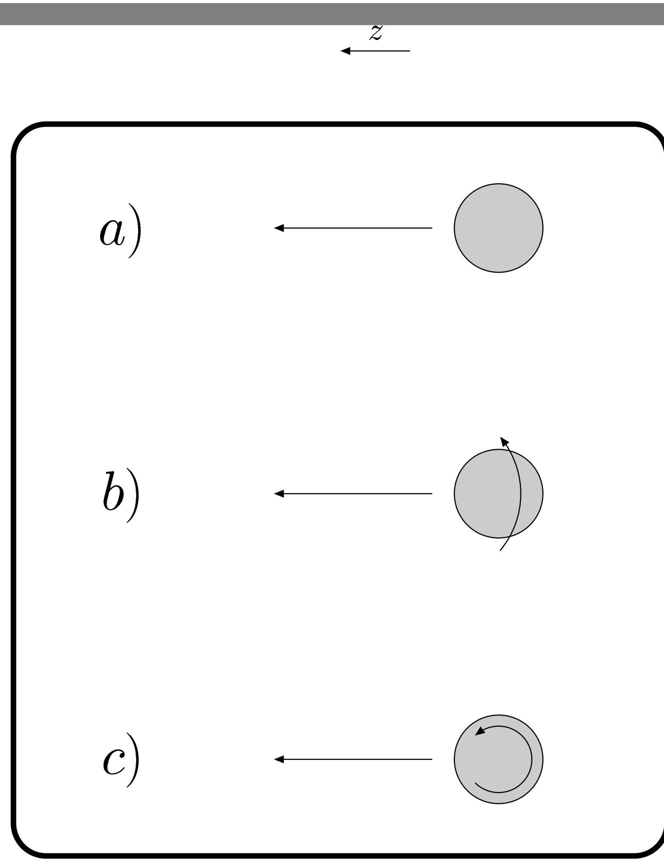
  with $\kappa^p_{u/d} \equiv F_2^{u/d}(0) = \mathcal{O}(1-2) \quad \Rightarrow \quad d_y^q = \mathcal{O}(0.2fm)$

- CM for flavor $q$ shifted relative to CM for whole proton by

$$\int dx \int d^2\mathbf{b}_\perp q_X(x,\mathbf{b}_\perp) x b_y = \frac{1}{2M}\int dx\, x E_q(x,0,0)$$

$\hookrightarrow$ not surprising to find that second moment of $E_q$ is related to angular momentum carried by flavor $q$

# physical origin for $\perp$ distortion




Comparison of a non-rotating sphere that moves in $z$ direction with a sphere that spins at the same time around the $z$ axis and a sphere that spins around the $x$ axis When the sphere spins around the $x$ axis, the rotation changes the distribution of momenta in the $z$ direction (adds/subtracts to velocity for $y > 0$ and $y < 0$ respectively). For the nucleon the resulting modification of the (unpolarized) momentum distribution is described by $E(x, 0, -\mathbf{\Delta}_\perp^2)$.

# simple model for $E_q(x, 0, -\mathbf{\Delta}_\perp^2)$

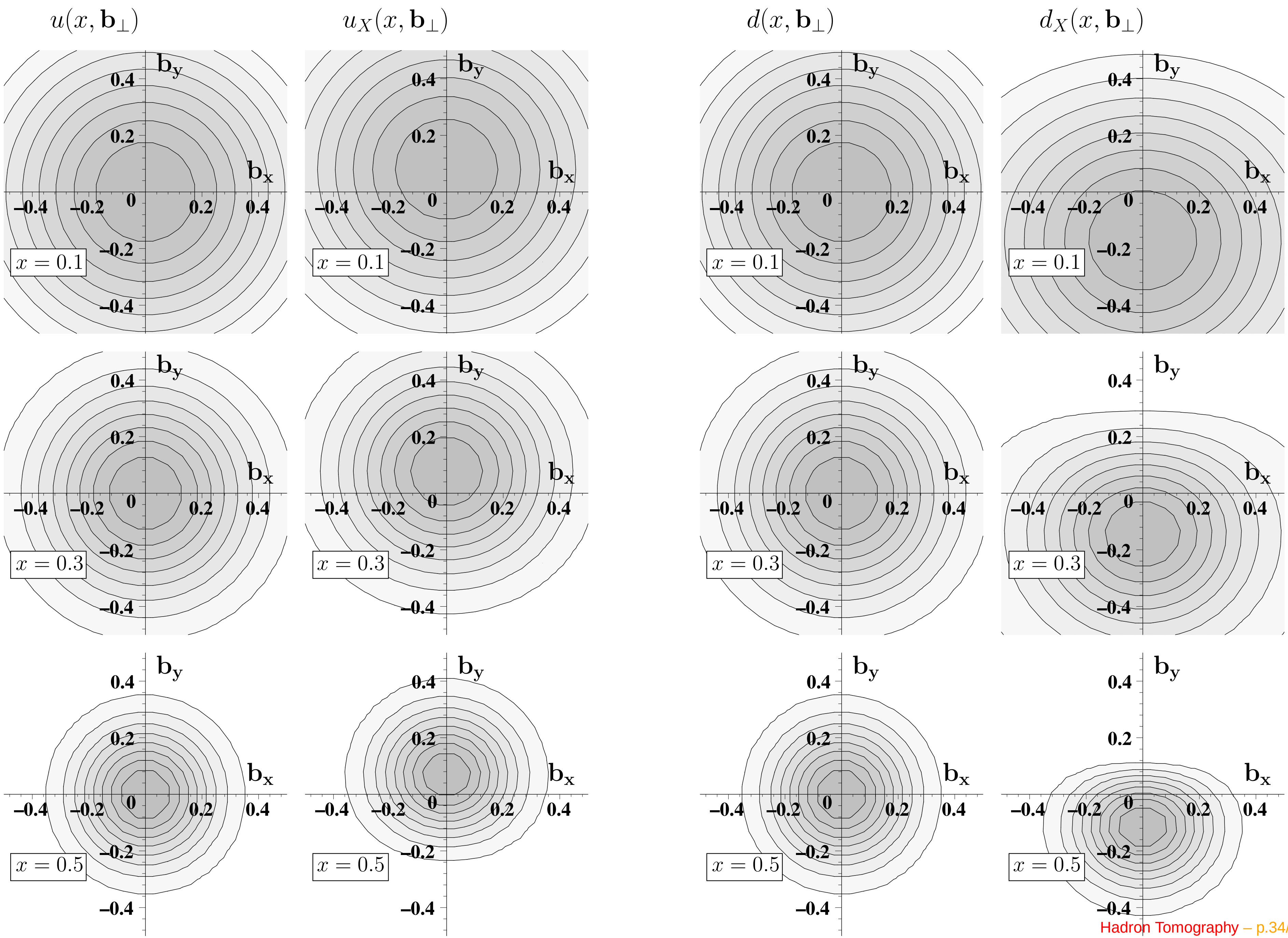
- For simplicity, make ansatz where $E_q \propto H_q$

$$\begin{aligned} E_u(x,0,-\mathbf{\Delta}_\perp^2) &= \frac{\kappa_u^p}{2} H_u(x,0,-\mathbf{\Delta}_\perp^2) \\ E_d(x,0,-\mathbf{\Delta}_\perp^2) &= \kappa_d^p H_d(x,0,-\mathbf{\Delta}_\perp^2) \end{aligned}$$

  with $H_q(x,0,-\mathbf{\Delta}_\perp^2) = q(x) e^{-a\mathbf{\Delta}_\perp^2 (1-x)\ln\frac{1}{x}}$ and

$$\kappa_u^p = 2\kappa_p + \kappa_n = 1.673 \qquad \kappa_d^p = 2\kappa_n + \kappa_p = -2.033.$$

- Satisfies: $\int dx E_q(x,0,0) = \kappa_q^P$
- Model too simple but illustrates that anticipated distortion is very significant since $\int dx E_q \sim \kappa_q$ known to be large!


$u(x, \mathbf{b}_\perp)$
$u_X(x, \mathbf{b}_\perp)$
$d(x, \mathbf{b}_\perp)$
$d_X(x, \mathbf{b}_\perp)$
$\mathbf{b_y}$
$\mathbf{b_x}$
0.4
0.2
0
−0.2
−0.4
$x = 0.1$
$x = 0.3$
$x = 0.5$

# $\perp$ single-spin asymmetry

- Example: left-right asymmetry in semi-inclusive $\gamma p \longrightarrow \pi^+ + X$ on a $\perp$ polarized target ($\vec{p}_\gamma \propto \vec{e}_z$, $\vec{S}_p \propto \vec{e}_x$, asymmetry $\propto \vec{e}_y$)
- Sivers mechanism: left-right asymmetry due to $\perp$ asymmetry of $\perp$-momentum dependent PDFs $f(x, \mathbf{k}_\perp)$

$$f(x, \mathbf{k}_\perp) = \int \frac{dy^- d^2\mathbf{y}_\perp}{16\pi^3} e^{-ixp^+y^- + i\mathbf{k}_\perp \cdot \mathbf{y}_\perp} \left\langle p \left| \bar{q}(0, y^-, \mathbf{y}_\perp)\gamma^+ q(0) \right| p \right\rangle .$$

  - gauge invariance $\rightarrow$ include Wilson line!
  - Naively, $f(x, \mathbf{k}_\perp) = f(x, -\mathbf{k}_\perp)$, due to time-reversal invariance, i.e. with above definition, Sivers asymmetry vanishes identically [$\vec{p}_q \cdot \left(\vec{p}_p \times \vec{S}_p\right)$ is T-odd]

# $\perp$ single-spin asymmetry

- However, Brodsky et al. $\Rightarrow$ Sivers asymmetry possible due to FSI! Formal argument: include FSI in eikonal approximation

$\hookrightarrow$ define $f(x, \mathbf{k}_\perp)$ gauge invariantly

$$f(x, \mathbf{k}_\perp) = \int \frac{dy^- d^2\mathbf{y}_\perp}{16\pi^3} e^{-ixp^+y^- + i\mathbf{k}_\perp \cdot \mathbf{y}_\perp} \left\langle p \left| \bar{q}(0, y^-, \mathbf{y}_\perp) W^\dagger_{y\infty} \gamma^+ W_{0\infty} q(0) \right| p \right\rangle$$

- $W_{y\infty} = P \exp\left(-ig \int_{y^-}^{\infty} dz^- A^+(y^+, z^-, \mathbf{y}_\perp)\right)$ indicates a Wilson-line operator going from point $y$ to infinity (FSI!).
- Wilson line not invariant under T

$\hookrightarrow$ Sivers asymmetry possibe [$f(x, \mathbf{k}_\perp) \neq f(x, -\mathbf{k}_\perp)$]

# $\perp$ single-spin asymmetry

- Presence of phase factors in definition of Sivers distribution do explain why SSA can be nonzero.
- does not obviously explain:
  - why these “phase factors” give rise to such large ‘stable’ polarization effects? (example: $\perp$ polarization in hyperon production)
  - which sign should one expect in which reaction?

# Physical origin of SSA

- mean $\perp$ momentum of active quark in semi-inclusive DIS contains term

$$\langle \mathbf{k}_\perp \rangle \sim \int dy^- e^{-ixp^+y^-} \left\langle p \left| \bar{q}(0,y^-,\mathbf{y}_\perp) W^\dagger_{y\infty} \partial_\perp W_{0\infty} \gamma^+ q(0) \right| p \right\rangle$$

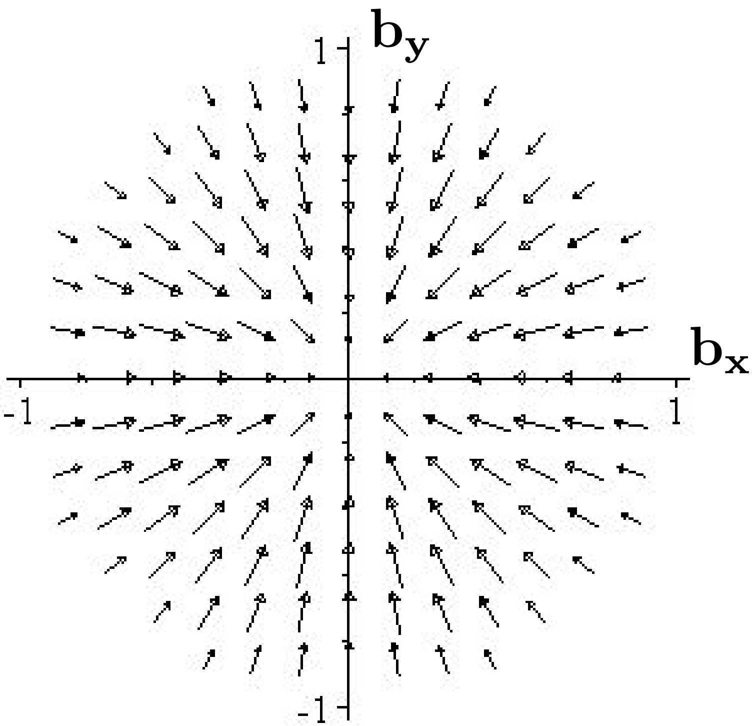
- Physics of this term (for simplicity abelian case):
  , this term simplifies as $\langle \mathbf{k}_\perp \rangle \sim ... - g \int_{z^-}^\infty dy^- \partial_\perp A^+(y^-,\mathbf{z}_\perp)$ which has semi-classical interpretation as impulse experienced by the active quark on its way out from $\perp$ position $\mathbf{z}_\perp$.

$\hookrightarrow$ mean $\perp$ momentum obtained as correlation between PDF and transverse impulse $\mathbf{I}_\perp(\mathbf{z}_\perp) = g \int_{z^-}^\infty dy^- \partial_\perp A^+(y^-,\mathbf{z}_\perp)$

- physics of this correlation $\longrightarrow$ switch to impact parameter representation

# Connection with $\perp$ distortion of $q(x, \mathbf{b}_\perp)$

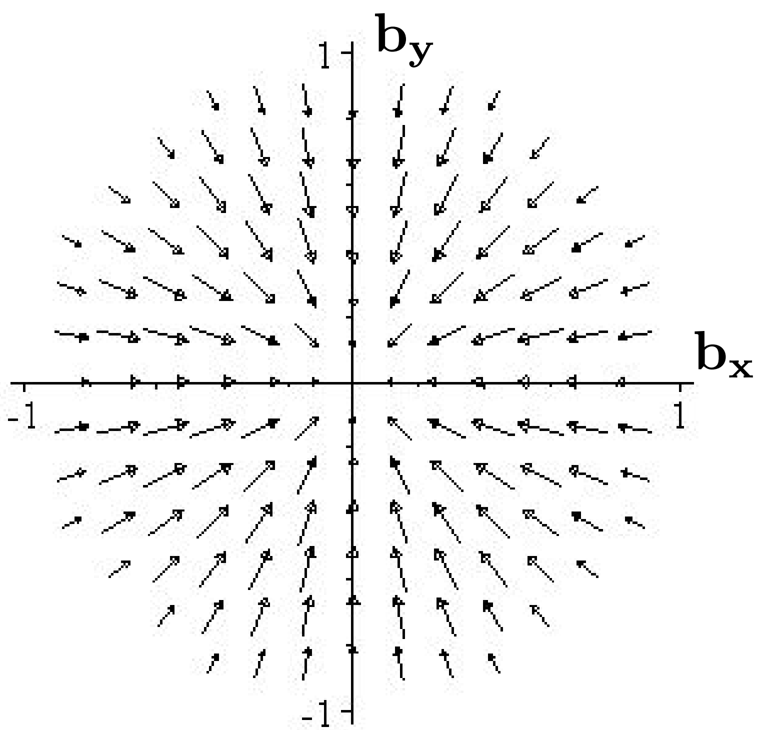
- use simple potential model to estimate $\mathbf{I}_\perp(\mathbf{z}_\perp) \equiv \partial_\perp \int dy^- A^+(y^-, \mathbf{z}_\perp)$ =mean $\perp$ impulse that the FSI exert on active quark on its way out as function of the separation from the CM



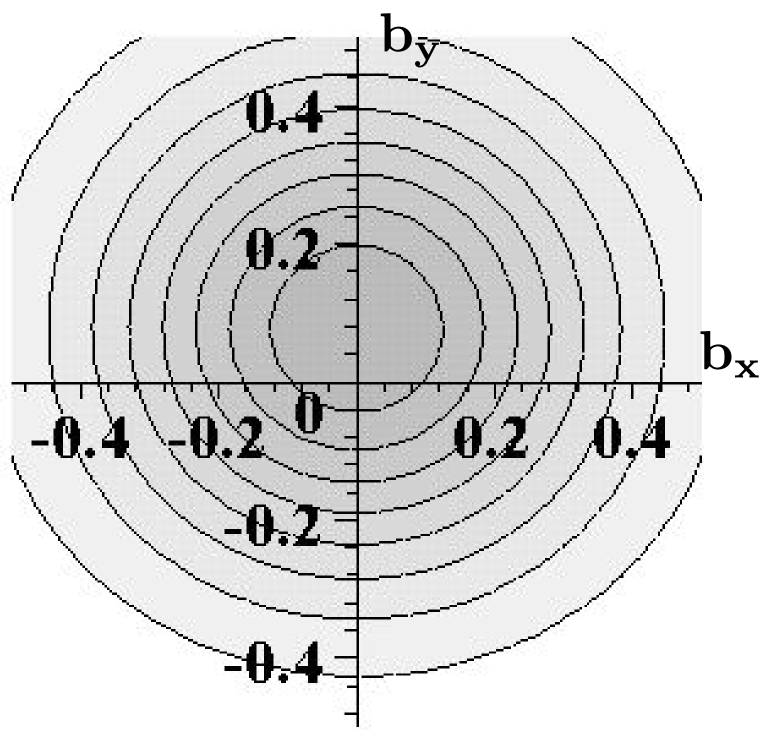

$\mathbf{I}_\perp$

# Connection with $\perp$ distortion of $q(x, \mathbf{b}_\perp)$

- use simple potential model to estimate
  $\mathbf{I}_\perp(zT) \equiv \partial_\perp \int dy^- A^+(y^-, \mathbf{z}_\perp)$=mean $\perp$ impulse that the FSI exert on active quark on its way out as function of the separation from the CM

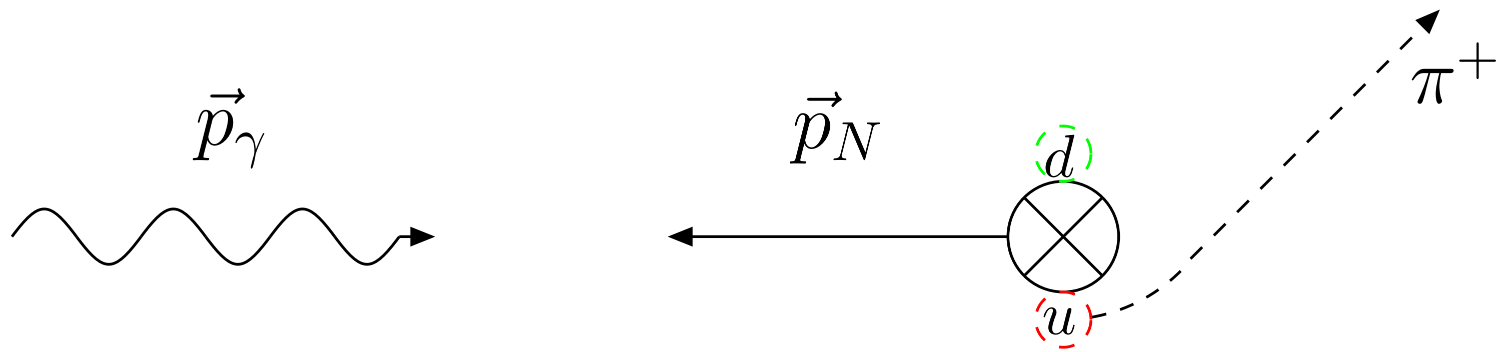



$\mathbf{I}_\perp$




$u_X(x, \mathbf{b}_\perp)$

# $\gamma p \to \pi X$ in Breit frame




- $u, d$ distributions in $\perp$ polarized proton have left-right asymmetry in $\perp$ position space (T-even!)
- attractive FSI deflects active quark towards the center of momentum

$\hookrightarrow$ FSI converts left-right position space asymmetry of leading quark into right-left asymmetry in momentum

- compare: convex lens that is illuminated asymmetrically

# Summary

- DVCS allows probing GPDS

$$\int \frac{dx^-}{2\pi} e^{ixp^+x^-} \left\langle p' \left| \bar{q}\left(-\frac{x^-}{2}\right) \gamma^+ q\left(\frac{x^-}{2}\right) \right| p \right\rangle$$

- GPDs resemble both PDFs and form factors: defined through matrix elements of light-cone correlation, but $\Delta \equiv p' - p \neq 0$.
- $t$-dependence of GPDs at $\xi = 0$ (purely $\perp$ momentum transfer) $\Rightarrow$ Fourier transform of impact parameter dependent PDFs $q(x, \mathbf{b}_\perp)$

$\hookrightarrow$ knowledge of GPDs for $\xi = 0$ provides novel information about nonperturbative parton structure of nucleons: distribution of partons in $\perp$ plane

$$q(x, \mathbf{b}_\perp) = \int \frac{d^2\mathbf{\Delta}_\perp}{(2\pi)^2} H(x, 0, -\mathbf{\Delta}_\perp^2) e^{-i\mathbf{b}_\perp \cdot \mathbf{\Delta}_\perp}$$

$$\Delta q(x, \mathbf{b}_\perp) = \int \frac{d^2\mathbf{\Delta}_\perp}{(2\pi)^2} \tilde{H}(x, 0, -\mathbf{\Delta}_\perp^2) e^{-i\mathbf{b}_\perp \cdot \mathbf{\Delta}_\perp}$$

- $q(x, \mathbf{b}_\perp)$, $\Delta q(x, \mathbf{b}_\perp)$ have probabilistic interpretation, e.g. $q(x, \mathbf{b}_\perp) > 0$ for $x > 0$

# Summary

- $\frac{\Delta_\perp}{2M} E(x, -\mathbf{\Delta}_\perp^2)$ describes how the momentum distribution of unpolarized partons in the $\perp$ plane gets transversely distorted when is nucleon polarized in $\perp$ direction.
- (attractive) final state interaction converts $\perp$ position space asymmetry into $\perp$ momentum space asymmetry

$\hookrightarrow$ simple physical explanation for sign of left-right asymmetry in semi-inclusive DIS

- Similar mechanism also applicable to many other semi-inclusive events, such as transverse polarizations in hyperon production.
- published in: M.B., PRD **62**, 71503 (2000), hep-ph/0105324, and hep-ph/0207047; see also D. Soper, PRD **15**, 1141 (1977).
- Connection to SSA in M.B., PRD **66**, 114005 (2002); hep-ph/0302144.

# extrapolating to $\xi = 0$

- bad news: $\xi = 0$ not directly accessible in DVCS since long. momentum transfer necessary to convert virtual $\gamma$ into real $\gamma$
- good news: moments of GPDs have simple $\xi$-dependence (polynomials in $\xi$)
  $\hookrightarrow$ should be possible to extrapolate!

even moments of $H(x,\xi,t)$:

$$\begin{aligned} H_n(\xi,t) &\equiv \int_{-1}^{1} dx x^{n-1} H(x,\xi,t) = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} A_{n,2i}(t)\xi^{2i} + C_n(t) \\ &= A_{n,0}(t) + A_{n,2}(t)\xi^2 + ... + A_{n,n-2}(t)\xi^{n-2} + C_n(t)\xi^n, \end{aligned}$$

i.e. for example

$$\int_{-1}^{1} dx x H(x,\xi,t) = A_{2,0}(t) + C_2(t)\xi^2.$$

- For $n^{th}$ moment, need $\frac{n}{2}+1$ measurements of $H_n(\xi,t)$ for same $t$ but different $\xi$ to determine $A_{n,2i}(t)$.
- GPDs @ $\xi = 0$ obtained from $H_n(\xi = 0, t) = A_{n,0}(t)$
- similar procedure exists for moments of $\tilde{H}$

back

# QCD evolution

So far ignored! Can be easily included because

- For $t \ll Q^2$, leading order evolution $t$-independent
- For $\xi = 0$ evolution kernel for GPDs
  same as DGLAP evolution kernel

likewise:

- impact parameter dependent PDFs evolve such that different $\mathbf{b}_\perp$
  do not mix (as long as $\perp$ spatial resolution much smaller than $Q^2$)

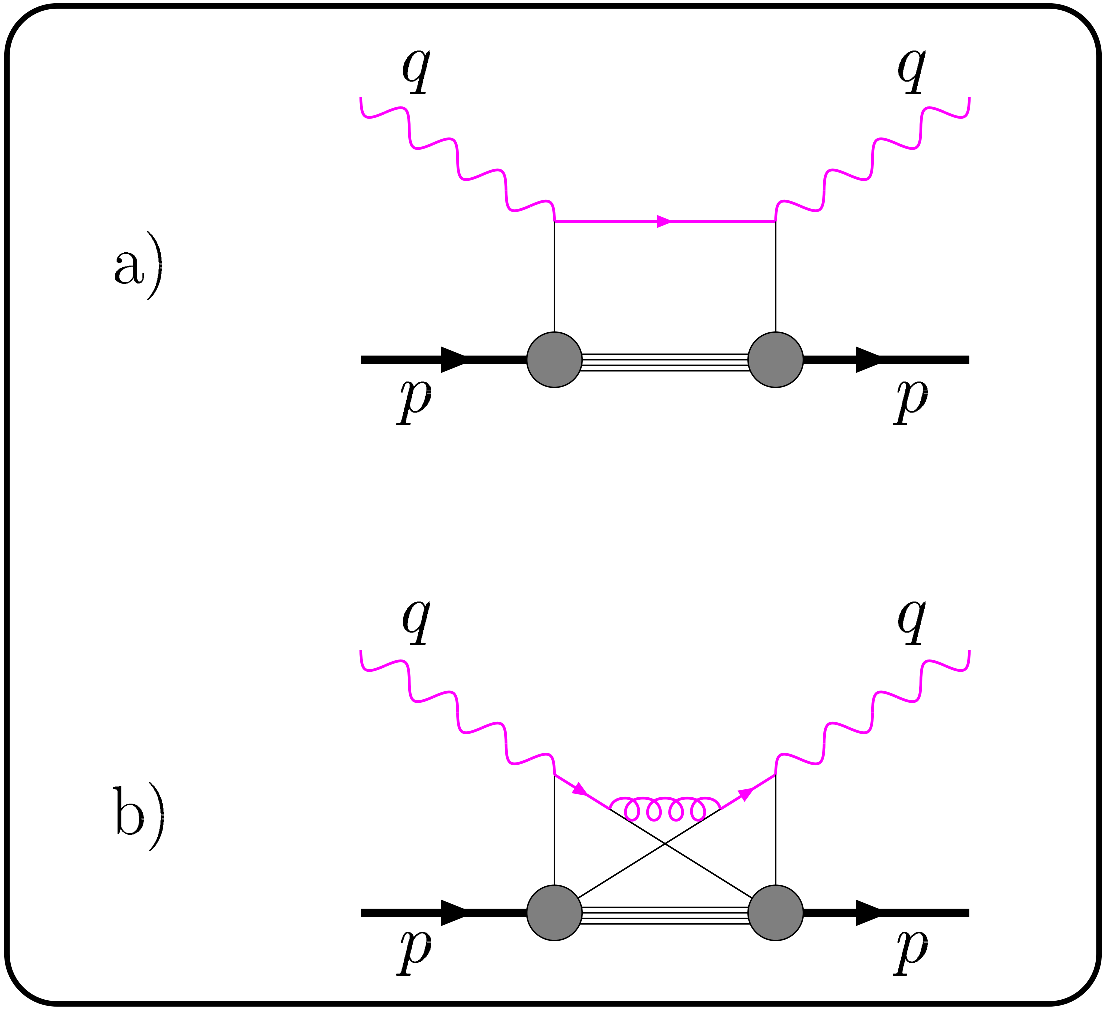
$\hookrightarrow$ above results consistent with QCD evolution:

$$H(x,0,-\mathbf{\Delta}_\perp^2,Q^2) = \int d^2b_\perp q(x,\mathbf{b}_\perp,Q^2)e^{-i\mathbf{b}_\perp\mathbf{\Delta}_\perp}$$
$$\tilde{H}(x,0,-\mathbf{\Delta}_\perp^2,Q^2) = \int d^2b_\perp \Delta q(x,\mathbf{b}_\perp,Q^2)e^{-i\mathbf{b}_\perp\mathbf{\Delta}_\perp}$$

where QCD evolution of $H, \tilde{H}, q, \Delta q$ is described by DGLAP and is independent on both $\mathbf{b}_\perp$ and $\mathbf{\Delta}_\perp^2$, provided one does not look at scales in $\mathbf{b}_\perp$ that are smaller than $1/Q$.

back

# suppression of crossed diagrams




Flow of the large momentum $q$ through typical diagrams contributing to the forward Compton amplitude. a) 'handbag' diagrams; b) 'cat's ears' diagram. Diagram b) is suppressed at large $q$ due to the presence of additional propagators.

back

# density interpretation for $q(x, \mathbf{b}_\perp)$

- express quark-bilinear in twist-2 GPD in terms of light-cone ‘good’ component $q_{(+)} \equiv \frac{1}{2}\gamma^-\gamma^+ q$

$$\bar{q}'\gamma^+ q = \bar{q}'_{(+)}\gamma^+ q_{(+)} = \sqrt{2} q'^{\dagger}_{(+)} q_{(+)}.$$

- expand $q_{(+)}$ in terms of canonical raising and lowering operators

$$q_{(+)}(x^-, \mathbf{x}_\perp) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \int \frac{d^2\mathbf{k}_\perp}{2\pi} \sum_s$$
$$\times \left[ u_{(+)}(k,s) b_s(k^+, \mathbf{k}_\perp) e^{-ikx} + v_{(+)}(k,s) d_s^\dagger(k^+, \mathbf{k}_\perp) e^{ikx} \right],$$

# density interpretation for $q(x,\mathrm{b}_\perp)$

with usual (canonical) equal light-cone time $x^+$ anti-commutation relations, e.g.

$$\left\{b_r(k^+,\mathbf{k}_\perp), b_s^\dagger(q^+,\mathbf{q}_\perp)\right\} = \delta(k^+ - q^+)\delta(\mathbf{k}_\perp - \mathbf{q}_\perp)\delta_{rs}$$

and the normalization of the spinors is such that

$$\bar{u}_{(+)}(p,r)\gamma^+ u_{(+)}(p,s) = 2p^+\delta_{rs}.$$

Note: $\bar{u}_{(+)}(p',r)\gamma^+ u_{(+)}(p,s) = 2p^+\delta_{rs}$ for $p^+ = p'^+$, one finds for $x > 0$

$$q(x,\mathbf{b}_\perp) = \mathcal{N}' \sum_s \int \frac{d^2\mathbf{k}_\perp}{2\pi} \int \frac{d^2\mathbf{k}'_\perp}{2\pi} \left\langle p^+,\mathbf{0}_\perp \right| b_s^\dagger(xp^+,\mathbf{k}'_\perp) b_s(xp^+,\mathbf{k}_\perp) \left| p^+,\mathbf{0}_\perp \right\rangle \times e^{i\mathbf{b}_\perp\cdot(\mathbf{k}_\perp - \mathbf{k}'_\perp)}.$$

# density interpretation for $q(x,\mathrm{b}_\perp)$

- Switch to mixed representation:
  momentum in longitudinal direction
  position in transverse direction

$$\tilde{b}_s(k^+,\mathbf{x}_\perp) \equiv \int \frac{d^2\mathbf{k}_\perp}{2\pi} b_s(k^+,\mathbf{k}_\perp) e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp}$$

$\hookrightarrow$

$$\begin{aligned} q(x,\mathbf{b}_\perp) &= \sum_s \left\langle p^+,\mathbf{0}_\perp \right| \tilde{b}_s^\dagger(xp^+,\mathbf{b}_\perp)\tilde{b}_s(xp^+,\mathbf{b}_\perp) \left| p^+,\mathbf{0}_\perp \right\rangle . \\ &= \sum_s \left| \tilde{b}_s(xp^+,\mathbf{b}_\perp) \left| p^+,\mathbf{0}_\perp \right\rangle \right|^2 \\ &\geq 0. \end{aligned}$$

back

# density interpretation for $q(x)$

- express quark-bilinear in twist-2 PDF in terms of light-cone 'good' component $q_{(+)} \equiv \frac{1}{2}\gamma^-\gamma^+ q$

$$\bar{q}'\gamma^+ q = \bar{q}'_{(+)}\gamma^+ q_{(+)} = \sqrt{2} q'^{\dagger}_{(+)} q_{(+)}.$$

- expand $q_{(+)}$ in terms of canonical raising and lowering operators

$$q_{(+)}(x^-, \mathbf{x}_\perp) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \int \frac{d^2\mathbf{k}_\perp}{2\pi} \sum_s$$
$$\times \left[ u_{(+)}(k,s) b_s(k^+, \mathbf{k}_\perp) e^{-ikx} + v_{(+)}(k,s) d_s^\dagger(k^+, \mathbf{k}_\perp) e^{ikx} \right],$$

# density interpretation for $q(x)$

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$$\left\{b_r(k^+,\mathbf{k}_\perp), b_s^\dagger(q^+,\mathbf{q}_\perp)\right\} = \delta(k^+ - q^+)\delta(\mathbf{k}_\perp - \mathbf{q}_\perp)\delta_{rs}$$

and the normalization of the spinors is such that

$$\bar{u}_{(+)}(p,r)\gamma^+ u_{(+)}(p,s) = 2p^+\delta_{rs}.$$

(Note: $\bar{u}_{(+)}(p',r)\gamma^+ u_{(+)}(p,s) = 2p^+\delta_{rs}$ for $p^+ = p'^+$)

- Insert in

$$q(x) = \int \frac{dx^-}{2\pi} \left\langle p\right|\overline{q}(0^-,\mathbf{0}_\perp)\gamma^+ q(x^-,\mathbf{0}_\perp)\left|p\right\rangle \, e^{ix^- x p^+}$$

# density interpretation for $q(x)$

- one finds for $x > 0$

$$q(x) = \mathcal{N}' \sum_s \int \frac{d^2\mathbf{k}_\perp}{2\pi} \int \frac{d^2\mathbf{k}'_\perp}{2\pi} \langle p| \, b_s^\dagger(xp^+, \mathbf{k}'_\perp) b_s(xp^+, \mathbf{k}_\perp) \, |p\rangle$$

$$= \mathcal{N}' \sum_s \left| \int \frac{d^2\mathbf{k}_\perp}{2\pi} b_s(xp^+, \mathbf{k}_\perp) \, |p\rangle \right|^2 \geq 0.$$

- antiquarks ($x < 0$) yield $q(x) < 0$

$\hookrightarrow$ usually define positive antiquark distribution

$$\bar{q}(x) \equiv -q(-x) \qquad (x > 0)$$

back

# Boosts in nonrelativistic QM

$$\vec{x}' = \vec{x} + \vec{v}t \qquad\qquad t' = t$$

purely kinematical (quantization surface $t = 0$ inv.)

$\hookrightarrow$ 1. boosting wavefunctions very simple

$$\Psi_{\vec{v}}(\vec{p}_1, \vec{p}_2) = \Psi_{\vec{0}}(\vec{p}_1 - m_1\vec{v}, \vec{p}_2 - m_2\vec{v}).$$

2. dynamics of center of mass

$$\vec{R} \equiv \sum_i x_i \vec{r}_i \qquad \text{with} \qquad x_i \equiv \frac{m_i}{M}$$

decouples from the internal dynamics

# Relativistic Boosts

$$t' = \gamma\left(t + \frac{v}{c^2}z\right), \qquad z' = \gamma\,(z + vt) \qquad \mathbf{x}'_\perp = \mathbf{x}_\perp$$

generators satisfy Poincaré algebra:

$$\begin{aligned}
[P^\mu, P^\nu] &= 0 \\
[M^{\mu\nu}, P^\rho] &= i\left(g^{\nu\rho}P^\mu - g^{\mu\rho}P^\nu\right) \\
\left[M^{\mu\nu}, M^{\rho\lambda}\right] &= i\left(g^{\mu\lambda}M^{\nu\rho} + g^{\nu\rho}M^{\mu\lambda} - g^{\mu\rho}M^{\nu\lambda} - g^{\nu\lambda}M^{\mu\rho}\right)
\end{aligned}$$

rotations: $M_{ij} = \varepsilon_{ijk}J_k$, boosts: $M_{i0} = K_i$.

# Galilean subgroup of $\perp$ boosts

introduce generator of $\perp$ 'boosts':

$$B_x \equiv M^{+x} = \frac{K_x + J_y}{\sqrt{2}} \qquad B_y \equiv M^{+y} = \frac{K_y - J_x}{\sqrt{2}}$$

Poincaré algebra $\Longrightarrow$ commutation relations:

$$\begin{aligned} \left[J_3, B_k\right] &= i\varepsilon_{kl} B_l & \left[P_k, B_l\right] &= -i\delta_{kl} P^+ \\ \left[P^-, B_k\right] &= -iP_k & \left[P^+, B_k\right] &= 0 \end{aligned}$$

with $k, l \in \{x, y\}$, $\varepsilon_{xy} = -\varepsilon_{yx} = 1$, and $\varepsilon_{xx} = \varepsilon_{yy} = 0$.

Together with $[J_z, P_k] = i\varepsilon_{kl} P_l$, as well as

$$\begin{aligned} \left[P^-, P_k\right] &= \left[P^-, P^+\right] = \left[P^-, J_z\right] = 0 \\ \left[P^+, P_k\right] &= \left[P^+, B_k\right] = \left[P^+, J_z\right] = 0. \end{aligned}$$

Same as commutation relations among generators of nonrel. boosts, translations, and rotations in x-y plane, provided one identifies

$$\begin{aligned} P^- &\longrightarrow \text{Hamiltonian} \\ \mathbf{P}_\perp &\longrightarrow \text{momentum in the plane} \\ P^+ &\longrightarrow \text{mass} \\ L_z &\longrightarrow \text{rotations around } z\text{-axis} \\ \mathbf{B}_\perp &\longrightarrow \text{generator of boosts in the plane,} \end{aligned}$$

back to discussion

# Consequences

- many results from NRQM carry over to $\perp$ boosts in IMF, e.g.
- $\perp$ boosts kinematical

$$
\begin{aligned}
\Psi_{\mathbf{\Delta}_\perp}(x,\mathbf{k}_\perp) &= \Psi_{\mathbf{0}_\perp}(x,\mathbf{k}_\perp - x\mathbf{\Delta}_\perp) \\
\Psi_{\mathbf{\Delta}_\perp}(x,\mathbf{k}_\perp,y,\mathbf{l}_\perp) &= \Psi_{\mathbf{0}_\perp}(x,\mathbf{k}_\perp - x\mathbf{\Delta}_\perp, y, \mathbf{l}_\perp - y\mathbf{\Delta}_\perp)
\end{aligned}
$$

- Transverse center of momentum $\mathbf{R}_\perp \equiv \sum_i x_i \mathbf{r}_{\perp,i}$ plays role similar to NR center of mass, e.g. $\int d^2\mathbf{p}_\perp \left|p^+,\mathbf{p}_\perp\right\rangle$ corresponds to state with $\mathbf{R}_\perp = \mathbf{0}_\perp$.

back

# $\perp$ Center of Momentum

- field theoretic definition

$$p^+ \mathbf{R}_\perp \equiv \int dx^- \int d^2\mathbf{x}_\perp T^{++}(x)\mathbf{x}_\perp = M^{+\perp}$$

- $M^{+\perp} = \mathbf{B}^\perp$ generator of transverse boosts
- parton representation:

$$\mathbf{R}_\perp = \sum_i x_i \mathbf{r}_{\perp,i}$$

($x_i =$ momentum fraction carried by $i^{th}$ parton)

back

# Poincaré algebra:

$$
\begin{aligned}
[P^\mu, P^\nu] &= 0 \\
[M^{\mu\nu}, P^\rho] &= i\left(g^{\nu\rho} P^\mu - g^{\mu\rho} P^\nu\right) \\
\left[M^{\mu\nu}, M^{\rho\lambda}\right] &= i\left(g^{\mu\lambda} M^{\nu\rho} + g^{\nu\rho} M^{\mu\lambda} - g^{\mu\rho} M^{\nu\lambda} - g^{\nu\lambda} M^{\mu\rho}\right)
\end{aligned}
$$

rotations: $M_{ij} = \varepsilon_{ijk} J_k$, boosts: $M_{i0} = K_i$. back

# Galilean subgroup of $\perp$ boosts

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with $k, l \in \{x, y\}$, $\varepsilon_{xy} = -\varepsilon_{yx} = 1$, and $\varepsilon_{xx} = \varepsilon_{yy} = 0$.

back

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back

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\Psi_{\boldsymbol{\Delta}_\perp}(x,\mathbf{k}_\perp) &= \Psi_{\mathbf{0}_\perp}(x,\mathbf{k}_\perp - x\boldsymbol{\Delta}_\perp) \\
\Psi_{\boldsymbol{\Delta}_\perp}(x,\mathbf{k}_\perp,y,\mathbf{l}_\perp) &= \Psi_{\mathbf{0}_\perp}(x,\mathbf{k}_\perp - x\boldsymbol{\Delta}_\perp, y, \mathbf{l}_\perp - y\boldsymbol{\Delta}_\perp)
\end{aligned}
$$

- Transverse center of momentum $\mathbf{R}_\perp \equiv \sum_i x_i \mathbf{r}_{\perp,i}$ plays role similar to NR center of mass, e.g. $|p^+, \mathbf{R}_\perp = \mathbf{0}_\perp\rangle \equiv \int d^2\mathbf{p}_\perp |p^+, \mathbf{p}_\perp\rangle$ corresponds to state with $\mathbf{R}_\perp = \mathbf{0}_\perp$.

back

# Proof that $\mathrm{B}_\perp|p^+,\mathbf{R}_\perp=\mathbf{0}_\perp\rangle=0$

- Use

$$e^{-i\mathbf{v}_\perp\cdot\mathbf{B}_\perp}|p^+,\mathbf{p}_\perp,\lambda\rangle=|p^+,\mathbf{p}_\perp+p^+\mathbf{v}_\perp,\lambda\rangle$$

$\hookrightarrow$

$$e^{-i\mathbf{v}_\perp\cdot\mathbf{B}_\perp}\int d^2\mathbf{p}_\perp|p^+,\mathbf{p}_\perp,\lambda\rangle=\int d^2\mathbf{p}_\perp|p^+,\mathbf{p}_\perp,\lambda\rangle$$

$\hookrightarrow$

$$\mathbf{B}_\perp\int d^2\mathbf{p}_\perp|p^+,\mathbf{p}_\perp,\lambda\rangle=0$$

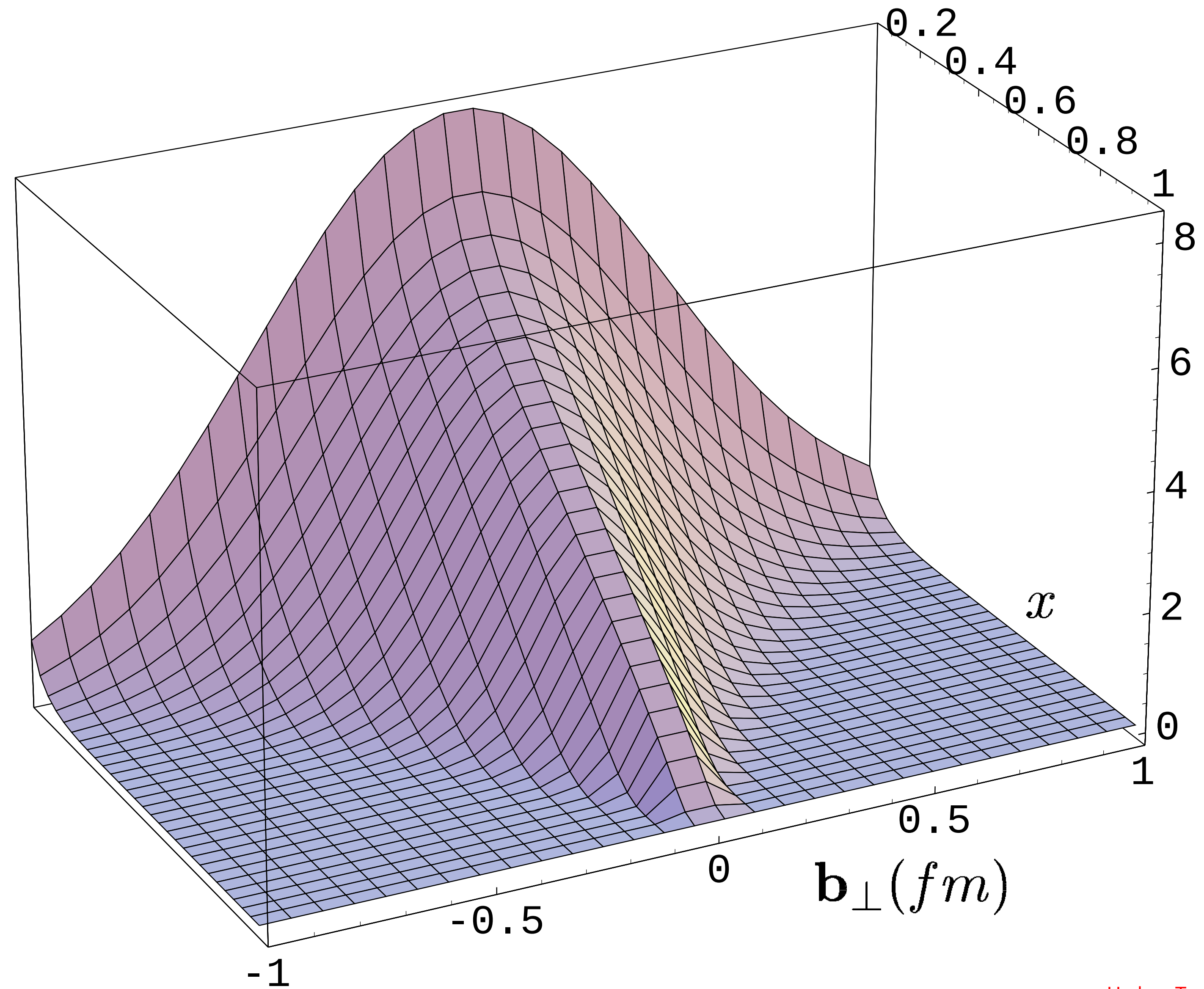
back

# Example

- Ansatz: $H_q(x,0,-\mathbf{\Delta}_\perp^2) = q(x) e^{-a\mathbf{\Delta}_\perp^2 (1-x)\ln\frac{1}{x}}$.

$$\hookrightarrow \quad q(x,\mathbf{b}_\perp) = q(x)\frac{1}{4\pi a(1-x)\ln\frac{1}{x}} e^{-\frac{\mathbf{b}_\perp^2}{4a(1-x)\ln\frac{1}{x}}}$$

# $q(x, \mathbf{b}_\perp)$

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# Center of Mass Frame in NRQM

- The boost operator in NRQM:

$$\vec{R} = \frac{1}{M} \sum_i m_i \vec{r}_i$$

- satisfies ($H = \sum_l \frac{\vec{p}_i^2}{2m_i} + V(\vec{r}_i)$)

$$[R_k, P_l] = -i\delta_{kl} \qquad \left[\vec{R}, H\right] = i\vec{P}$$

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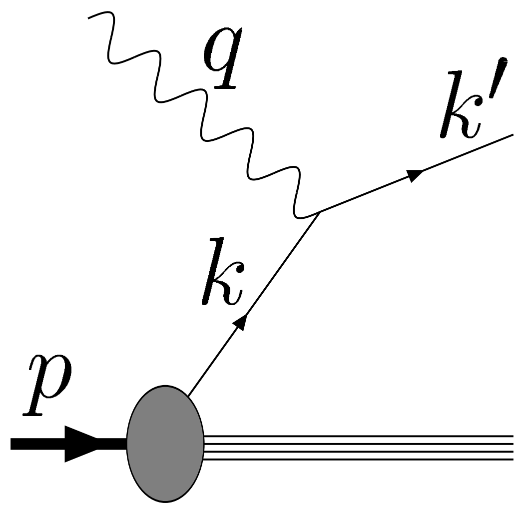
# Physical Meaning of $x_{Bj} = \frac{Q^2}{2p\cdot q}$




- Go to frame where $\mathbf{q}_\perp = 0$, i.e.

$$Q^2 = -q^2 = -2q^+q^- \qquad\qquad 2p\cdot q = 2q^-p^+ + 2q^+p^-$$

- Bjorken limit: $\quad q^- \to \infty \quad , \quad q^+$ fixed

$\hookrightarrow$

$$x_{Bj} = -\frac{q^+q^-}{q^-p^+ + q^+p^-} \to -\frac{q^+}{p^+}$$

# Physical Meaning of $x_{Bj} = \frac{Q^2}{2p\cdot q}$

- $x_{Bj} = -\frac{q^+}{p^+}$
- LC energy-momentum dispersion relation

$$k^- = \frac{m^2 + \mathbf{k}_\perp^2}{2k^+}$$

$\hookrightarrow$ struck quark with $k^{-\prime} = k^- + q^- \to \infty$ can only be on mass shell if $k^{+\prime} = k^+ + q^+ \approx 0$

$\hookrightarrow$

$$k^+ = -q^+ \qquad \Rightarrow \qquad x \equiv \frac{k^+}{p^+} = x_{Bj}$$

$\hookrightarrow$ $x_{Bj}$ has physical meaning of light-cone momentum fraction carried by struck quark before it is hit by photon

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