Deterministic Randomness in Relativistic Hydrodynamics

M. Stephanov

with Y. Yin, 1712.10305;
with X. An, G. Basar and H.-U. Yee,
1902.09517, 1912.13456, 2009.10742;

BEST COLLABORATION

Hydrodynamics

- Hydrodynamics is an effective theory, i.e., relies on separation of scales.

  Assumes local equilibrium and describes evolution of a fluid towards global equilibrium.

  This evolution is slow, because of conservation laws.

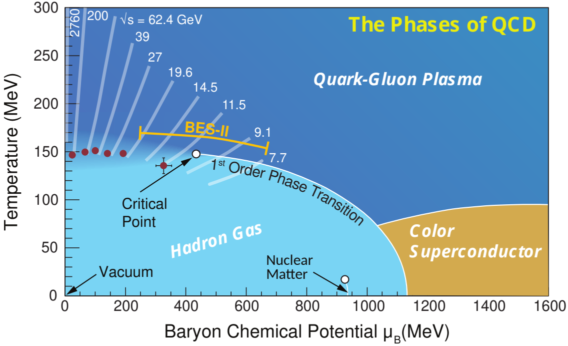
- Is hydrodynamics a deterministic theory?

  No. This would violate fluctuation-dissipation theorem.

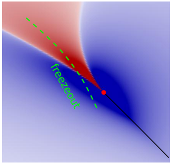
- What is the role of the randomness and how do we describe it?

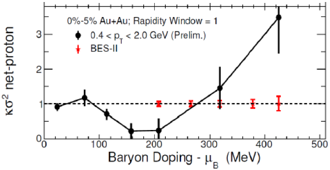
Critical point: intriguing hints

Where on the QCD phase boundary is the CP?



Equilibrium κ_4 vs T and μ_B:





"intriguing hint" (2015 LRPNS)

Motivation for phase II of BES at RHIC and BEST topical collaboration.

Theory/experiment gap: predictions assume equilibrium, but

Non-equilibrium physics is essential near the critical point.

Challenge: develop hydrodynamics *with fluctuations* capable of describing *non-equilibrium* effects on critical-point signatures.

Also notable:
Fluctuations are the first step to extend hydro to smaller systems.

Stochastic hydrodynamics

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- Non-linearities + locality $\Rightarrow$ UV divergences.
In numerical simulations – cutoff dependence.

Deterministic approach

- Variables are one- and two-point functions:
  $\psi = \langle\breve{\psi}\rangle$ and $G = \langle\breve{\psi}\breve{\psi}\rangle - \langle\breve{\psi}\rangle\langle\breve{\psi}\rangle$ – equal-time correlator

  Nonlinearities lead to dependence of flux on G.

$$\partial_t \psi = -\nabla \cdot \mathsf{Flux}[\psi, G]; \qquad \text{(conservation)}$$

$$\partial_t G = \mathsf{L}[G; \psi]. \qquad \text{(relaxation)}$$

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- In Bjorken flow by Akamatsu *et al*, Martinez-Schaefer.
  For arbitrary relativistic flow – by An *et al* (this talk).
  Earlier, in *nonrelativistic* context, – by Andreev in 1970s.
- Advantage: equations are *deterministic*.

  "Infinite noise" causes UV renormalization of EOS and transport coefficients – can be taken care of *analytically* *(1902.09517)*
- Two- and higher-pt functions systematically describe deviations from local equilibrium.

Yin, MS, 1712.10305

Rajagopal et al, 1908.08539

Du et al, 2004.02719

- Hydro requires $\tau_{\rm expnsn} \gg \tau_{\rm eqlbrtn}$.

  "Critical slowing down": $\tau_{\rm eqlbrtn} \sim \xi^3 \to \infty$.

  - $\xi_{\rm max} \sim \tau_{\rm expnsn}^{1/3}$ – magnitude of flucts. determined by dynamics.

Fluctuation dynamics near CP: Hydro+

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  - Critical fluctuations ($\xi \to \infty$);
  - Slow relaxation mode with $\tau_{\rm eqlbrtn} \sim \xi^3$.

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- Dynamics near CP requires two main ingredients:

  - Critical fluctuations ($\xi \to \infty$);

  - Slow relaxation mode with $\tau_{\rm eqlbrtn} \sim \xi^3$.

- Both described by the same object: the two-point function of the slowest hydrodynamic mode $m \equiv (s/n)$, i.e., $\langle\, \delta m(x_1)\, \delta m(x_2)\, \rangle$.

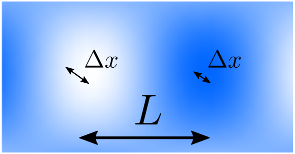
Additional variables in Hydro+

- Hydro+ extends Hydro with new *non-hydrodynamic* d.o.f..
- At the CP the *slowest* (i.e., most out of equilibrium) new d.o.f. is the 2-pt function $\langle\delta m\delta m\rangle$ of the slowest hydro variable:

$$\phi_{\boldsymbol{Q}}(\boldsymbol{x}) = \int_{\Delta\boldsymbol{x}} \langle\delta m\,(\boldsymbol{x}_+)\ \delta m\,(\boldsymbol{x}_-)\rangle\ e^{i\boldsymbol{Q}\cdot\Delta\boldsymbol{x}}$$

where $\boldsymbol{x} = (\boldsymbol{x}_+ + \boldsymbol{x}_-)/2$ and $\Delta\boldsymbol{x} = \boldsymbol{x}_+ - \boldsymbol{x}_-$.

- Wigner transformed b/c dependence on $\boldsymbol{x}$ ($\sim L$) is slow and relevant $\Delta\boldsymbol{x} \ll L$. Scale separation similar to kinetic theory.



Relaxation of fluctuations towards equilibrium

- As usual, equilibration maximizes entropy $S = \sum_i p_i \log(1/p_i)$:

$$s_{(+)}(\epsilon, n, \phi_{\boldsymbol{Q}}) = s(\epsilon, n) + \frac{1}{2}\int_{\boldsymbol{Q}} \left(\log \frac{\phi_{\boldsymbol{Q}}}{\bar{\phi}_{\boldsymbol{Q}}} - \frac{\phi_{\boldsymbol{Q}}}{\bar{\phi}_{\boldsymbol{Q}}} + 1 \right)$$

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- The equation for $\phi_{\boldsymbol{Q}}$ is a relaxation equation with rate

$$\Gamma(\boldsymbol{Q}) \approx 2DQ^2 \quad \text{for} \quad Q \ll \xi^{-1}, \quad D \sim 1/\xi.$$

- Impact of fluctuations on hydrodynamics:
  - "Renormalization" of bulk viscosity $\zeta \sim 1/\Gamma_\xi \sim \xi^3$.
  - (Non-analytic) frequency dependence of $\zeta(\omega)$ at $\omega \ll \Gamma_\xi$.
    "Long-time tails" (Kovtun-Yaffe 2003)
- Impact on observables: "memory" effects

  Berdnikov-Rajagopal, Mukherjee-Venugopalan-Yin, . . .

Full relativistic hydrodynamics

An, Basar, Yee, MS, 1902.09517,1912.13456

- To embed Hydro+ into a unified theory for critical as well as non-critical fluctuations we develop a general *deterministic* (*hydro-kinetic*) formalism.

- Expand stochastic hydro eqs. in $\{\delta m, \delta p, \delta u^{\mu}\} \sim \phi_A$ and then average, using equal-time correlator

$$G_{AB}(x,y) \stackrel{?}{=} \langle\, \phi_A(x+y/2)\, \phi_B(x-y/2)\, \rangle.$$

  - What is “equal-time” in *relativistic* hydro?
  - $\langle\phi(x)\phi(x)\rangle$ is singular (cutoff dependent). Renormalization?

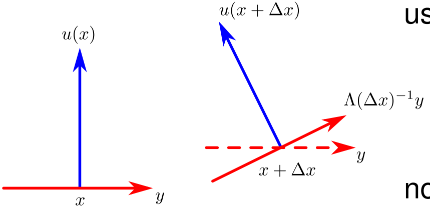
Equal time and *confluent* derivative

- We need equal-time correlator $G = \langle \phi(t, \boldsymbol{x}_+)\phi(t, \boldsymbol{x}_-)\rangle$.

  But what does "equal time" mean? Requires a frame choice.

  The most natural choice is local $u(x)$ (at $x = (x_+ + x_-)/2$).

- *Confluent* derivative wrt x at "y-fixed" takes this into account:



using $\Lambda(\Delta x)u(x+\Delta x) = u(x)$:

$$\Delta x \cdot \bar{\nabla} G(x,y) \equiv G(x+\Delta x, \Lambda(\Delta x)^{-1}y) - G(x,y)\,.$$

not $G(x+\Delta x, y) - G(x,y)\,.$

- We define *confluent* equal time correlator $\bar{G}_{AB}(x,y)$ and its Wigner transform $W_{AB}(x,q)$

more

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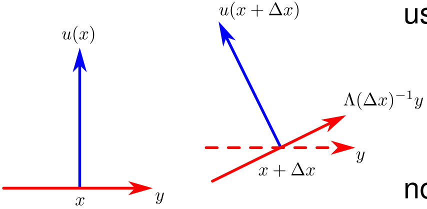

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Sound-sound correlation and phonon kinetic equation

- After many *miraculous* cancellations we arrive at "hydro-kinetic" equations for components of W.

  The longitudinal components, corresponding to p and u^μ fluctuations at $\delta(s/n) = 0$, obey the following eq. ($N_L \equiv W_L/(wc_s|q|)$)

$$\underbrace{\left[(u+v)\cdot\bar{\nabla} + f\cdot\frac{\partial}{\partial q}\right] N_L}_{\mathcal{L}[N_L]\text{ – Liouville op.}} = -\gamma_L q^2 \left(N_L - \underbrace{\frac{T}{c_s|q|}}_{N_L^{(0)}} \right)$$

- Kinetic eq. for phonons with $E = c_s|q|$, $v = c_s q/|q|$ ($q \cdot u = 0$)

$$f_\mu = \underbrace{-E(a_\mu + 2v^\nu\omega_{\nu\mu})}_{\text{inertial + Coriolis}} \underbrace{-q^\nu \partial_{\perp\mu} u_\nu}_{\text{“Hubble”}} - \bar{\nabla}_{\perp\mu} E\,.$$

- $N_L^{(0)}$ is equilibrium Bose-distribution.

Diffusive mode fluctuations

- Fluctuations of $m \equiv s/n$ and transverse components of u^μ obey

$$\text{(entropy-entropy)} \qquad \mathcal{L}[N_{mm}] = -2\Gamma_\lambda \left(N_{mm} - \frac{c_p}{n}\right) + \dots$$

$$\text{(entropy-velosity)} \qquad \mathcal{L}[N_{mi}] = -(\Gamma_\eta + \Gamma_\lambda) N_{mi} + \dots$$

$$\text{(velocity-velocity)} \qquad \mathcal{L}[N_{ij}] = -2\Gamma_\eta \left(N_{ij} - \frac{Tw}{n}\right) + \dots$$

  - $\mathcal{L}$ is Liouville operator with $v = f = 0$, i.e., no propagation, but relaxation: $\Gamma_X = \gamma_X q^2$, where $\gamma_\lambda = \lambda/c_p$ and $\gamma_\eta = \eta/w$.
  - "$\dots$" are terms $\sim$ background grads, mixing $N_{mm} \leftrightarrow N_{mi} \leftrightarrow N_{ij}$.

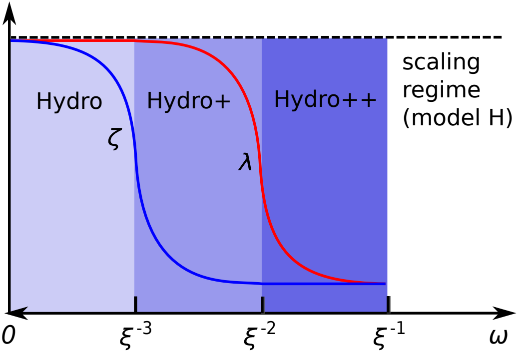
- Near critical point Γ_λ is smallest, $\gamma_\lambda = \lambda/c_p \sim 1/\xi \to 0$.

  N_{mm} equation decouples and matches Hydro+ ($\phi_{\boldsymbol{Q}} = nN_{mm}$). Very nontrivially! *An, Basar, Yee, MS, 1912.13456*

Beyond Hydro+

- Hydro+ breaks down when hydro frequency/rate is of order ξ^{-2} due to next-to-slowest modes (N_{mi} and N_{ij}).
- The formalism extends Hydro+ to higher frequencies, i.e., shorter hydrodynamic scales (all the way to ξ.)

  Fluctuations (N_{mi}) enhance conductivity for small ω.



Renormalization

- Expansion of $\langle T^{\mu\nu}\rangle$ in fluctuations ϕ contains

more

$$\langle\phi(x)\phi(x)\rangle = G(x,0) = \int \frac{d^3q}{(2\pi)^3}\, W(x,q).$$

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- Such short-distance singularities can be absorbed into redefinion of EOS (i.e., pressure) and transport coefficients:

$$\begin{aligned}\langle T^{\mu\nu}(x)\rangle &= \epsilon u^\mu u^\nu + p(\epsilon,n)\Delta^{\mu\nu} + \Pi^{\mu\nu} + \Big\{G(x,0)\Big\}\\ &= \epsilon_R u_R^\mu u^\nu + p_R(\epsilon_R,n_R)\Delta_R^{\mu\nu} + \Pi_R^{\mu\nu} + \Big\{\tilde G(x,0)\Big\}\ .\end{aligned}$$

Constraints of 2nd law, conformality satisfied *nontrivially*. more

Theory/experiment gap: we have been discussing linearized fluctuations and two-point correlators, but

non-Gaussian fluctuations are sensitive signatures of the critical point

Nonlinearity and multiplicative noise

An et al 2009.10742

- Now nonlinearity and multiplicative noise matter even more:

$$\partial_t \breve{\psi} = -\nabla \cdot \Big(\text{Flux}[\breve{\psi}] + \text{ Noise} \Big) , \qquad \langle \text{Noise Noise} \rangle \sim 2Q[\breve{\psi}] \, .$$

- General multivariable Langevin equation:

$$\frac{d\breve{v}_i}{dt} = F_i[\breve{v}] + H_{ij}[\breve{v}]\xi_j \, , \quad \text{ambiguous}$$

Fokker-Plank equation (Ito calculus):

$$\partial_t P = \Big(-F_i P + \left(Q_{ij} P \right)_{,j} \Big)_{,i} \, , \quad \text{unambiguous}$$

$(Q \equiv HH^T)$.

- Different (e.g., Stratonovich) calculus/scheme $\equiv$ different F_i.

Physical (unambiguous) formulation

- Equilibrium solution obeys:

$$\text{Probability flux} = F_i P_{\mathrm{eq}} - \left(Q_{ij} P_{\mathrm{eq}}\right)_{,j} = \left(\Omega_{ij} P_{\mathrm{eq}}\right)_{,j} \ ,$$

  I.e., $F_i = P_{\mathrm{eq}}^{-1} \left(M_{ij} P_{\mathrm{eq}}\right)_{,j}$, where $M \equiv Q + \Omega$.

- Define the problem in terms of *physical properties*

$$\text{Onsager matrix } M \text{ and entropy } S \equiv \log P_{\mathrm{eq}},$$

  rather than $F_i = M_{ij} S_{,j} + M_{ij,j}$ (Ito).

Perturbation theory

- *Small* fluctuations are Gaussian
- Introduce expansion parameter: ε.

  Power counting: $S'' \sim \varepsilon^{-1}$, so that $\delta v \sim \sqrt{\varepsilon}$.

  Then $G_n^{\mathsf{c}} \equiv \langle \delta v_{i_1} \dots \delta v_{i_n} \rangle^{\mathsf{c}} \sim \varepsilon^{n-1}$.
- In hydrodynamics, small parameter is $(q/\Lambda)^3$: $1/q$ – wavelength of fluctuations $\gg 1/\Lambda$ – size of hydro cell (UV cutoff).

Evolution equations for cumulants

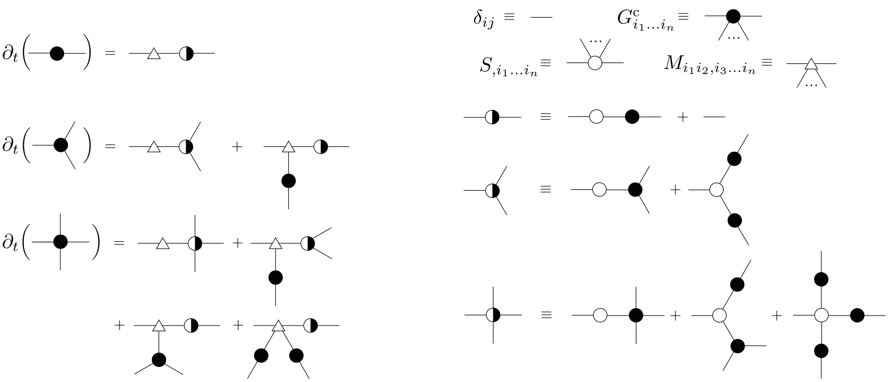
Systematically truncate each equation to leading order:

$$\partial_t G^{\mathsf{c}}_{i_1 i_2} = \left[M_{i_1 j} \left(S_{,jk} G^{\mathsf{c}}_{k i_2} + \delta_{j i_2} \right) \right]_{\mathrm{P} i_1 i_2} + \mathcal{O}(\varepsilon),$$

$$\begin{aligned}\partial_t G^{\mathsf{c}}_{i_1 i_2 i_3} = \Big[\frac{1}{2} M_{i_1 j} & \left(S_{,jk} G^{\mathsf{c}}_{k i_2 i_3} + S_{,jk\ell} G^{\mathsf{c}}_{k i_2} G^{\mathsf{c}}_{\ell i_3} \right) \\ & + M_{i_1 j, m} G^{\mathsf{c}}_{m i_2} \left(S_{,jk} G^{\mathsf{c}}_{k i_3} + \delta_{j i_3} \right) \Big]_{\mathrm{P} i_1 i_2 i_3} + \mathcal{O}(\varepsilon^2).\end{aligned}$$

$$\vdots$$

Diagrammatic representation



- Tree diagrams at leading order in ε.
- At higher-orders loops describe feedback of fluctuations (e.g., long-time tails).

Diffusion (nonlinear and stochastic)

- The prototype of hydrodynamics:

$$\partial_t \breve{n} = -\boldsymbol{\nabla} \cdot \breve{\boldsymbol{J}}\,,$$

$$\breve{\boldsymbol{J}} = -\breve{\lambda}\boldsymbol{\nabla}\breve{\alpha} + \sqrt{\breve{\lambda}}\,\boldsymbol{\xi},$$

$\breve{\lambda} = \lambda(\breve{n})$ is conductivity and $\breve{\alpha} = \alpha(\breve{n})$ is chem. potential ($T = 1$).

- Translate: $i, j \to \boldsymbol{x}, \boldsymbol{y}$,

$$S_{,ij} \to \frac{\delta^2 S}{\delta n_{\boldsymbol{x}} \delta n_{\boldsymbol{y}}} = -\alpha'(n_{\boldsymbol{x}})\delta_{\boldsymbol{xy}} \quad \text{and} \quad M_{ij} \to -\boldsymbol{\nabla}_{\boldsymbol{x}}\lambda(n_{\boldsymbol{x}})\boldsymbol{\nabla}_{\boldsymbol{x}}\delta_{\boldsymbol{xy}}\,.$$

Generalizing Wigner transform

- $$W_n(\boldsymbol{x}, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) \equiv \int d\boldsymbol{y}_1^3 \ldots \int d\boldsymbol{y}_n^3 \, G_n\left(\boldsymbol{x} + \boldsymbol{y}_1, \ldots, \boldsymbol{x} + \boldsymbol{y}_n\right) \delta^{(3)}\left(\frac{\boldsymbol{y}_1 + \ldots + \boldsymbol{y}_n}{n}\right) e^{-i(\boldsymbol{q}_1 \cdot \boldsymbol{y}_1 + \ldots + \boldsymbol{q}_n \cdot \boldsymbol{y}_n)};$$

  $$G_n\left(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\right) = \int \frac{d\boldsymbol{q}_1^3}{(2\pi)^3} \ldots \int \frac{d\boldsymbol{q}_n^3}{(2\pi)^3} \, W_n(\boldsymbol{x}, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)(2\pi)^3 \delta^{(3)}\left(\boldsymbol{q}_1 + \ldots + \boldsymbol{q}_n\right) e^{i(\boldsymbol{q}_1 \cdot \boldsymbol{x}_1 + \ldots + \boldsymbol{q}_n \cdot \boldsymbol{x}_n)}.$$

  - Properties similar to the usual ($n = 2$) Wigner transform.
  - Takes advantage of the scale separation:
    long-scale x-dependence and short-scale $\boldsymbol{y}_n$-dependence.
  - One wavevector is redundant: $\boldsymbol{q}_1 + \ldots + \boldsymbol{q}_n = 0$.

Equations

- $$\partial_t W_2(\boldsymbol{q}_1) = -\left[\gamma \boldsymbol{q}_1^2 W_2(\boldsymbol{q}_2) + \lambda \boldsymbol{q}_1 \cdot \boldsymbol{q}_2\right]_{\mathrm{P}\boldsymbol{q}_1\boldsymbol{q}_2},$$

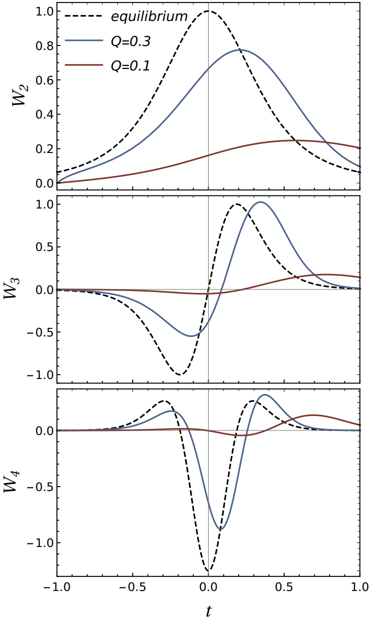
$$\partial_t W_3(\boldsymbol{q}_1,\boldsymbol{q}_2) = -\left[\frac{1}{2}\gamma \boldsymbol{q}_1^2 W_3(\boldsymbol{q}_2,\boldsymbol{q}_3) + \frac{1}{2}\gamma' \boldsymbol{q}_1^2 W_2(\boldsymbol{q}_2) W_2(\boldsymbol{q}_3) + \lambda' \boldsymbol{q}_1 \cdot \boldsymbol{q}_2 W_2(\boldsymbol{q}_3)\right]_{\mathrm{P}\boldsymbol{q}_1\boldsymbol{q}_2\boldsymbol{q}_3},$$

$$\partial_t W_4^{\mathsf{c}}(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3) = -\left[\frac{1}{6}\gamma \boldsymbol{q}_1^2 W_4^{\mathsf{c}}(\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4) + \frac{1}{6}\gamma'' \boldsymbol{q}_1^2 W_2(\boldsymbol{q}_2) W_2(\boldsymbol{q}_3) W_2(\boldsymbol{q}_4) + \frac{1}{2}\gamma' \boldsymbol{q}_1^2 W_2(\boldsymbol{q}_2) W_3(\boldsymbol{q}_3,\boldsymbol{q}_4) + \frac{1}{2}\lambda' \boldsymbol{q}_1 \cdot \boldsymbol{q}_2 W_3(\boldsymbol{q}_3,\boldsymbol{q}_4) + \frac{1}{2}\lambda'' \boldsymbol{q}_1 \cdot \boldsymbol{q}_2 W_2(\boldsymbol{q}_3) W_2(\boldsymbol{q}_4)\right]_{\mathrm{P}\boldsymbol{q}_1\boldsymbol{q}_2\boldsymbol{q}_3\boldsymbol{q}_4},$$

  $(\gamma \equiv \lambda\alpha')$
- Map from diagrams:
  $S_{,ij} \to -\alpha'$, $S_{,ijk} \to -\alpha''$, $M_{ij} \to -\lambda \boldsymbol{q}_1 \cdot \boldsymbol{q}_2$, $M_{ij,k} \to -\lambda' \boldsymbol{q}_1 \cdot \boldsymbol{q}_2$, , etc..

Example: expansion through a critical region

- Two main features:
  - Lag, or "memory".
  - Smaller Q – slower evolution. Conservation laws.
- The magnitude of the observed critical point signatures depends on the scale of fluctuations probed.



Work in progress and outlook

- *Non-gaussian* fluctuations in *full* relativistic hydrodynamics.
- Connect *fluctuating* hydro with freezeout kinetics and implement in full hydrodynamic code and event generator.

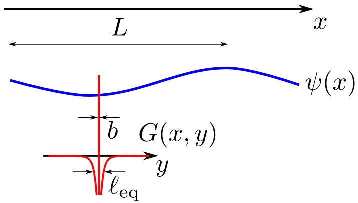
  Compare with experiment.
- First-order transition in fluctuating hydrodynamics?
- Loops, long-time tails, renormalization.
- Connection to path integral (SK) formulation.

More

Separation of scales

$$G(x,y) = \langle\, \phi(x+y/2)\, \phi(x-y/2)\, \rangle$$

depends on x slowly (L), but on y – fast ($\ell_{\rm eq} \sim \sqrt{L} \ll L$).



Similar to separation of scales in QFT in kinetic regime. ($q \gg k$)

Scales

- Hydro cell size b: coarse-grain quantum operators over scale $b \gg \ell_{\rm mic}$ to leave only slow modes for which quantum fluctuatu-ations are negligible compared to thermal, i.e., $\hbar\omega \ll kT$.
  $\ell_{\rm mic} \sim \ell_{\rm mfp},\, c_s/T$.
  $\breve\psi = (\,\breve T^{i0},\, \breve J^0\,)$ are *classical* stochastic variables.
- Hydrodynamic gradients scale L: must be $L \gg b$. back
- Size of local equlibrium cell $\ell_{\rm eq} \equiv \ell_*$: diffusion length in evolution time scale, typically $\tau_{\rm ev} \sim L/c_s$

$$\ell_* \sim \sqrt{\gamma\tau_{\rm ev}} \sim \sqrt{\gamma L/c_s}.$$

- $b \ll L$ implies the hierarchy:

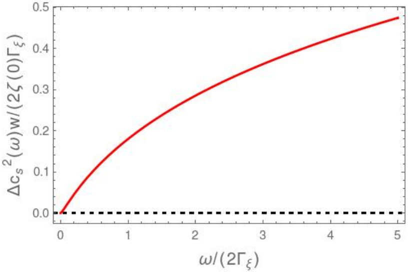
$$\ell_{\rm mic} \ll b < \ell_* \ll L \quad \text{or} \quad T/c_s \gg \Lambda > q_* \gg k \quad (\gamma q_*^2 = c_s k)$$

Hydro+ vs Hydro: real-time bulk response

Hydrodynamics breaks down for processes faster than $\Gamma_\xi \sim \xi^{-3} \to$ Hydro+

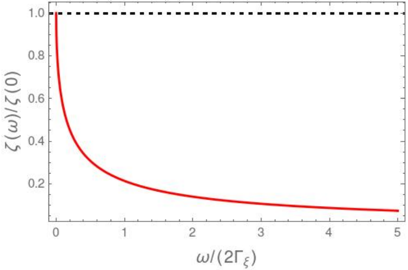
- Stiffness of eos (sound speed) is underestimated in hydro (- - -):

  $c_s \to 0$ at CP, but
  only modes with $\omega \ll \Gamma_\xi$ are critically soft.



- Dissipation during expansion is overestimated in hydro (- - -):

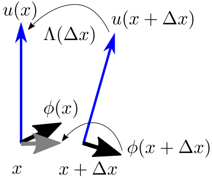
  $\zeta \to \infty$ at CP, but
  only modes with $\omega \ll \Gamma_\xi$ experience large ζ.



Confluent derivative, connection and correlator

Take out dependence of *components* of ϕ due to change of $u(x)$:

$$\Delta x \cdot \bar{\nabla}\phi = \Lambda(\Delta x)\phi(x+\Delta x) - \phi(x)$$



Confluent two-point correlator:

$$\bar{G}(x,y) = \Lambda(y/2)\,\langle\phi(x+y/2)\,\phi(x-y/2)\rangle\,\Lambda(-y/2)^T$$

(boost to $u(x)$ – rest frame at midpoint)

$$\bar{\nabla}_\mu \bar{G}_{AB} = \partial_\mu \bar{G}_{AB} - \bar{\omega}^C_{\mu A}\bar{G}_{CB} - \bar{\omega}^C_{\mu B}\bar{G}_{AC} - \mathring{\omega}^b_{\mu a}\, y^a \frac{\partial}{\partial y^b}\bar{G}_{AB}\,.$$

Connection $\bar{\omega}$ corresponds to the boost Λ.

back

Connection $\mathring{\omega}$ makes sure derivative is independent of the choice of local space triad e_a needed to express $\boldsymbol{y} \equiv \boldsymbol{x}_+ - \boldsymbol{x}_-$.

We then define the Wigner transform $W_{AB}(x,q)$ of $\bar{G}_{AB}(x,y)$.

Renormalization

Expansion of $\langle T^{\mu\nu} \rangle$ contains $\langle \phi(x)\phi(x) \rangle = G(x, 0) = \int \frac{d^3q}{(2\pi)^3} W(x, q)$.

This integral is divergent (equilibrium $G^{(0)}(x, y) \sim \delta^3(y)$).

back

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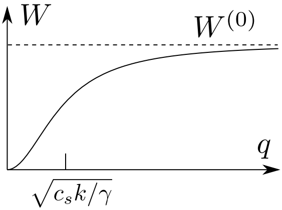

$$W(x,q) \;\sim\; \underbrace{W^{(0)}}_{Tw} \;+\; \underbrace{W^{(1)}}_{\partial u/q^2} \;+\; \widetilde{W}$$

(∼“OPE” or gradient expansion)

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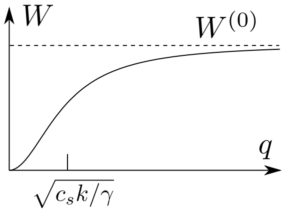

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($\sim$"OPE" or gradient expansion)

$$G(x,0) \sim \underbrace{\Lambda^3}_{\text{ideal (EOS)}} + \underbrace{\Lambda\,\partial u}_{\text{visc. terms}} + \underbrace{\widetilde{G}}_{\text{finite "}\partial^{3/2}\text{"}}$$

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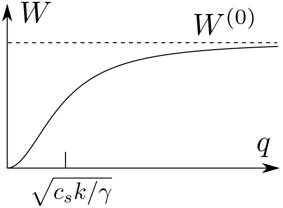

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$$\begin{aligned}\langle T^{\mu\nu}(x)\rangle &= \epsilon u^\mu u^\nu + p(\epsilon,n)\Delta^{\mu\nu} + \Pi^{\mu\nu} + \Big\{G(x,0)\Big\}\\ &= \epsilon_R u_R^\mu u^\nu + p_R(\epsilon_R,n_R)\Delta_R^{\mu\nu} + \Pi_R^{\mu\nu} + \Big\{\tilde{G}(x,0)\Big\}\ .\end{aligned}$$

Renormalized e.o.s. and transport coefficients

- Fluctuation corrections to kinetic coefficients are positive. back
- Corrections to pressure and bulk viscosity vanish for conformal e.o.s.

$$p_R(\epsilon_R, n_R) = p(\epsilon_R, n_R) + \frac{T\Lambda^3}{6\pi^2}\left((1 - c_s^2 - 2\dot{T} + \dot{c}_s) + \frac{1}{2}(1 - \dot{c}_p)\right),$$

$$\eta_R = \eta + \frac{T\Lambda}{30\pi^2}\left(\frac{1}{\gamma_L} + \frac{7}{2\gamma_\eta}\right),$$

$$\zeta_R = \zeta + \frac{T\Lambda}{18\pi^2}\left(\frac{1}{\gamma_L}(1 - 3\dot{T} + 3\dot{c}_s)^2 + \frac{2}{\gamma_\eta}\left(1 - \frac{3}{2}(\dot{T} + c_s^2)\right)^2 + \frac{9}{4\gamma_\lambda}(1 - \dot{c}_p)^2\right),$$

$$\lambda_R = \lambda + \frac{T^2 n^2 \Lambda}{3\pi^2 w^2}\left(\frac{c_p T}{(\gamma_\eta + \gamma_\lambda) w} + \frac{c_s^2}{2\gamma_L}\right).$$

$$\gamma_\eta \equiv \frac{\eta}{w}, \quad \gamma_\zeta \equiv \frac{\zeta}{w}, \quad \gamma_\lambda \equiv \frac{\kappa}{c_p} = D, \quad \dot{X} \equiv \left(\frac{\partial \log X}{\partial \log s}\right)_m .$$