UnNuclear Physics: Conformal Symmetry in Nuclear Reactions

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S@INT Seminar

April 19, 2022

References

- H. Hammer and D.T. Son, arXiv:2103.12610 (PNAS 2021)
- T. Schäfer and G. Baym, arXiv:2109.06924 (PNAS 2021)

Role of symmetry in physics

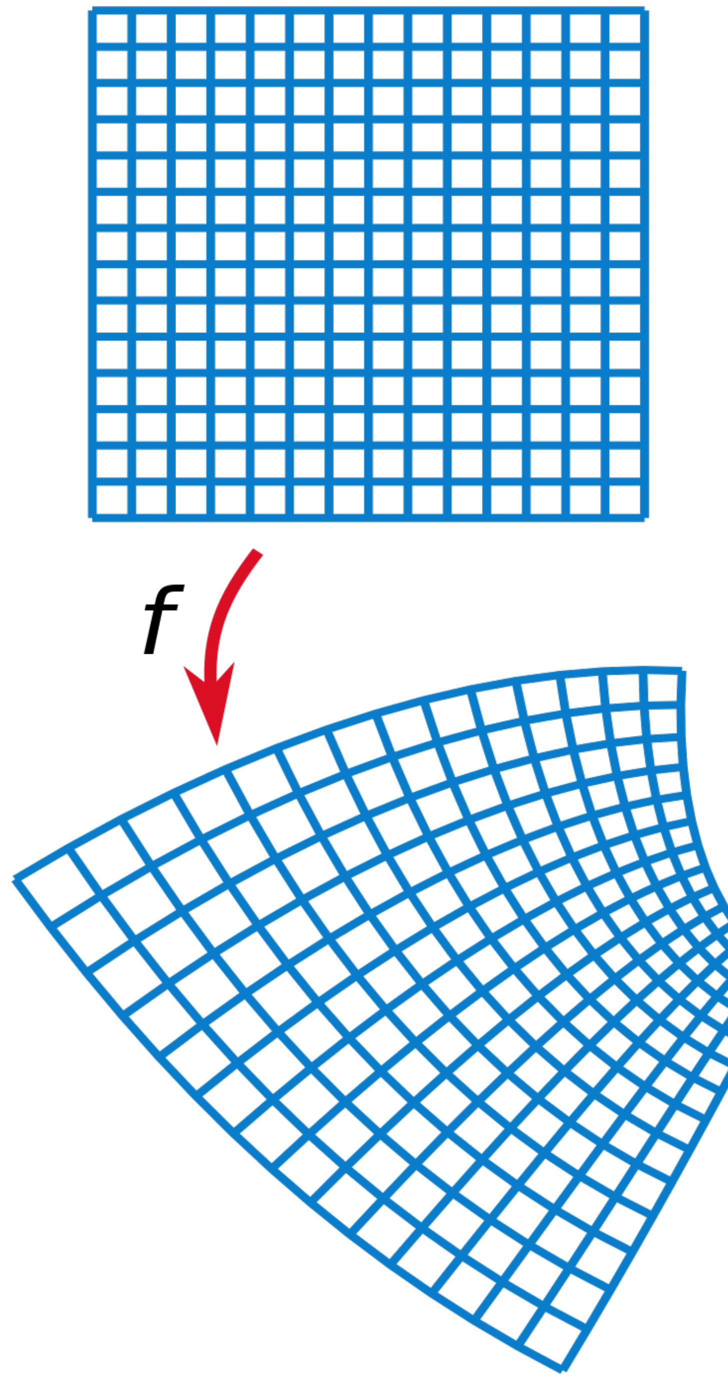
- Symmetries play a very important role in physics
- Spacetime symmetry is key to understanding of elementary particles and matter
- In particle physics, Lorentz and Poincare symmetry
- Conformal symmetry are important for quantum field theory, theory of phase transitions

Poincaré symmetry

- Time and spatial translations: $x^\mu \to x^\mu + a^\mu$
- Rotations and Lorentz boosts
- Elementary particles: irreducible representations of the Poincaré group (Wigner)
  - mass and spin when $m \neq 0$
  - when $m = 0$: helicity instead of spin

Conformal symmetry

- An extension of Poincaré group: conformal symmetry
- All transformations that preserve angle
- include: dilatation $x^\mu \rightarrow \lambda x^\mu$
- and 4 “proper conformal transformations”
- Field theory with this symmetry: conformal field theory
- applications in theoretical physics including phase transitions



Scale invariance

- Consider the Schrödinger equation of a free particle

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi$$

- From a solution $\psi(t, \mathbf{x})$ one can construct a new one

$$\tilde{\psi}(t, \mathbf{x}) = \lambda^{3/2}\psi(\lambda^2 t, \lambda\mathbf{x})$$

- Persists for system of N noninteracting particles:

$$\tilde{\psi}(t, \mathbf{x}_1, \ldots, \mathbf{x}_N) = \lambda^{3N/2}\psi(\lambda^2 t, \lambda\mathbf{x}_1, \ldots, \mathbf{x}_N)$$

Interacting systems with scale invariance

- Consider two interacting particles. The Schrödinger equation for the relative coordinate

$$i\frac{\partial\psi}{\partial t} = \left(-\frac{\nabla^2}{2m} + V(r)\right)\psi$$

- The requirement that $\tilde{\psi}(t,\mathbf{x}) = \lambda^{3/2}\psi(\lambda^2 t, \lambda\mathbf{x})$ is a solution requires

$$V(r) = \frac{\alpha}{r^2}$$

- A system of N particles interacting through α/r^2 potential is expected to be scale-invariant

- In fact, the symmetry of the free time-dependent Schrödinger equation is larger.

- If $\psi(t, \mathbf{x})$ is a solution, then

$$\tilde{\psi}(t, \mathbf{x}) = \frac{1}{(1+\alpha t)^{3/2}} \exp\left(\frac{i}{2}\frac{m\alpha x^2}{1+\alpha t}\right)\psi\left(\frac{t}{1+\alpha t}, \frac{\mathbf{x}}{1+\alpha t}\right)$$

  is also a solution

- The full symmetry group of the Schrödinger equation is called the Schrödinger symmetry

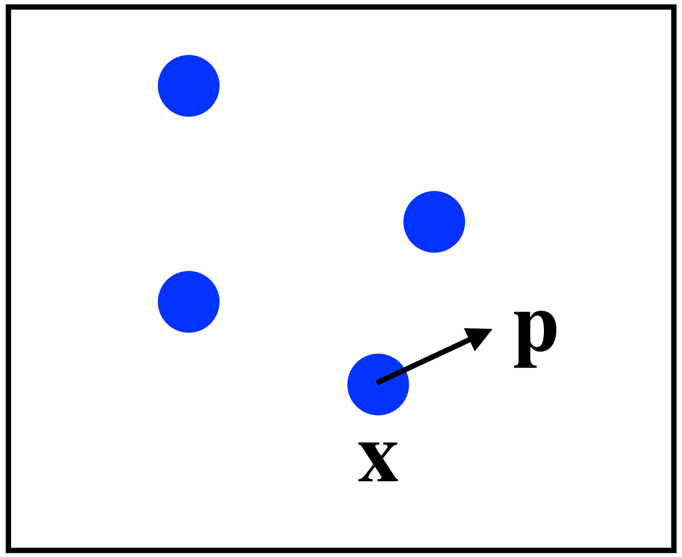
Schrödinger symmetry

- Time and spatial translations
- Galilean boost

$$\tilde{\psi}(t, \mathbf{x}) = e^{im\mathbf{v}\cdot\mathbf{x} - \frac{i}{2}mv^2 t}\,\psi(t, \mathbf{x} - \mathbf{v}t)$$

- Scaling
- Proper conformal transformation
- This group of symmetries is the non-relativistic version of conformal symmetry, so is sometimes called "nonrelativistic conformal symmetry"

Schrödinger algebra



- Free particles $(\mathbf{x}_a, \mathbf{p}_a), a = 1,2,\ldots N$
- $\mathbf{P} = \sum_a \mathbf{p}_a \qquad H = \sum_a \frac{p_a^2}{2m}$
- $\mathbf{K} = \sum m\mathbf{x}_a$ Galilean boosts
- $D = \sum \frac{1}{2}(\mathbf{x}_a \cdot \mathbf{p}_a + \mathbf{p}_a \cdot \mathbf{x}_a)$ dilatation
- $C = \frac{1}{2}m\sum x_a^2$ proper conformal transformation
- Angular momentum
- Mass $M = Nm$

Schrödinger algebra

Using $[x, p] = i$ we can compute the commutators

$X \setminus Y$	P_j	K_j	D	C	H
P_i	0	$-i\delta_{ij}M$	$-iP_i$	$-iK_i$	0
K_i	$i\delta_{ij}M$	0	iK_i	0	iP_i
D	iP_j	$-iK_j$	0	$-2iC$	$2iH$
C	iK_j	0	$2iC$	0	iD
H	0	$-iP_j$	$-2iH$	$-iD$	0

$$[J_{ij}, N] = [J_{ij}, D] = [J_{ij}, C] = [J_{ij}, H] = 0,$$
$$[J_{ij}, P_k] = i(\delta_{ik}P_j - \delta_{jk}P_i), \qquad [J_{ij}, K_k] = i(\delta_{ik}K_j - \delta_{jk}K_i),$$
$$[J_{ij}, J_{kl}] = i(\delta_{ik}J_{jl} + \delta_{jl}J_{ik} - \delta_{il}J_{jk} - \delta_{jk}J_{il}).$$

Beyond free theory

- Is the Schrödinger symmetry good only for non-interacting theory and $1/r^2$ interaction?
- Are there scale-invariant systems with short-ranged interaction?
- Answer: yes! the unitarity regime

Unitarity regime: Zeldovich's 1960 paper

SOVIET PHYSICS JETP VOLUME 11, NUMBER 4 OCTOBER, 1960

THE EXISTENCE OF NEW ISOTOPES OF LIGHT NUCLEI AND THE EQUATION OF STATE OF NEUTRONS

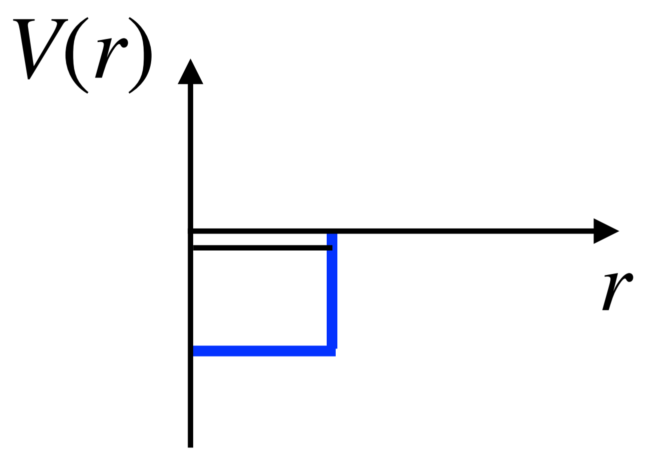
Ya. B. ZEL'DOVICH

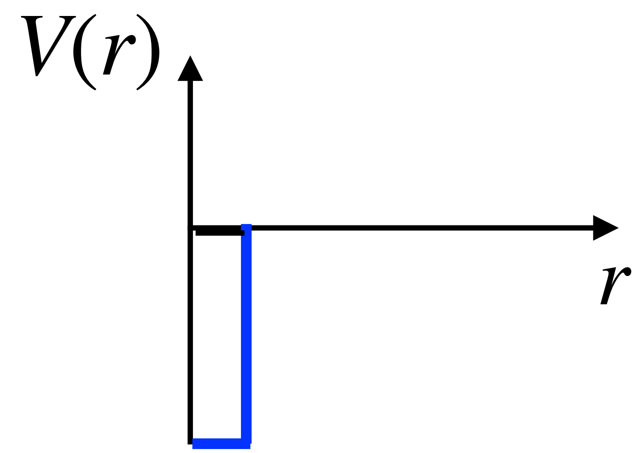
Submitted to JETP editor October 22, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **38**, 1123-1131 (April, 1960)

The limits of stability (relative to nucleon emission) of light nuclei are considered. The existence (in the sense of stability against decay with emission of a nucleon) of the following nuclei is predicted: He^8, Be^{12}, O^{13}, $B^{15,17,19}$, C^{16-20}, N^{18-21}, Mg^{20}. The problem of the possibility of existence of heavy nuclei composed of neutrons only is considered. The problem is reduced to that of a Fermi gas with a resonance interaction between the particles. The energy of such a gas is proportional to $\omega^{2/3}$, where ω is its density. The accuracy of the calculations is not sufficient to determine the sign of the energy and answer the question as to the existence of neutron nuclei.

“Unitarity regime”





- Take a potential of a certain shape, e.g.,
  $V(r) = -V_0$ for $r < r_0$, 0 for $r > r_0$

- shrink the range, adjusting the depth so that there is one almost bound state at zero energy

$$V_0 = \frac{\pi^2 \hbar^2}{8m} \frac{1}{r_0^2}$$

- In the limit $r_0 \to 0$: “unitarity regime”

Scattering length

- When the scattering is large and positive, the potential has a shallow bound state

$$E = -\frac{\hbar^2}{ma^2}$$

- The bound state disappears when $a \to \infty$

- $a \to \infty, r_0 \to 0$ limit: boundary condition

$$\psi(\mathbf{x}, \mathbf{y}) = \frac{C}{|\mathbf{x} - \mathbf{y}|} + b(\mathbf{x}, \mathbf{y}), \quad b(\mathbf{x}, \mathbf{x}) = 0$$

Systems with large scattering length

- Helium-4 atoms $a \sim 100$ Å
- Neutrons $a \sim -20$ fm
- Ultracold trapped atoms: a can be tuned by a magnetic field
- In all these cases, interaction is short-ranged but particles “feel” each other at much larger distance

Problem of unitary fermions

- Two types of particles: $\mathbf{x}_a$ and $\mathbf{y}_b$ (spin-up and spin-down fermions)

$$H = -\frac{1}{2m}\sum_{a=1}^{N_1}\nabla^2_{\mathbf{x}_a} - \frac{1}{2m}\sum_{b=1}^{N_2}\nabla^2_{\mathbf{y}_b}$$

- When 2 particles of different spins approach each other, the wave function has the asymptotic form

$$\psi(\mathbf{x},\mathbf{y}) = \frac{C}{|\mathbf{x}-\mathbf{y}|} + b(\mathbf{x},\mathbf{y}), \quad b(\mathbf{x},\mathbf{x}) = 0$$

- ψ changes sign when exchanging two $\mathbf{x}$'s or two $\mathbf{y}$'s

An interaction with no free parameter!

Properties of unitary gas

- A gas of spin-1/2 particles with short-ranged interaction fine-tuned to unitarity
- Scale invariance: physical quantities can be figured out by scaling arguments
- Example: Bertsch parameter ξ $(T = 0)$
- $\frac{E}{N} = \xi \frac{3}{5} \varepsilon_F, \quad \varepsilon_F = \frac{1}{2m} (3\pi^2 n)^{2/3}$

$$\xi \approx 0.37$$

Nonrelativistic CFT

Y. Nishida, DTS, 2007

- One can build up the formalism of nonrelativistic conformal field theory in analogy with the relativistic theory
- Many notions can be extended
  - operator dimensions
  - operator-state correspondence

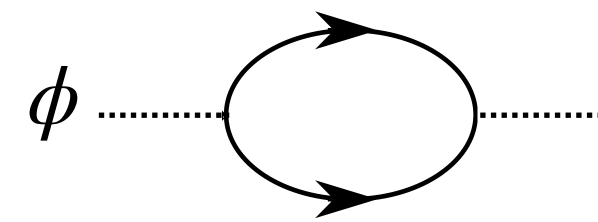
Fermions at unitarity as a NRCFT

- $$L = i\psi^\dagger\left(\partial_t + \frac{\nabla^2}{2m}\right)\psi - c_0\psi_\uparrow^\dagger\psi_\downarrow^\dagger\psi_\downarrow\psi_\uparrow \qquad \Delta[\psi] = \tfrac{3}{2}$$
- Introducing auxiliary field ϕ
- $$L = i\psi^\dagger\left(\partial_t + \frac{\nabla^2}{2m}\right)\psi - \psi_\uparrow^\dagger\psi_\downarrow^\dagger\phi - \phi^\dagger\psi_\downarrow\psi_\uparrow + \frac{\phi^\dagger\phi}{c_0}$$
    - Propagator of ϕ

$$G_\phi(\omega, \mathbf{p}) = \frac{1}{\sqrt{\frac{p^2}{4m} - \omega}} \qquad \Delta[\phi] = 2 \neq 2 \times \tfrac{3}{2}$$

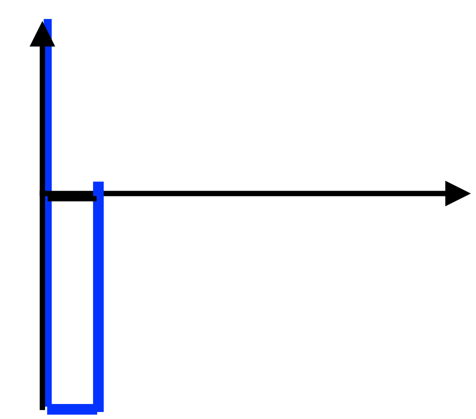
Renormalization

- $G_\phi^{-1}(\omega, \mathbf{p}) = c_0^{-1}$ + one-loop integral



- $= c_0^{-1} + \Lambda + \left(\dfrac{p^2}{4m} - \omega\right)^{1/2}$
- Unitarity: fine-tuning so that $c_0 + \Lambda = 0$
- (scattering length: $c_0 + \Lambda = \dfrac{1}{a}$)
- Physically: fine-tune the attractive short-range potential to have a bound state at threshold

$$G_\phi(\omega, \mathbf{p}) = \frac{1}{\sqrt{\frac{p^2}{4m} - \omega}}$$



Operator-state correspondence

Y. Nishida, DTS, 2007

- Dimension of a primary operator = energy of a state in a harmonic potential
- Example:

1 particle in h.p. $E = \frac{3}{2}\hbar\omega$ $\qquad [\psi] = \frac{3}{2}$

2 particles at unitarity in h.p.

$E = 2\hbar\omega$ $\qquad [\psi] = 2$

Operator-state correspondence

- Dimension of a primary operator = energy of a state in a harmonic potential

N	S	L	O	Δ
2	0	0	$\psi_\uparrow\psi_\downarrow$	2
3	$1/2$	1	$\psi_\downarrow\psi_\uparrow\boldsymbol{\nabla}\psi_\uparrow$	4.273
3	$1/2$	0	$\psi_\downarrow\boldsymbol{\nabla}\psi_\uparrow\cdot\boldsymbol{\nabla}\psi_\uparrow$	4.666
4	0	0	$\psi_\downarrow\psi_\uparrow\boldsymbol{\nabla}\psi_\downarrow\cdot\boldsymbol{\nabla}\psi_\uparrow$	5.0–5.1

“UnNuclear physics”

A nonrelativistic version of unparticle physics

field in NRCFT: “unnucleus”

H.-W. Hammer and DTS, 2103.12610

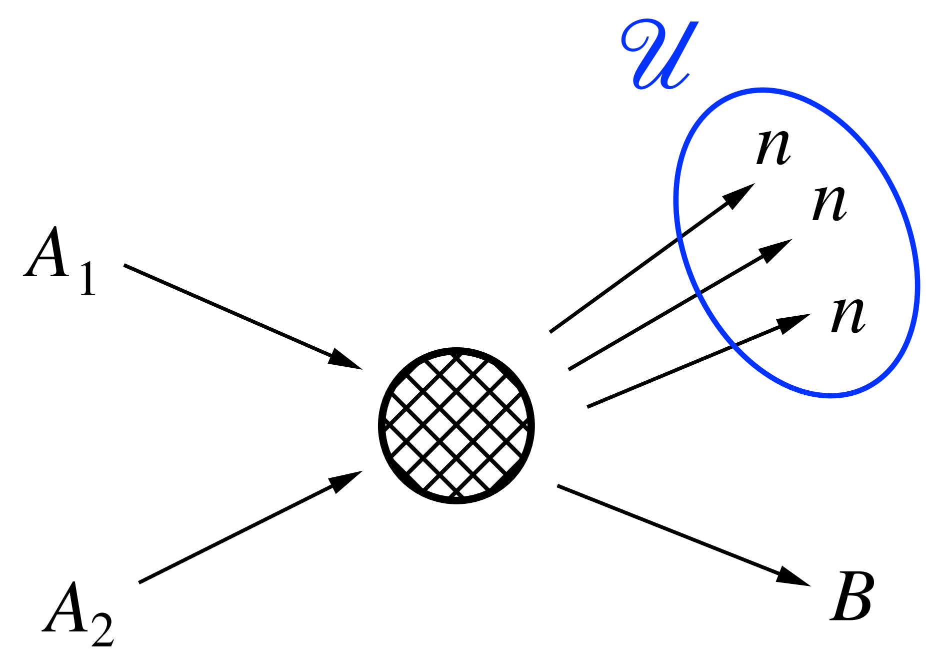
Few-neutron systems as unnuclei

- Neutrons have anomalously large scattering length: $a_{nn} \approx -19$ fm $\gg r_0 \approx 2.8$ fm
- In a wide range of energy is neutrons are fermions at unitarity

Nuclear reactions

- Many nuclear reactions with emissions of neutrons:
  - $^3H + {}^3H \to {}^4He + 2n$
  - $^7Li + {}^7Li \to {}^{11}C + 3n$
  - $^4He + {}^8He \to {}^8Be + 4n$
- Final-state neutrons can be considered as forming an "unnucleus" - a field in NRCFT
  - Regime of validity: kinetic energy of neutrons in their c.o.m. frame between $\hbar^2/ma^2 \sim 0.1$ MeV $\hbar^2/mr_0^2 \sim 5$ MeV

Few-neutron systems as unnuclei

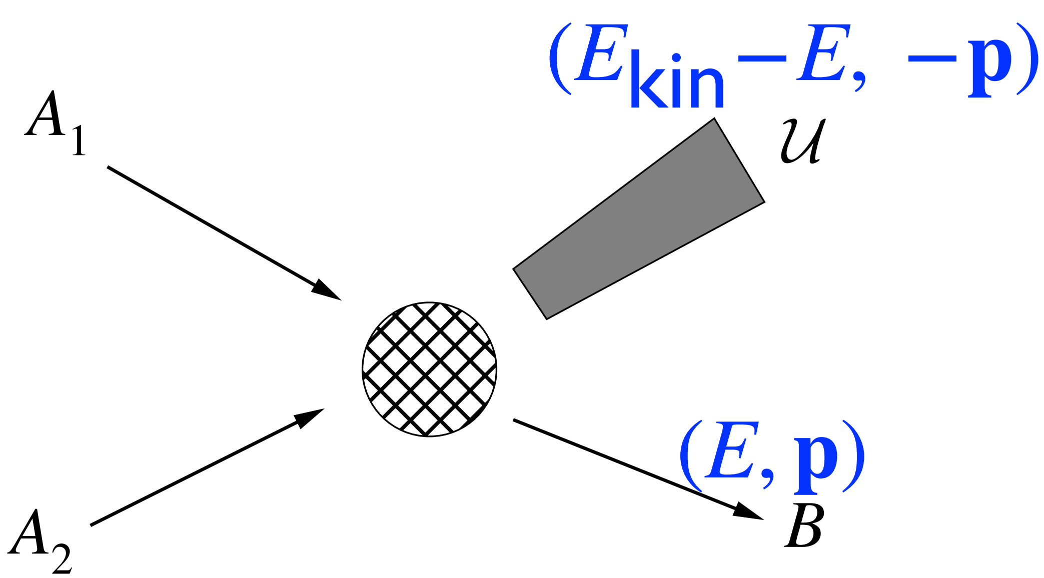


Factorization:

$$\frac{d\sigma}{dE} \sim |\mathscr{M}|^2 \sqrt{E_B} \times \mathrm{Im}\, G_{\mathscr{U}}(E_{\mathscr{U}}, \mathbf{p})$$

primary reaction has larger energy than final-state interaction

Rates of processes involving an unnucleus



$$E_{\mathrm{kin}} = E + E_{\mathscr{U}}$$

- $$\frac{d\sigma}{dE} \sim |\mathscr{M}|^2 \sqrt{E} \times \underbrace{\mathrm{Im}\, G_{\mathscr{U}}(E_{\mathrm{kin}} - E, \mathbf{p})}$$

$$\left(E_{\mathrm{kin}} - E - \frac{p^2}{2M_{\mathscr{U}}}\right)^{\Delta - \frac{5}{2}}$$

- Near end point: $\dfrac{d\sigma}{dE} \sim (E_0 - E)^{\Delta - \frac{5}{2}}$

Nuclear reactions

- $^3H + {^3H} \rightarrow {^4He} + 2n$
- $^7Li + {^7Li} \rightarrow {^{11}C} + 3n$
- $^4He + {^8He} \rightarrow {^8Be} + 4n$

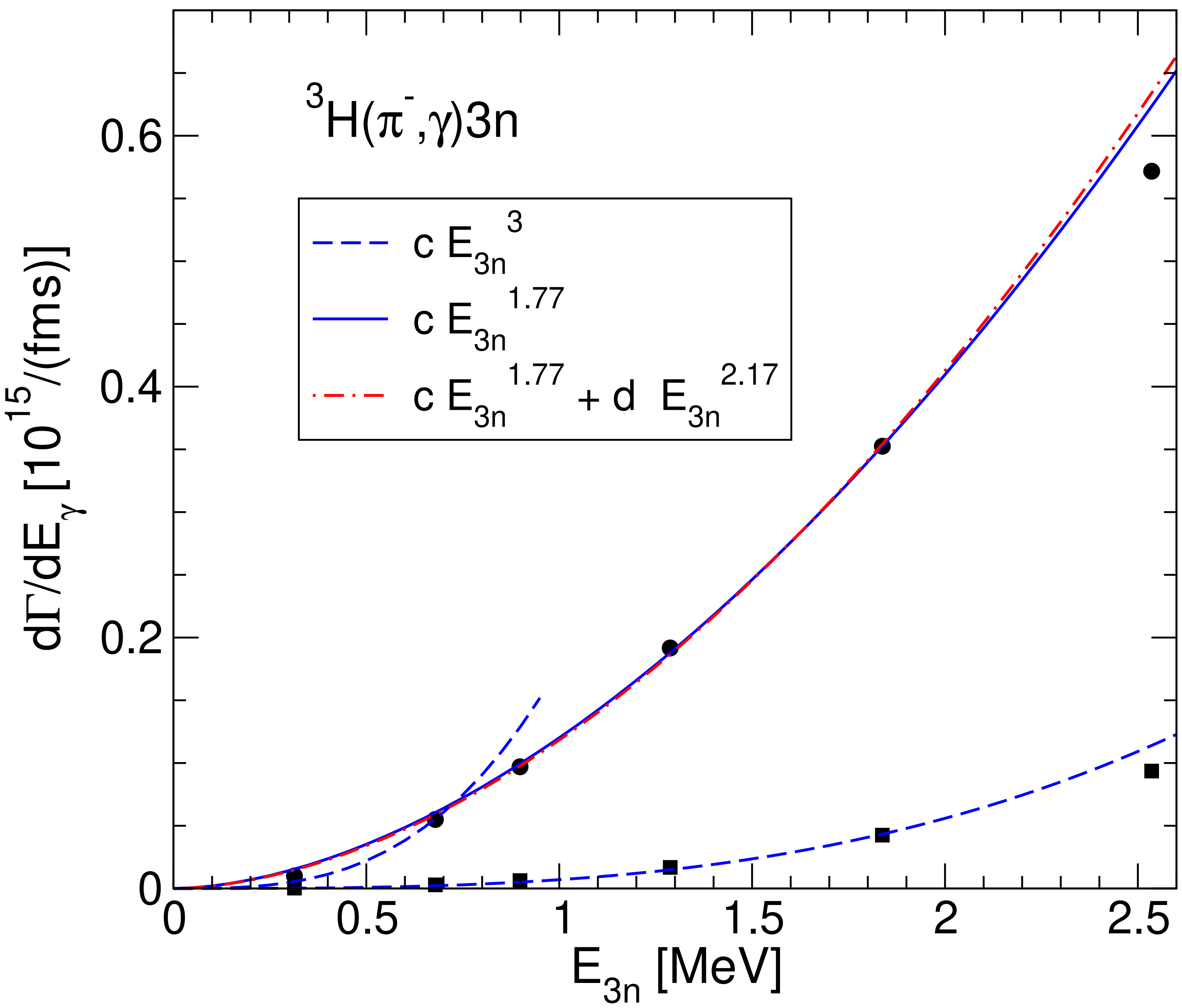
$\alpha = -0.5$

$\alpha = 1.77$

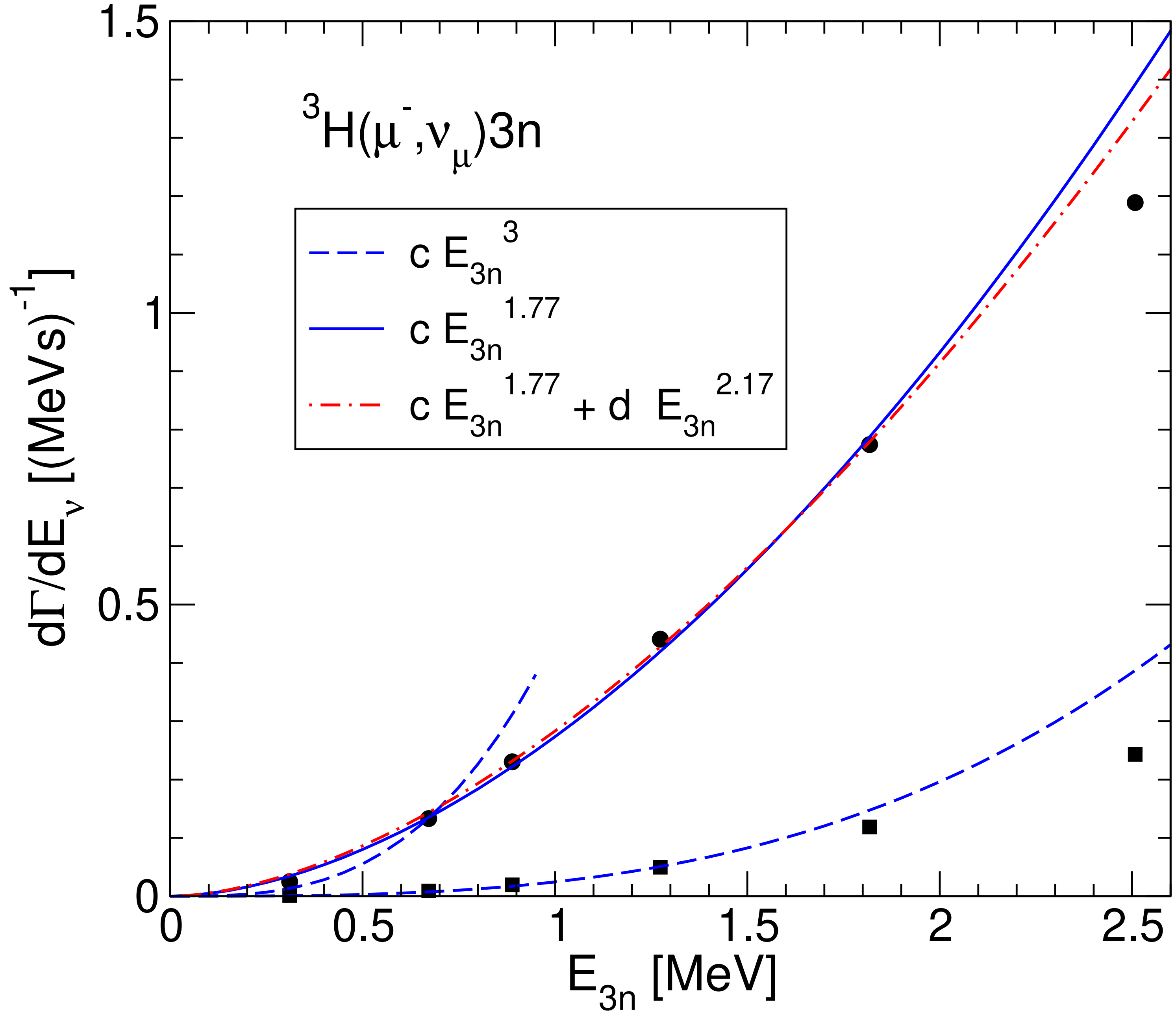
$\alpha = 2.5 - 2.6$

- Prediction:
  - $\dfrac{d\sigma}{dE} \sim (E_0 - E)^\alpha$
- Regime of validity: kinetic energy of neutrons in their c.o.m. frame between $\hbar^2/ma^2 \sim 0.1$ MeV $\hbar^2/mr_0^2 \sim 5$ MeV

Comparison with microscopic models



$$\pi^- + {}^3\mathrm{H} \rightarrow \gamma + 3n$$



$$\mu^- + {}^3\mathrm{H} \rightarrow \nu_\mu + 3n$$

Conclusion

- There is a nonrelativistic version of conformal field theory
- Example: fermions at unitarity
- Approximately realized by neutrons; leads to "unnuclear behavior" of differential cross sections near threshold
- (also in decay of multi-particle resonances Son, Stephanov, Yee 2212.03318)
- Possible extension to other systems
  X(3872) Braaten and Hammer

Thank you