Model Hamiltonians for the Fractional Quantum Hall Effect

Natalia Toro
Advisor: Wick Haxton

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1 Introduction

The fractional quantum Hall effect (FQHE), discovered by Stui and Störmer in 1982, has been the subject of much theoretical and experimental interest since. Whereas the integer case (IQHE) can be understood in terms of single-electron behavior in the presence of a magnetic field, the FQHE appears to be the result of Coulomb interactions between electrons in the system. Laughlin [1] explained the origin of the simplest class of observed FQHE features, those at Landau Levels (LLs) with fractional fillings $1/m$ for m an odd integer. Several others, most notably Jain [2], have proposed extensions to this scheme that describe a more general class of fillings. While Jain's description is remarkably consistent with experimental observations, his results follow from a numerical procedure for which there is no apparent physical justification. Ginocchio and Haxton have proposed a theory which yields the same states found by Jain but through more physically transparent arguments [3]. According to this formulation, the features at fillings with numerators other than unity result from changes in the n-particle correlations allowed by the system.

We note that the integer fillings at which a plateau is reached in the IQHE are precisely the fillings at which the Hamiltonian has a unique ground state, which is characterized by its symmetry. Moving any electron from a full LL would require placing it in a higher LL, raising the energy. It is, in fact, this energy gap that leads to the IQHE. We will see that similar gaps in the eigenvalues of the inter-electron Coulomb Hamiltonian at partially filled LLs lead to the FQHE. At $1/m$ filling, the unique ground states are in fact the Laughlin states to high accuracy. We will see that these, as well as the states constructed by Ginocchio and Haxton at other fillings, are also characterized by a symmetry of the state, specifically in the correlations among electrons in the system.

To better understand these correlations and their effect on the FQHE features, we examine a collection of n-particle operators which measure the presence of various inter-particle correlations and locally mimic the behavior of the Hamiltonian. We construct a two-particle operator associated with each $1/m$-filled state, the eigenvalue spectrum of which is closely matched to that of the Coulomb potential in that subspace. These model Hamiltonians have a zero eigenvalue in the $1/m$-filled space to which they correspond, and for all less densely filled systems, while they have only non-zero eigenvalues in more dense systems. Thus, these operators can be used to index a state according to the presence of various correlations. They are also reminiscent of order parameters in second-order phase transitions, though the physical relevance of this similarity has not been explored.

2 Toy Model

To demonstrate the role of inter-electron interactions in producing the FQHE and the possibility of creating partially-filled incompressible states, we will discuss a toy model constructed by Ginocchio and Haxton. Let us consider a system of N identical charges on a lattice of m sites, as shown in Figure 1. Clearly, the charges repel each other, so configurations in which two particles are near one another are energetically unfavorable. Of course, the most energetically favorable configuration would be one in which the particles are equidistant from one another. But in the general case, the discrete lattice makes this impossible. Thus, in cases such as that with 8 slots, there is a degeneracy of ground states (two are shown in the figure). In the case of a system with 7 slots, however, there is a unique ground state precisely because of its symmetry. Indeed this is the most dense state that can be formed in which each charge is equidistant from each neighboring charge, and corresponds to the 1/3-filled ground state. Any motion of electrons or any shrinking of the system requires that a new interaction—two charges next to one another—be added, breaking the symmetry. The next state in which this condition is satisfied is that with 13 slots, analogous to the 1/5-filled ground state in the FQHE. Similarly, a series of states with equally spaced electrons is produced, in progressively less dense states. These correspond to the Laughlin states. But additional symmetric configurations are possible. The state with 5 slots, for instance, can be seen as a configuration involving two clusters of two charges, held as far from one another as possible. Moving one of the charges would, again, increase the energy.

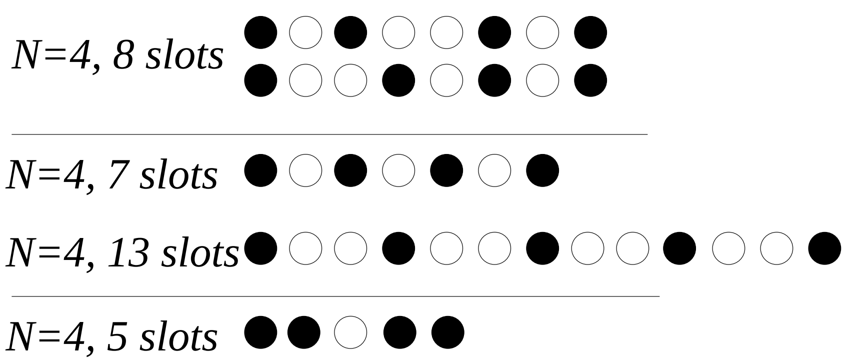

Figure 1: Toy Model states for 4 particles in 8, 7, 13, and 5 slots. The two 8-slot states are degenerate ground states, whereas the 7, 13, and 5-slot states are unique, symmetric ground states analogous to the 1/3-, 1/5-, and 2/5-filled states in the true system.

3 Wavefunctions

We will follow Haldane's choice of geometry [4], and consider electrons on a spherical surface containing a Dirac monopole of magnetic flux $2Shc/e$, where $2S$ is an integer. The single-particle wavefunctions in the first LL are given by Wigner D functions

$$D^{(S)}_{q,S}(\phi,\theta,0) = \left[\frac{}{(2S)!(S+q)!(S-q)!}\right]^{1/2} u^{S+q}_{1/2} u^{S-q}_{-1/2} \equiv u^{2S}_q, \tag{1}$$

where the elementary spinors $u_{\pm 1/2}$ are given by

$$\begin{aligned} u_q(\phi,\theta) &= D^{(1/2)}_{1/2,q}(\phi,\theta) \\ &= \begin{cases} cos(\theta/2)e^{i\phi/2}, & q = 1/2, \\ sin(\theta/2)e^{i\phi/2}, & q = -1/2. \end{cases} \end{aligned}$$

There are thus $2S+1$ degenerate single-particle wavefunctions in the lowest LL. In the limit $S- > \infty$, this spherical geometry and a more realistic infinite planar geometry become equivalent. We choose this geometry for the convenient form of the wavefunctions. The $(2S+1)$-particle closed-shell wavefunction is given by

$$L_1(2S+1) = \prod_{i<j}^{2S+1} u(i)\cdot u(j) \tag{2}$$

where the correlations $u(i).u(j) = u_{+1/2}(i)u_{-1/2}(j) - u_{-1/2}(i)u_{+1/2}(j)$ ensure that each pair of electrons' angular momenta couple to $J_{ij} \leq 2S-1$, the maximum allowed for fermions by symmetry. The Laughlin wavefunction for the $1/m$-filled state is a generalization of this wavefunction, namely

$$\begin{aligned} L_m(N) &= \left[\prod_{i<j}^{N} u(i)\cdot u(j)\right]^m \qquad (3) \\ &= \left[\prod_{i<j}^{N} u(i)\cdot u(j)\right] L^{\rm sym}_{m-1}(N), \qquad (4) \end{aligned}$$

where $L^{\rm sym}_{m-1}(N)$ is defined similarly, but is symmetric because m is even.

Thus, the Laughlin wavefunction is a uniformly spread closed-shell wavefunction on N particles. Each spinor should appear $2S$ times in this expression, so the state exists when $2S = m(N-1)$ (for large N, this is a filling of $1/m$).

Ginocchio and Haxton, to extend this to a larger class of filling factors, have considered dividing the N particles into clusters of N_c particles (wavefunctions are later antisymmetrized in over all particles). We define $L^1_d(I) \equiv \prod_{i<j}^{N_c} d(i).d(j)$ where $d_q = (-1)^{1/2+q} d/du_{-q}$. Further, $U_-(I)$ denotes the product of wavefunctions $u_-(i)$ for i in the I'th cluster, and $U_+(I)$ denotes the product of wavefunctions $u_+(i)$ for i in the I'th cluster. With these definitions, we can define a state with N particles in N_c clusters given by

$$\mathcal{A}\left[\prod_{I<J}^{N/N_c} U_-(I)U_+(J) \prod_{I=1}^{N/N_c} L^1_d(I)\right] L^{\rm sym}_{m-1}(N). \tag{5}$$

In this expression, the L_d^1 operators form clusters of the particles in group I by reducing the magnetic flux between each pair by one unit, while the $U(I)$ product spreads the clusters further from one another. This wavefunction is valid for

$$2S = (m-1)(N-1) + \frac{N}{N_c} - N_c.$$

The wavefunction appears to be identical to that found by Jain **??**, but, unlike Jain's wavefunction, does not require numerical projection from higher LLs.

4 Model Hamiltonians

The Ginocchio-Haxton wavefunctions imply that the energy gap between the ground state and excited states at these fillings is the result of introducing new n-body angular momentum correlations into the system. Because of its symmetry, any change in an incompressible state results in the introduction of a less energetically favorable correlation, producing the gap between ground and excited states. It is not, however, immediately clear which correlations are involved at which fillings, particularly in states with $N_c \neq 1$. Thus, we construct operators which "count" the presence of a given correlation. If we assume that this correlation is the dominant cause of the energy gap between the ground state and excited states for p/m-filling, then its spectrum and that of the Hamiltonian should, to a good approximation, differ only by a constant shift and a normalization factor. Moreover, any system less dense than p/m will have zero eigenvalue of the correlation operator (because it is possible to create a state with no such correlations), while eigenvalues for more dense states must be non-zero.

In the case of the Laughlin $1/m$-filled states, the logical correlation to consider is a pair of electrons coupled to angular momentum $2S-m$. The two-body operators $\nabla_{12}^{2k}\delta(\vec{r}_{12})$, for $k = m-2$ or $m-1$ (these two operators differ only by a constant multiplicative factor on the subspace considered), are nonzero only between pairs of electrons coupled to $J_{12} = 2S-m$ or greater. These operators are zero for the $1/m$-filled ground state, as well as for all less dense ground states, but they become nonzero in any more dense state. In fact, for each N their behavior resembles that of an order parameter in a 2nd-order phase transition (Figure 2). Moreover, the eigenvalue spectra of the Coulomb and model Hamiltonians in the 1/3-filled case are very nearly identical (Figure3). Similar results are found for the 1/5-filled state.

A more complex problem, with which we have made some progress, concerns the relevant model Hamiltonians for systems with $N_c > 1$. Because we believe the interactions in question are fundamentally between clusters, it appears that the omdel Hamiltonians must be n-body operators, rather than merely two-body as in the $1/m$-filled case. For example, in the 2/5-filled state ($m = 3, N_c = 2$), interactions occur between clusters of two electrons. Thus, we conjecture that the simplest operator we can construct which will show analogous behavior in the 2/5-filled state is a three-body operator, corresponding to interaction between each cluster of two electrons and each "spectator" electron outside the cluster. We have found that, at $N = 6$, the ground state operator $\nabla_{12}^2\nabla_{13}^2\nabla_{23}^6\delta(\vec{r}_{12})\delta(\vec{r}_{13})$ is indeed zero for 2/5-filling and all less dense systems, and nonzero for more dense systems, and for all N. Further analysis of the behavior of this operator will determine whether it is, in fact, the proper model Hamiltonian for the 2/5-filled state.

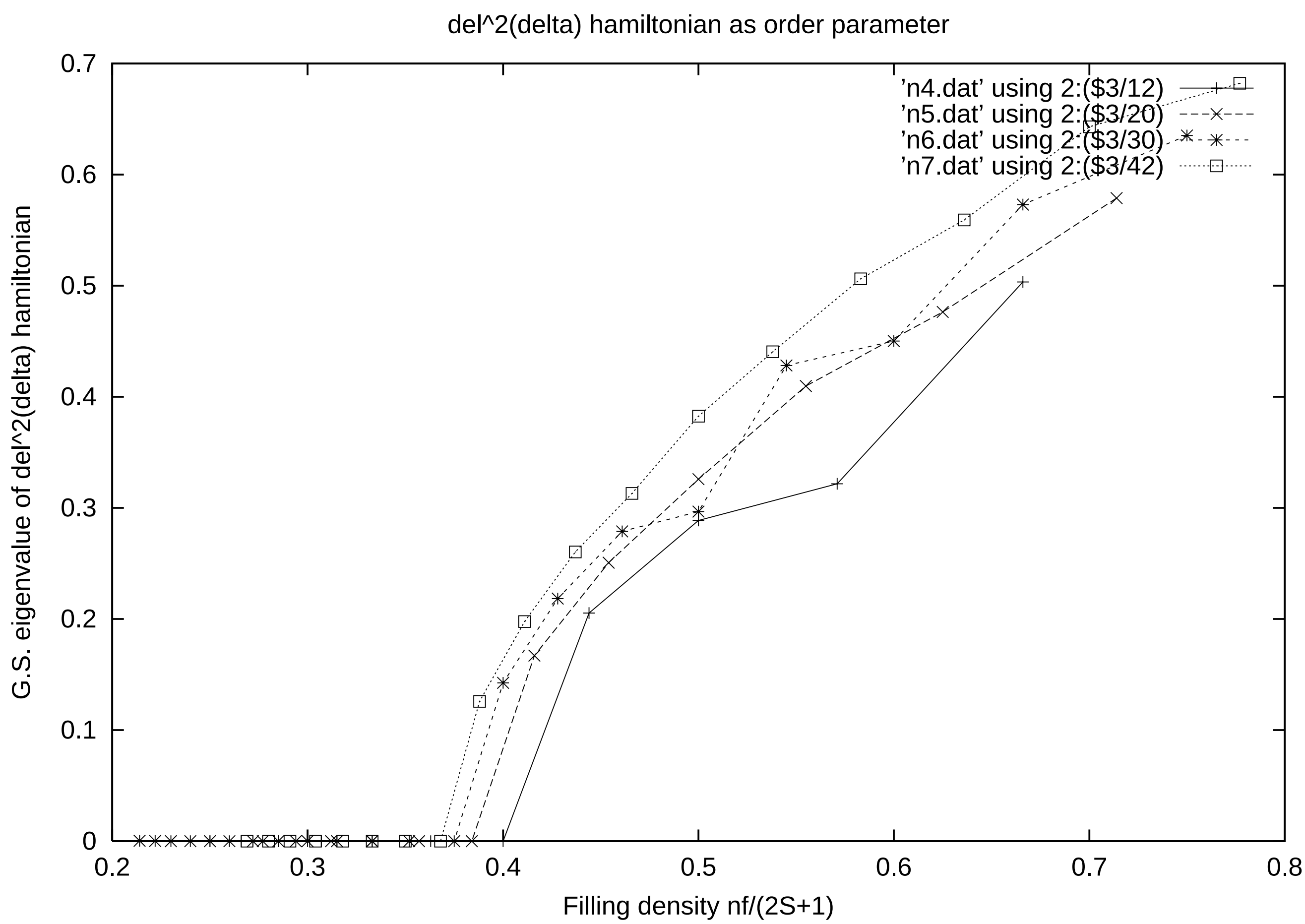

Figure 2: $\nabla^2(\delta(r_{12}))$ ground state eigenvalues for different numbers of particles, filling fractions

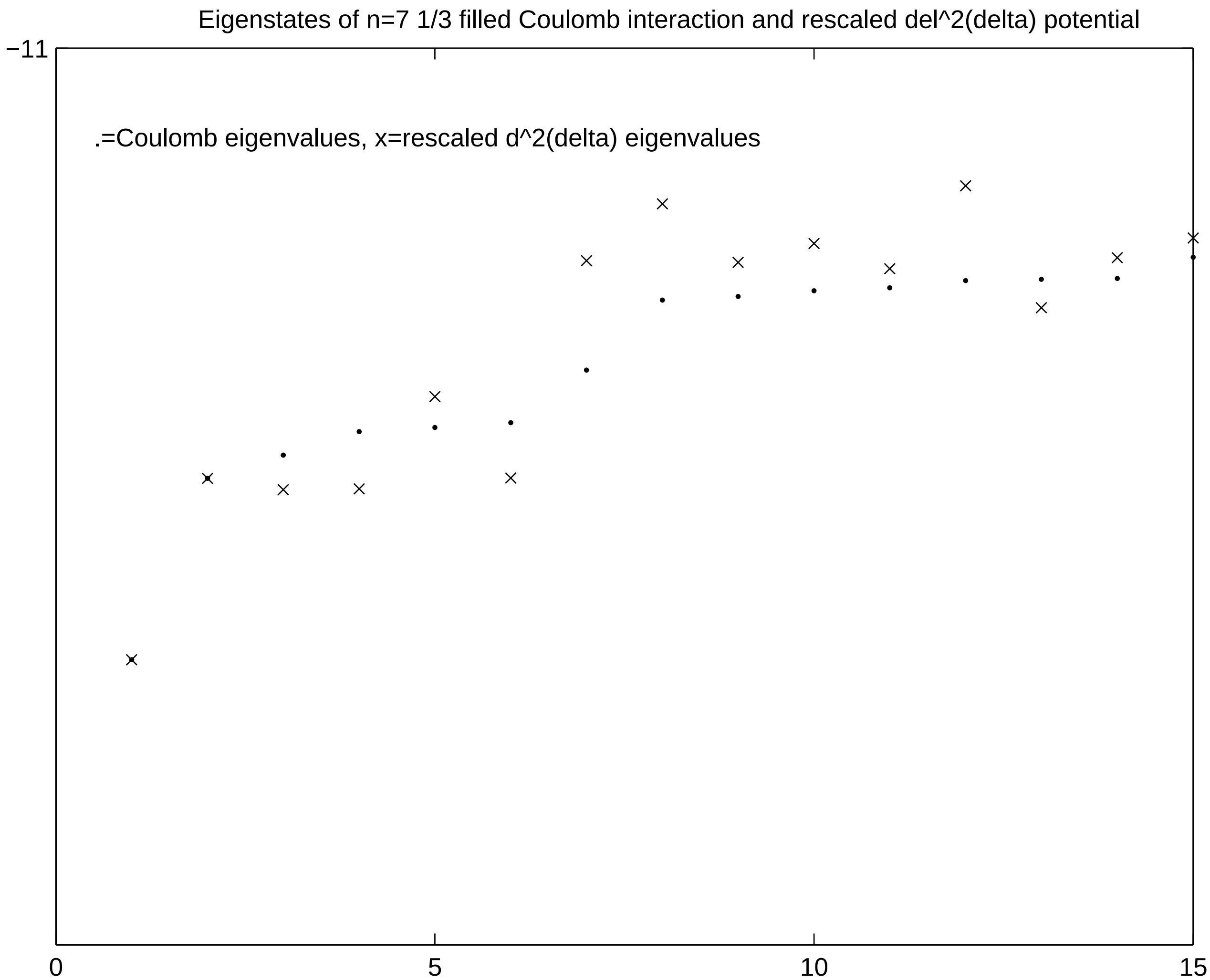

Figure 3: $\nabla^2(\delta(r_{12}))$ eigenvalue spectrum in $N = 8$ 1/3-filled state

5 Conclusion

Much work remains to be done in describing model Hamiltonians for the many FQHE features. Even within the hierarchy described by the Jain and Ginocchio-Haxton wavefunctions, we do not yet understand the general form of the desired operator. Even further results could be obtained when we generalize to the larger class of states involving filling of higher LLs. Moreover, we do not yet understand the significance, if any, of the resemblance of Figure 2 and analogous graphs for other operators to the plots of order parameters near 2nd-order phase transitions. If it is possible to associate a model Hamiltonian with each feature, this will demonstrate precisely what correlations between particles are involved in breaking the symmetry of the ground states at each filling. If the features are, indeed, consequences of multi-electron interactions, this will be clear from the functional form of the model Hamiltonians.

6 Acknowledgements

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