Light Front Quantum Mechanics and the Form Factor

Or: Waiting for Mathematica

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The Light Front Variables

- Light front variables were "invented" by Dirac in 1949

- The Light Front (LF) variables (in natural units, where $c = \hbar = 1$):

$$\begin{aligned} x^+ &= t + z \\ x^- &= t - z \\ \vec{x}_\perp &= x\hat{x} + y\hat{y} \end{aligned}$$

- With momentum variables defined similiarly:

$$m^2 = p_\mu p^\mu = p^0 p^0 - p^3 p^3 - p_\perp^2 = p^+ p^- - p_\perp^2$$

- Which gives a really cool equation

$$\boxed{p^- = \frac{p_\perp^2 + m^2}{p^+}}$$

What Do the LF Variables Mean?

- QM commutation relations are:

$$\left[x^j, p^k\right] = i\delta_{jk} \qquad \text{for} \qquad j, k = 1, 2, 3$$

- Light front commutators are:

$$\begin{aligned} \left[x^+, p^+\right] &= 0 \\ \left[x^-, p^-\right] &= 0. \\ \left[x^+, p^-\right] &= -2i \\ \left[x^-, p^+\right] &= -2i \end{aligned}$$

- So we think of $-\frac{x^-}{2}$ as the third spatial variable and p^+ as the third momentum variable, while $\frac{x^+}{2}$ is likened to time and p^- to energy.

The Two-Particle Proton

- We're thinking about the proton, which is a 3 particle problem...not something I can do yet.

- For the sake of simplicity we work with the 2 particle problem.

- The process we care about:

The Energy Squared Operator

- Variables of particle 1 (or 2) are denoted by a subscript. Big letters denote the sum of both particles, ex. $P^\mu = p_1^\mu + p_2^\mu$.

- To find relativistically correct wave functions we need the energy squared operator, which we call s:

$$s = \frac{p_{1,\perp}^2 + m_1^2}{x} + \frac{p_{2,\perp}^2 + m_2^2}{(1-x)}.$$

- **Another** x?

- The Bjorken variable, $x = \frac{p_1^+}{P^+}$ gives the fraction of the total plus momentum that particle one has.

- Now what?

More Useful Variables

- Assume equal mass particles and work in CM frame, which means

$$m_1 = m_2 \equiv m \qquad \text{and} \qquad \vec{p}_1 = -\vec{p}_2 \equiv \vec{p}.$$

- Using a new variable $\vec{p}_\perp = (1-x)\vec{p}_{1,\perp} - x\vec{p}_{2,\perp}$ we can rewrite s:

$$s = \frac{p_\perp^2 + m^2}{x(1-x)}$$

- Now define

$$\begin{aligned} p_3 &= (x - \frac{1}{2})\sqrt{\frac{p_\perp^2 + m^2}{x(1-x)}} \\ p &= \sqrt{p_\perp^2 + (p_3)^2} \\ E &= \sqrt{p^2 + m^2} \end{aligned}$$

A Relativistic Equation for Ψ

- A starting equation for relativistic QM is $M^2\Psi = s\Psi + W\Psi$, where W is the interaction.

- Letting $M = 2m - \epsilon$ we can rewrite this, using our nifty variables, as:

$$(\frac{\epsilon^2}{4} - m\epsilon)\Psi = (p^2 + V)\Psi$$

- Our eigenvalue equation...doesn't it look a lot like the Schrödinger equation?

Plan of Attack

- Model quark interactions as simple central potentials studied in elementary QM

- Use $\Psi(r)$ solution to SE to find $\Psi(p)$

- Make substitution in $\Psi(p)$

$$p^2 \rightarrow p_\perp^2 + (p_3)^2 = \frac{p_\perp^2 + m^2}{4x(1-x)} - m^2$$

- We now have a relativistic Ψ.

- Set constants in Ψ so that we get $< r^2 >= 1.7 fm^2$

The Form Factor-What and Why?

- $F(q_\perp^2)$ tells us the probability that the system will stay in a state Ψ after one of the constituent quarks is given a momentum boost $\vec{q}$.

- It is an important quantity because F can be measured experimentally since:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{pc}} |F(q^2)|^2$$

Three Different equations for F

- The LF equation (it's relativistically correct, mind you):

$$F(q_\perp^2) = \int \frac{\mathrm{d}^2 p_\perp \mathrm{d}x}{x(1-x)} \Psi^*(\vec{p}_\perp + (1-x)\vec{q}_\perp, x)\Psi(\vec{p}_\perp, x)$$

- The non-relativistic equation (the following Ψ's are solutions to the SE):

$$F_{nr}(q^2) = \int \mathrm{d}^3 r |\Psi(\vec{r})|^2 e^{\frac{i\vec{q}\cdot\vec{r}}{2}}$$

- F in the Breit Frame:

$$F_{bf}(q^2) = \frac{1}{\tau} \int \mathrm{d}x\mathrm{d}y\mathrm{d}z |\Psi(x,y,z)|^2 e^{\frac{iqz}{2\tau}}$$

- Where $\tau = \sqrt{1 + \frac{q^2}{4M^2}}$

- Note: Normalization requires F(0)=1

The Harmonic Oscillator

- The ground state solution to the SE with $V(r) = \frac{1}{2}m\omega^2 r^2$ is $\psi(p) = Ae^{-\frac{b^2p^2}{2}}$.

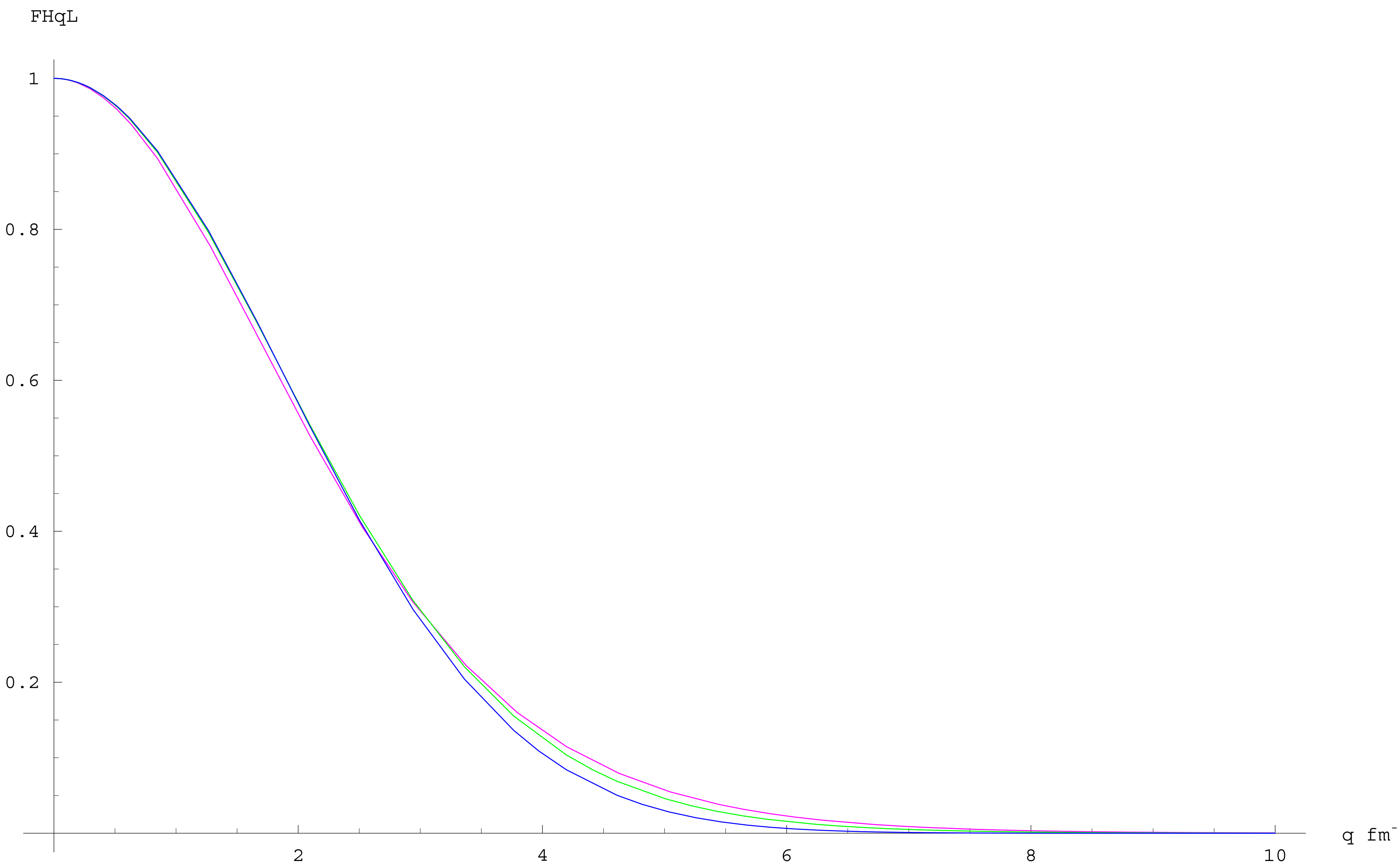
- $F_{nr}(q^2) = e^{-\frac{q^2}{16b^2}}$

- $F(q^2) = N \int_0^1 \mathrm{d}x e^{\frac{-4m^2(1-2x)^2+q^2(1-x)^2}{16b^2(1-x)x}} e^{\frac{-(1-x)q^2}{8b^2x}}$

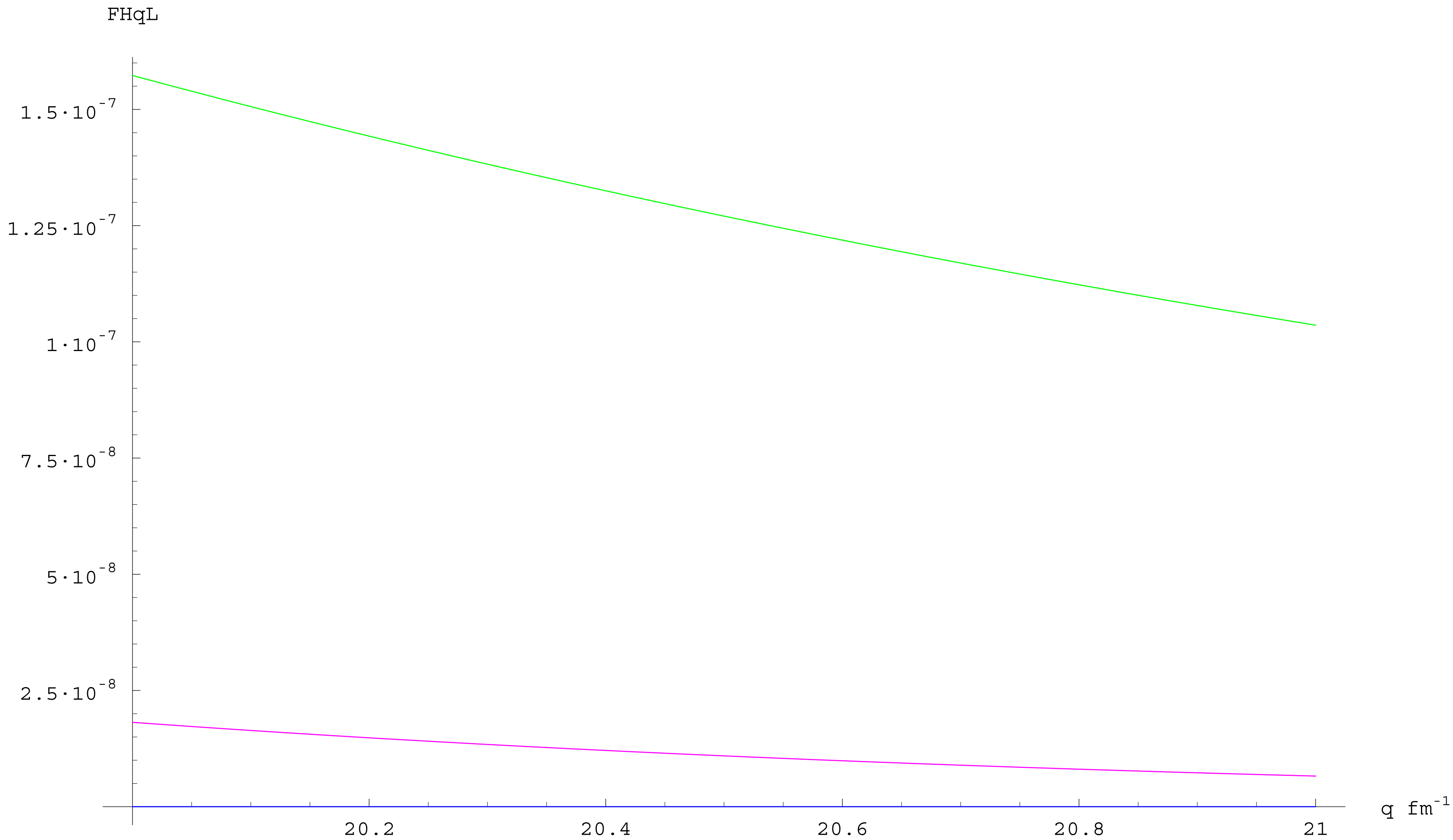
- $F_{bf} = \frac{e^{-\frac{q^2}{16b^2(1+\frac{q^2}{4M^2})}}}{\sqrt{1+\frac{q^2}{4M^2}}}$

- Note: $\hbar c = 197.329 MeV fm \equiv 1$.

Plot of the three F's



HO Form Factors at High q

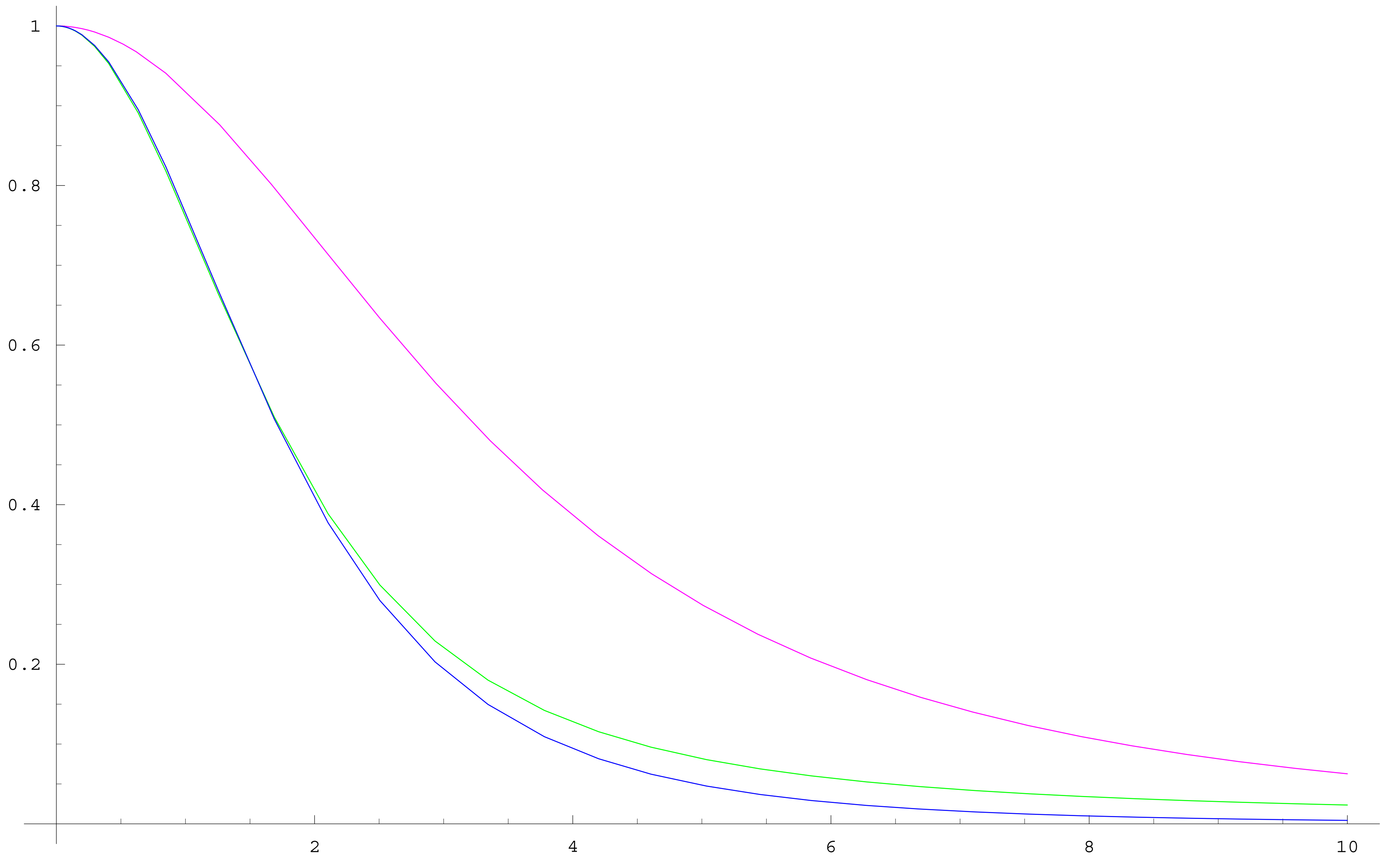


The Coulombic Potential

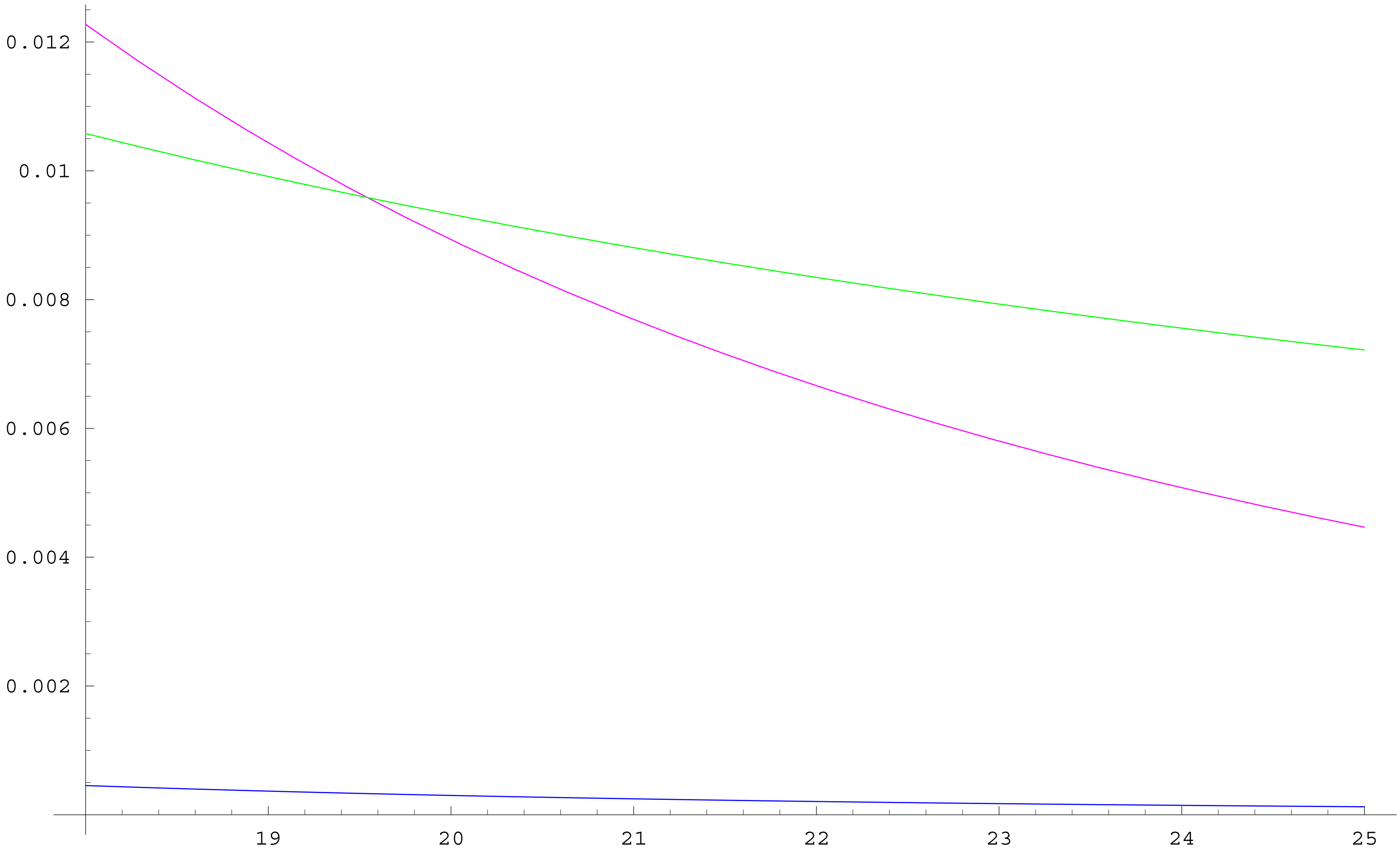
- Potential: $V(r) = -\frac{e^2}{r}$

- Ground state solution: $\Psi(p) = B\frac{1}{(1+a^2p^2)^2}$

Coulomb F's



High q



The NR limit of F

- In the limit $\frac{p^2}{m^2} << 1$ then $F \rightarrow F_{nr}$
- Keeping more terms gives a better $F_{nr}(q^2)$
- Work in progress...sorry, no pictures yet.

The Search for an Inverse Transformation

- It would be nice to be able to calculate $|\Psi(r)|^2$ given $F(q^2)$.

- I've been working on finding a way

- So far:

$$\tilde{F}(\vec{b}) = \int_0^1 \mathrm{d}x |\Psi(\frac{\sqrt{x(1-x)}\vec{b}}{1-x}, x)|^2$$

- More work is being done...might not be possible?

Conclusions: What I've gained

- LF coordinates can simplify relativistic QM
- Now know what is and how to calculate the form factor
- Calculated lots of form factors
- Got to work on problems that are unsolved
- Learned how to use Mathematica
- Learned Latex