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References: This code of conduct is based heavily on that of the [INT](#) and the [APS](#). We are also grateful to Roxanne Springer for valuable discussion and guidance.

# Universal critical dynamics in the QCD phase diagram

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## Based on

JR, Yunxin Ye, Sören Schlichting, and Lorenz von Smekal, arXiv:2403.04573 & 2409.14470

JR and Lorenz von Smekal, JHEP **10**, 065 (2023)

**Rising Researchers Seminar Series**, December 10, 2024

# Motivation: matter under extreme conditions

CRC-TR 211

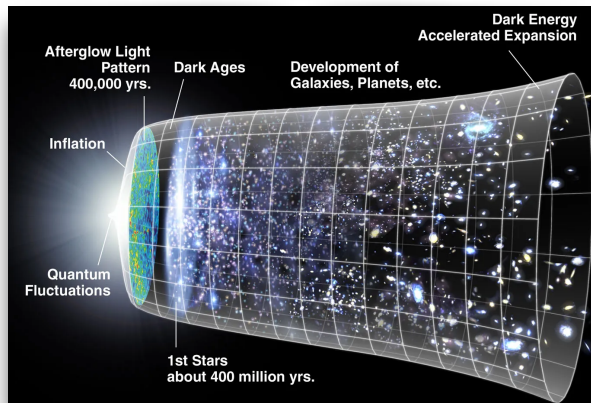
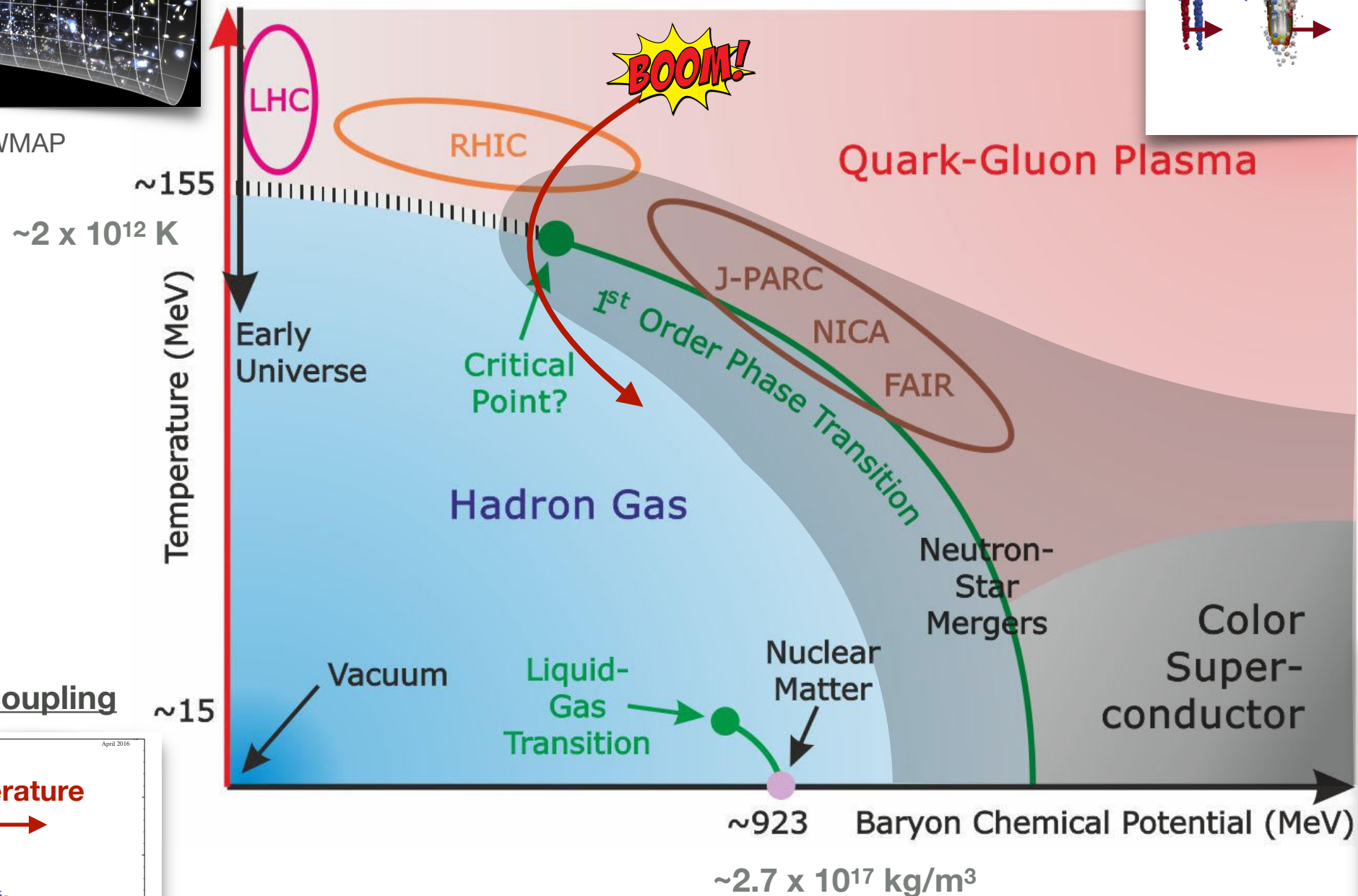


Image: WMAP

## Phase structure of QCD

$\sim 10 \mu\text{s}$  after Big Bang



## Heavy-ion collision

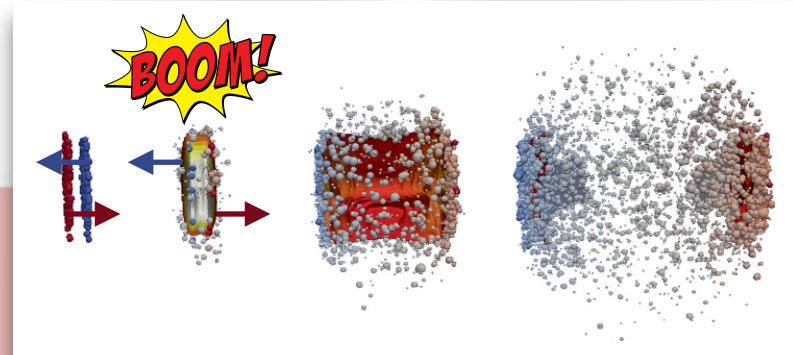
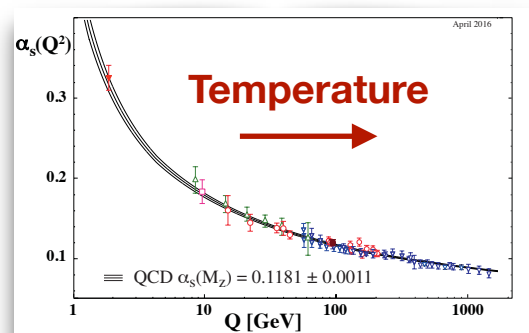


Image: MADAL collaboration

## Running coupling



PDG, Review of Particle Physics (2020)

Figure: A. Steidl, CRC-TR 211

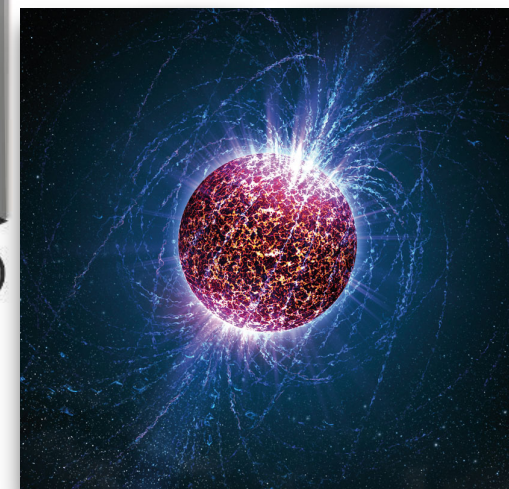
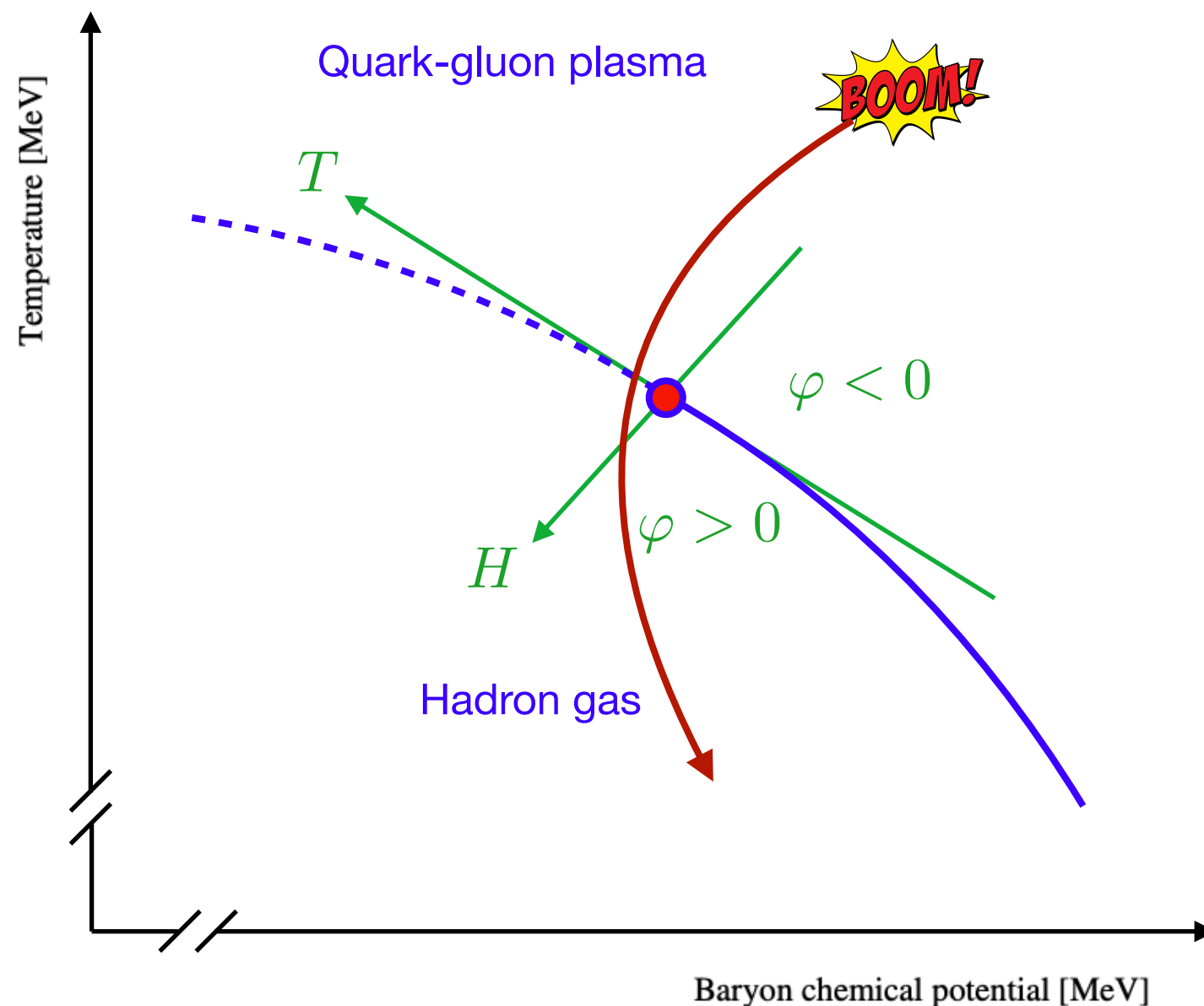


Image: Penn State University

1. Introduction to critical dynamics
2. Critical dynamics near the QCD critical point
3. Critical dynamics near the chiral phase transition



## 1. Introduction to critical dynamics



**Chiral order parameter:**  $\varphi \sim \bar{q}q - \langle \bar{q}q \rangle_c$

**Critical point:** Correlation length  $\xi$  diverges

- Microscopic details irrelevant
- Critical properties **fully** determined by symmetries, number of dimensions, ...
- For example, observables behave as power laws with **universal exponents**:

$$\xi = f^+ \tau^{-\nu} + \dots$$

$\nu \approx 0.63$

reduced temperature  
 $\tau = (T - T_c)/T_c$

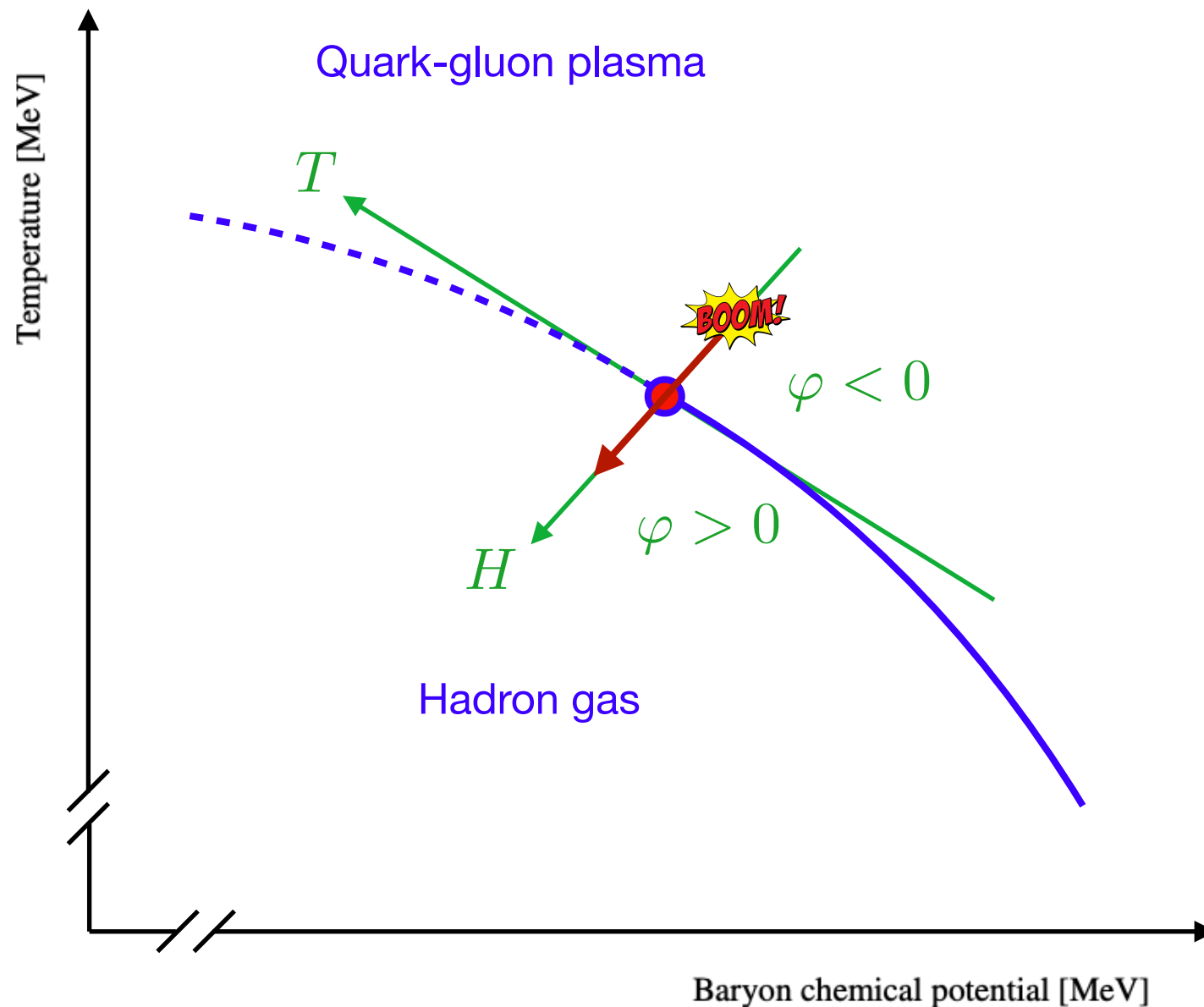
→ **map to  $Z_2$  (Ising) model**

see Parotto et al., PRC **101**, 034901 (2020)

**Ising model**  $\Rightarrow$  equilibrium universal properties of **QCD's critical point**

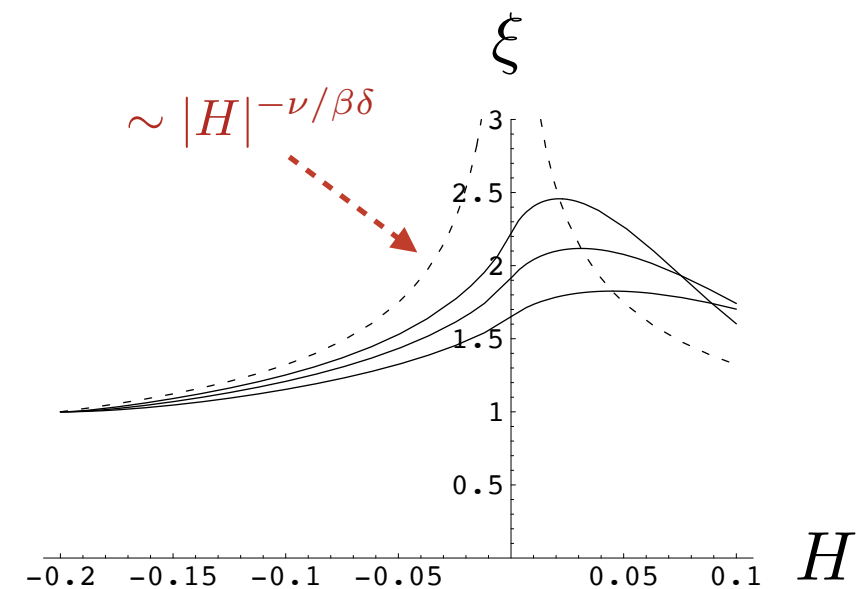
but we want **dynamics...**

Stephanov, Rajagopal & Shuryak,  
PRL **81** (1998) 4816-4819



**Chiral order parameter:**  $\varphi \sim \bar{q}q - \langle \bar{q}q \rangle_c$

Time evolution of correlation length  $\xi$ :



Berdnikov & Rajagopal, PRD **61**, 105017 (2000)

**critical slowing down:**

$\xi_t \sim \xi^z$

correlation time

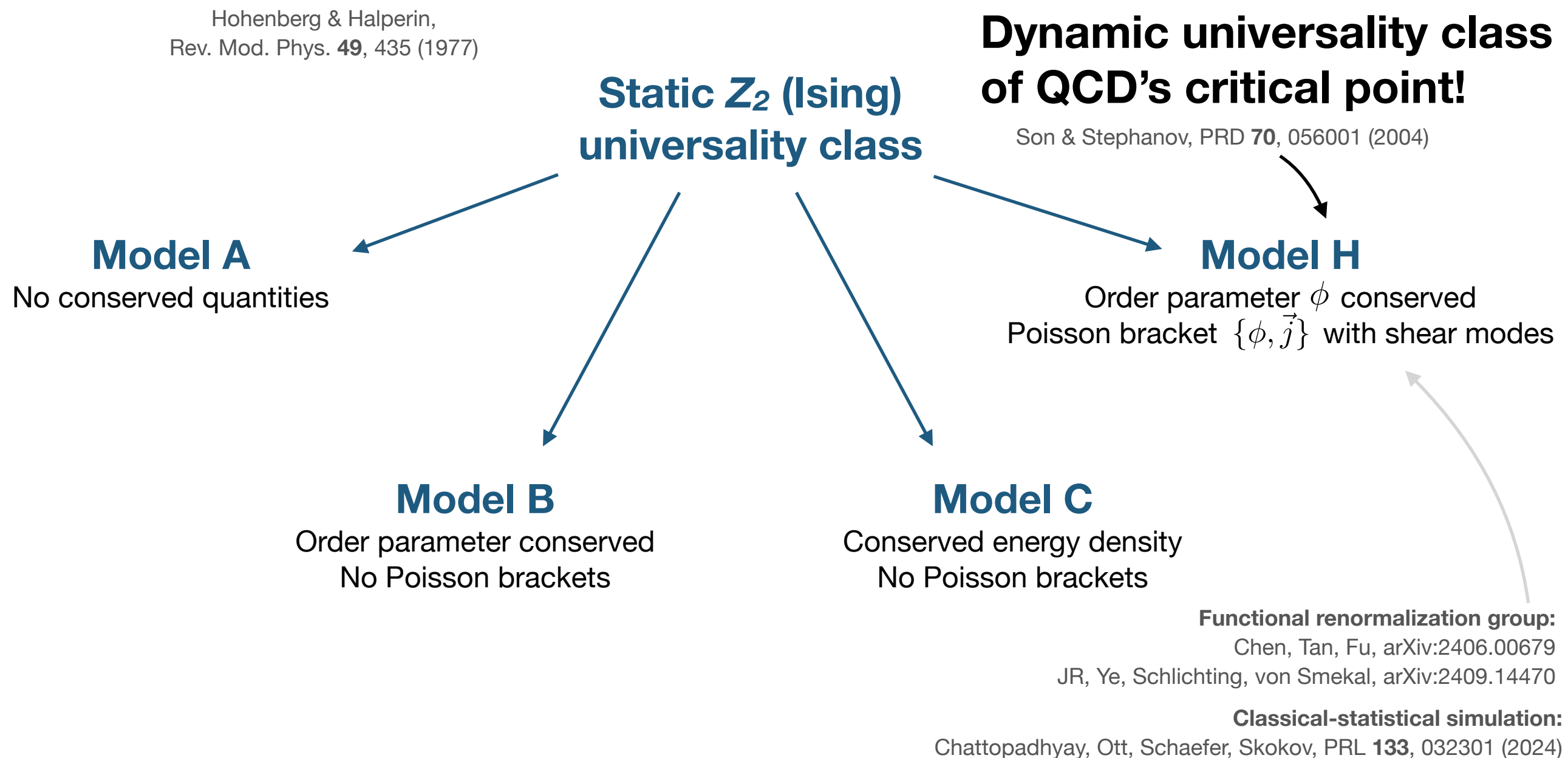
correlation length

**dynamic critical exponent**

Hohenberg & Halperin,  
Rev. Mod. Phys. **49**, 435 (1977)

$z$  depends on conservation laws, Poisson bracket relations, ...

→ Static universality classes split up into **dynamic** universality classes:



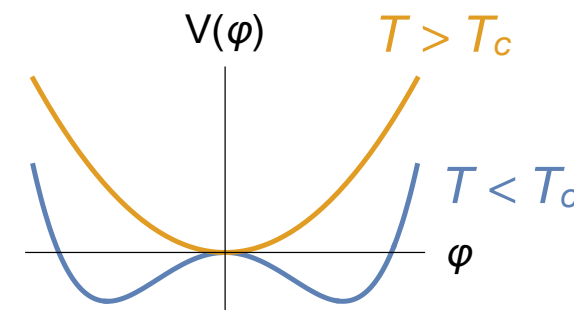
**Order parameter:**  $\varphi \sim \bar{q}q - \langle \bar{q}q \rangle_c$  (chiral condensate)

**Model A**

$$z = 2 + c\eta_{\perp}$$

- **Statics:** Landau-Ginzburg-Wilson (LGW) free energy

$$F = \int d^d x \left\{ \frac{1}{2} (\vec{\nabla} \varphi)^2 + V(\varphi) \right\}$$



- **Dynamics:** Langevin equation of motion

$$\frac{\partial \varphi}{\partial t} = -\Gamma_0 \frac{\delta F}{\delta \varphi} + \xi$$

relaxation to minimal  
free energy

random forces from  
thermal fluctuations

→ modelled as Gaussian white noise:

$$\langle \xi(t, \vec{x}) \rangle = 0$$

$$\langle \xi(t, \vec{x}) \xi(t', \vec{x}') \rangle = 2\Gamma_0 T \delta(t - t') \delta(\vec{x} - \vec{x}')$$

Einstein relation

- **If** there are no (other) conservation laws → **Model A**
- **But:** in QCD there are!



## Model B

$$z = 4 - \eta_{\perp}$$

Son & Stephanov,  
PRD **70**, 056001 (2004)

**Order parameter:**  $\varphi \sim \bar{q}q - \langle \bar{q}q \rangle_c$  (chiral condensate)

**coupled to:**  $n \sim \bar{q}\gamma^0 q - \langle \bar{q}\gamma^0 q \rangle_c$  (baryon density)

- **Statics:** LGW free energy

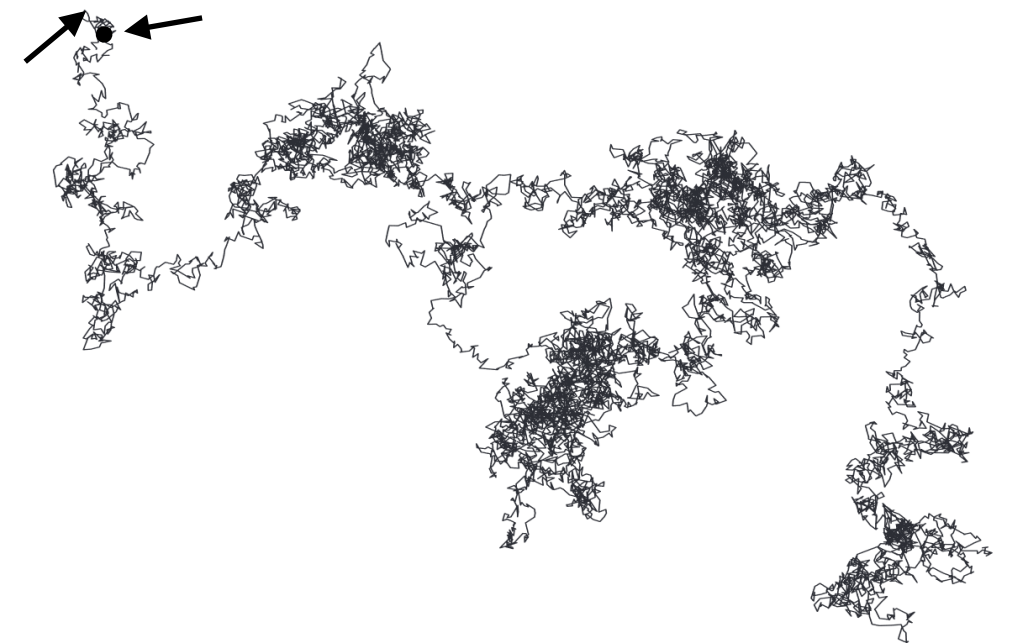
$$F = \int d^d x \left\{ \frac{1}{2} (\vec{\nabla} \varphi)^2 + V(\varphi) + B\varphi n + \frac{n^2}{2\chi_0} \right\}$$

- **Dynamics:** Langevin equations of motion

$$\begin{aligned} \partial_t \varphi &= -\Gamma_0 \frac{\delta F}{\delta \varphi} + \xi \\ \partial_t n &= \bar{\lambda} \vec{\nabla}^2 \frac{\delta F}{\delta n} + \vec{\nabla} \cdot \vec{\zeta} \end{aligned}$$

Gaussian white noises

**diffusive!**



- Critical dynamics dominated by diffusion  $\leadsto$  **Model B**

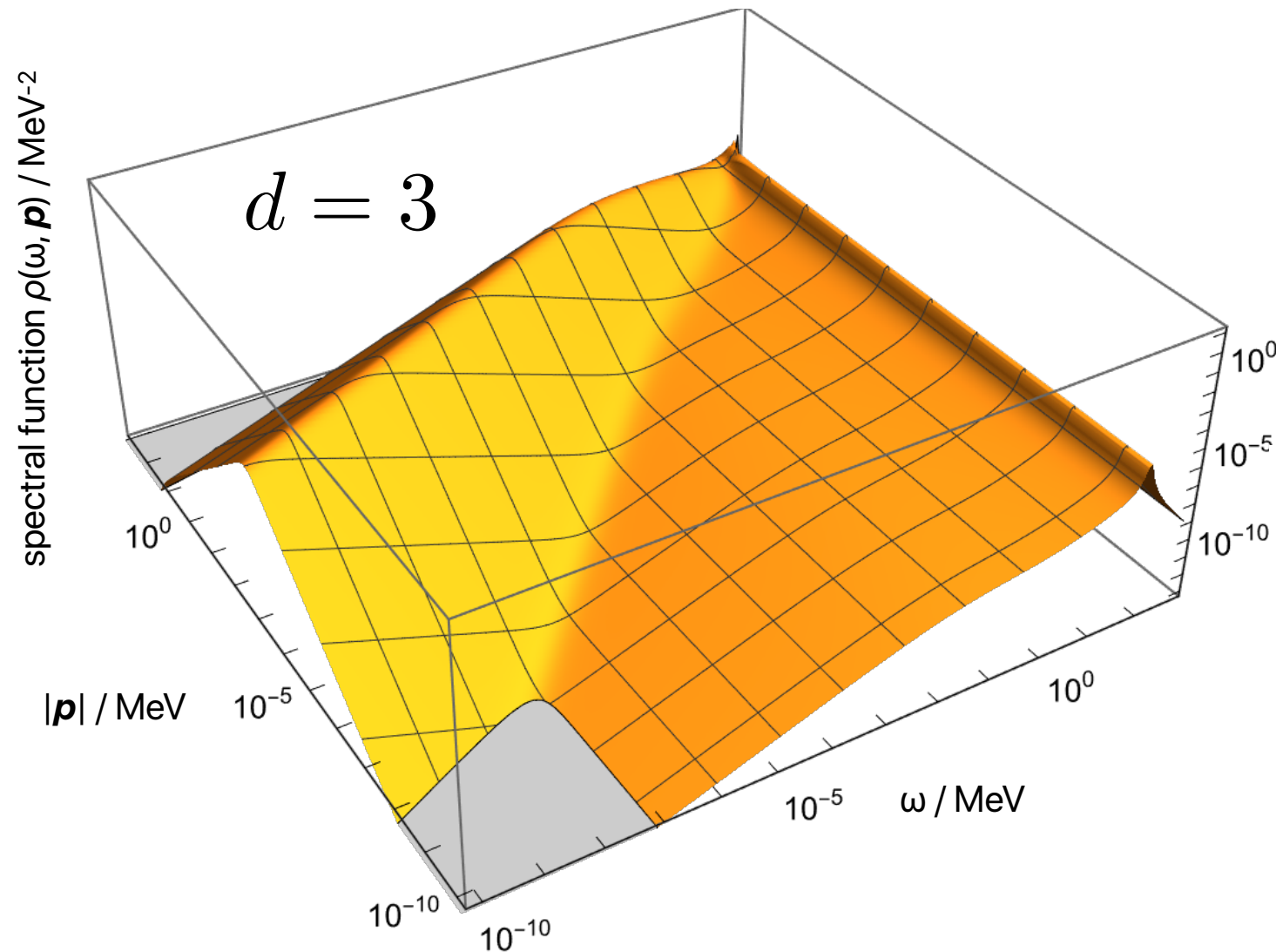
P. Mörters, Y. Peres, *Brownian Motion*  
(Cambridge University Press, 2010)

- spectral function of chiral order parameter:

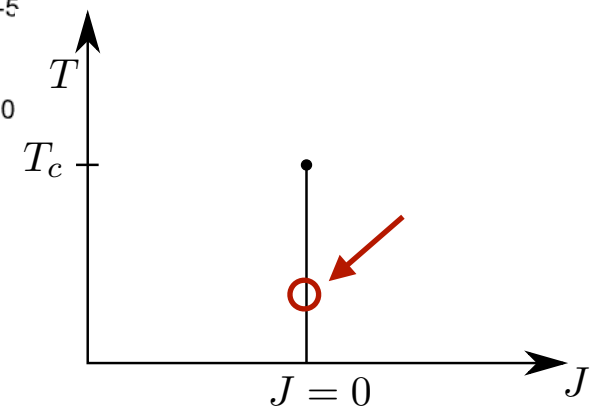
$$\rho(\omega, \mathbf{p}) \equiv \frac{1}{2\pi i} \int dt e^{i\omega t} \int d^d x e^{-i\mathbf{p} \cdot \mathbf{x}} i \langle [\varphi(t, \mathbf{x}), \varphi(0, \mathbf{0})] \rangle$$

**Model B**

$$z = 4 - \eta_{\perp}$$



$\tau = -0.356$   
(non-critical)



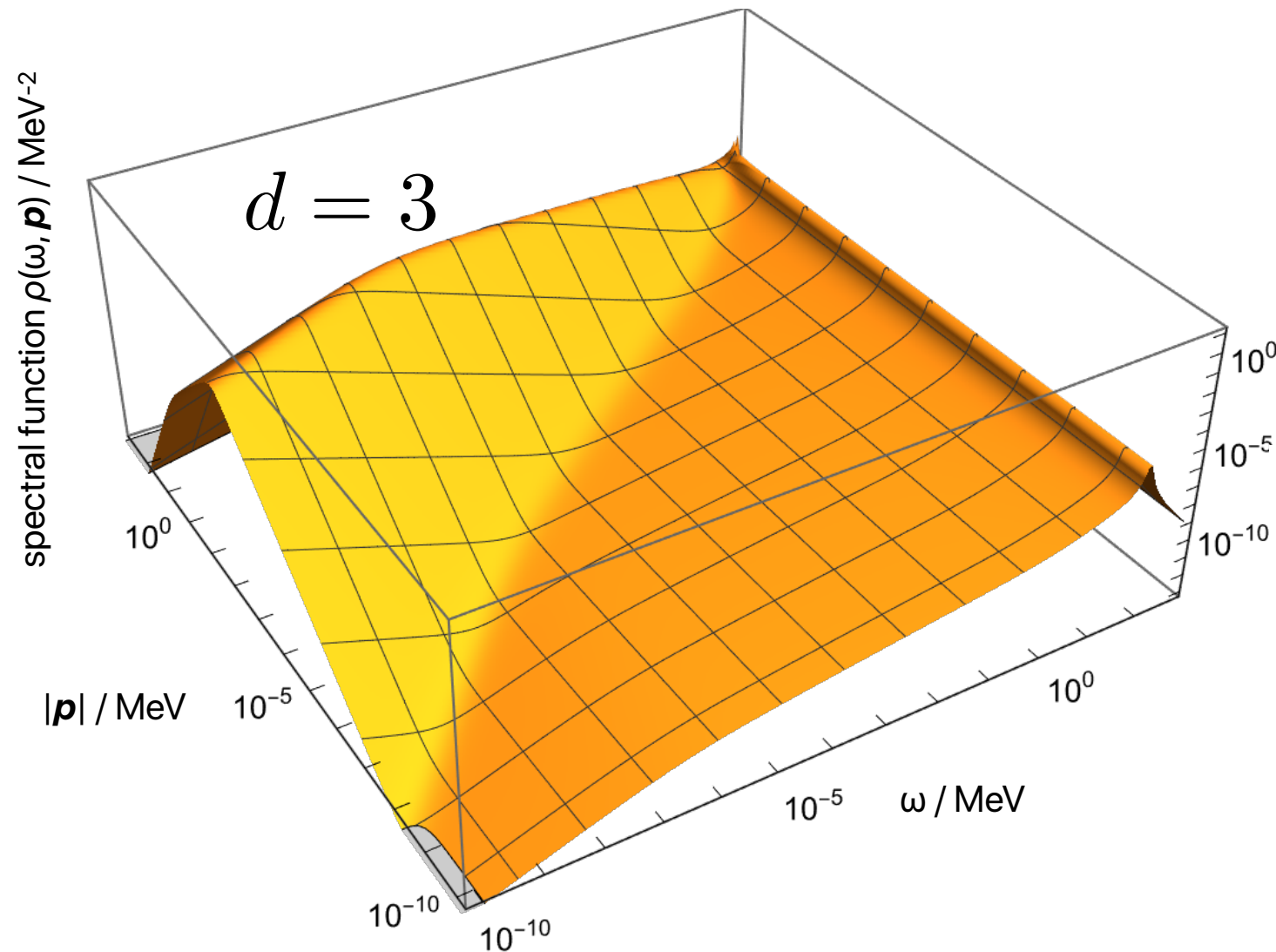
obtained with real-time functional renormalization group  
JR, L. von Smekal, JHEP **10**, 065 (2023)

- spectral function of chiral order parameter:

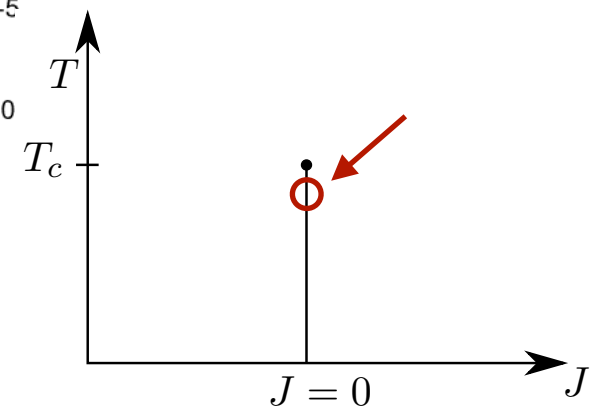
$$\rho(\omega, \mathbf{p}) \equiv \frac{1}{2\pi i} \int dt e^{i\omega t} \int d^d x e^{-i\mathbf{p} \cdot \mathbf{x}} i \langle [\varphi(t, \mathbf{x}), \varphi(0, \mathbf{0})] \rangle$$

**Model B**

$$z = 4 - \eta_{\perp}$$



$\tau = -0.0021$   
(near criticality)



obtained with real-time functional renormalization group  
JR, L. von Smekal, JHEP **10**, 065 (2023)

- spectral function of chiral order parameter:

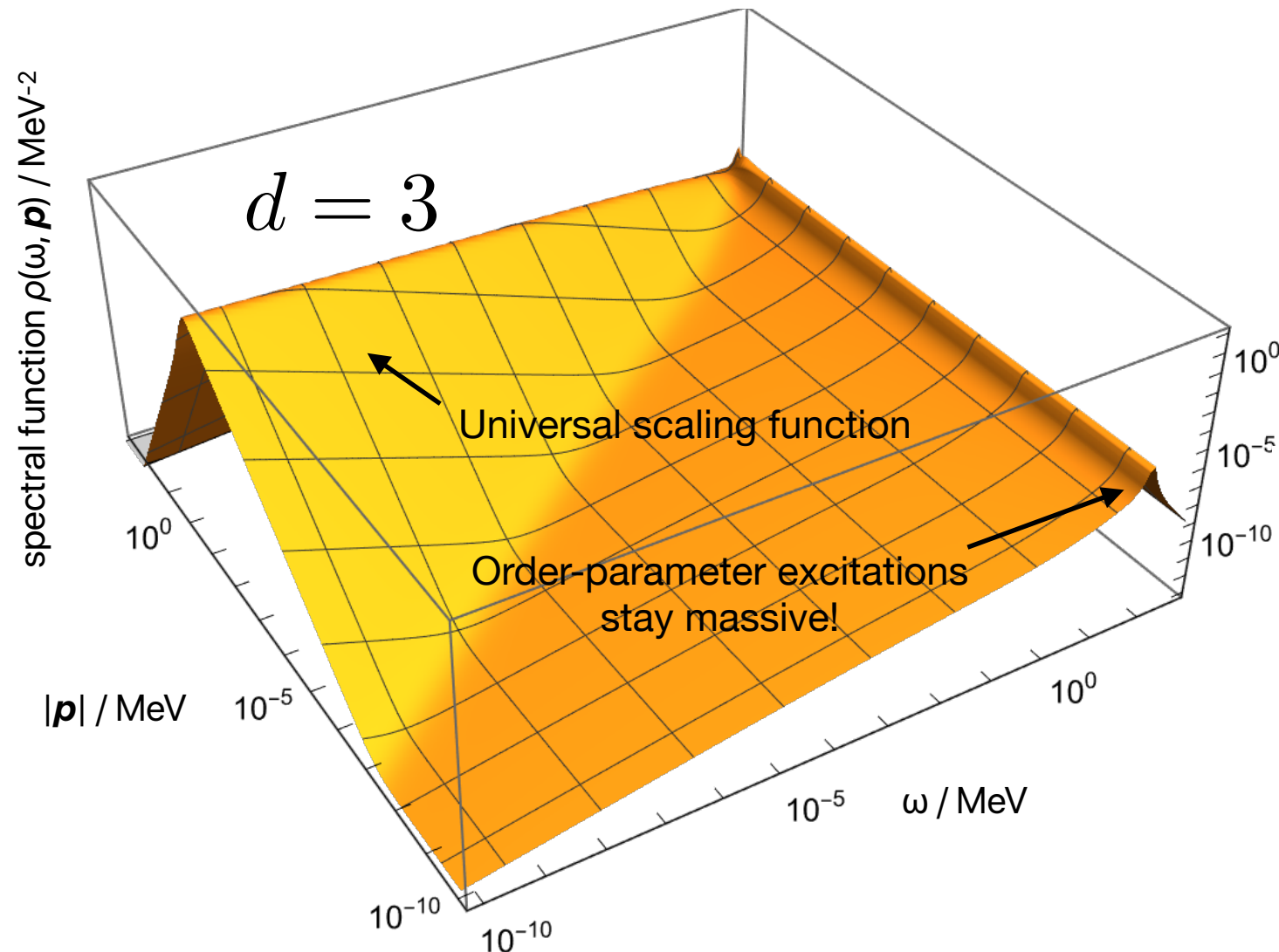
$$\rho(\omega, \mathbf{p}) \equiv \frac{1}{2\pi i} \int dt e^{i\omega t} \int d^d x e^{-i\mathbf{p} \cdot \mathbf{x}} i \langle [\varphi(t, \mathbf{x}), \varphi(0, \mathbf{0})] \rangle$$

**Model B**

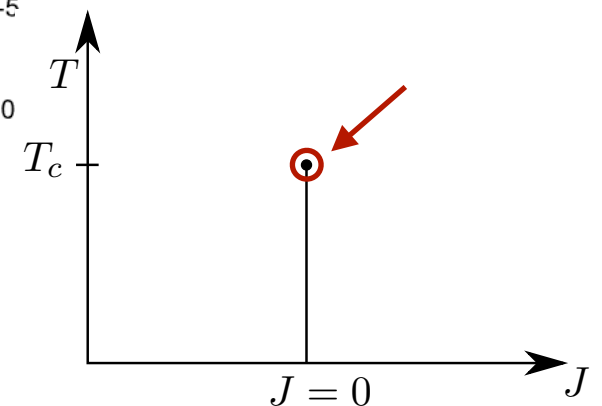
$$z = 4 - \eta_{\perp}$$

$$\omega \sim |\mathbf{p}|^z$$

$z = 4$   
(here)



$\tau \approx 0$   
(critical)



Universal scaling functions computed in:

Model A, C: Schweitzer, Schlichting, von Smekal, Nucl. Phys. B **960**, 115165 (2020)

Model B, BC: Schweitzer, Schlichting, von Smekal, Nucl. Phys. B **984**, 115944 (2022)

obtained with real-time functional renormalization group  
JR, L. von Smekal, JHEP **10**, 065 (2023)

## 2. Critical dynamics near the QCD critical point

Son & Stephanov, PRD **70**, 056001 (2004)

- integrate out 'fast' chiral order parameter  
 $\leadsto$  form effective (low-frequency) theory for baryon density  $n$
- Now introduce coupling to energy-momentum tensor:  $T^{00} \equiv \varepsilon$      $T^{0l} \equiv j^l$
- Five conserved charges  $\leadsto$  five eigenmodes:
  - energy density  $\varepsilon$
  - momentum density  $\vec{j}$
  - baryon density  $n$

$\left. \begin{array}{l} \varepsilon \\ \vec{j} \\ n \end{array} \right\} \begin{array}{l} \text{fast!} \\ \text{two sound modes } \omega \sim |\vec{p}| \\ \text{two shear modes } \omega \sim \vec{p}^2 \\ \text{diffusion of entropy per baryon } s/n \\ \text{(at constant pressure) } \omega \sim \vec{p}^2 \end{array}$
- $\leadsto$  **three modes relevant for critical dynamics:**
- Order parameter  $\phi \sim \delta(s/n)$ , transverse part  $\vec{j}$  of momentum density

**Order parameter:**  $\phi \sim \delta(s/n)$  coupled to  $\vec{j} \equiv \vec{j}_\perp$   
 entropy per baryon                      transverse momentum density

**Statics:**  $Z_2$  LGW free energy

$$F = \int d^d x \left\{ \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{24} \phi^4 + \frac{\vec{j}^2}{2\rho} \right\} \quad \text{(transverse) fluid velocity}$$

$$\leadsto \vec{v} = \frac{\delta F}{\delta \vec{j}} = \frac{\vec{j}}{\rho}$$

**Dynamics:**

- Reversible (ideal) part:** Poisson bracket technique  
 conserve  $F$  exactly

see Dzyaloshinskii and Volovick,  
 Annals of Physics **125**, 67 (1980)

$-\vec{v} \cdot \vec{\nabla} \phi$  **advection**

$$\{\phi(\vec{x}), j_l(\vec{x}')\} = \phi(\vec{x}') \frac{\partial}{\partial x'_l} \delta(\vec{x} - \vec{x}')$$

$$\frac{\partial \phi}{\partial t} = \{\phi, F\} = \underbrace{\{\phi, \vec{j}\} \cdot \frac{\delta F}{\delta \vec{j}}}_{\text{reversible mode couplings}}$$

$$\frac{\partial j_l}{\partial t} = \mathcal{T}[\{j_l, F\}] = \mathcal{T} \left[ \underbrace{\{j_l, \phi\} \frac{\delta F}{\delta \phi}}_{\text{reversibility}} + \underbrace{\{j_l, j_m\} \frac{\delta F}{\delta j_m}}_{\text{convection}} \right]$$

$-(\partial_l \phi) \vec{\nabla}^2 \phi$  **reversibility**                       $-(\vec{v} \cdot \vec{\nabla}) j_l$  **convection**

**'reversible mode couplings'**



- Beyond ideal evolution: **dissipative** terms

see Landau & Lifshitz, *Statistical Physics Part 1* (Butterworth-Heinemann, Oxford, 1980)

order-parameter mobility

$$\frac{\partial \phi}{\partial t} = \underbrace{g\{\phi, \vec{j}\}}_{\text{mode coupling}} \cdot \frac{\delta F}{\delta \vec{j}} + \underbrace{\sigma \vec{\nabla}^2 \frac{\delta F}{\delta \phi}}_{\text{shear viscosity}} + \underbrace{\theta}_{\text{stochastic forces}}$$

mode coupling

$$\frac{\partial j_l}{\partial t} = \mathcal{T}_{lm} \left[ g\{j_m, \phi\} \frac{\delta F}{\delta \phi} + g\{j_m, j_n\} \frac{\delta F}{\delta j_n} + \underbrace{\eta \vec{\nabla}^2 \frac{\delta F}{\delta j_m}}_{\text{shear viscosity}} \right] + \underbrace{\xi_l}_{\text{stochastic forces}}$$

shear viscosity

• diffusion

• **stochastic forces**

$$\sim P_{eq} \sim e^{-F[\phi, \vec{j}]/T}$$

## Model H

$$z = 4 - \eta_{\perp} - x_{\sigma}$$

Liquid-gas critical point in pure fluid

- connected by fluctuation-dissipation relations:

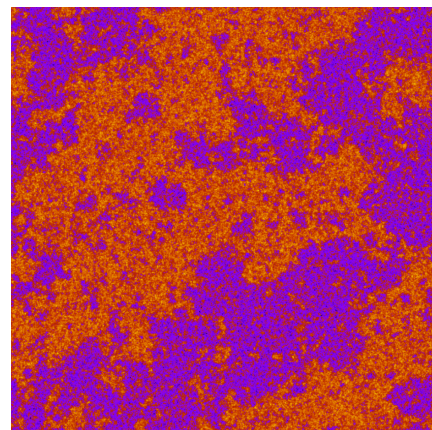
$$\langle \theta(t, \vec{x}) \theta(t', \vec{x}') \rangle = -2\sigma \vec{\nabla}^2 \delta(\vec{x} - \vec{x}') \delta(t - t'),$$

$$\langle \xi_l(t, \vec{x}) \xi_m(t', \vec{x}') \rangle = -2\eta \mathcal{T}_{lm} \vec{\nabla}^2 \delta(\vec{x} - \vec{x}') \delta(t - t').$$

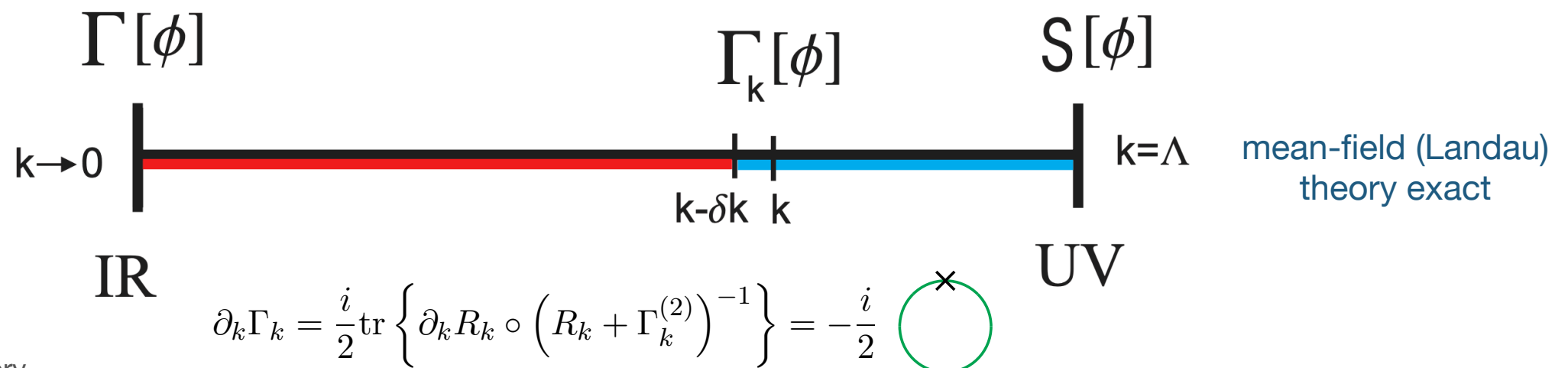
- Difficulty: large long-wavelength ('infrared') fluctuations at critical point
- Wilson: introduce **infrared cutoff** to suppress (critical) fluctuations  $p \lesssim k$

$$S \rightarrow S + \Delta S_k \quad \Delta S_k = \int_{xx'} \phi(x) R_k(x, x') \phi(x')$$

integrate (critical) fluctuations  
'momentum shell by momentum shell'

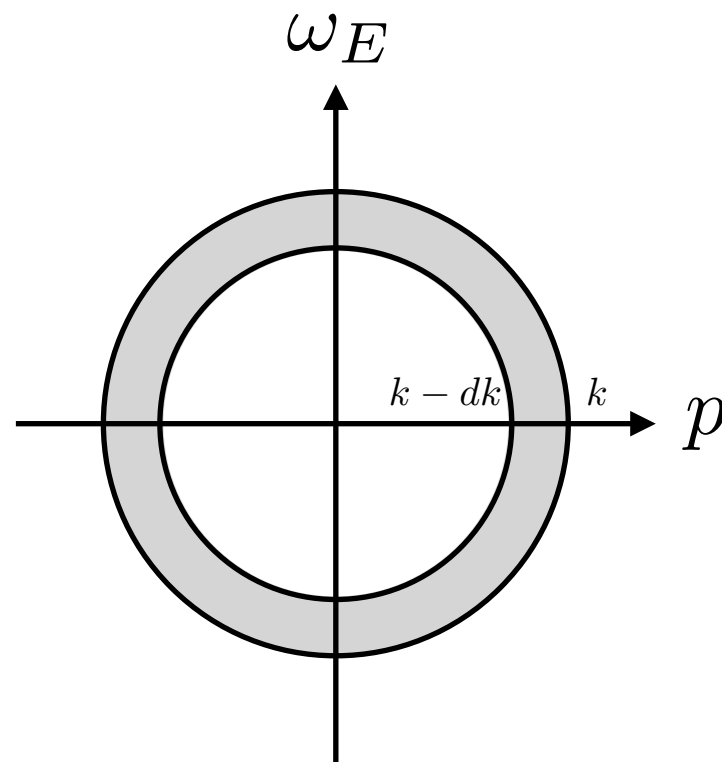


Critical configuration of 2d  $\phi^4$  theory  
(Courtesy of D. Schweitzer)



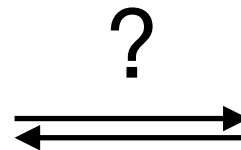
C. Wetterich, Phys. Lett. B **301** (1993) 90-94

- Criticality: Scale invariance  $\leadsto$  RG fixed point  $k \leftrightarrow \xi^{-1}$

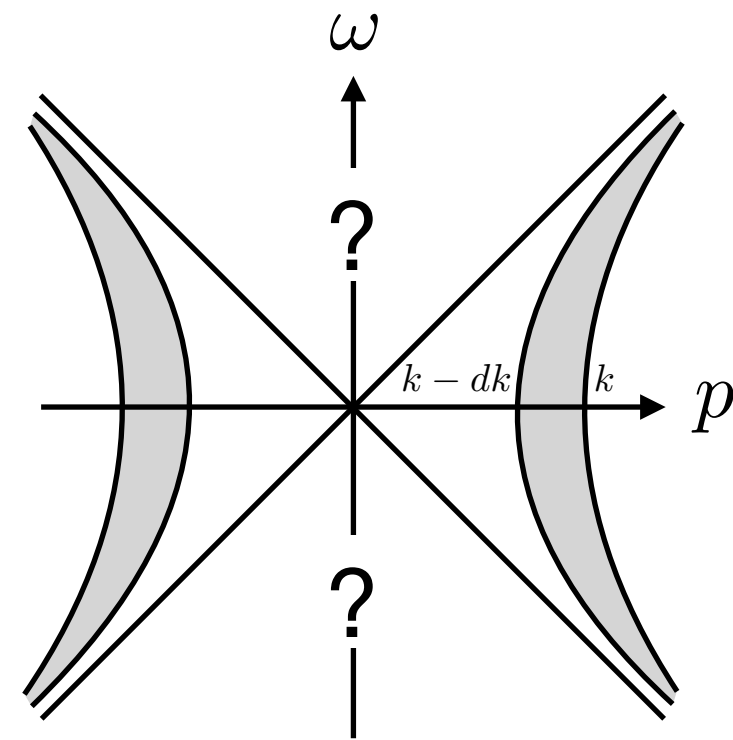


Wilsonian renormalization in  
Euclidean spacetime

**Conceptually straightforward:**  
imaginary time, all changes adiabatic



vs.



Wilsonian renormalization in  
Minkowski spacetime

**Conceptually intricate:**  
timelike momenta  $\sim$  causal structure?

**Problem:** Frequency-dependent regulators  
usually **violate causal structure**

**Solution:** Interpret regulator as  
fictitious scale-dependent heat bath  
 $\Rightarrow$  **Spectral representation**

JR, L. von Smekal, JHEP **10**, 065 (2023)

Regulator has spectral representation

⇓ JR, L. von Smekal, JHEP **10**, 065 (2023)

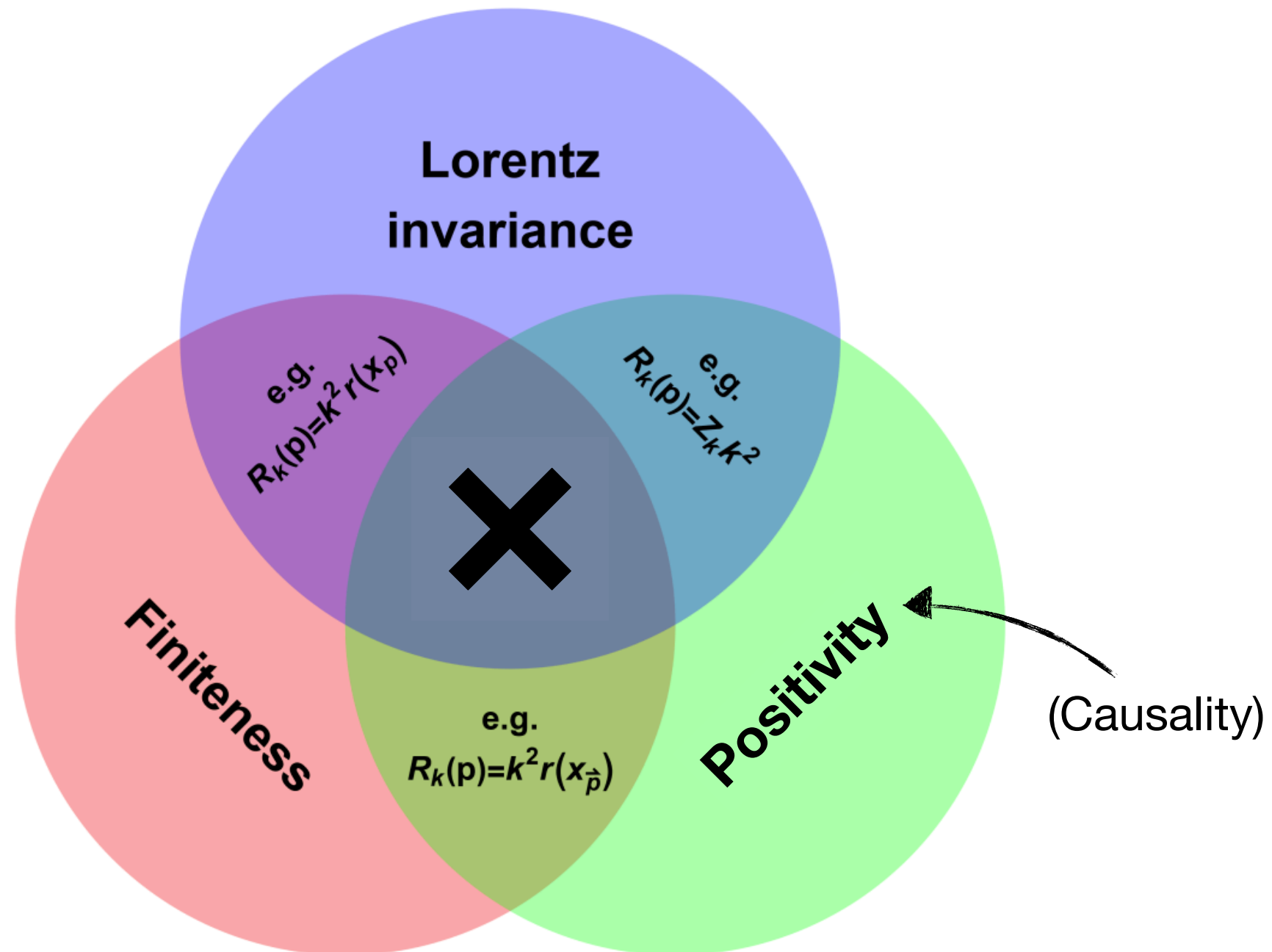


Figure adapted from fQCD collaboration, SciPost Phys. Core 6, 061 (2023)

- add regulators on level of **LGW free energy**  $F$

$$F \rightarrow F + \frac{1}{2} \int \psi R_k \psi$$

JR, Ye, Schlichting, von Smekal, arXiv:2403.04573

**+ symmetries intact:**

thermal equilibrium symmetry,

Sieberer, Chiocchetta, Gambassi, Täuber, Diehl, PRB **92** (2015) 13, 134307

temporal (non-Abelian) gauge symmetry,

Canet, Delamotte, Wschebor, PRE **93** (2016) 6, 063101

BRST symmetry

Crossley, Glorioso, Liu, JHEP **09** (2017) 095

**– regulators couple to composite response fields  $\tilde{\Psi}$**

- **Exact results:**

- Ward identities: flow of  $g$  vanishes
- Flow of statics (i.e.,  $F_k$ ) **independent** of dynamics

## Model H

$$z = 4 - \eta_{\perp} - x_{\sigma}$$

JR, Ye, Schlichting, von Smekal,  
arXiv:2409.14470

- at fixed point:

$$\sigma \sim k^{-x_{\sigma}}$$

order-parameter mobility

$$\eta \sim k^{-x_{\eta}}$$

shear viscosity

- weak-scaling relation:

$$k \partial_k f_H = (d - 4 + \eta_{\perp} + x_{\sigma} + x_{\eta}) f_H \stackrel{!}{=} 0 \quad \Longrightarrow \quad x_{\sigma} + x_{\eta} = 4 - d - \eta_{\perp}$$

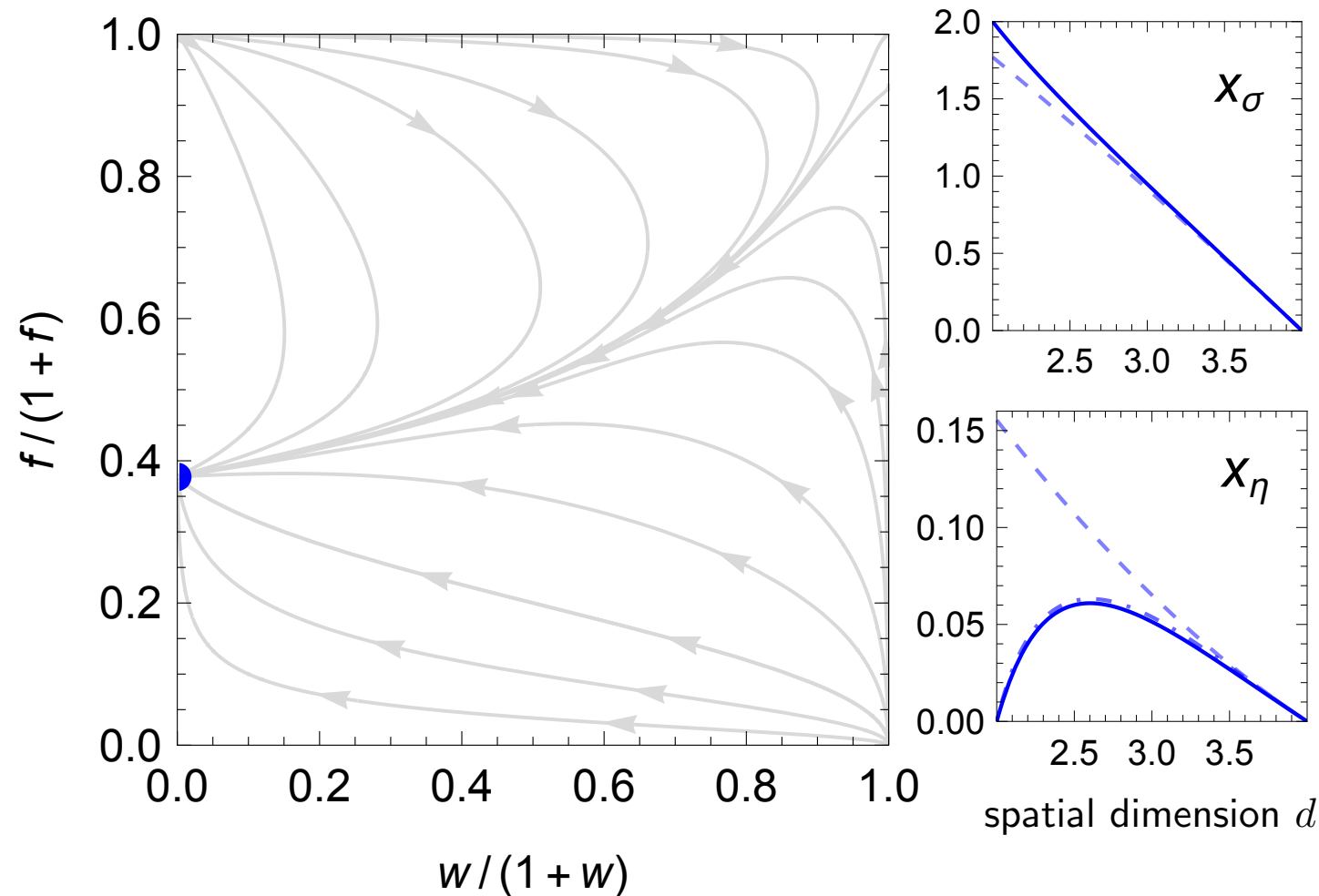
$f_H^* \neq 0$

see also Täuber, *Critical Dynamics* (Cambridge University Press, 2014)

flow of (dimensionless) dynamic couplings:

$$w_H \equiv \rho \frac{\sigma_k k^2}{\eta_k}$$

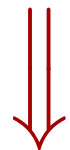
$$f_H \equiv \frac{d \Omega_d g^2 T}{(2\pi)^d} \frac{k^{d-4}}{\sigma_k \eta_k}$$



**$d = 3$ :**

$$x_{\sigma} \approx 0.949$$

$$x_{\eta} \approx 0.051$$

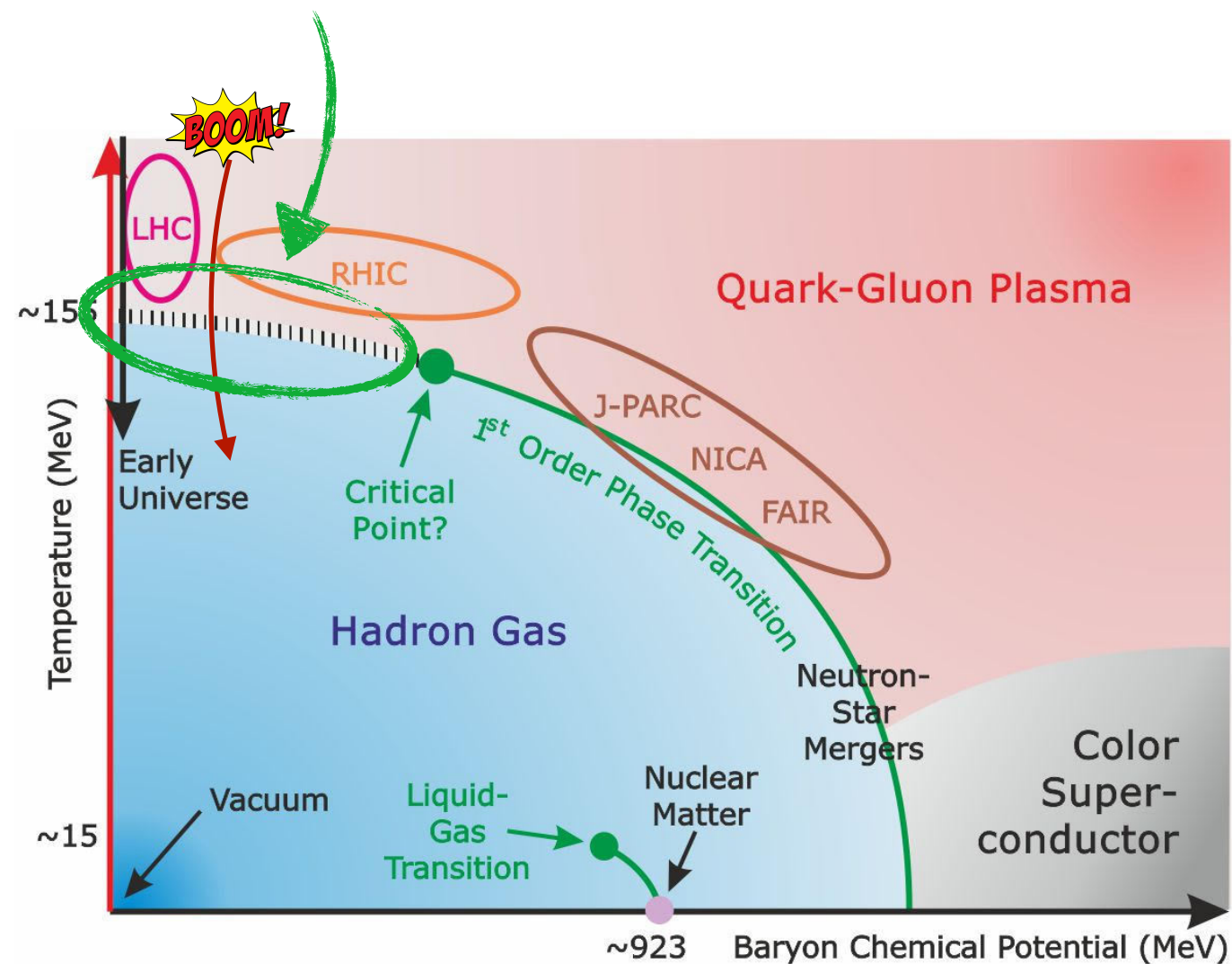


$$z_{\phi} \approx 3.051$$

## 3. Critical dynamics near the chiral phase transition

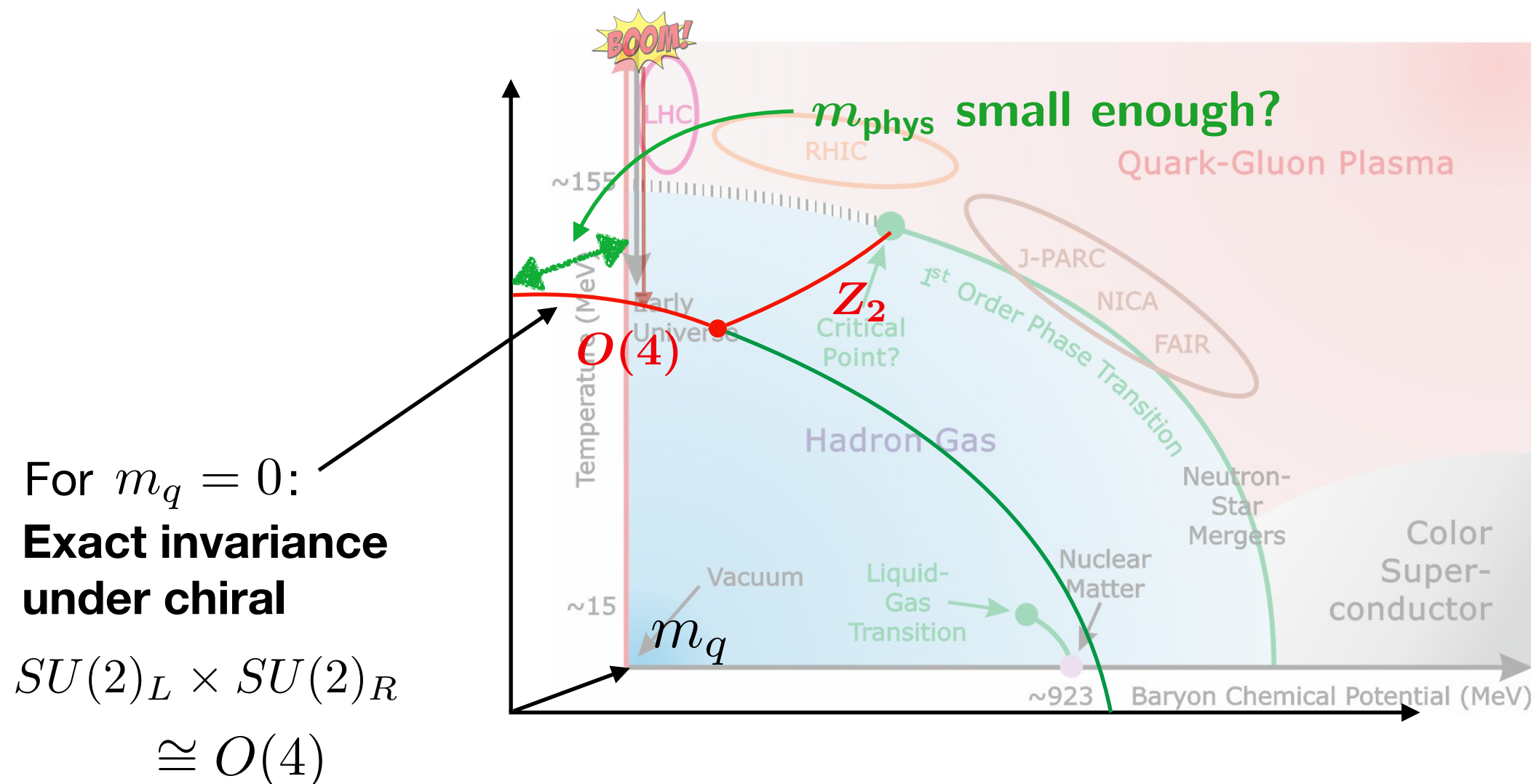


is there a second-order phase transition?



Where is the second-order phase transition here?

Add third axis (quark masses  $m_q$ )



**2<sup>nd</sup> order phase transition**

Dynamic universality class: **Model G**

**Chiral order parameter:**  $\phi = (\sigma, \vec{\pi})$

Rajagopal & Wilczek, NPB **399** (1993) 395-425

Son & Stephanov, PRL **88** (2002) 202302

Florio, Grossi, Soloviev & Teaney, PRD **105** (2022) 5, 054512

**Statics:**  $O(4)$  LGW free energy

$$F = \int d^d x \left\{ \frac{1}{2} (\partial^i \phi_a) (\partial^i \phi_a) + \frac{m^2}{2} \phi_a \phi_a + \frac{\lambda}{4! N} (\phi_a \phi_a)^2 + \frac{1}{4\chi} n_{ab} n_{ab} \right\}$$

conserved iso-vector & iso-axial-vector charges

**Equations of motion:**

**dissipation & noise**  $\leadsto$  relaxation towards thermal equilibrium  $e^{-F/T}$  **ideal time evolution**

$$\partial_t \phi_a = -\Gamma_0 \frac{\delta F}{\delta \phi_a} + \theta_a + \frac{g}{2} \{ \phi_a, n_{bc} \} \frac{\delta F}{\delta n_{bc}}$$

$$\partial_t n_{ab} = \gamma \vec{\nabla}^2 \frac{\delta F}{\delta n_{ab}} + \vec{\nabla} \cdot \vec{\zeta}_{ab} + g \{ n_{ab}, \phi_c \} \frac{\delta F}{\delta \phi_c} + \frac{g}{2} \{ n_{ab}, n_{cd} \} \frac{\delta F}{\delta n_{cd}}$$

Larmor precession

reversibility

**Poisson brackets:**

$$\{ \phi_a, n_{bc} \} = \phi_b \delta_{ac} - \phi_c \delta_{ab}$$

(non-linear) reversible mode couplings

$$\{ n_{ab}, n_{cd} \} = \delta_{ac} n_{bd} + \delta_{bd} n_{ac} - \delta_{ad} n_{bc} - \delta_{bc} n_{ad}$$

**Model G**

$$z = d/2$$

$O(4)$  Heisenberg antiferromagnet

## Model G

$$z = d/2$$

JR, Ye, Schlichting, von Smekal,  
arXiv:2409.14470

- at fixed points:

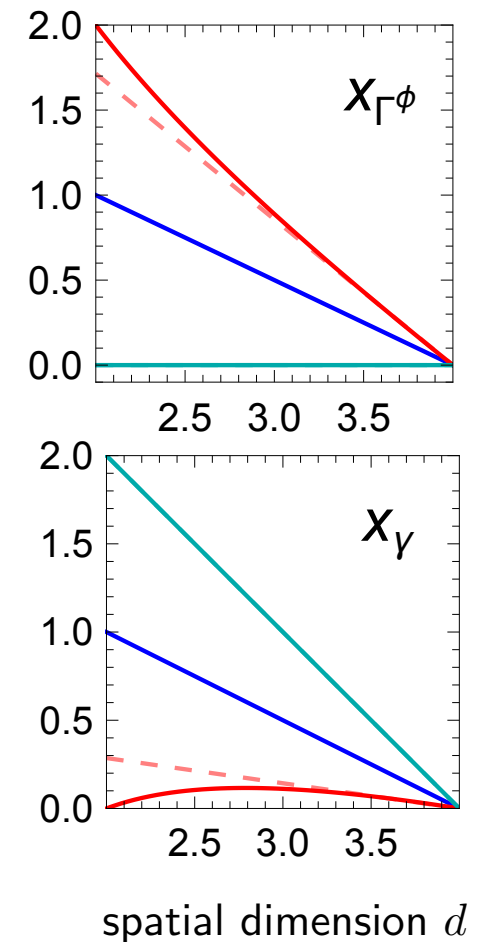
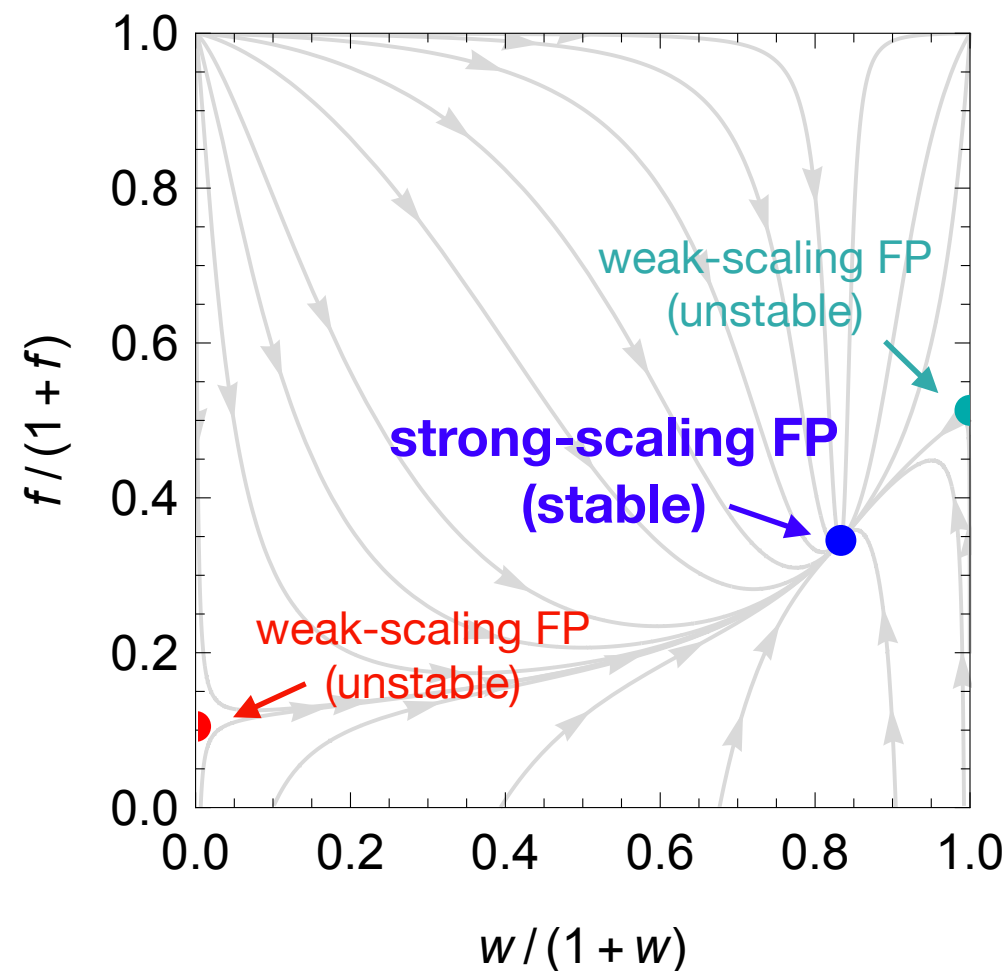
$$\Gamma^\phi \sim k^{-x_{\Gamma^\phi}}$$

sigma/pion damping rate

$$\gamma \sim k^{-x_\gamma}$$

iso-(axial-)vector charge mobility

$$w_G \equiv \chi \frac{\Gamma_k^\phi}{\gamma_k} \quad f_G \equiv \frac{d \Omega_d g^2 T}{(2\pi)^d} \frac{k^{d-4}}{\Gamma_k^\phi \gamma_k}$$



- at all fixed points: weak-scaling relation

$$x_{\Gamma^\phi} + x_\gamma = 4 - d - \eta_\perp$$

(as in Model H)

- at **strong-scaling fixed point**, in addition:

$$k \partial_k w_G = (x_\gamma - x_{\Gamma^\phi} - \eta_\perp) w_G \stackrel{!}{=} 0$$

$$w_G^* \neq 0 \implies$$

$$\underline{x_{\Gamma^\phi} = x_\gamma - \eta_\perp}$$

**exactly fixes**

$$z_\phi = z_n = d/2$$

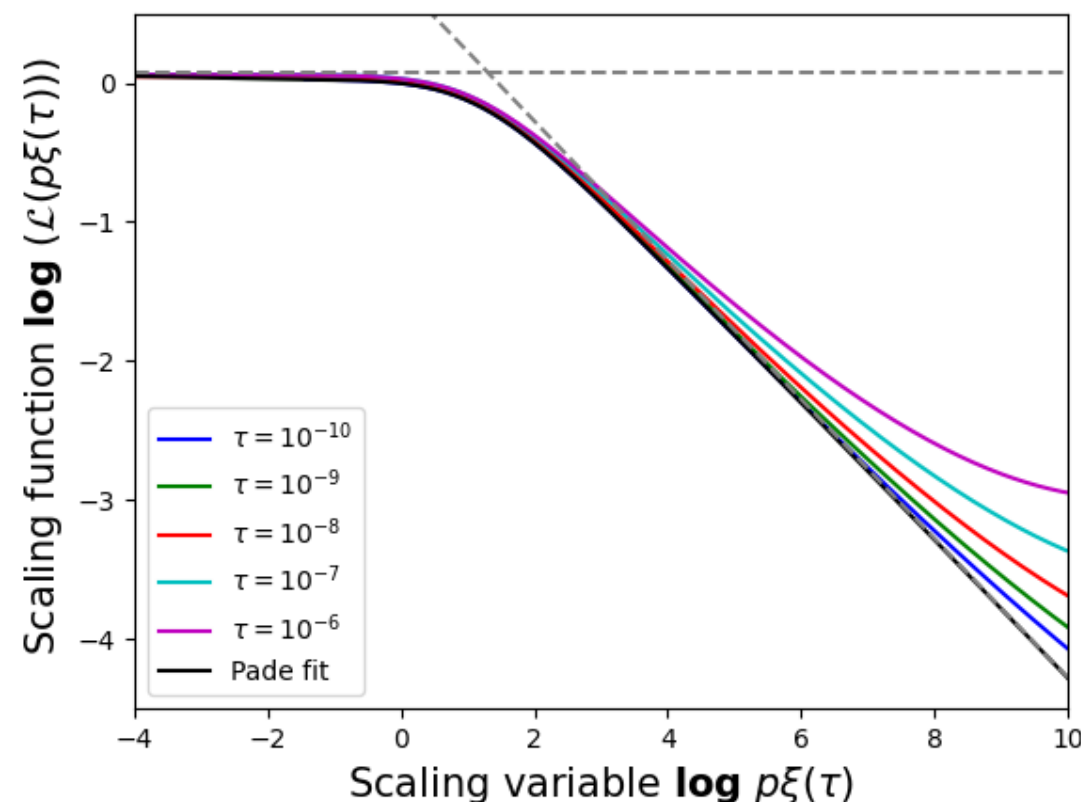
see also Täuber, *Critical Dynamics* (Cambridge University Press, 2014)

- **Strong-scaling** of charge diffusion coefficient
- Dynamic critical exponent:

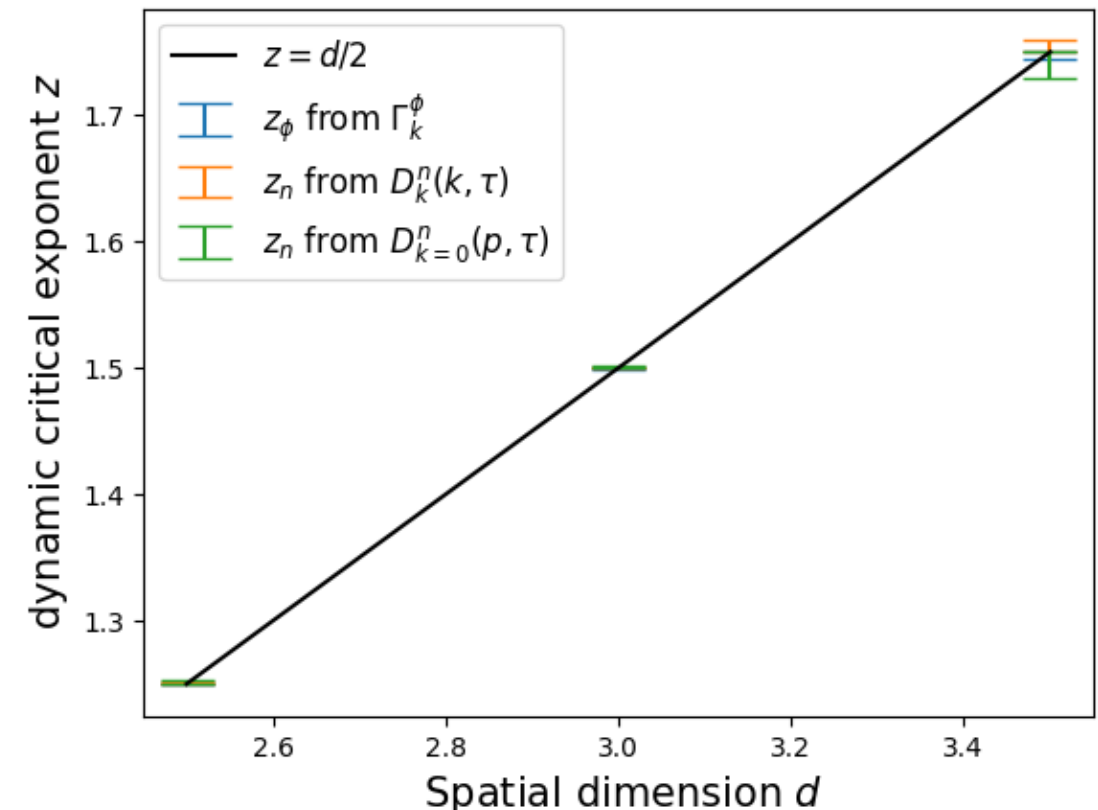
$$D_n(\mathbf{p}, \tau) = s^{2-z} D_n(s\mathbf{p}, s^{1/\nu} \tau)$$

$$\leadsto D_n(p, \tau) \sim \tau^{-\nu(2-z)} \mathcal{L}(\tau^{-\nu} \bar{p}) \quad \bar{p} = f^+ p$$

$$z_\phi = z_n = \frac{d}{2}$$



dynamic universal scaling function



strong scaling verified ✓

## Summary:

- Critical spectral functions of **Model A, B & C**  
JR, L. von Smekal, JHEP **10**, 065 (2023)
- FRG flow for systems with reversible mode couplings (**Model G & H**)  
JR, Ye, Schlichting, von Smekal, arXiv:2403.04573 & 2409.14470

## Outlook:

- dynamics of **Model G** for non-vanishing external fields (quark masses)
- dynamic universal scaling functions of **Model H**
- real-time dynamics of quark-meson model  
JR, Ye, Schlichting, von Smekal, in prep.

**Thank you for your attention!**

Backup



**Goal:** compute **non-equilibrium** correlation functions

→ Path integral requires **doubling number of fields:**

L.V. Keldysh, Sov. Phys. JETP **20** (1965) 1018

$$\langle O(t) \rangle = \text{tr} (O(t) \rho_0) \quad (\text{Heisenberg picture})$$

$$= \text{tr} (U(-\infty, t) O U(t, -\infty) \rho_0)$$

(extend evolution to  $t = +\infty$ )

$$= \int_{\rho_0} \mathcal{D}\phi^+ \mathcal{D}\phi^- e^{i(S[\phi^+] - S[\phi^-])} O(\phi^+(t), \phi^-(t))$$

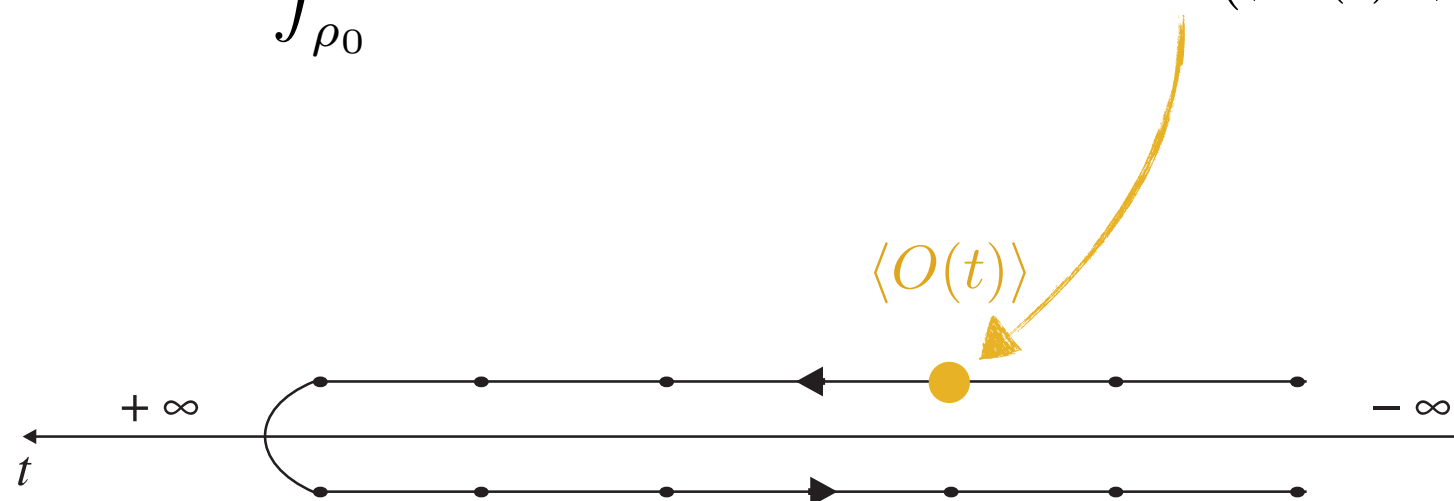
→ in particular: **direct access to real-time** Green functions

$$G^K(t, t') = i \langle \{ \phi(t), \phi(t') \} \rangle$$

$$G^R(t, t') = i \theta(t - t') \langle [\phi(t), \phi(t')] \rangle$$

$$G^A(t, t') = i \theta(t' - t) \langle [\phi(t'), \phi(t)] \rangle$$

$$G^{\tilde{K}}(t, t') = 0$$



→ **Causal structure** built into the formalism!

Figure adapted from  
Kamenev, *Field Theory of Non-Equilibrium Systems*  
(Cambridge University Press, 2011)

**closed-time path**

**Model C**

$$z = 2 + a/\nu$$

**Statics:** Landau-Ginzburg-Wilson functional

$$F = \int d^d x \left\{ \frac{1}{2} (\vec{\nabla} \varphi)^2 + V(\varphi) + \frac{g}{2} \varphi^2 n + \frac{n^2}{2\chi_0} \right\}$$

- **Dynamics:** Langevin equations of motion

$$\partial_t^2 \varphi + \gamma \partial_t \varphi = -\frac{\delta F}{\delta \varphi} + \xi$$

Gaussian white noises

$$\partial_t n = \bar{\lambda} \vec{\nabla}^2 \frac{\delta F}{\delta n} + \vec{\nabla} \cdot \vec{\zeta}$$

**diffusive!**

- Order parameter not conserved but interacts non-linearly with conserved (energy) density  $\leadsto$  **Model C**

in classical simulations:  
**solve Langevin equation**

$$\partial_t^2 \varphi + \gamma \partial_t \varphi = -\frac{\delta F}{\delta \varphi} + \xi(x)$$

$$\begin{aligned} \langle \xi(x) \rangle &= 0 \\ \langle \xi(x) \xi(x') \rangle &= 2\gamma T \delta(x - x') \end{aligned}$$

Introduce (Hubbard)  
response field  $\tilde{\varphi}$

integrate  $\tilde{\varphi}$

However, we need:

$$Z = \int \mathcal{D}\tilde{\varphi} \mathcal{D}\varphi e^{iS[\tilde{\varphi}, \varphi]} \quad S[\tilde{\varphi}, \varphi] = \int_x \left[ -\tilde{\varphi} \left( \partial_t^2 \varphi + \gamma \partial_t \varphi + \frac{\delta F}{\delta \varphi} \right) + i\gamma T \tilde{\varphi}^2 \right]$$

fluctuations

deterministic part of eom's

**Path-integral formulation**  
for (real-time) FRG

**Solution:** Observe that regulator is a self-energy

- Self-energies generally inherit **causal structure**

→ **Spectral representation** from (subtracted) Kramers-Kronig relations

mass-like part (trivially causal) → 'spectral density'

$$R_k^{R/A}(\omega, \mathbf{p}) = R_k^{R/A}(0, \mathbf{p}) - \int_0^\infty \frac{d\omega'}{2\pi} \frac{2\omega'^2 J_k(\omega', \mathbf{p})}{\omega'((\omega \pm i\varepsilon)^2 - \omega'^2)}$$

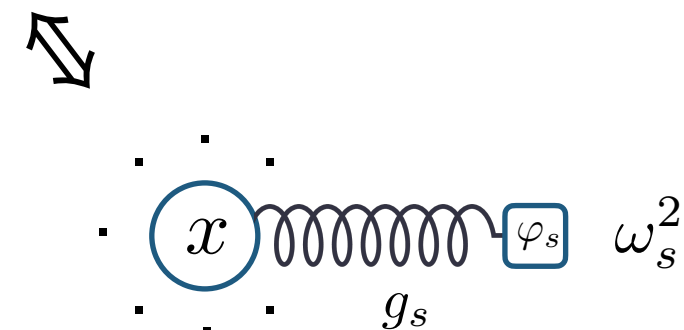
$$J_k(\omega, \mathbf{p}) = 2 \operatorname{Im} R_k^R(\omega, \mathbf{p})$$

JR, von Smekal, JHEP **10**, 065 (2023)

JR, Schweitzer, Sieke, von Smekal, Phys. Rev. D **105**, 116017 (2022)

- Interpret as coupling to **fictitious heat bath:**

(Hubbard-Stratonovich transformation)



here:  $J_k(\omega) = \pi \sum_s \frac{g_s^2(k)}{\omega_s(k)} (\delta(\omega - \omega_s(k)) - \delta(\omega + \omega_s(k)))$

→ **Physical** only for **positive-semidefinite** spectral densities  $J_k(\omega, \mathbf{p}) \geq 0 \quad (\omega > 0)$

→ Spectral density encodes **spectrum of bath oscillators**

$$R_k^{R/A}(\omega) = R_k^{R/A}(0) - \int_0^\infty \frac{d\omega'}{2\pi} \frac{2\omega^2 J_k(\omega')}{\omega'((\omega \pm i\varepsilon)^2 - \omega'^2)} \quad \text{in} \quad \Gamma_k^{(2)R}(\omega) = (\omega + i\varepsilon)^2 - m^2 + R_k^R(\omega)$$

- spectral density:  $\leadsto$  **Regulator** (retarded part):

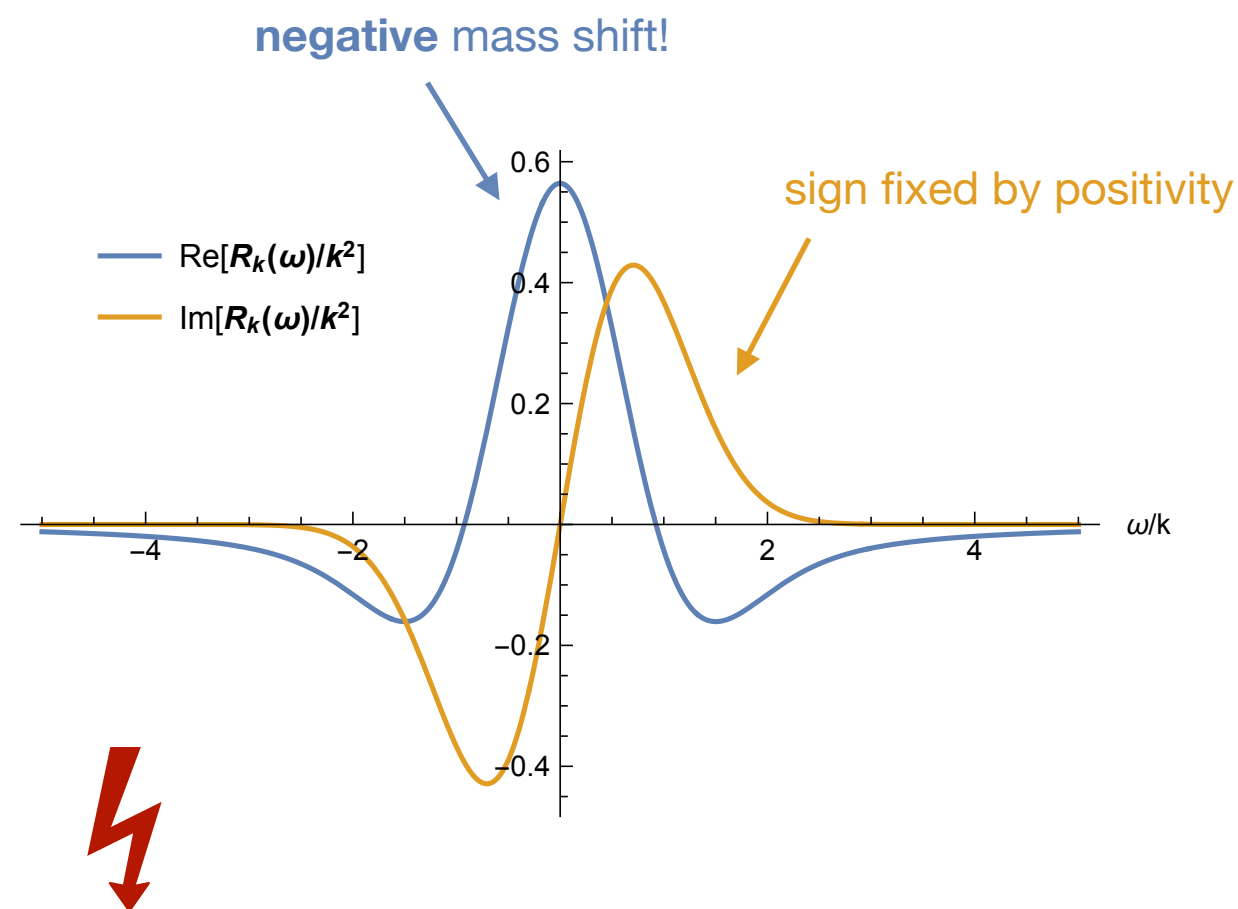
$$J_k(\omega) = 2k\omega e^{-\omega^2/k^2} = 2 \operatorname{Im} R_k^R(\omega)$$

- assume UV finiteness:

$$\Delta M_{UV}^2(k) = -R_k^{R/A}(0) + \underbrace{\int_0^\infty \frac{d\omega'}{\pi} \frac{J_k(\omega')}{\omega'}}_{\geq 0 \text{ (positivity)}} \stackrel{!}{=} 0$$

$\Rightarrow$  IR mass shift:

$$\Delta M_{IR}^2(k) = -R_k^{R/A}(0) < 0 \quad \text{is negative!}$$



**Solution:** choose IR mass shift  $\Delta M_{IR}^2(k) > 0$  positive (at cost of **UV finiteness**)

First step towards **dynamic universality class**:  
identify relevant **slow** (hydrodynamic) modes

Rajagopal & Wilczek, NPB **399** (1993) 395-425

Son & Stephanov, PRL **88** (2002) 202302

Grossi, Soloviev, Teaney & Yan, PRD **104** (2021) 3, 034025

- Order parameter:  $\phi = (\sigma, \pi)$  **slow (critical slowing down)**
- Six iso-(axial-)vector charge densities:

$$V_i^0 = \frac{1}{2} \bar{q} \gamma^0 \tau_i q$$

$$A_i^0 = \frac{1}{2} \bar{q} \gamma^0 \gamma^5 \tau_i q$$

**conserved in chiral limit**  
 $\leadsto$  **hydrodynamic modes**

$\leadsto$  combine:

$$(n_{ab}) = \left( \begin{array}{c|ccc} 0 & A_1^0 & A_2^0 & A_3^0 \\ \hline -A_1^0 & 0 & V_3^0 & -V_2^0 \\ -A_2^0 & -V_3^0 & 0 & V_1^0 \\ -A_3^0 & V_2^0 & -V_1^0 & 0 \end{array} \right)$$

**find O(4) Lie algebra:**

$$[n_{ab}, n_{cd}] = i (\delta_{ac} n_{bd} + \delta_{bd} n_{ac} - \delta_{ad} n_{bc} - \delta_{bc} n_{ad})$$

**order parameter:**

$$[\phi_a, n_{bc}] = i (\delta_{ac} \phi_b - \delta_{ab} \phi_c)$$

**( $n$ 's generate O(4) transformations)**

- Energy-momentum, baryon charge: **also slow, but irrelevant (in RG sense)**

see Nakano, Skokov, Friman, PRD **85** (2012) 096007

## Model G

$$z = d/2$$

- Add regulators to LGW functional:

$$F \rightarrow F + \frac{1}{2} \int_{xy} \left( \phi_a(\mathbf{x}) R_k^\phi(\mathbf{x}, \mathbf{y}) \phi_a(\mathbf{y}) + \frac{1}{2} n_{ab}(\mathbf{x}) R_k^n(\mathbf{x}, \mathbf{y}) n_{ab}(\mathbf{y}) \right)$$

+ symmetries intact

– regulators cubic in fields

- Truncation for effective average (Martin-Siggia-Rose) action:

$$\Gamma_k = \int_x \left[ -\tilde{\phi}_{a,k} \left( \frac{\partial \phi_a}{\partial t} + \gamma_{\phi,k}(\nabla) \frac{\delta F_k}{\delta \phi_a} - \frac{g}{2} \{ \phi_a, n_{bc} \} \frac{\delta F_k}{\delta n_{bc}} \right) \right. \\ \left. - \frac{1}{2} \tilde{n}_{ab,k} \left( \frac{\partial n_{ab}}{\partial t} + \gamma_{n,k}(\nabla) \frac{\delta F_k}{\delta n_{ab}} - g \{ n_{ab}, \phi_c \} \frac{\delta F_k}{\delta \phi_c} - \frac{g}{2} \{ n_{ab}, n_{cd} \} \frac{\delta F_k}{\delta n_{cd}} \right) \right. \\ \left. + iT \tilde{\phi}_{a,k} \gamma_{\phi,k}(\nabla) \tilde{\phi}_{a,k} + \frac{1}{2} iT \tilde{n}_{ab,k} \gamma_{n,k}(\nabla) \tilde{n}_{ab,k} \right]$$

kinetic coefficients:

- Important (exact) results:

$$\gamma_{\phi,k}(\mathbf{p}) = \Gamma_k^\phi + \mathcal{O}(\mathbf{p}^2)$$

$$\gamma_{n,k}(\mathbf{p}) = D_k^n(\mathbf{p}) \mathbf{p}^2$$

- Ward identities: flow of  $g$  vanishes
- Flow of statics (i.e.,  $F_k$ ) independent of dynamics



- Transform to accelerated frame of reference:

$$t \rightarrow t$$

$$\vec{x} \rightarrow \vec{x} - \vec{\alpha}(t) \quad \text{'time-gauged Galilean boost'}$$

$$\phi(t, \vec{x}) \rightarrow \phi(t, \vec{x} - \vec{\alpha}(t))$$

$$\vec{j}(t, \vec{x}) \rightarrow \vec{j}(t, \vec{x} - \vec{\alpha}(t)) + \rho \dot{\vec{\alpha}}(t)$$

- Transformed EoM's (here for simplicity without dissipation):

$$\frac{\partial \phi}{\partial t} + \frac{\delta F}{\delta \vec{j}} \cdot \vec{\nabla} \phi = 0 \quad \text{invariant}$$

$$\frac{\partial \vec{j}}{\partial t} + \rho \ddot{\vec{\alpha}}(t) + \mathcal{T} \left[ \left( \frac{\delta F}{\delta \vec{j}} \cdot \vec{\nabla} \right) \vec{j} - (\vec{\nabla} \phi) \frac{\delta F}{\delta \phi} \right] = 0$$

additional fictitious force...

**but:** field-independent!

↪ interpret as part of external source

**Definition: 'Extended' symmetry**  
Change  $\delta S$  of action  $S$  is linear in fields

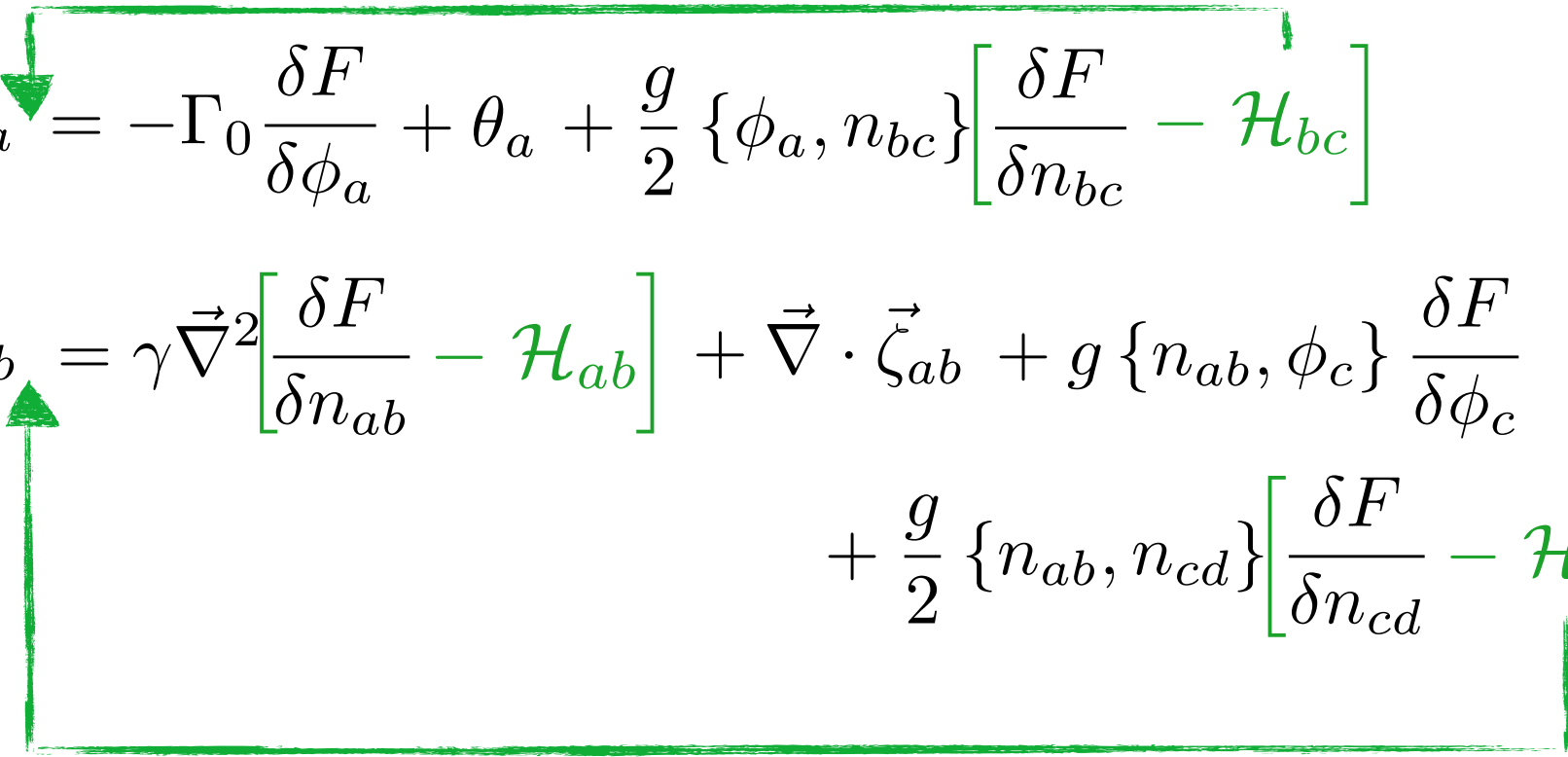
poses **just as strong restrictions** on  
one-particle irreducible (1PI) effective action  
as **normal symmetry**

see Canet, Delamotte, Wschebor, PRE **93** (2016) 6, 063101

Floerchinger, Grossi, PRD **105** (2022) 8, 085015

- Apply external ‘magnetic’ field  $\mathcal{H}_{ab}$ :  $F \rightarrow F - \frac{1}{2} \int d^d x \mathcal{H}_{ab} n_{ab}$   

‘magnetization’
- Equations of motion change according to:

$$\begin{aligned} \partial_t \phi_a &= -\Gamma_0 \frac{\delta F}{\delta \phi_a} + \theta_a + \frac{g}{2} \{ \phi_a, n_{bc} \} \left[ \frac{\delta F}{\delta n_{bc}} - \mathcal{H}_{bc} \right] \\ \partial_t n_{ab} &= \gamma \vec{\nabla}^2 \left[ \frac{\delta F}{\delta n_{ab}} - \mathcal{H}_{ab} \right] + \vec{\nabla} \cdot \vec{\zeta}_{ab} + g \{ n_{ab}, \phi_c \} \frac{\delta F}{\delta \phi_c} \\ &\quad + \frac{g}{2} \{ n_{ab}, n_{cd} \} \left[ \frac{\delta F}{\delta n_{cd}} - \mathcal{H}_{cd} \right] \end{aligned}$$


- Apply external ‘magnetic’ field  $\mathcal{H}_{ab}$ :  $F \rightarrow F - \frac{1}{2} \int d^d x \mathcal{H}_{ab} n_{ab}$   
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- Equations of motion change according to:

$$\begin{aligned} \partial_t \phi_a - g(\mathcal{H}\phi)_a &= -\Gamma_0 \frac{\delta F}{\delta \phi_a} + \theta_a + \frac{g}{2} \{ \phi_a, n_{bc} \} \frac{\delta F}{\delta n_{bc}} \\ \partial_t n_{ab} - g[\mathcal{H}, n]_{ab} &= \gamma \vec{\nabla}^2 \left[ \frac{\delta F}{\delta n_{ab}} - \mathcal{H}_{ab} \right] + \vec{\nabla} \cdot \vec{\zeta}_{ab} + g \{ n_{ab}, \phi_c \} \frac{\delta F}{\delta \phi_c} \\ &\quad + \frac{g}{2} \{ n_{ab}, n_{cd} \} \frac{\delta F}{\delta n_{cd}} \end{aligned}$$

**Covariant time-derivatives,**  
in which  $\mathcal{H}_{ab}$  = zero-component  
of **external  $O(4)$  gauge field**

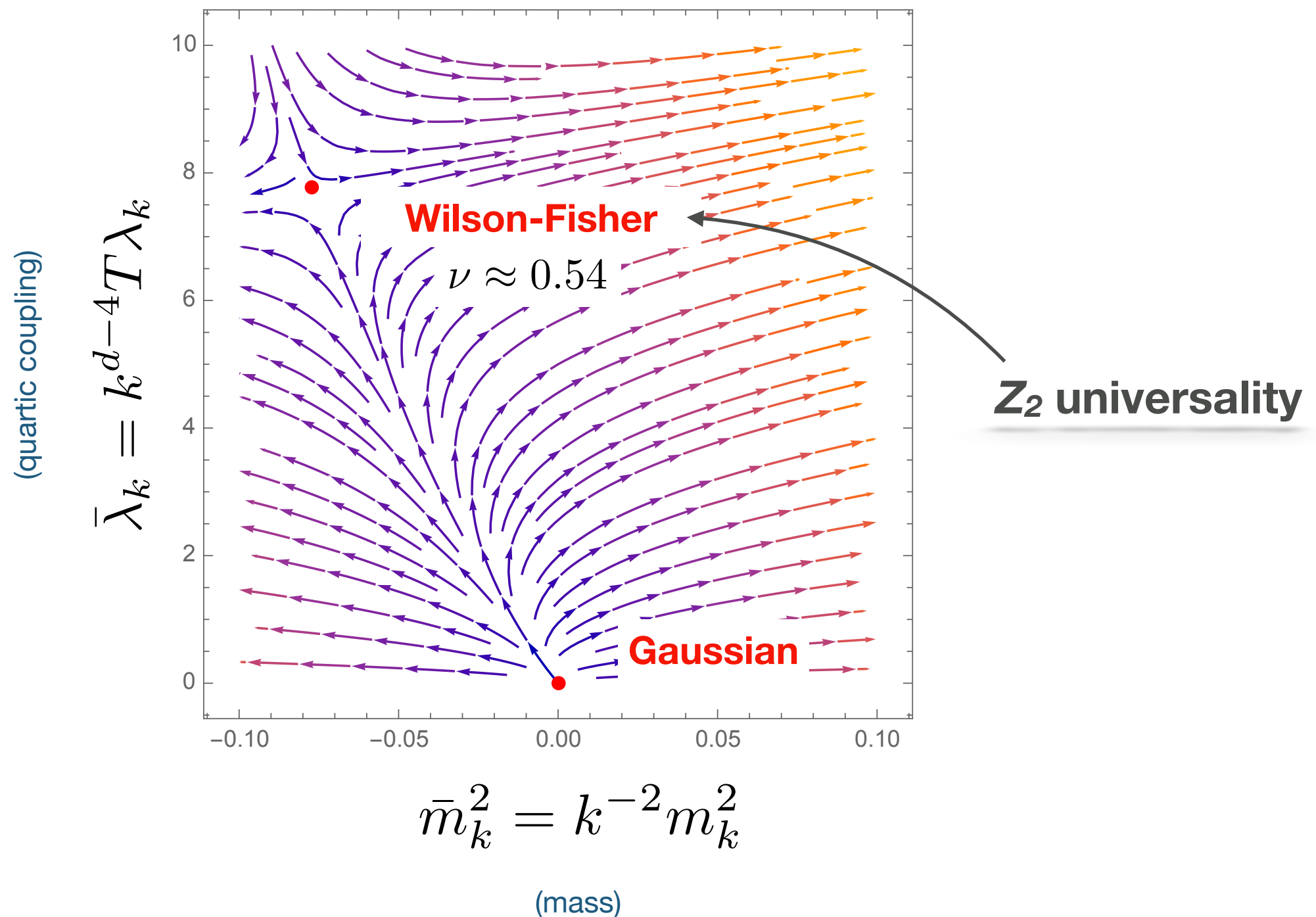
Inhomogeneous  
(time dependent) term  
drops out here

- Equations of motion **invariant** under **time-gauged**  $O(4)$  transformations

$$\begin{aligned} \phi(t, \vec{x}) &\rightarrow O(t) \phi(t, \vec{x}) \\ n(t, \vec{x}) &\rightarrow O(t) n(t, \vec{x}) O^T(t) \end{aligned} \quad \mathcal{H}(t, \vec{x}) \rightarrow O(t) \mathcal{H}(t, \vec{x}) O^T(t) + \frac{1}{g} O(t) \partial_t O^T(t)$$

Second-order phase transitions correspond to non-trivial **RG fixed points**:

First look at flow of LGW free energy  $\sim$  flow of (static) couplings  $m_k^2, \lambda_k$



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First look at flow of LGW free energy  $\sim$  flow of (static) couplings  $m_k^2$ ,

