# Rising Researchers Seminar Series

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References: This code of conduct is based heavily on that of the INT and the APS. We are also grateful to Roxanne Springer for valuable discussion and guidance.

# A consistent approach to the phenomenology of TMD distributions in the application to hadron structure studies

**Tommaso Rainaldi** – Old Dominion University

Rising Researchers Seminar Series

December 3, 2024

Jefferson Lab

Rising Researchers Seminar Series

OLD DOMINION UNIVERSITY

# Based on

- Phenomenology of TMD parton distributions in Drell-Yan and $Z^0$ boson production in a hadron structure oriented approach

(PhysRevD.110.074016)

- (F. Aslan, M. Boglione, J. O. Gonzalez-Hernandez, T. Rainaldi, T. C. Rogers, A. Simonelli )

- The resolution to the problem of consistent large transverse momentum in TMDs

(PhysRevD.107.094029)

- (J. O. Gonzalez-Hernandez, T. Rainaldi, T. C. Rogers )

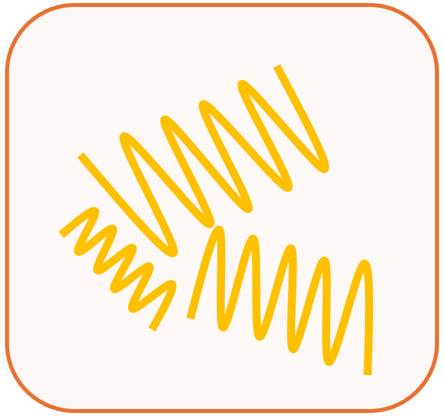
- Combining nonperturbative transverse momentum dependence with TMD evolution

(PhysRevD.106.034002)

- (J. O. Gonzalez-Hernandez, T. C. Rogers, N. Sato )

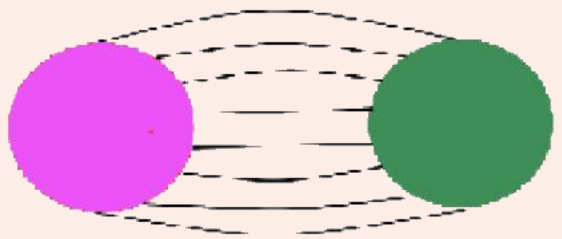
# The Lagrangian of QCD

$$\mathcal{L}_{QCD} = \underbrace{\sum_f \bar{\psi}_{i,f}\left(i\gamma^\mu (D_\mu)_{ij} - m_f \delta_{ij}\right)\psi_{j,f}}_{quarks\ \&\ antiquarks} - \underbrace{\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a}_{gluons}$$

# The difficulties with QCD

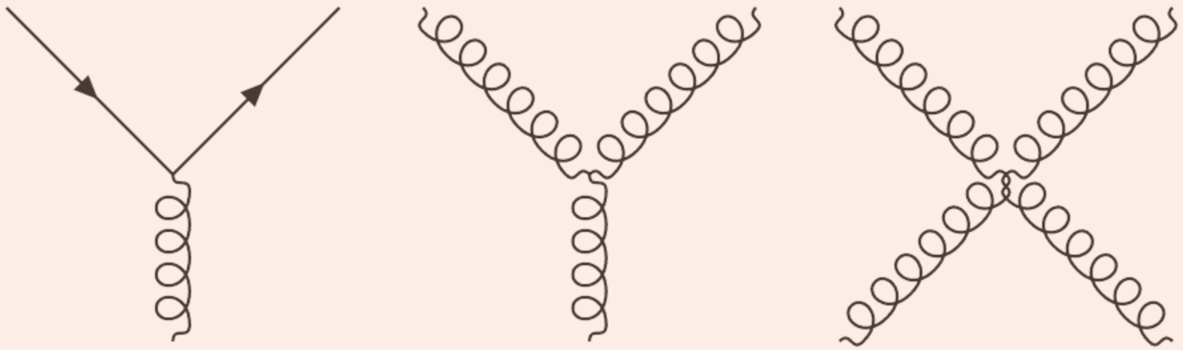
1 Quark and gluons are never observed
( color confinement )



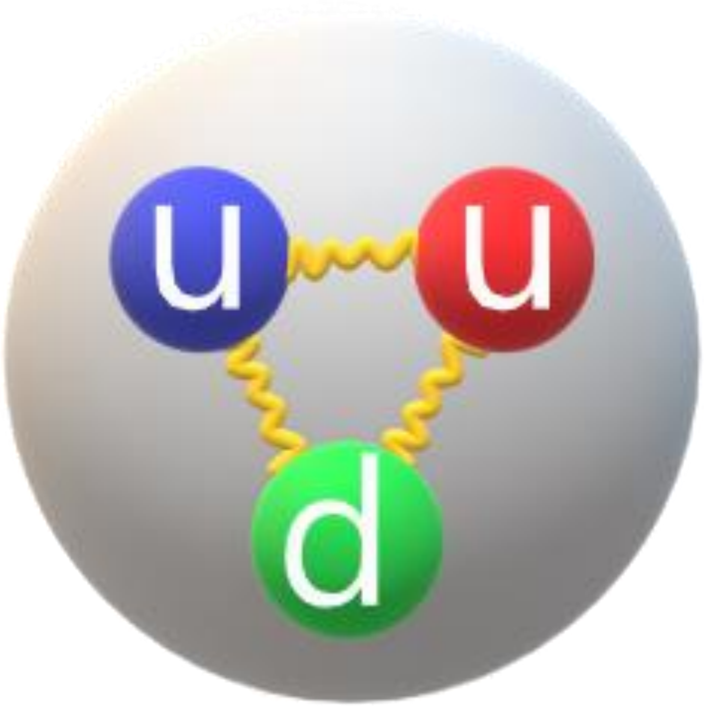
2 The interaction is **strong**
(of order 1)

| | |
|---|---|
| **Strong** | **1** |
| Electromagnetic | 1/137 |
| Weak | $10^{-6}$ |
| Gravity | $10^{-39}$ |

3 Unlike photons, gluons **interact with themselves**



# Hadronic Structure: Proton



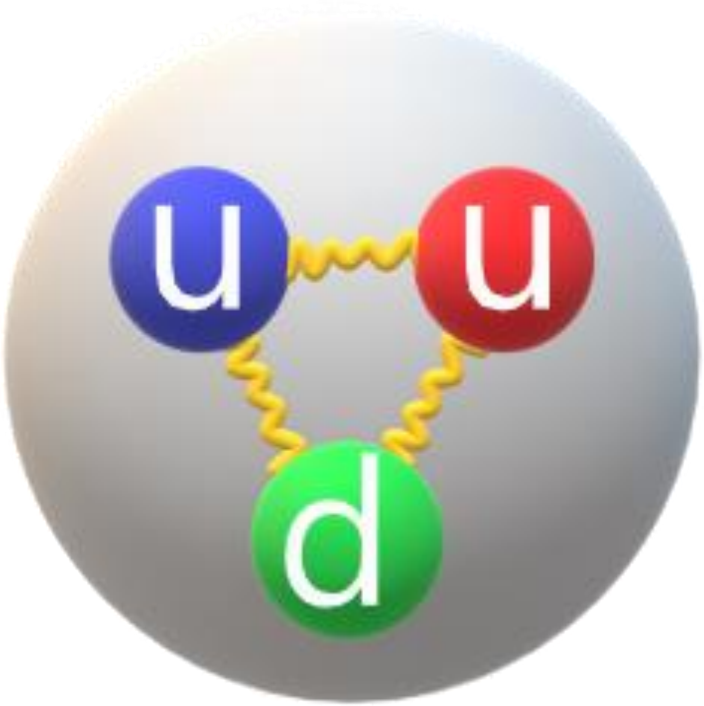

Flavor $: 2\,\mathrm{u} + 1\,\mathrm{d}$

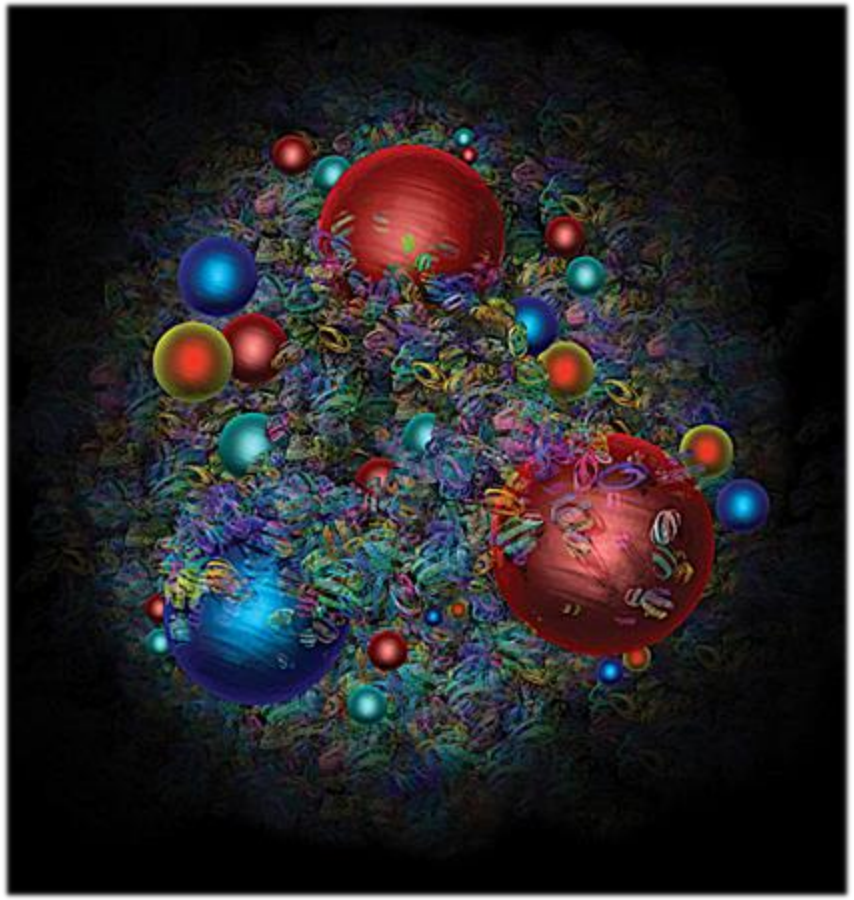
Charge $: 2 \cdot \left(\frac{2}{3}\right) + 1 \cdot \left(-\frac{1}{3}\right) = 1$

Spin $: 1/2$

Reality is much more difficult

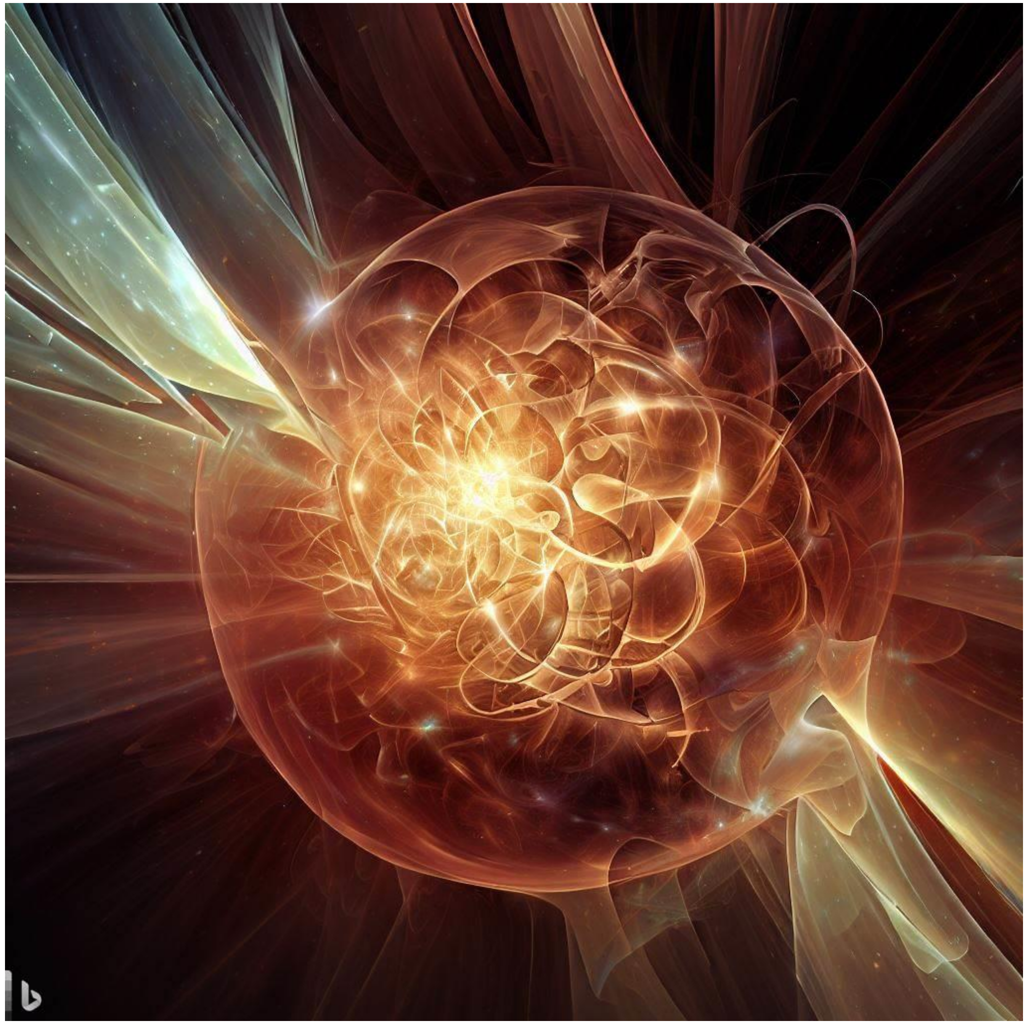
# Hadronic Structure: Proton

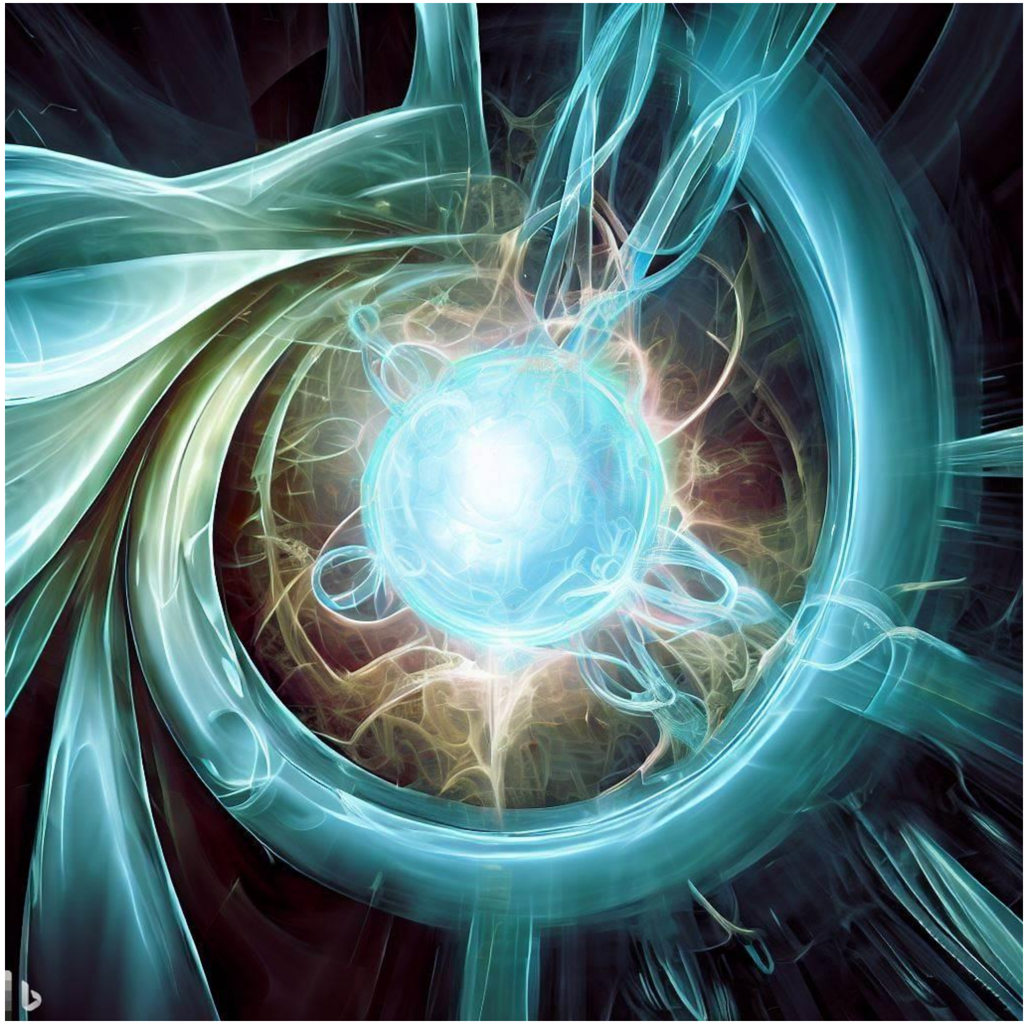


Credits: CERN

# According to “AI” (Bing image creator)





# Many ways to tackle QCD

## Lattice QCD

Exact calculations on a lattice (discrete space-time)

## Factorization

Nonperturbative and perturbative physics combined

→ Suited for hadronic structure

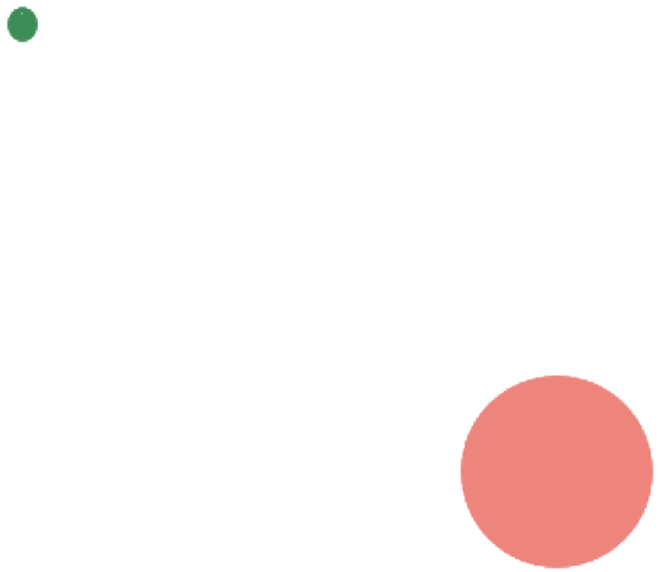
## Effective QFTs

Chiral perturbation theory,
Topological solitons,
....

## Others...

# Why factorization is useful

We look at scattering processes (some examples)



| | |
|---|---|
| DIS | $l + H \to l + X$ |
| SIDIS | $l + H_A \to l + H_B + X$ |
| SIA | $l + \bar{l} \to H + X$ |
| DY | $H_A + H_B \to l + \bar{l} + X$ |

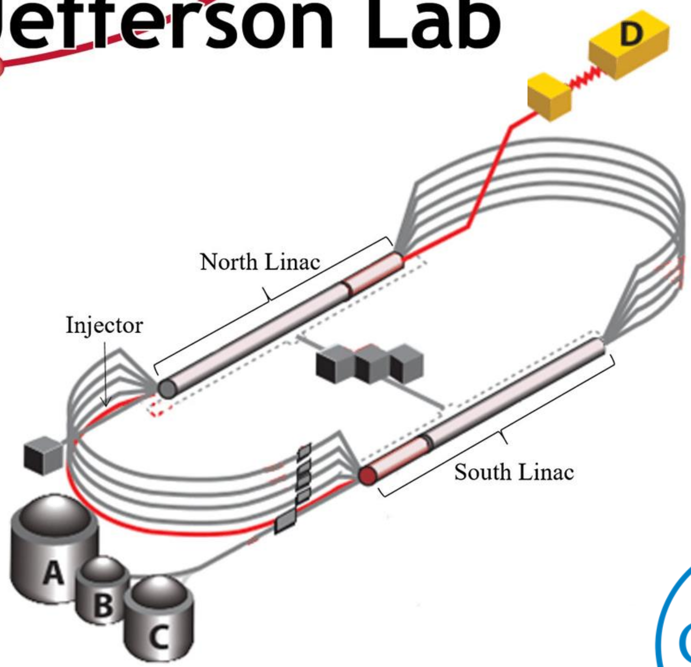
There are different factorization theorems for different processes but the idea is the same:

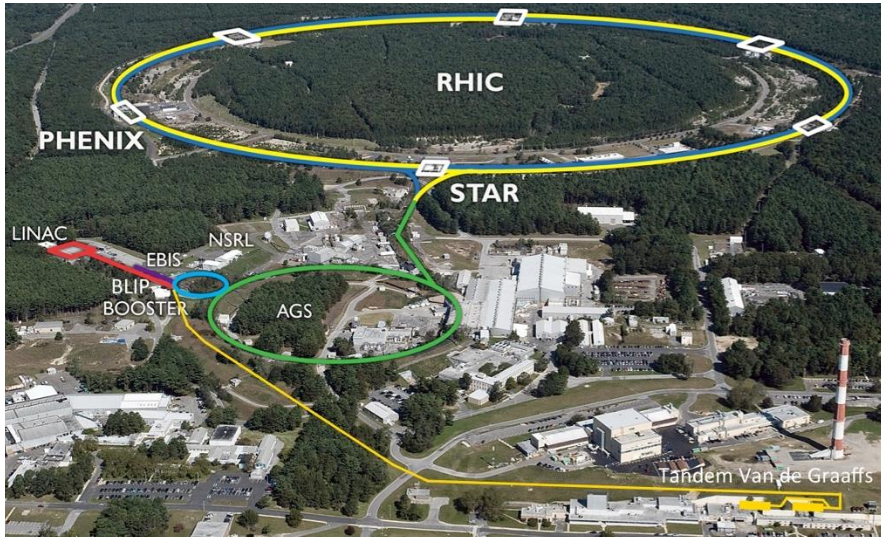
disentangle perturbative and nonperturbative (intrinsic) physics

# Where the experiments take place (some examples)

Brookhaven National Laboratory

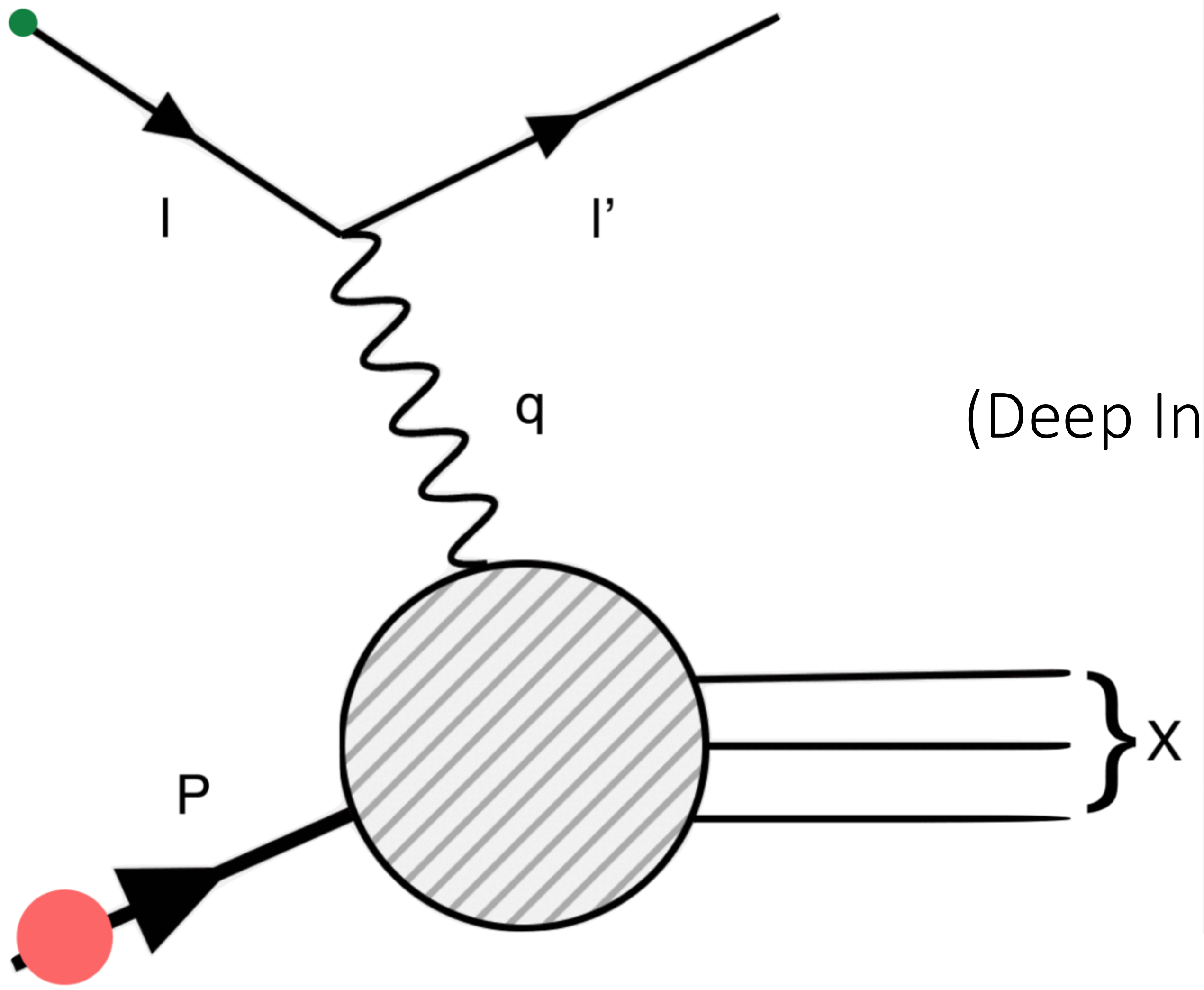
Jefferson Lab




CERN

# Typical scattering experiment




DIS

(Deep Inelastic Scattering)

# Factorization theorem for DIS

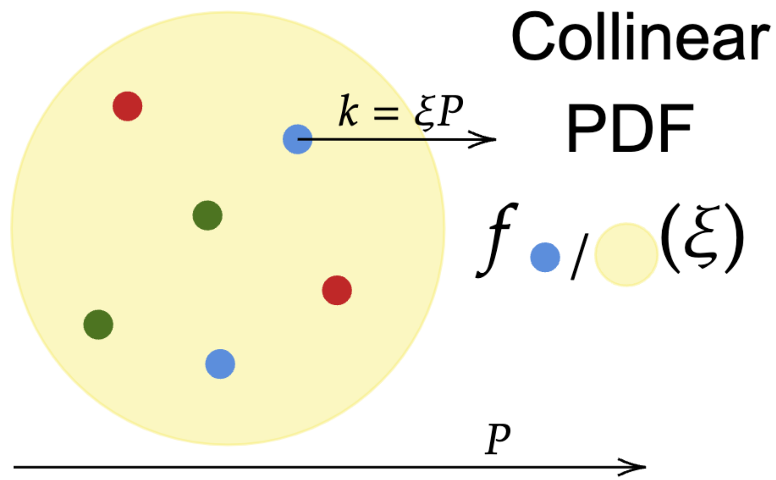
$$\sigma_{\text{DIS}} = \sum_{\text{constituents}} \hat{\sigma}_{ij} \otimes f_j + \mathcal{O}\left(\frac{m^2}{Q^2}\right)$$

Perturbatively calculable "hard" part

Mellin convolution

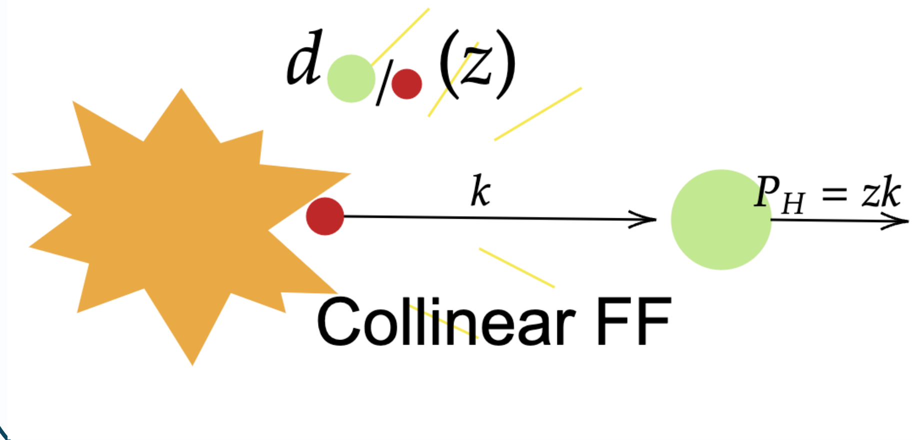
Nonperturbative part (intrinsic)

# What is this intrinsic factor? (intuitively)




**Parton Distribution Function (PDF)**:
"Behaves" as the probability distribution to find parton j in hadron H carrying collinear momentum fraction ξ

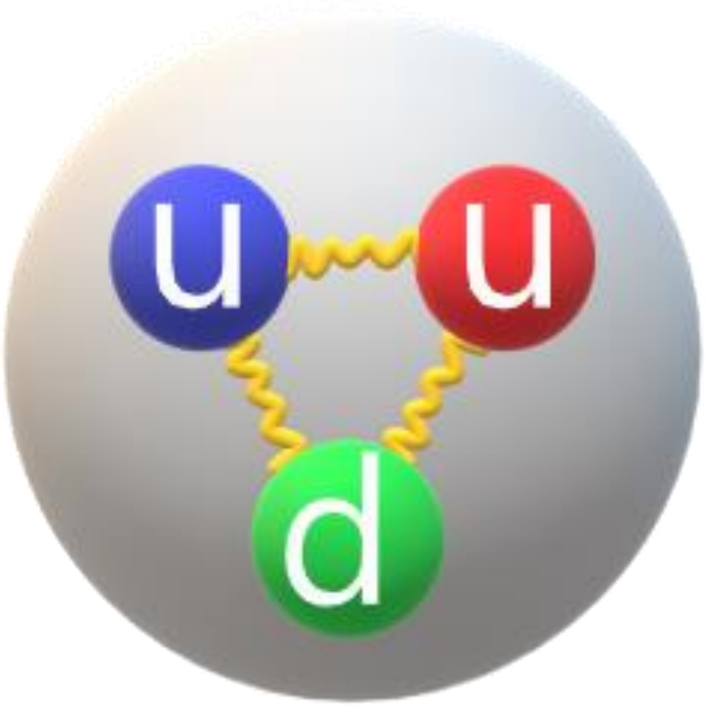
$$f_{j/H}(\xi)$$




**Fragmentation Function (FF)**:
"Behaves" as the probability distribution to find hadron H from parton j carrying collinear momentum fraction z

$$d_{H/j}(z)$$

# What would you expect the pdfs of the quarks in a proton to be?




The UP PDF should be situated at 1/3 but twice the size actually

# PDF: operator and renormalization

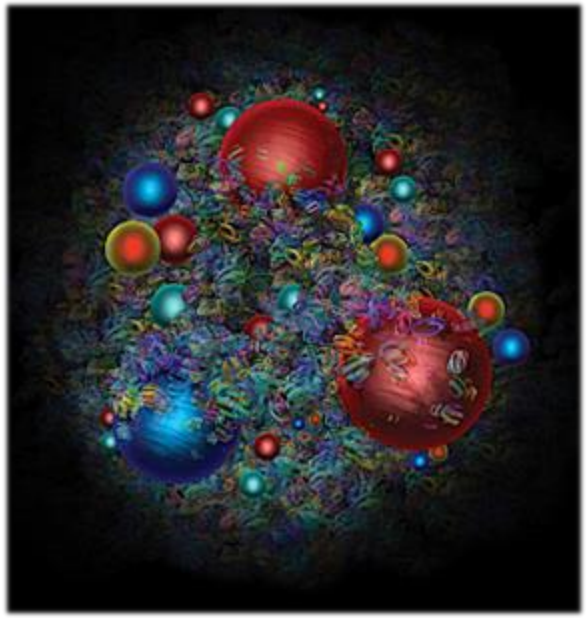
$$f_{(0)j/H}(\xi) = \int \frac{\mathrm{d}w^-}{2\pi} e^{-i\xi P^+ w^-} \langle P|\overline{\psi}_{(0)j}(0, w^-, \mathbf{0}_\mathrm{T}) W_{(0)}(w^-, 0) \frac{\gamma^+}{2} \psi_{(0)j}|P\rangle_c = \infty$$

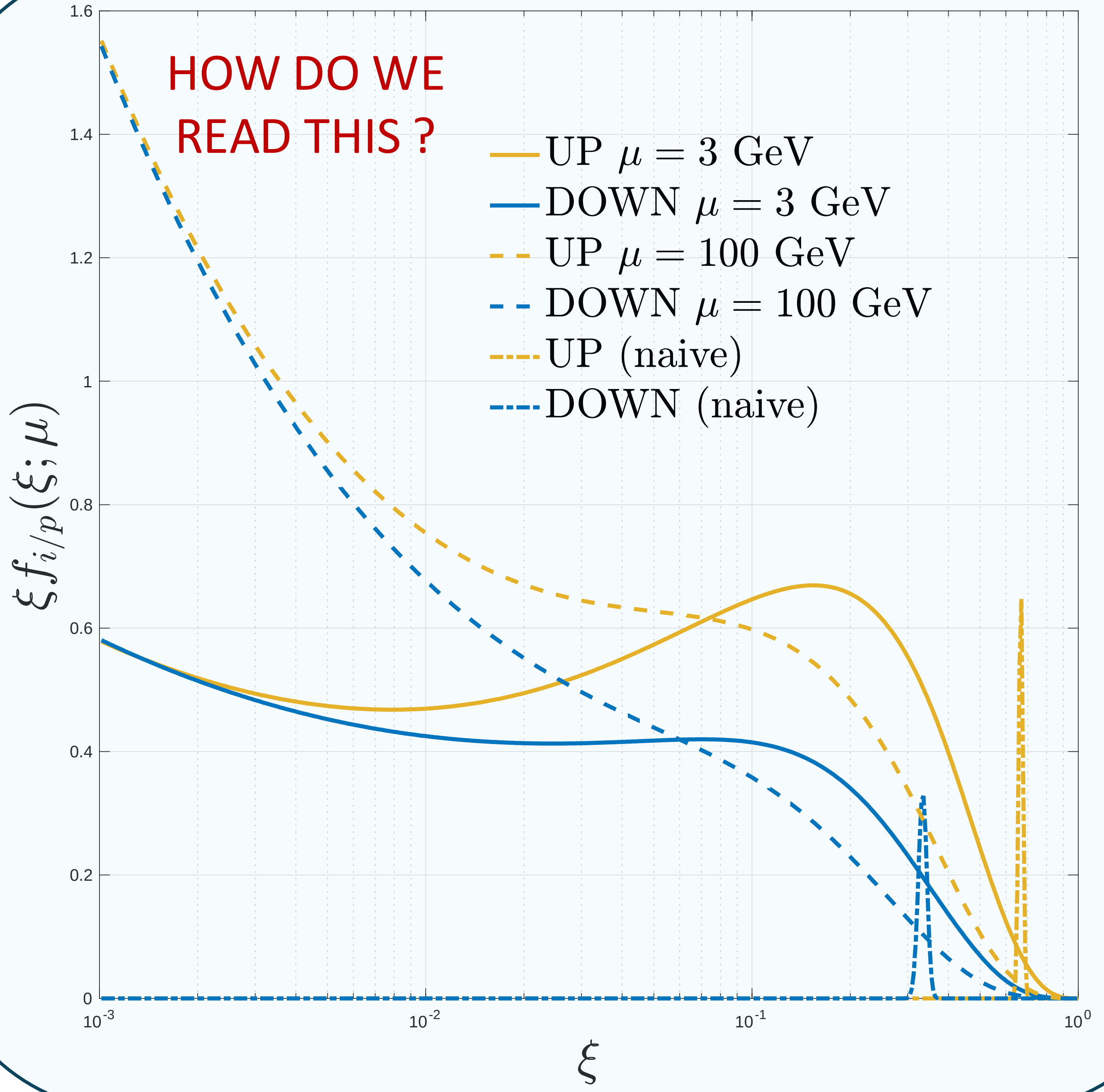
Like the running coupling of QCD they vary with the energy scale (DGLAP equations)

$$f_{i/H}(\xi;\mu) = Z_{ij} \otimes f_{(0)j/H}$$

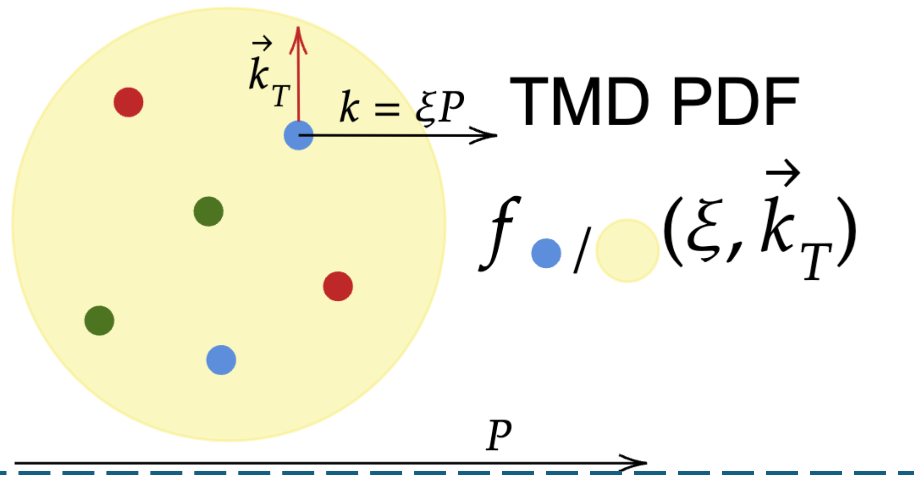
$$\frac{\mathrm{d} f_{i/H}(\xi;\mu)}{\mathrm{d}\log\mu} = 2P_{ij} \otimes f_{j/H}$$

# PDFs from experiments



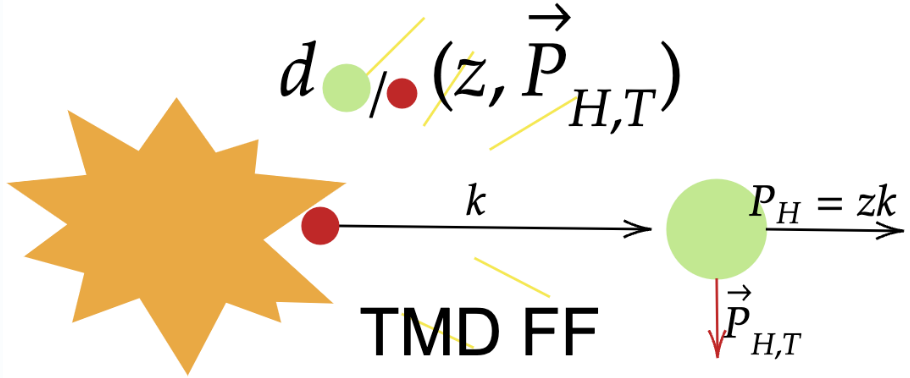


# Generalization to 3D motion? (intuitively)




**Transverse Momentum Dependent (TMD) PDF**:
The parton will generally also move in the transverse direction

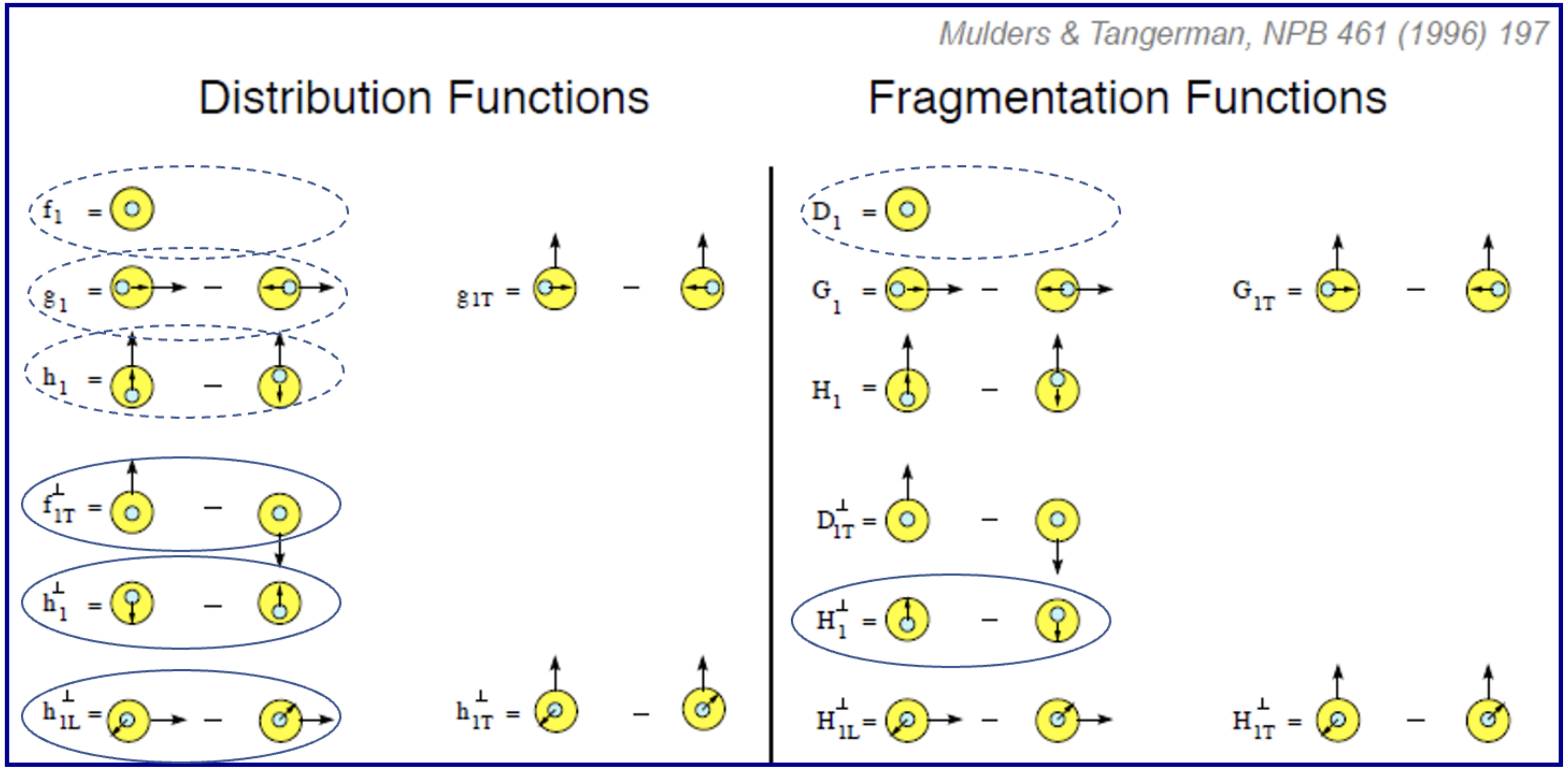
$$f_{j/H}(\xi, \mathbf{k}_T)$$




**Transverse Momentum Dependent (TMD) FF**:
The hadron will generally also move in the transverse direction

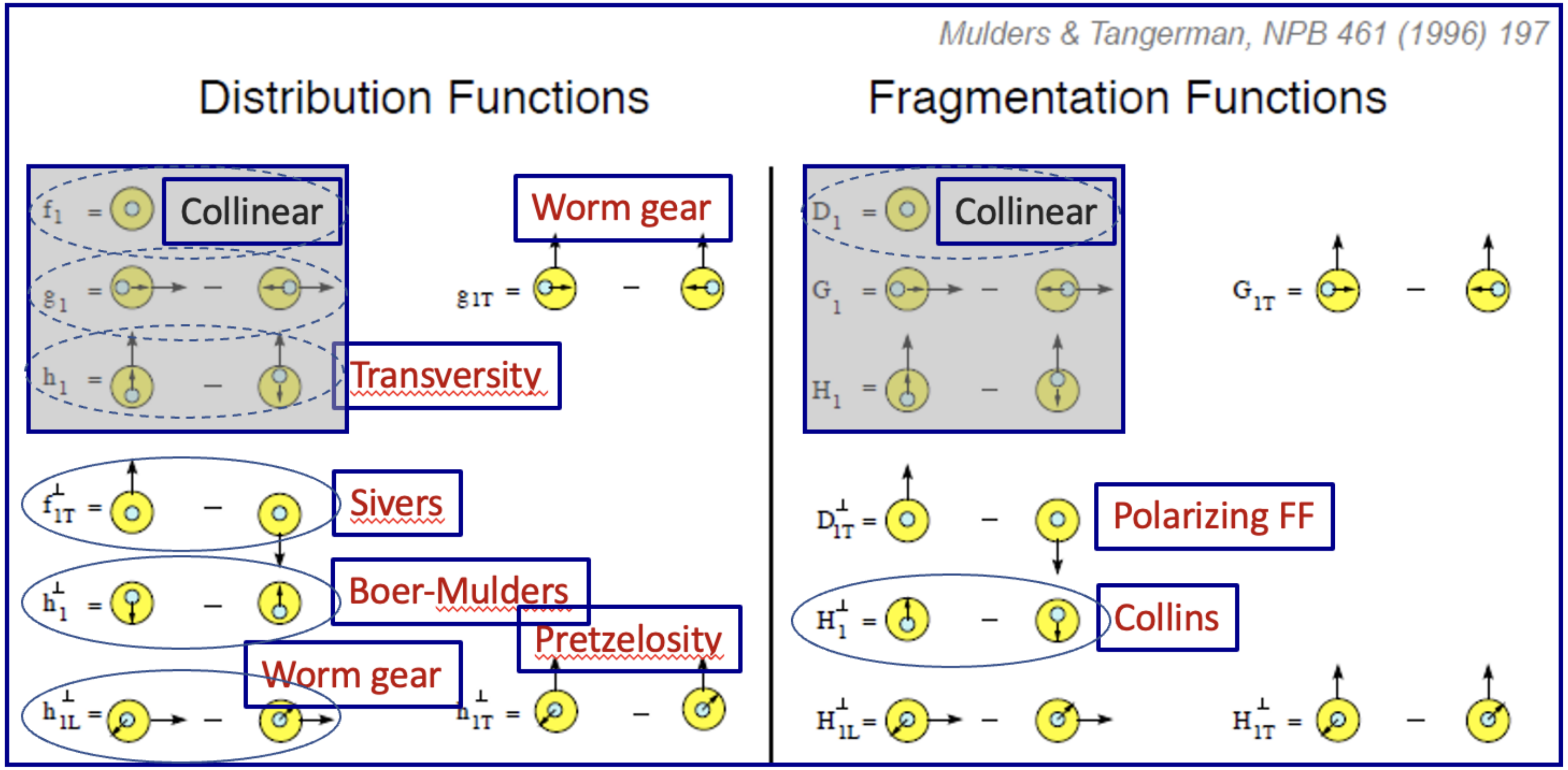
$$D_{H/j}(z, \mathbf{P}_H)$$

# Zoo of distributions




*(P. Mulders, R. Tangerman (1996))*

# Zoo of distributions

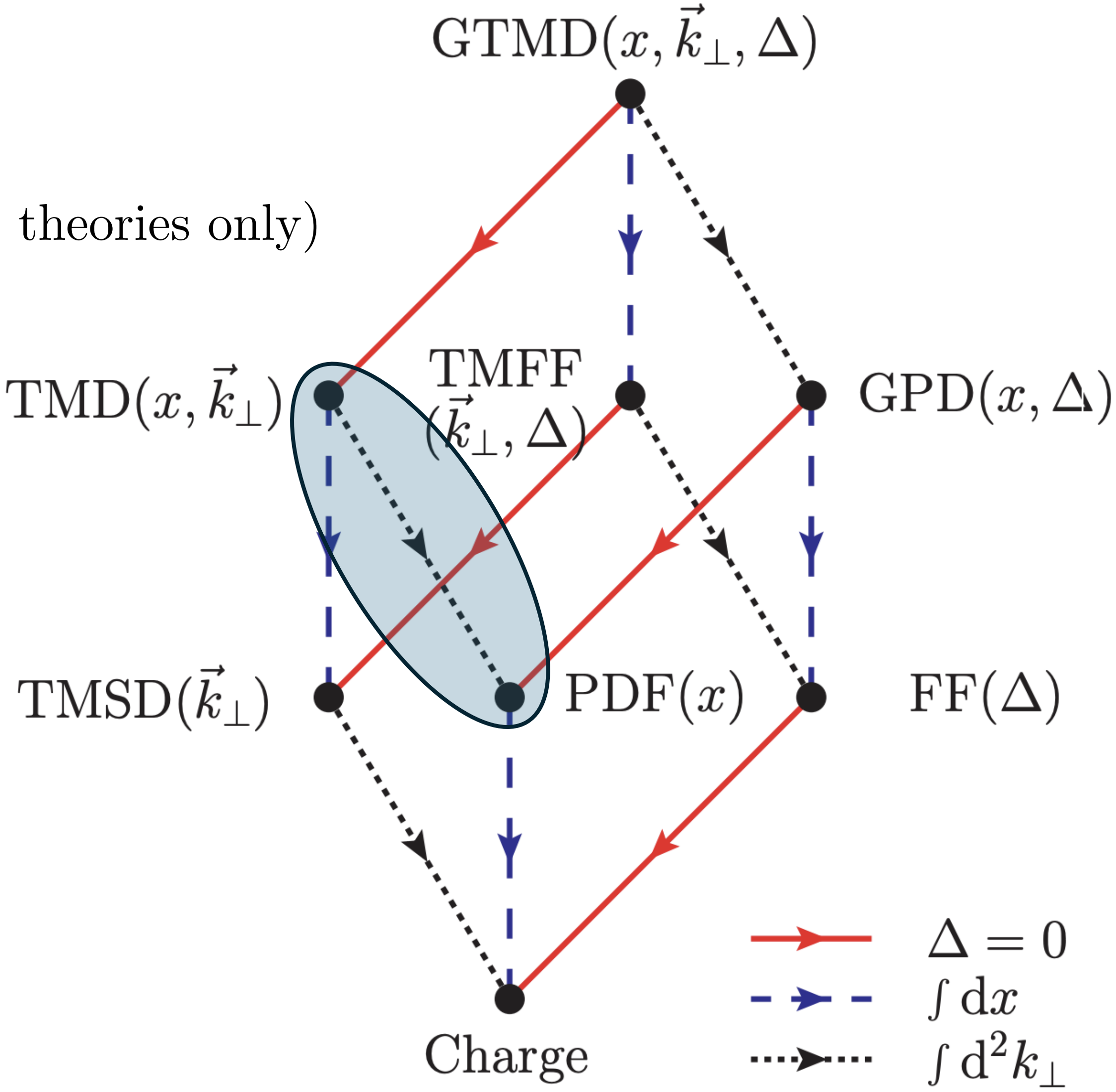



*(P. Mulders, R. Tangerman (1996))*

*They all have operator definitions (regardless of factorization)

# More complications …

$$f_{i/H}(x), f_{i/H}^{\mathrm{TMD}}(x, \boldsymbol{k}_{\mathrm{T}}) \geq 0, \quad \text{(superrenormalizable non gauge theories only)}$$



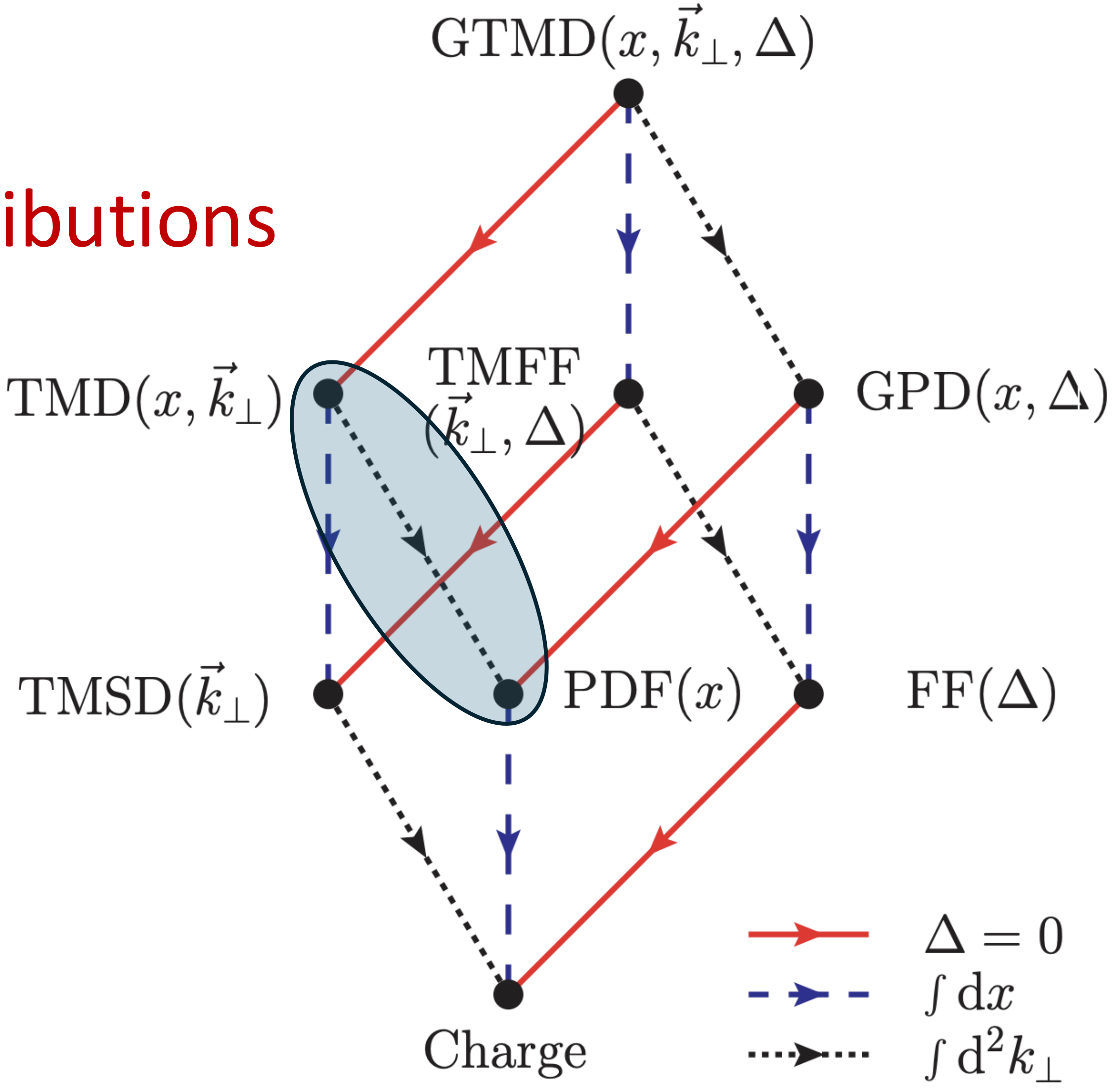

Credits: Lorcé, Pasquini and Vanderhaeghen

# More complications …

They are not (exactly) probability distributions

They can be negative

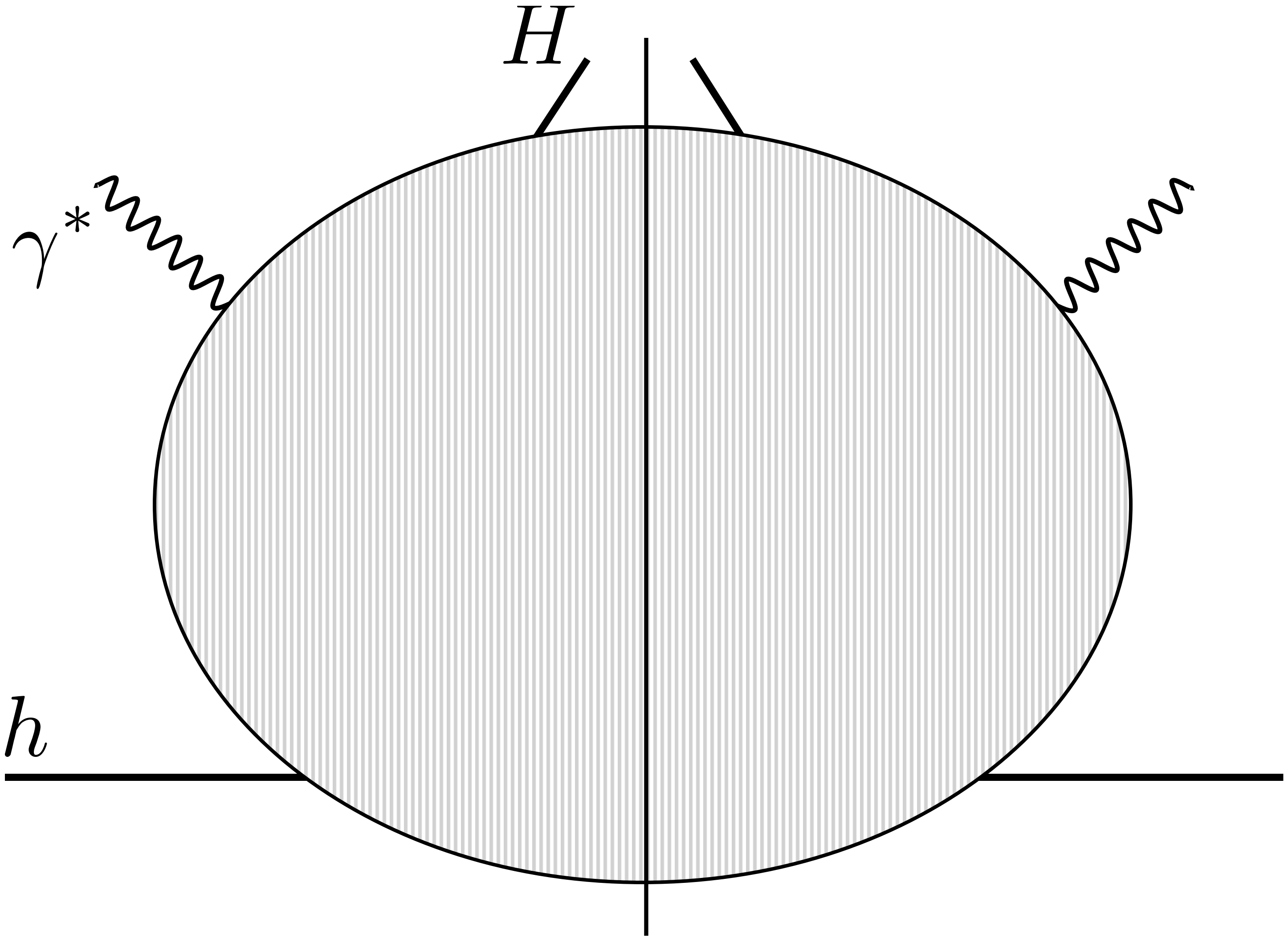
$$f_{i/H}(x;\mu) \neq \int \mathrm{d}^2\boldsymbol{k}_\mathrm{T} f_{i/H}^{\mathrm{TMD}}(x,\boldsymbol{k}_\mathrm{T};\mu,\zeta)$$



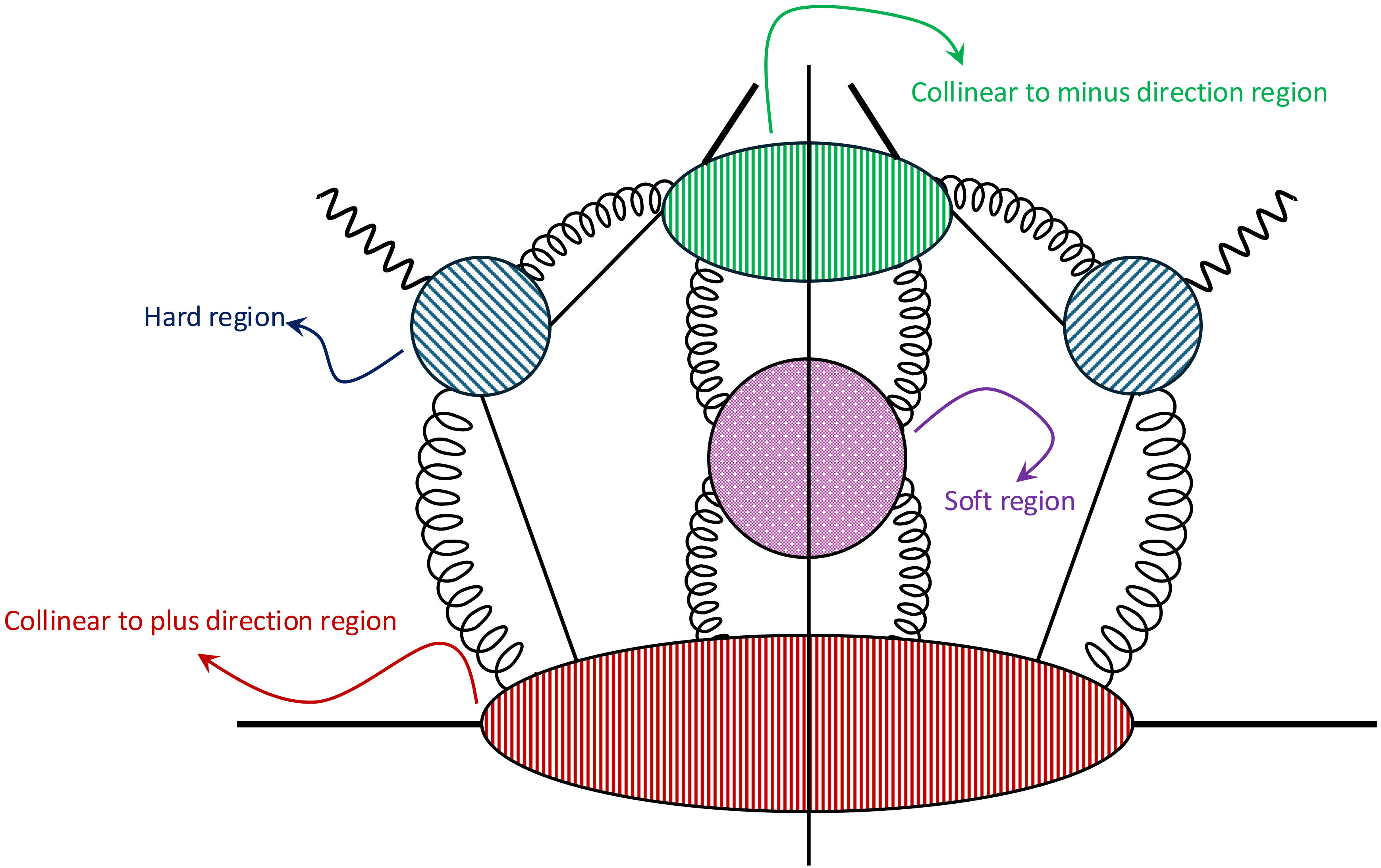

Credits: Lorcé, Pasquini and Vanderhaeghen

# Where we encounter TMDs

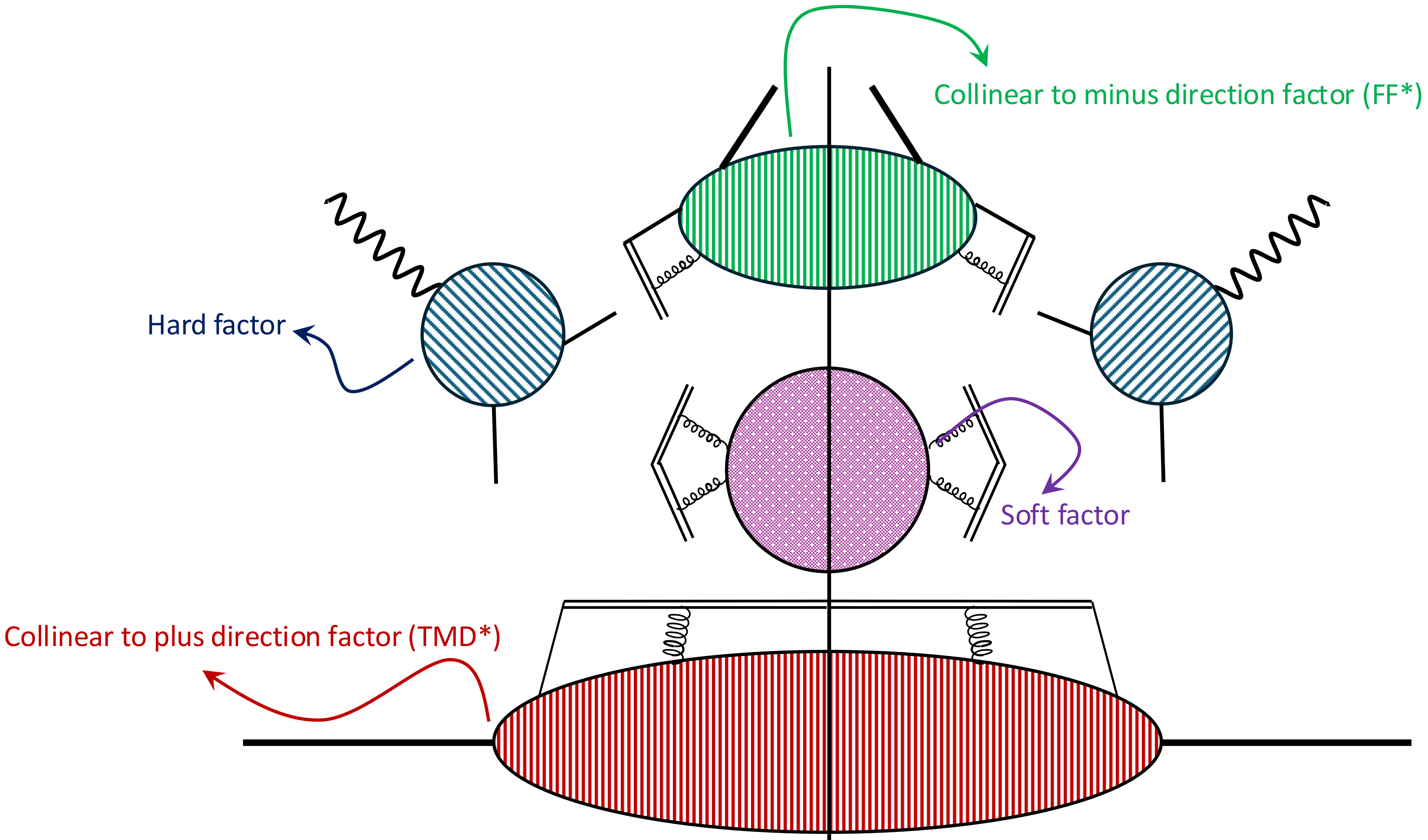
# SIDIS: Cross section $l + h \to \gamma^* + h \to l + H + X$



# SIDIS: leading power regions



# SIDIS: leading power factorization




$$H \int \mathrm{d}^2\boldsymbol{k}_{1\mathrm{T}} \mathrm{d}^2\boldsymbol{k}_{2\mathrm{T}} f_{j/p}\left(x, \boldsymbol{k}_{1\mathrm{T}}; \mu, \sqrt{\zeta}\right) D_{h/j}\left(z, z\boldsymbol{k}_{2\mathrm{T}}; \mu, \sqrt{\zeta}\right) \delta^{(2)}\left(\boldsymbol{q}_{\mathrm{T}} + \boldsymbol{k}_{1\mathrm{T}} - \boldsymbol{k}_{2\mathrm{T}}\right)$$

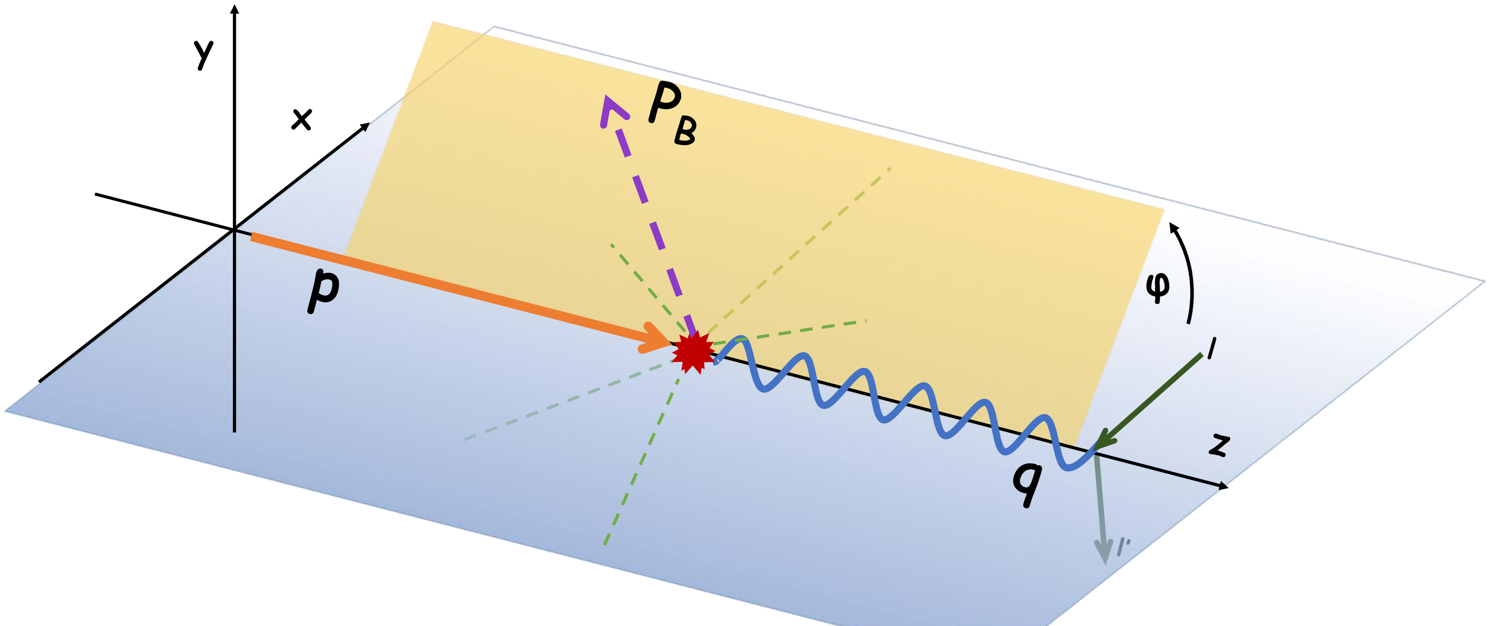
# SIDIS

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}q_{\mathrm{T}}^2} = \underbrace{W_{\mathrm{SIDIS}}}_{q_{\mathrm{T}} \ll Q} + \underbrace{Y_{\mathrm{SIDIS}}}_{q_{\mathrm{T}} \sim Q} + \mathcal{O}\left(\frac{m^2}{Q^2}\right)$$

FO collinear factorization

$$\mathrm{FO}_{\mathrm{SIDIS}} - \mathrm{ASY}_{\mathrm{SIDIS}}$$

$$H \int \mathrm{d}^2\boldsymbol{k}_{1\mathrm{T}} \mathrm{d}^2\boldsymbol{k}_{2\mathrm{T}} f_{j/p}\left(x, \boldsymbol{k}_{1\mathrm{T}}; \mu, \sqrt{\zeta}\right) D_{h/j}\left(z, z\boldsymbol{k}_{2\mathrm{T}}; \mu, \sqrt{\zeta}\right) \delta^{(2)}\left(\boldsymbol{q}_{\mathrm{T}} + \boldsymbol{k}_{1\mathrm{T}} - \boldsymbol{k}_{2\mathrm{T}}\right)$$



# What we know

# We like to work with the Fourier transform of the TMD

$$\tilde{f}_{i,H}(x;\boldsymbol{b}_T;\mu,\zeta) = \int \mathrm{d}^2\boldsymbol{k}_T e^{-i\boldsymbol{k}_T\cdot\boldsymbol{b}_T} f_{i,H}(x,\boldsymbol{k}_T;\mu,\zeta)$$

$\boldsymbol{b}_T$-space

$\boldsymbol{k}_T$-space

# 1: The tail of the TMDs

At large TM ($k_T$)/ small $b_T$ the TMDs are **uniquely determined** by an OPE expansion in terms of collinear PDFs/FFs

$$f_{i/H}(x, b_T; \mu, \zeta) = \widetilde{C}_{ij}(x, b_T; \mu, \zeta) \otimes f_{j/H}(x; \mu) + \mathcal{O}(m b_T)$$

Perturbatively calculable
(3D info)

Usual PDFs
(1D info)

# 2: The RG equations

$$\frac{\partial \ln \tilde{f}_{j/p}\left(x, \boldsymbol{b}_\mathrm{T}; \mu, \sqrt{\zeta}\right)}{\partial \ln \sqrt{\zeta}} = \tilde{K}\left(\boldsymbol{b}_\mathrm{T}; \mu\right)$$

$$\frac{\mathrm{d} \ln \tilde{f}_{j/p}\left(x, \boldsymbol{b}_\mathrm{T}; \mu, \sqrt{\zeta}\right)}{\mathrm{d} \ln \mu} = \gamma\left(\alpha_S(\mu); \mu/\sqrt{\zeta}\right)$$

$$\frac{\mathrm{d}\tilde{K}\left(\boldsymbol{b}_\mathrm{T}; \mu\right)}{\mathrm{d} \ln \mu} = -\gamma_K\left(\alpha_S(\mu)\right)$$

**Perturbative**

**Perturbative only at small $b_T$**

$$\tilde{f}_{j/p}\left(x, \boldsymbol{b}_\mathrm{T}; \mu, \sqrt{\zeta}\right) = \tilde{f}_{j/p}\left(x, \boldsymbol{b}_\mathrm{T}; \mu_0, \sqrt{\zeta_0}\right) \times$$
$$\times \exp\left\{\int_{\mu_0}^{\mu} \frac{\mathrm{d}\mu'}{\mu'}\left[\gamma\left(\alpha_S(\mu'); 1\right) - \ln\left(\frac{\sqrt{\zeta}}{\mu'}\right)\gamma_K\left(\alpha_S(\mu')\right)\right] + \ln\left(\frac{\sqrt{\zeta}}{\sqrt{\zeta_0}}\tilde{K}\left(\boldsymbol{b}_\mathrm{T}; \mu_0\right)\right)\right\}$$

# 3: The integral relation

The integral over 2 dimensions of a 3D distribution doesn't give back the 1D distribution

(3D info) (1D info) violation

$$\int_0^{\mu_c} \mathrm{d}^2\boldsymbol{k}_T\, f(x, k_T; \mu, \mu^2) = f(x;\mu) + \Delta(x;\mu,\mu_c) + p.s.$$

→ Pseudoprobability interpretation

$$\Delta(x;\mu,\mu_c) = C_\Delta(x;\mu,\mu_c) \otimes f(x;\mu)$$

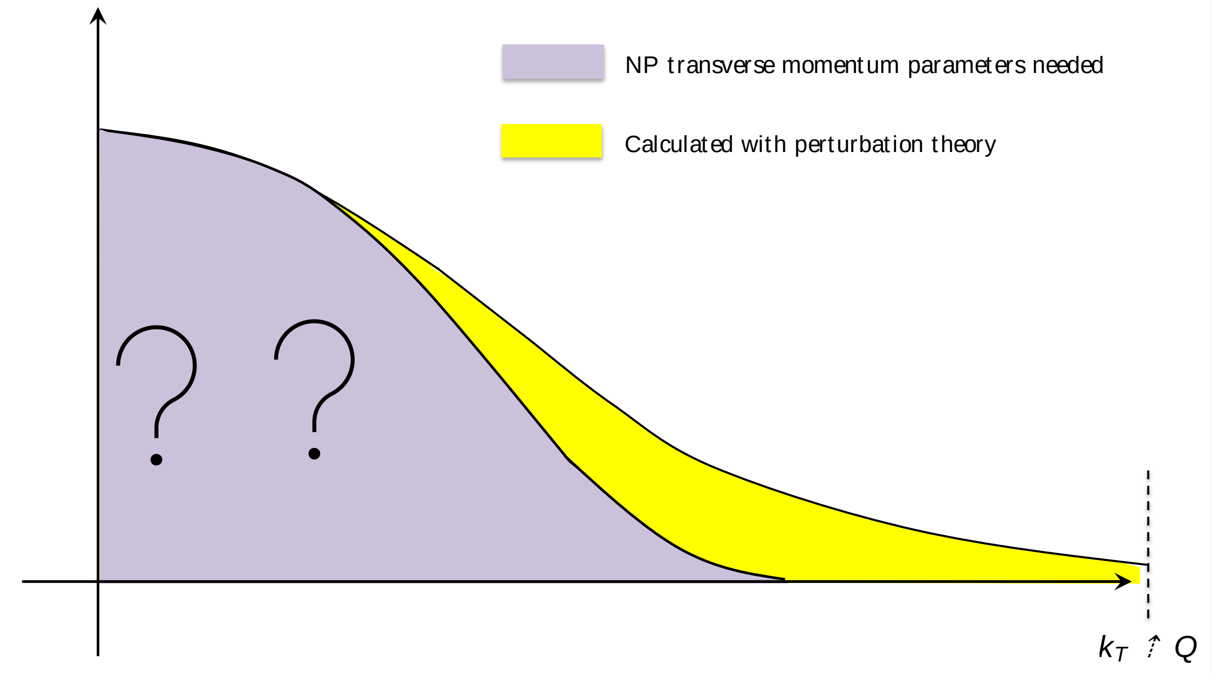
The violation is calculable using Feynman diagrams

→ Perturbatively calculable

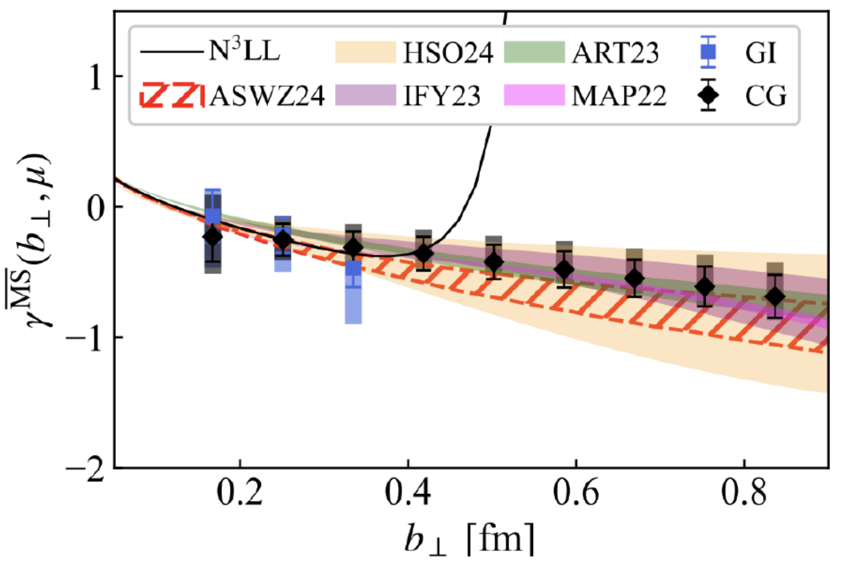
→ $\overline{\mathrm{MS}}$ PDF

# What we don't know

# Nonperturbative content of the TMDs




Small $k_T$ (large $b_T$) region




*Lattice calculation from*
*Bollweg, Gao, Mukherjee, Zhao, (2024), 2403.00664 [hep-lat]*

The Collins-Soper kernel at large $b_T$

# Conventional approach

# Parametrization

$$\tilde{f}_{j/p}(x;\boldsymbol{b}_\mathrm{T};\mu_Q,Q) = \tilde{f}^{\mathrm{OPE}}_{j/p}(x;\boldsymbol{b}_*;\mu_{b_*},\mu_{b_*}) \times$$

$$\times \exp\left\{\int_{\mu_{b_*}}^{\mu_Q}\frac{\mathrm{d}\mu'}{\mu'}\left[\gamma(\alpha_S(\mu');1)-\ln\left(\frac{Q}{\mu'}\right)\gamma_K(\alpha_S(\mu'))\right]+\ln\left(\frac{Q}{\mu_{b_*}}\right)\tilde{K}(\boldsymbol{b}_*;\mu_{b_*})\right\}$$

$$\times \exp\left\{-g_{j/p}(x,\boldsymbol{b}_\mathrm{T})-g_K(\boldsymbol{b}_\mathrm{T})\ln\left(\frac{Q}{Q_0}\right)\right\}$$

Nonperturbative

Perturbatively calculable

Drop this

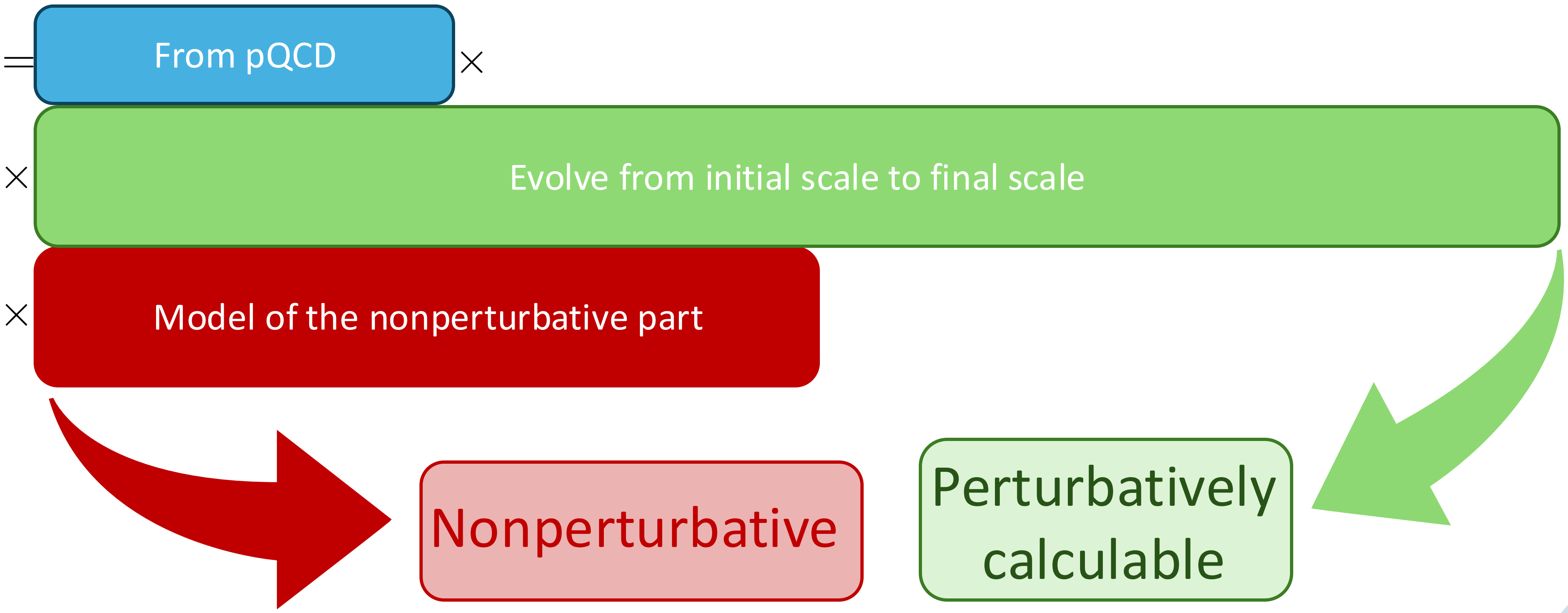
$$\tilde{f}^{\mathrm{OPE}}_{j/p}(x,\boldsymbol{b}_*;\mu_{b_*},\mu_{b_*}) = \widetilde{C}_{j/j'}(x/\xi,\boldsymbol{b}_*;\mu_{b_*},\mu_{b_*})\otimes\tilde{f}_{j'/p}(\xi;\mu_{b_*}) + \mathcal{O}\left(m^2 b^2_{\max}\right)$$

Same for FF

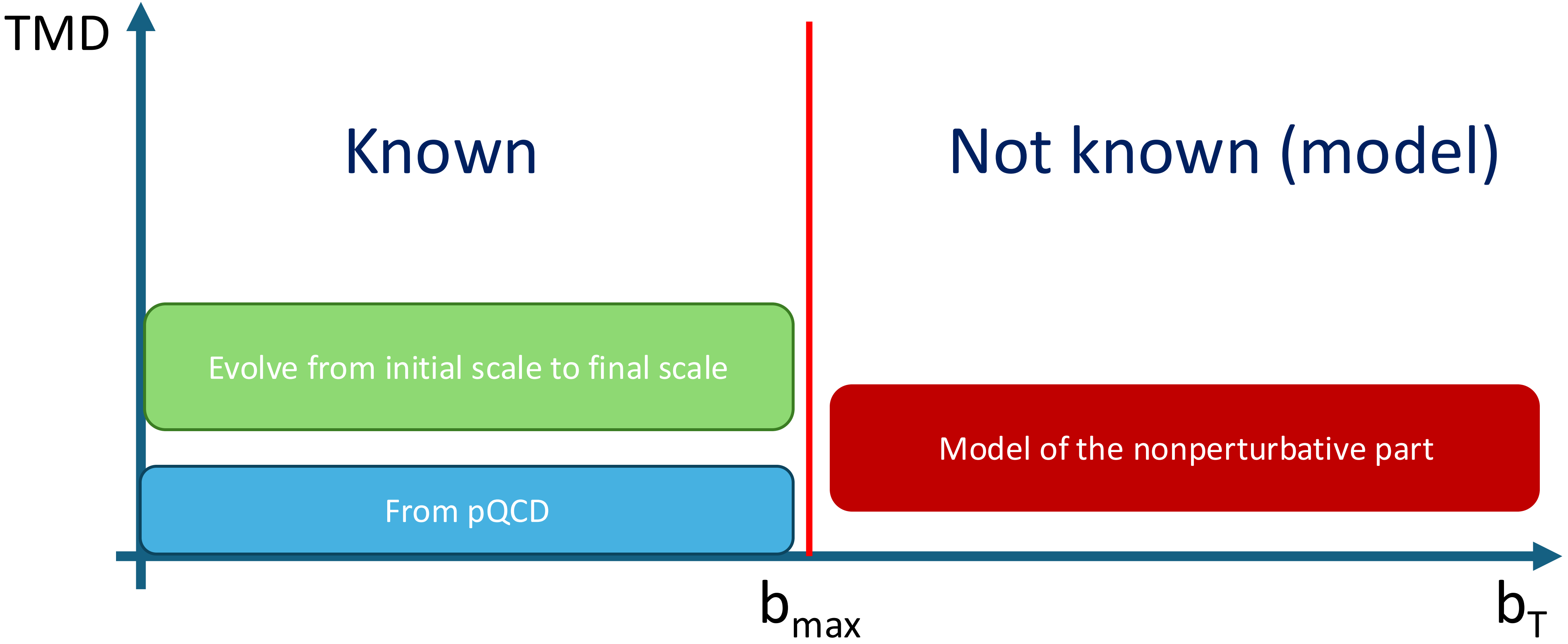
Fixed order collinear factorization

# Parametrization

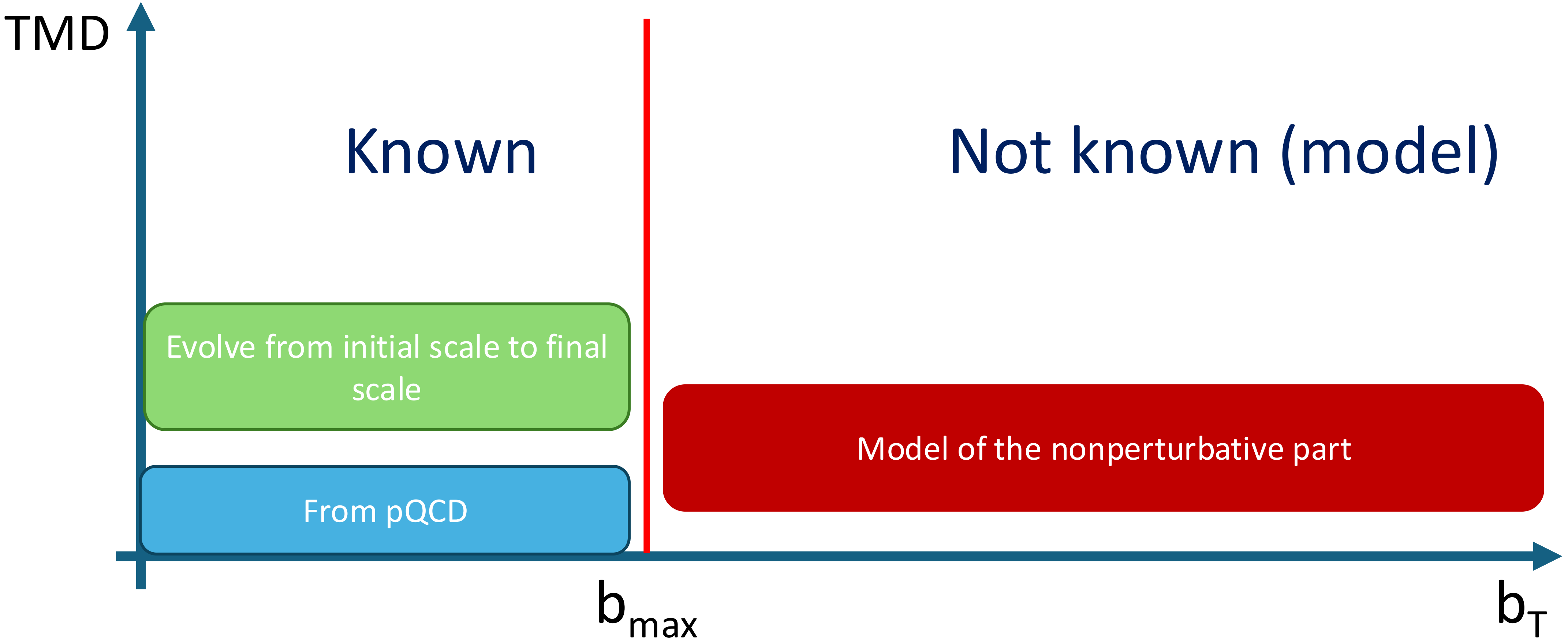
$$\tilde{f}_{j/p}\left(x; \boldsymbol{b}_{\mathrm{T}}; \mu_Q, Q\right) =$$



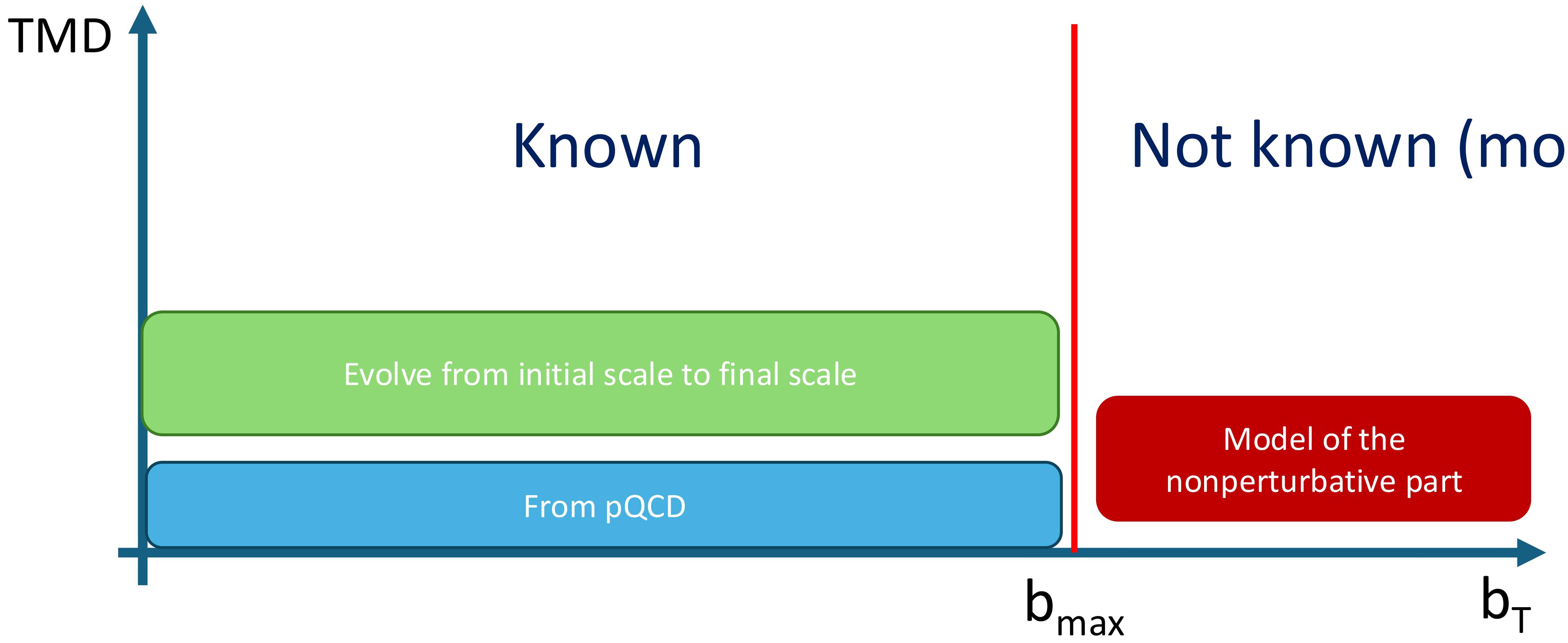
# Separation of contributions



# Separation of contributions



# Separation of contributions

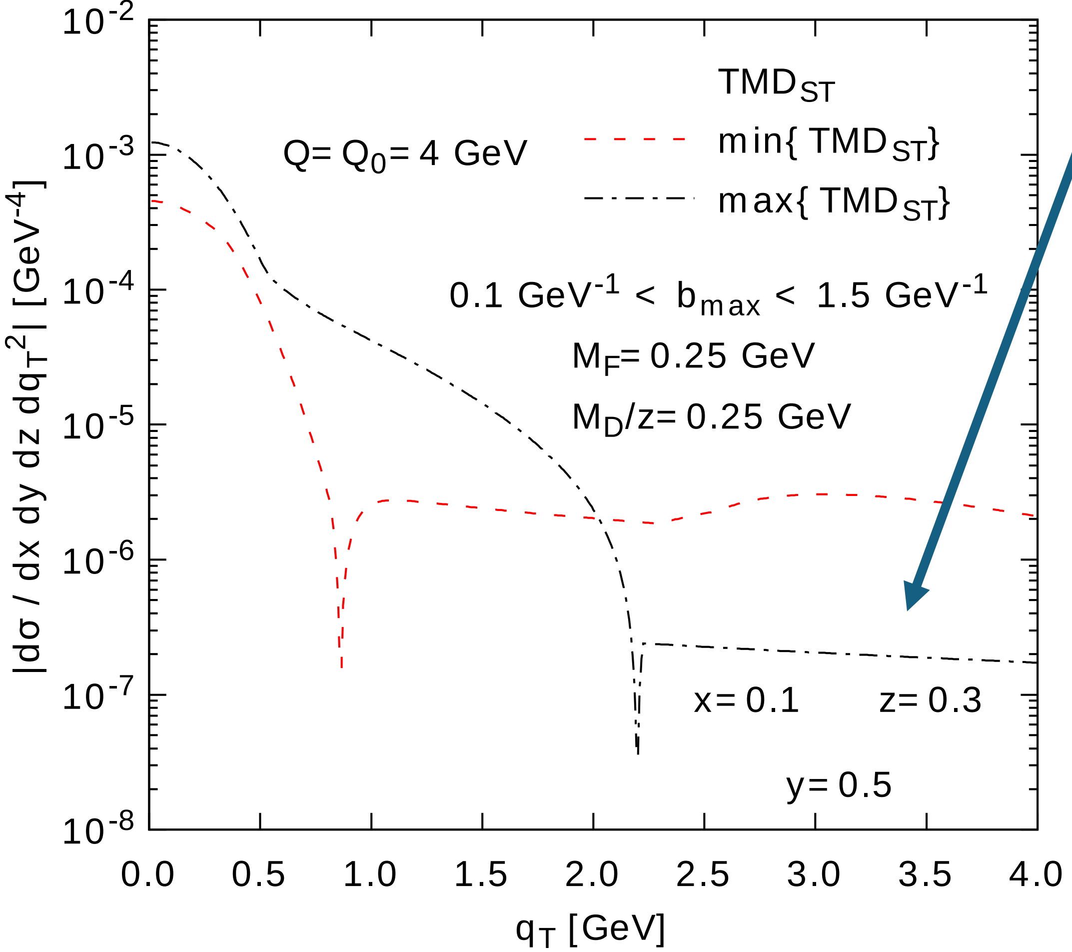


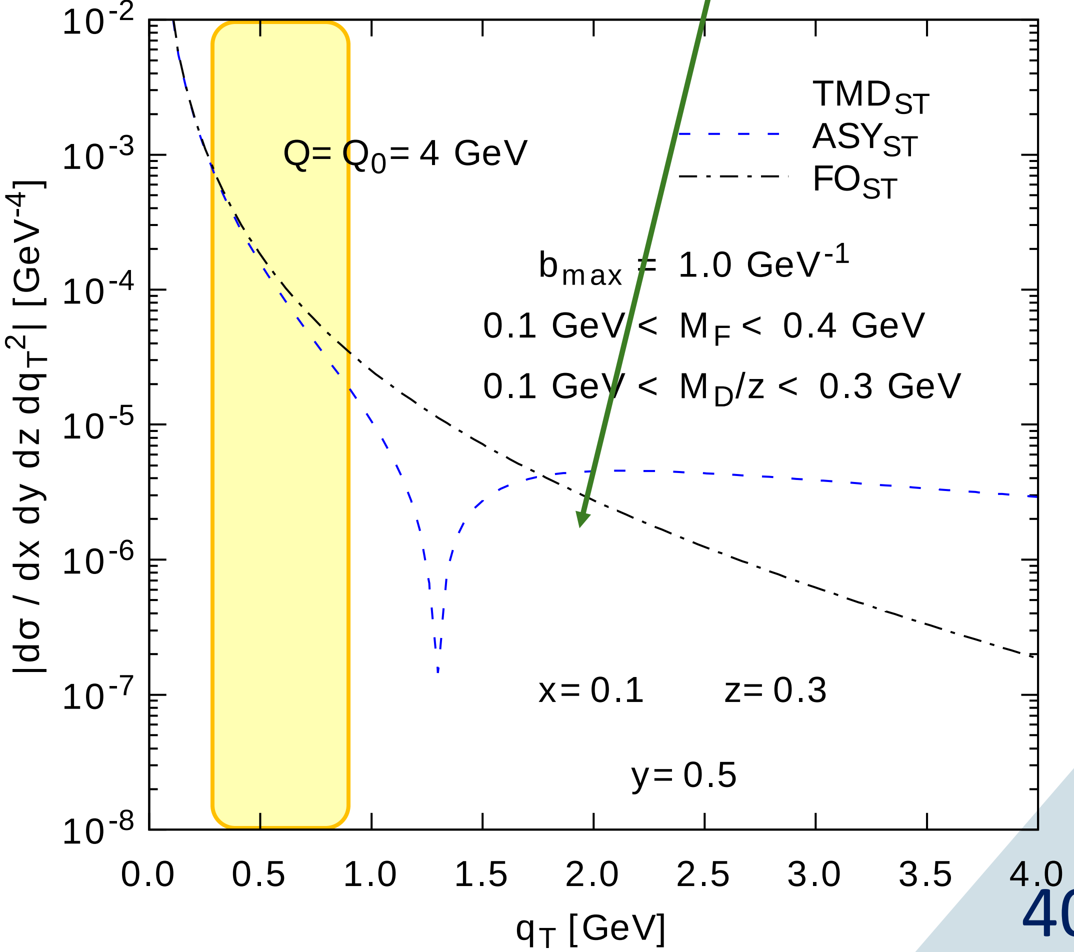
# (Some) Issues with conventional approach

Large $b_{max}$ dependence

What is going on?

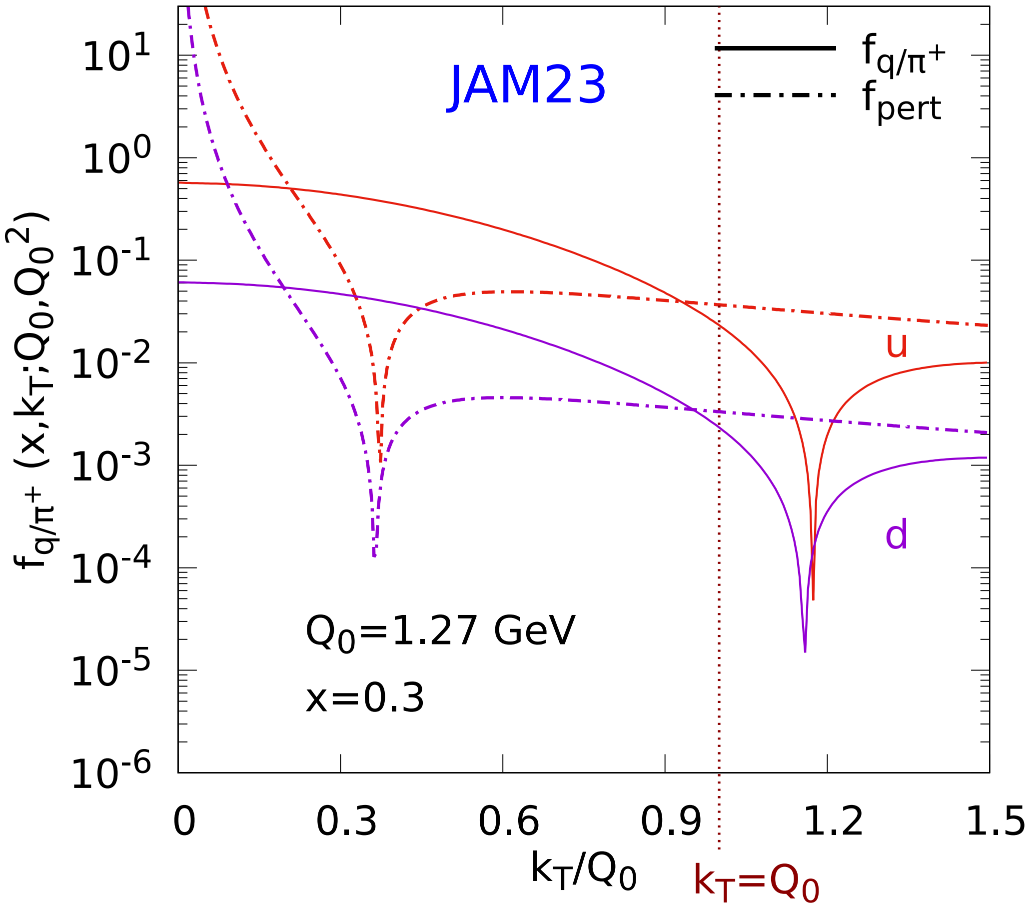
Large $q_T$ inconsistency

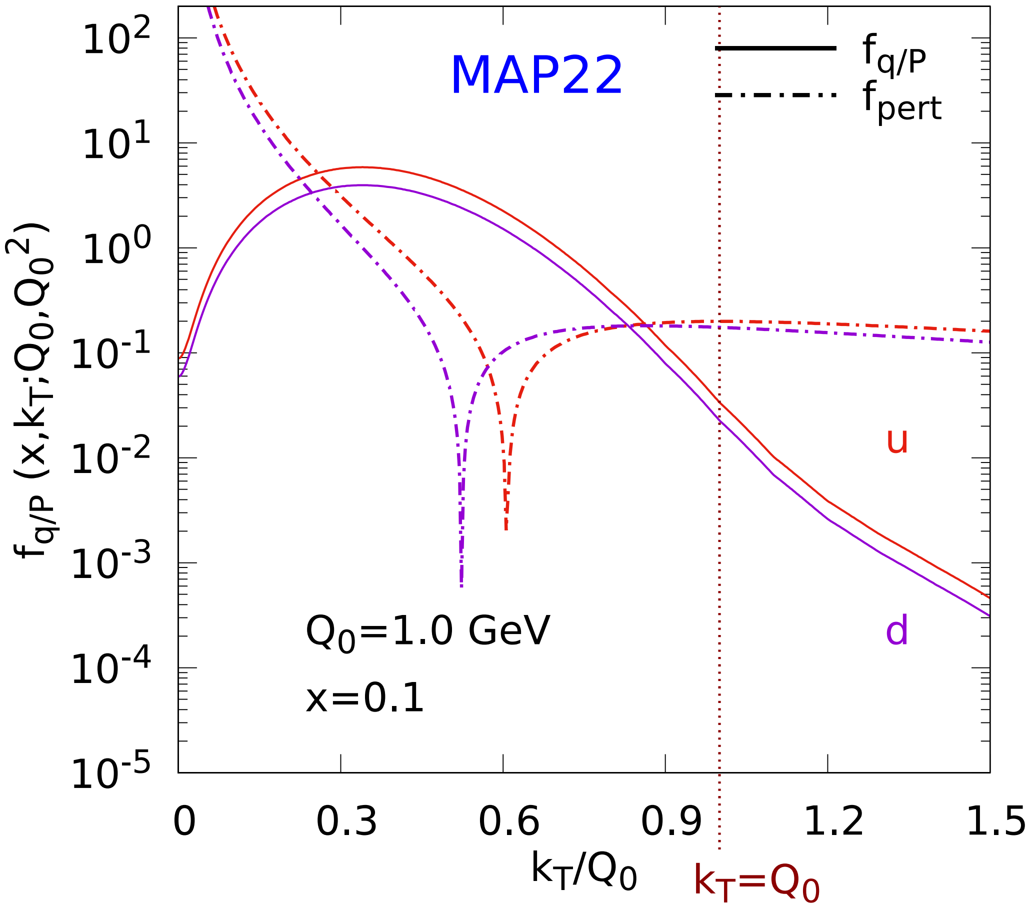


# Other issues

Solid and dash-dotted lines should be the same at $k_T/Q$ of order 1



# Some disadvantages

- What are the effects of the assumptions, ansatz and auxiliary parameters?
- Can we actually tell whether or not we are being consistent with theory?
- Can we maximize the predictive power while minimizing the theoretical uncertainties?

In the standard approach this is either hard or not possible

# Hadron Structure Oriented approach

We build an input scale parametrization that already satisfies the constraints the theory gives us

**OPE expansion at small $b_T$** ( equivalently at large $k_T$ )

**Integral relation** (quasi probabilistic interpretation)

**Derivable from pQCD**

We can do it without the $b_{max}$ or $b_{min}$ issues

Bypassed by imposing integral relation

Solved by using renormalization group improvement

# Hadron Structure Oriented approach

# The main features

- Preservation of all theoretical constraints from the outset
- No need of auxiliary parameters
- Clear distinction between perturbative and nonperturbative
- Easily swappable models

# TMD PDF HSO parametrization at input scale

Fixed order collinear factorization

$\mathcal{O}(\alpha_S)$

Large $k_T$ OPE coefficients

$$f_{\text{inpt,i/p}}(x, \boldsymbol{k}_T; \mu_{Q_0}; Q_0^2) = \frac{1}{2\pi}\frac{1}{k_T^2+m^2}\left[A^f_{i/p}(x;\mu_{Q_0}) + B^f_{i/p}(x;\mu_{Q_0}) \ln\frac{Q_0^2}{k_T^2+m^2}\right] + \frac{1}{2\pi}\frac{1}{k_T^2+m^2} A^{f,g}_{i/p}(x;\mu_{Q_0})$$
$$+ C^f_{i/p} f_{\text{core,i/p}}(x, \boldsymbol{k}_T; Q_0^2)$$

Small $k_T$ model

NP parameters

Such that

$$f^c_{j/p}(x;\mu_Q) \equiv 2\pi \int_0^{k_c} \mathrm{d}k_\mathrm{T}\, k_\mathrm{T} f_{j/p}\left(x, \boldsymbol{k}_\mathrm{T}; \mu_Q, \sqrt{\zeta}\right)$$
$$= f_{j/p}(x;\mu_Q) + \Delta_{j/p}(x;\mu_Q, k_c) + \text{p.s.}$$

# TMD PDF HSO parametrization at input scale

Fixed order collinear factorization

$\mathcal{O}\left(\alpha_S\right)$

Large $k_T$ OPE coefficients

$f_{\text{inpt,i/p}}(x, \boldsymbol{k}_T; \mu_{Q_0}; Q_0^2) =$ From pQCD (modified)

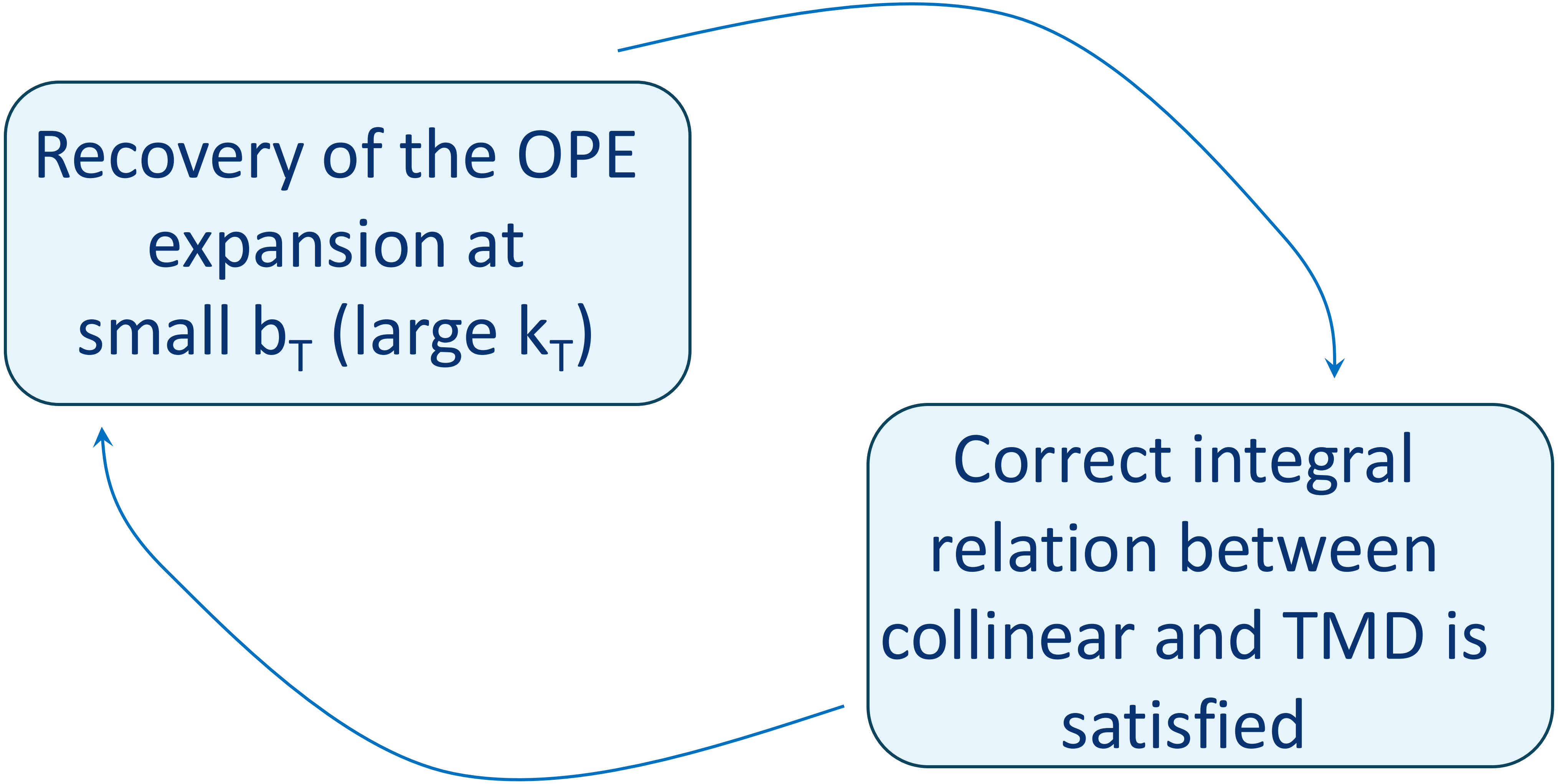
$+ C_{i/p}^{f}$ NP model

Such that

$$f^c_{j/p}\left(x;\mu_Q\right) \equiv 2\pi \int_0^{k_c} \mathrm{d}k_\mathrm{T}\, k_\mathrm{T} f_{j/p}\left(x, \boldsymbol{k}_\mathrm{T}; \mu_Q, \sqrt{\zeta}\right)$$

$$= f_{j/p}\left(x;\mu_Q\right) + \Delta_{j/p}\left(x;\mu_Q, k_c\right) + \text{p.s.}$$

# The role of C (no matter the NP model)



# Choose "core" models (examples)

$$f^{\text{Gauss}}_{\text{core},i/p}\left(x,\boldsymbol{k}_{\text{T}};Q_0^2\right)=\frac{e^{-k_{\text{T}}^2/M_F^2}}{\pi M_F^2}$$

$$D^{\text{Gauss}}_{\text{core},h/j}\left(z,z\boldsymbol{k}_{\text{T}};Q_0^2\right)=\frac{e^{-z^2k_{\text{T}}^2/M_D^2}}{\pi M_D^2}$$

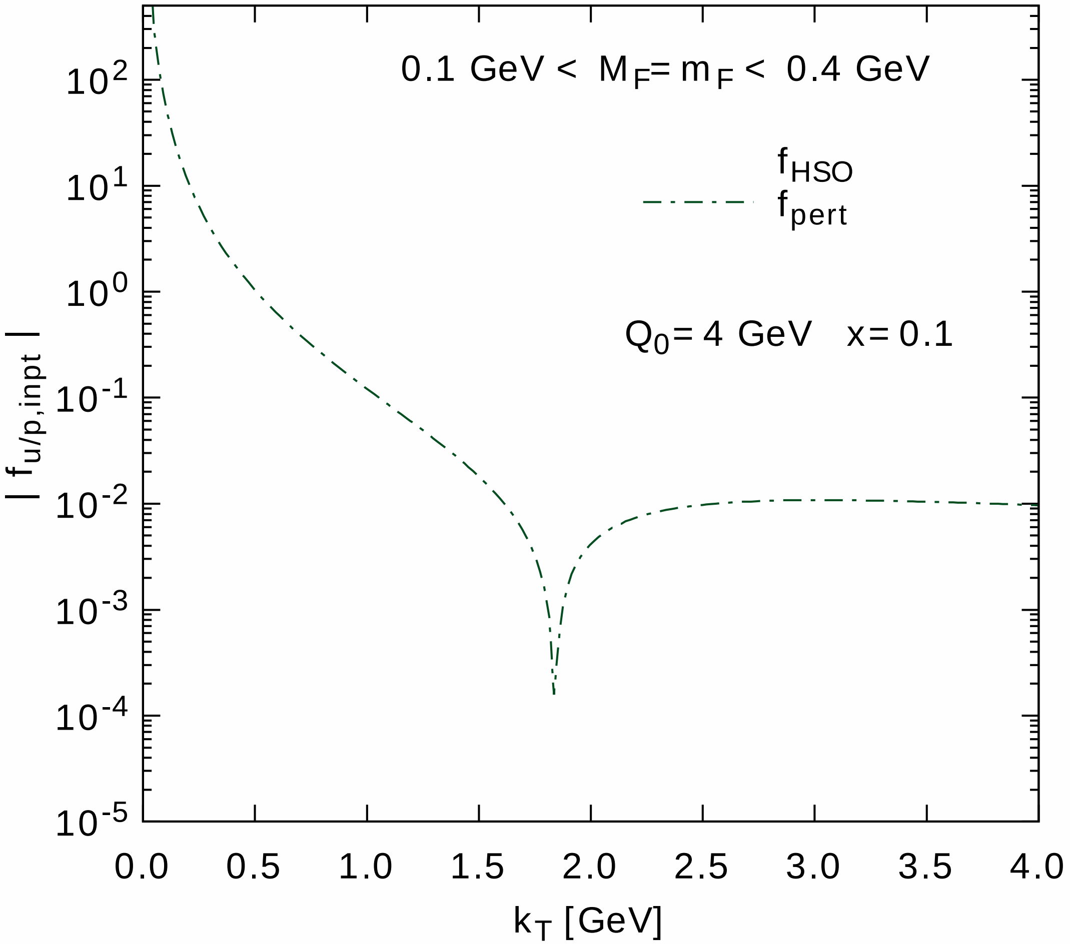
Gaussian "core" models

Spectator-like "core" models

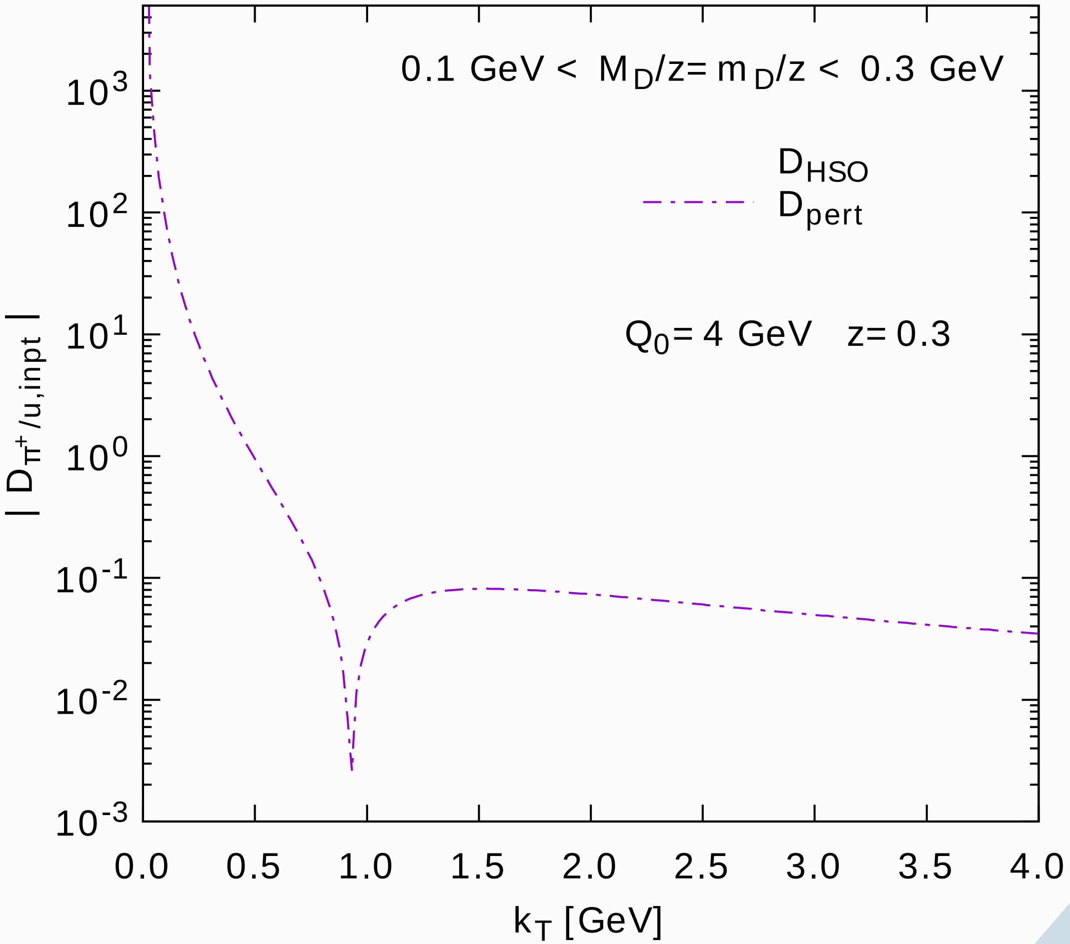
$$f^{\text{Spect}}_{\text{core},j/p}\left(x,\boldsymbol{k}_{\text{T}};Q_0^2\right)=\frac{6M_{0F}^6}{\pi\left(2M_F^2+M_{0F}^2\right)}\frac{M_F^2+k_{\text{T}}^2}{\left(M_{0F}^2+k_{\text{T}}^2\right)^4}$$

$$D^{\text{Spect}}_{\text{core},h/j}\left(z,z\boldsymbol{k}_{\text{T}};Q_0^2\right)=\frac{2M_{0D}^4}{\pi\left(M_D^2+M_{0D}^2\right)}\frac{M_D^2+z^2k_{\text{T}}^2}{\left(M_{0D}^2+z^2k_{\text{T}}^2\right)^3}$$

## Up-quark from Proton TMD pdf



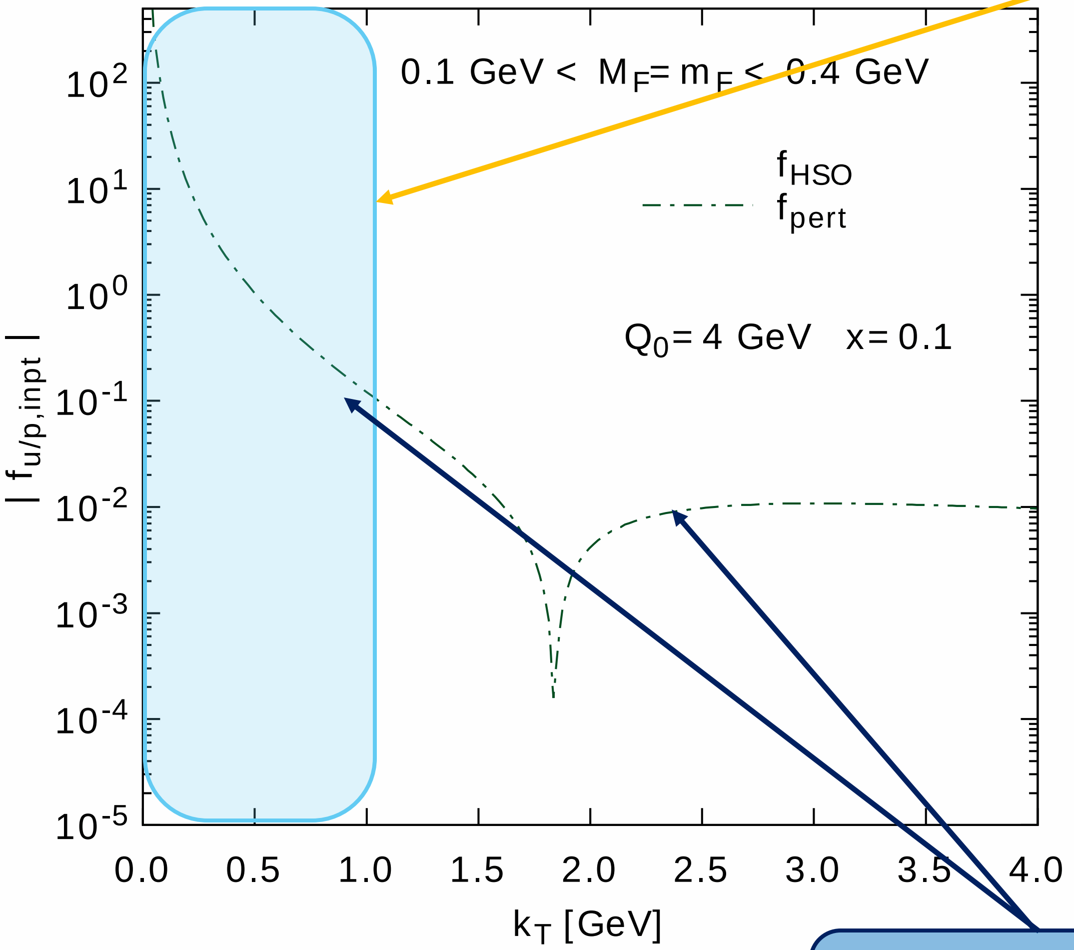

## $\pi^+$ from Up-quark TMD ff

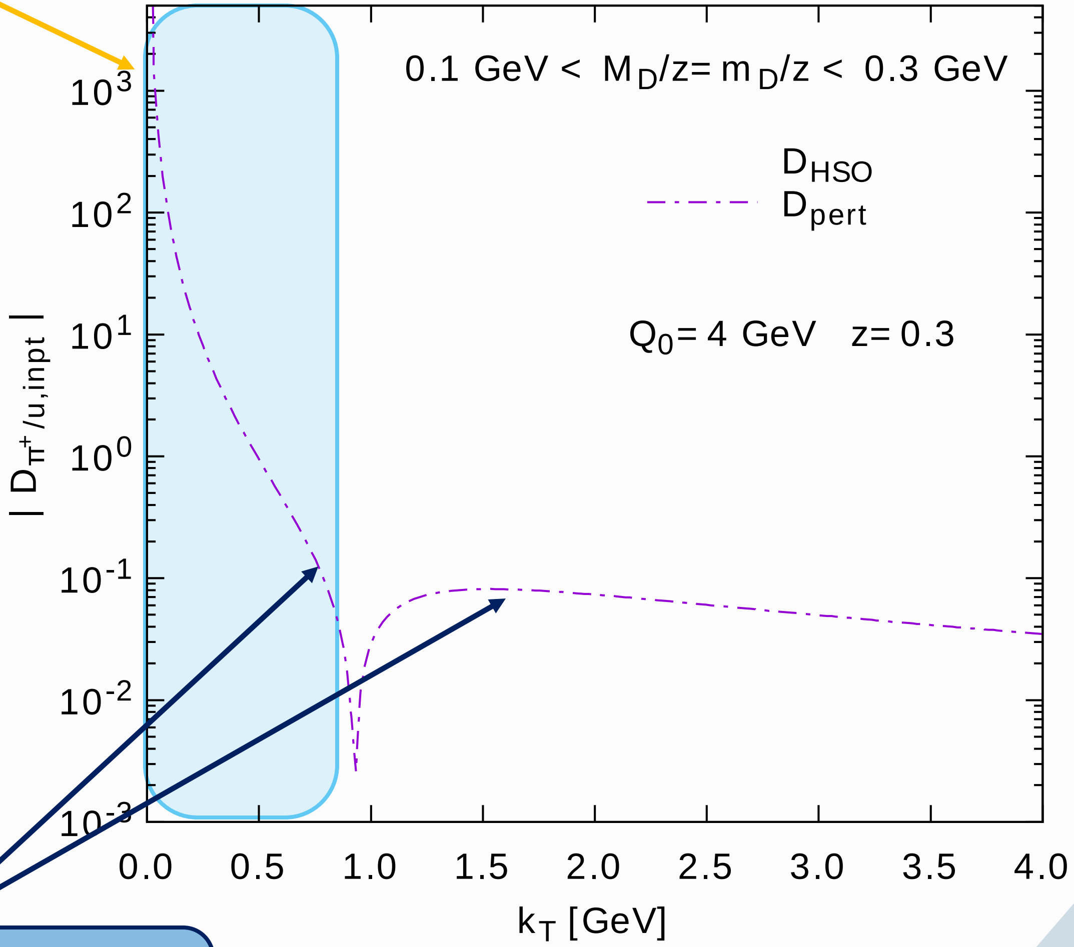


# Up-quark from Proton TMD pdf

# $\pi^+$ from Up-quark TMD ff

Model dependence

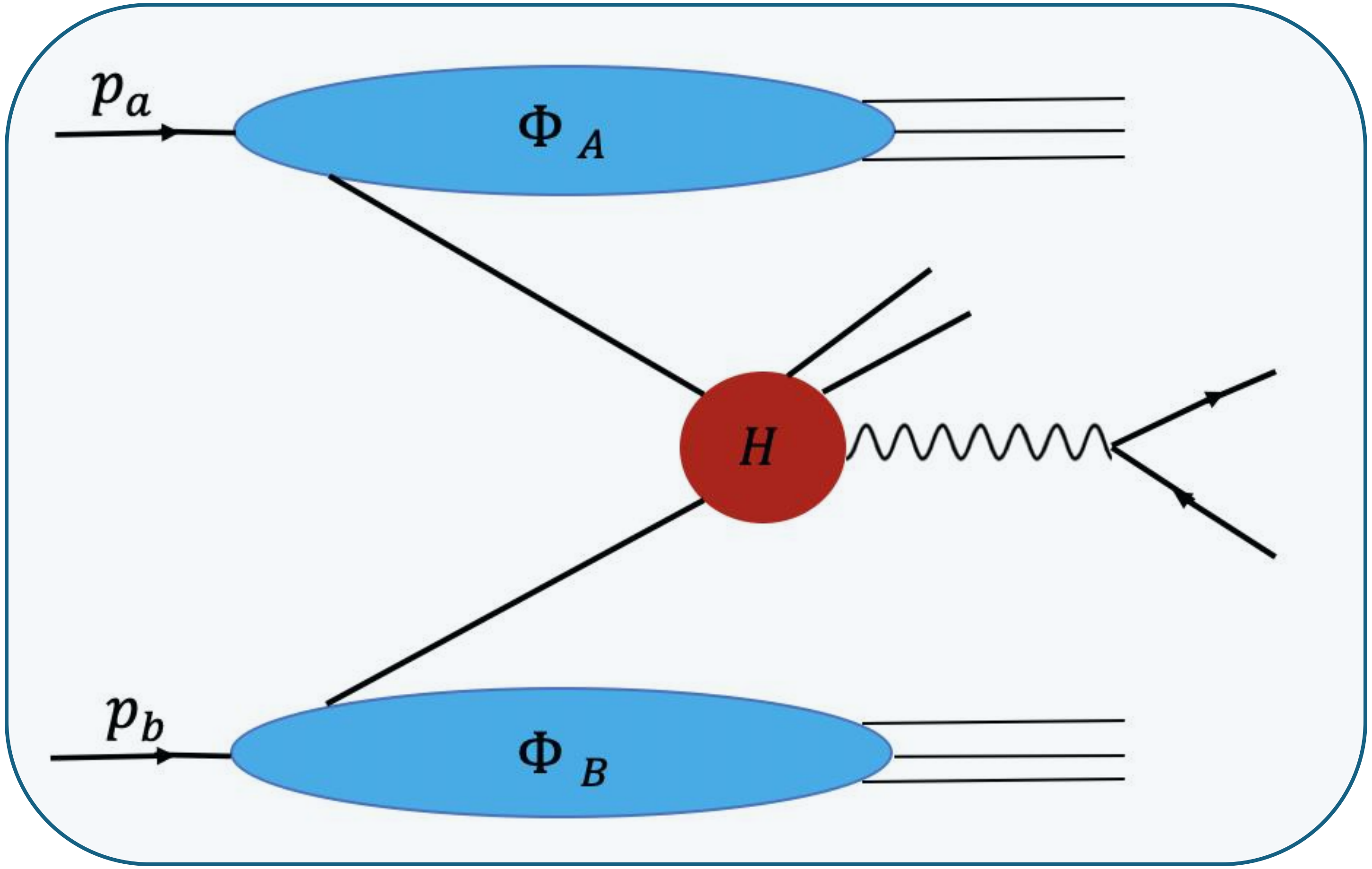



Smooth transition

# A first phenomenological implementation

# Drell-Yan

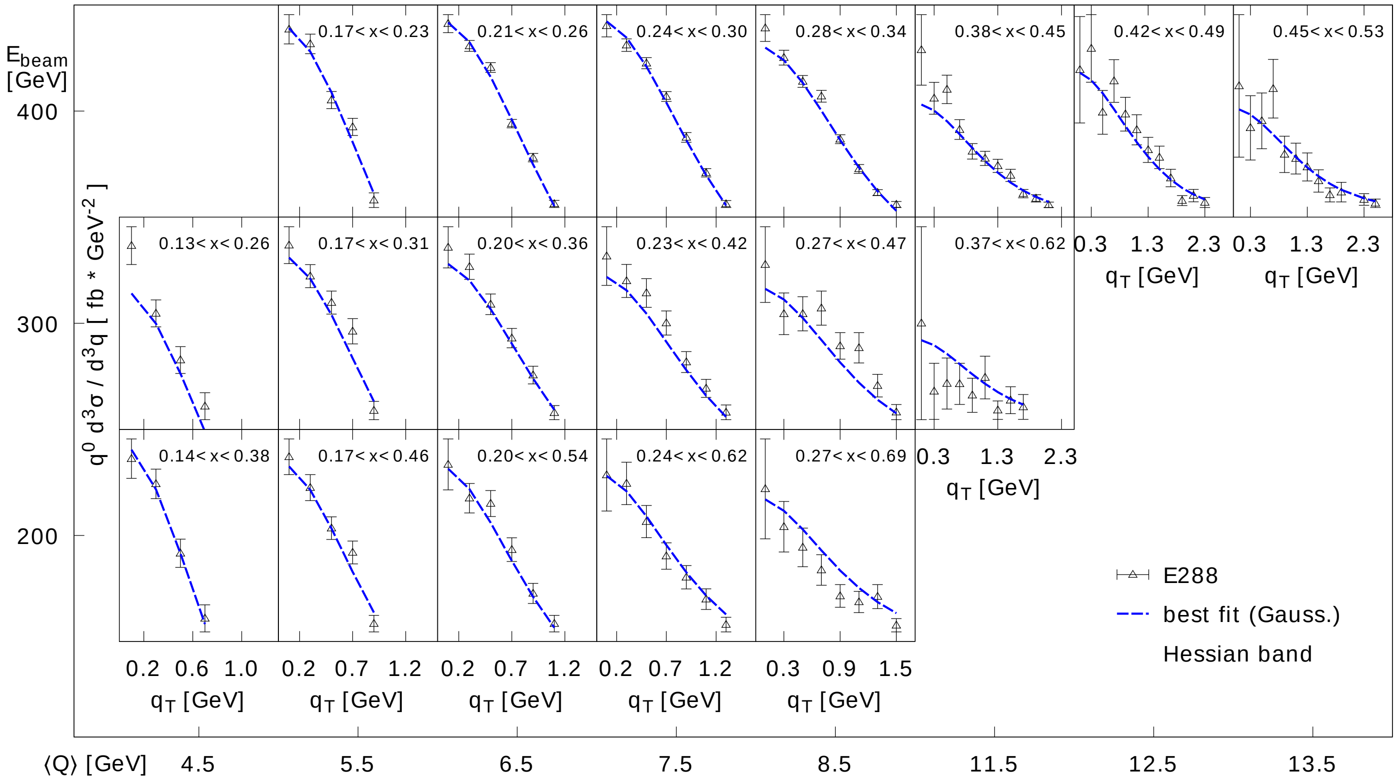



$$W^{\mu\nu}_{\mathrm{DY}}\left(x_a, x_b, Q, \boldsymbol{q}_T\right) =$$

$$= \sum_j H^{\mu\nu}_{j,\mathrm{DY}} \int \frac{\mathrm{d}^2\boldsymbol{b}_T}{(2\pi)^2} e^{i\boldsymbol{q}_T\cdot\boldsymbol{b}_T}\, \tilde{f}_{j/h_a}\left(x_a, \boldsymbol{b}_T; \mu_Q; Q^2\right) \tilde{f}_{\bar{j}/h_b}\left(x_b, \boldsymbol{b}_T; \mu_Q; Q^2\right)$$

$$+ \left(a \longleftrightarrow b\right) + \mathcal{O}\left(\frac{q_T}{Q}, \frac{m}{Q}\right)$$

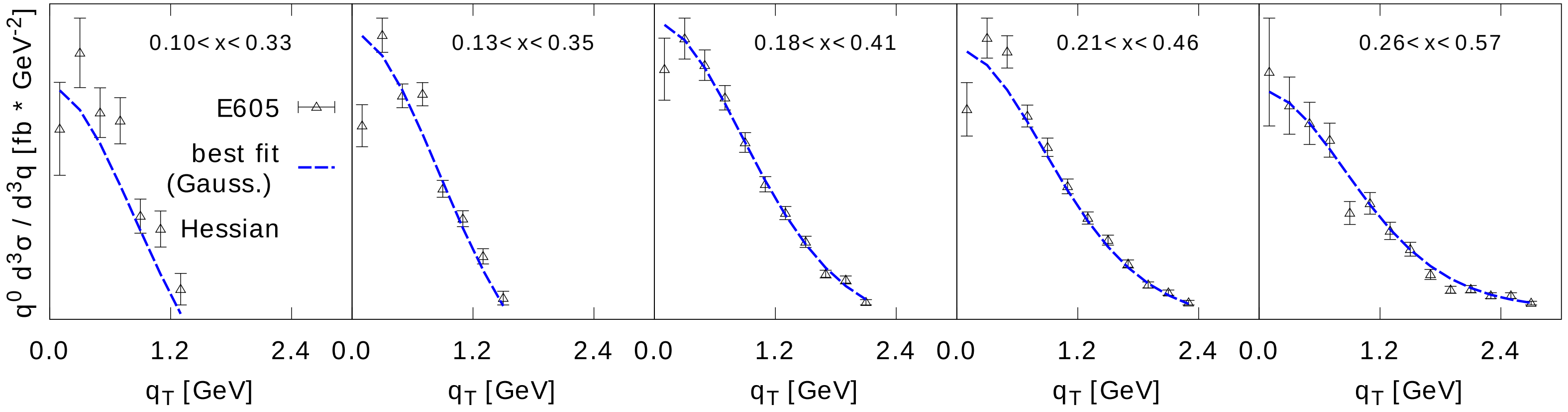
# Low Q fit results




**Gaussian fits**

| | E288 (130 pts.) | E605 (52 pts.) |
|---|---|---|
| $\chi^2_{\text{dof}}$ | 1.04 | 1.68 |

Just 4 parameters for now




Spectator model too:

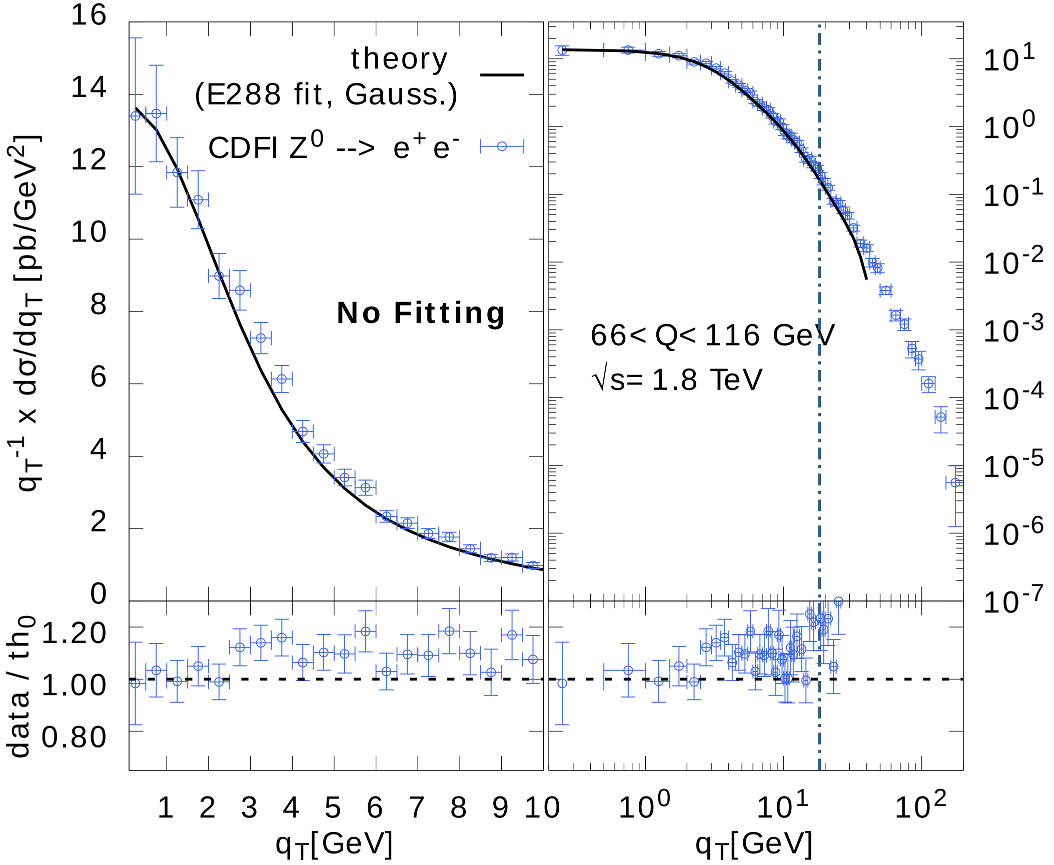
**Spectator model fit**

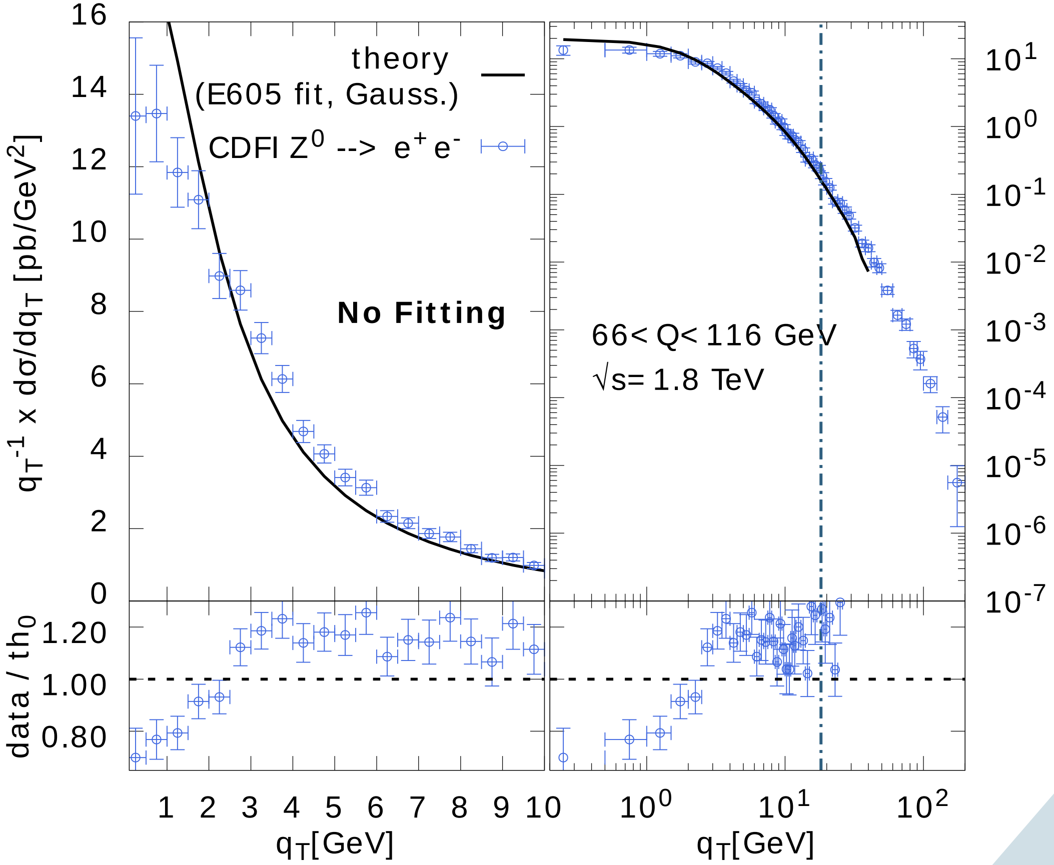
| | E288 (130 pts.) |
|---|---|
| $\chi^2_{\text{dof}}$ | 1.04 |

# Higher Q postdictions: Testing the predictive power

A postdiction of CDFI with just E288 or E605 data

**Just 3+1 parameters**

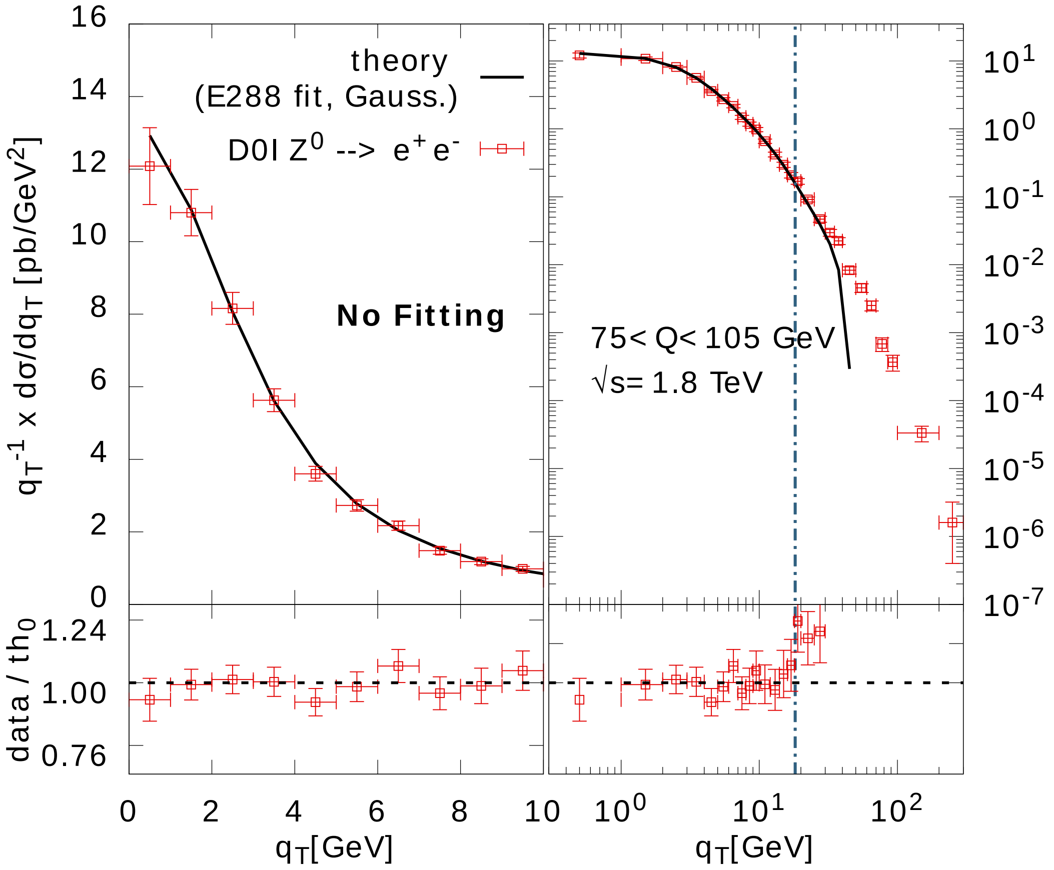


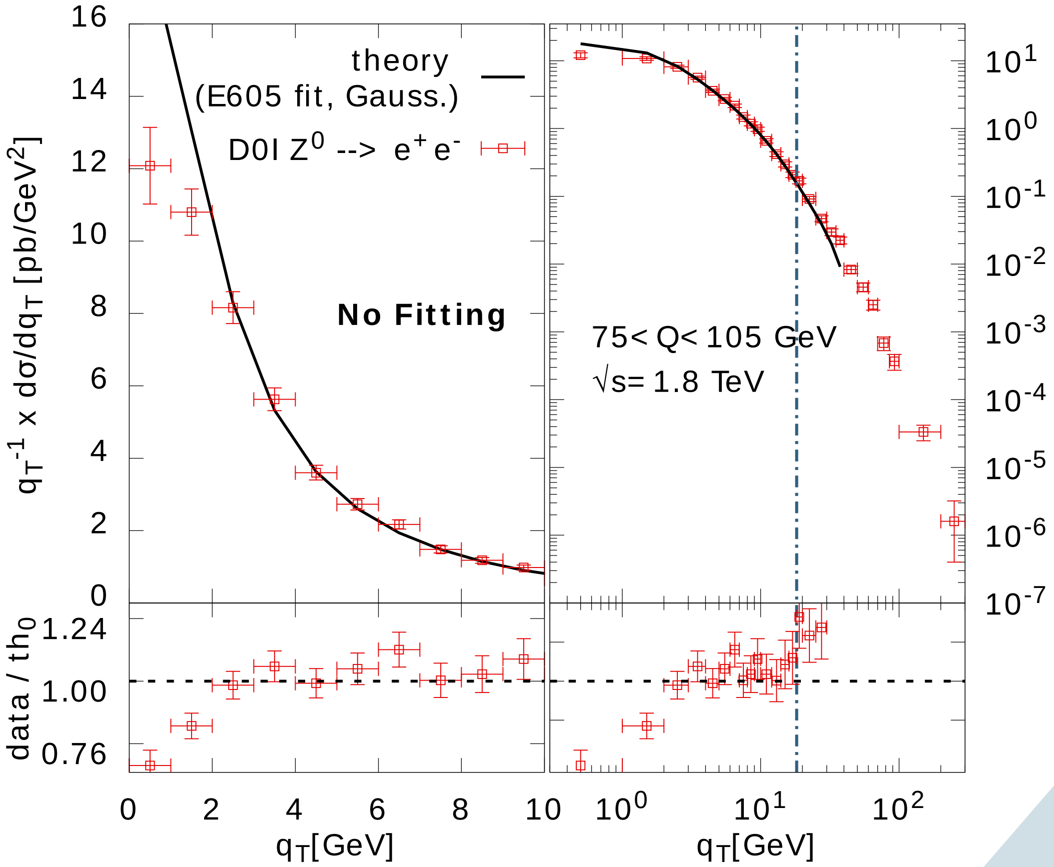



# Higher Q postdictions: test different fits on the same experiment

A postdiction of D0I with just E288 or E605 data

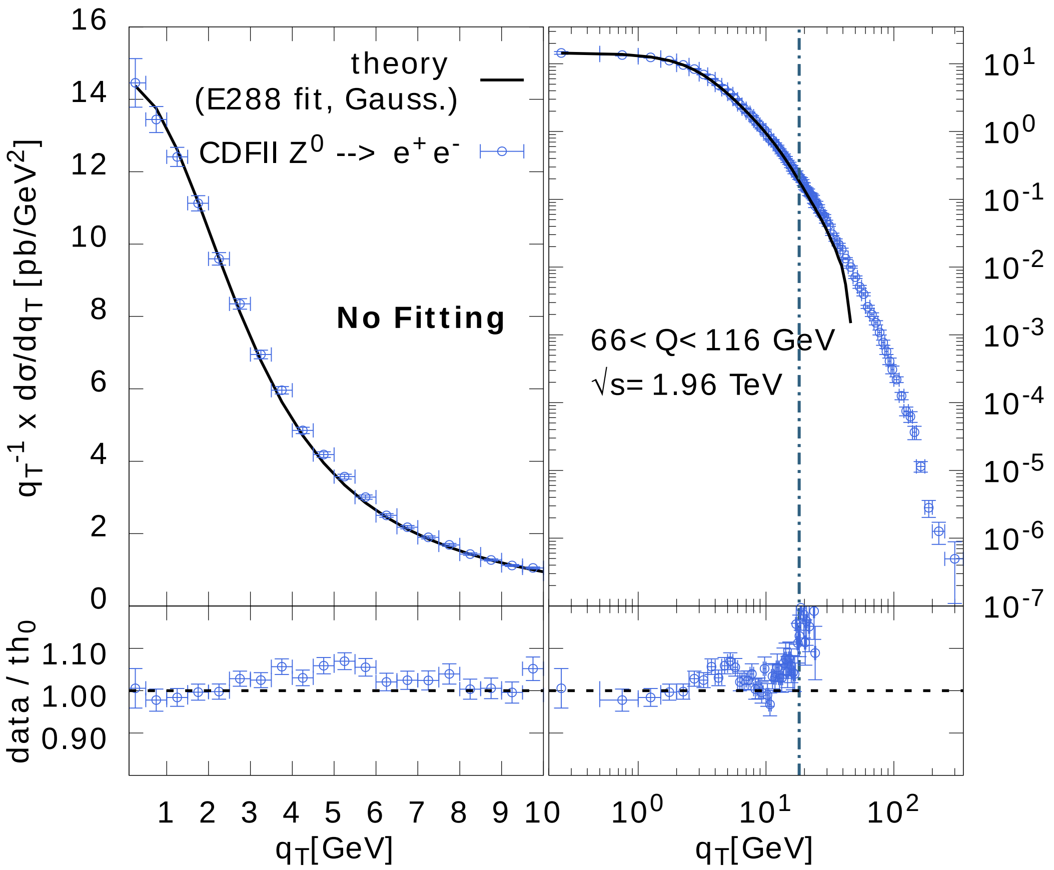
**Just 3+1 parameters**



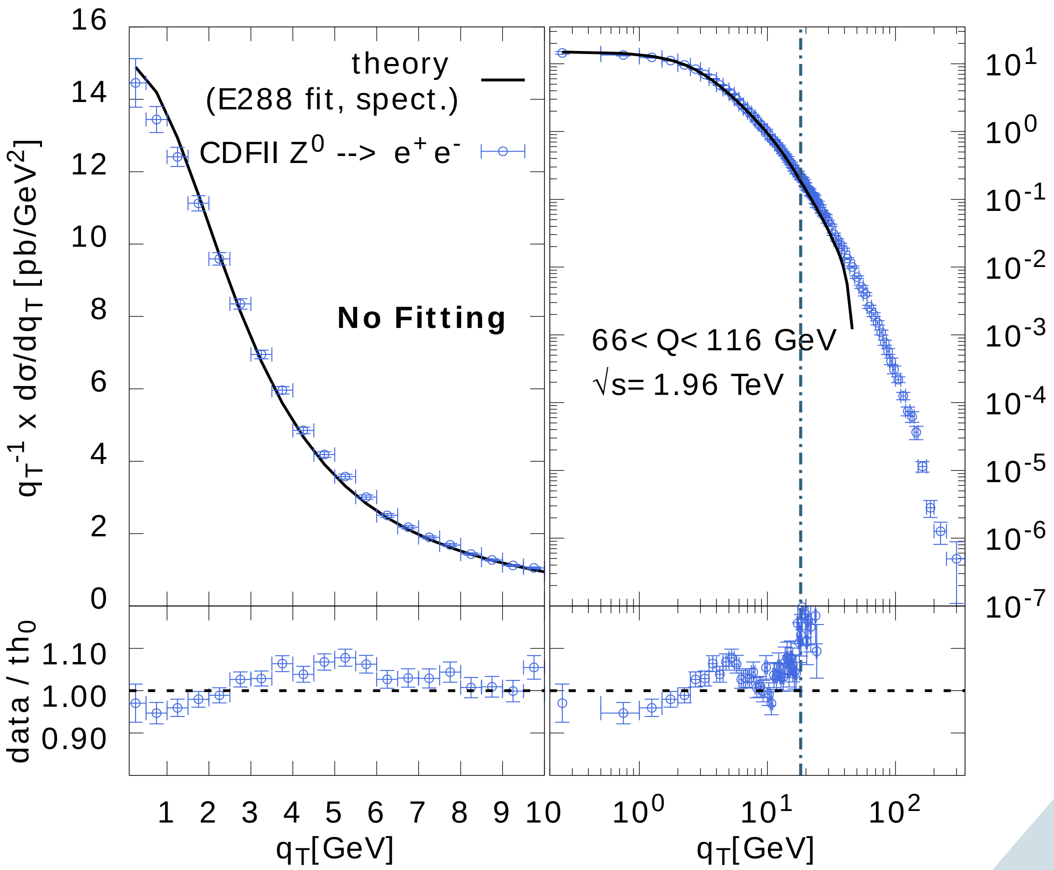



# Higher Q postdictions: test different models on the same experiment

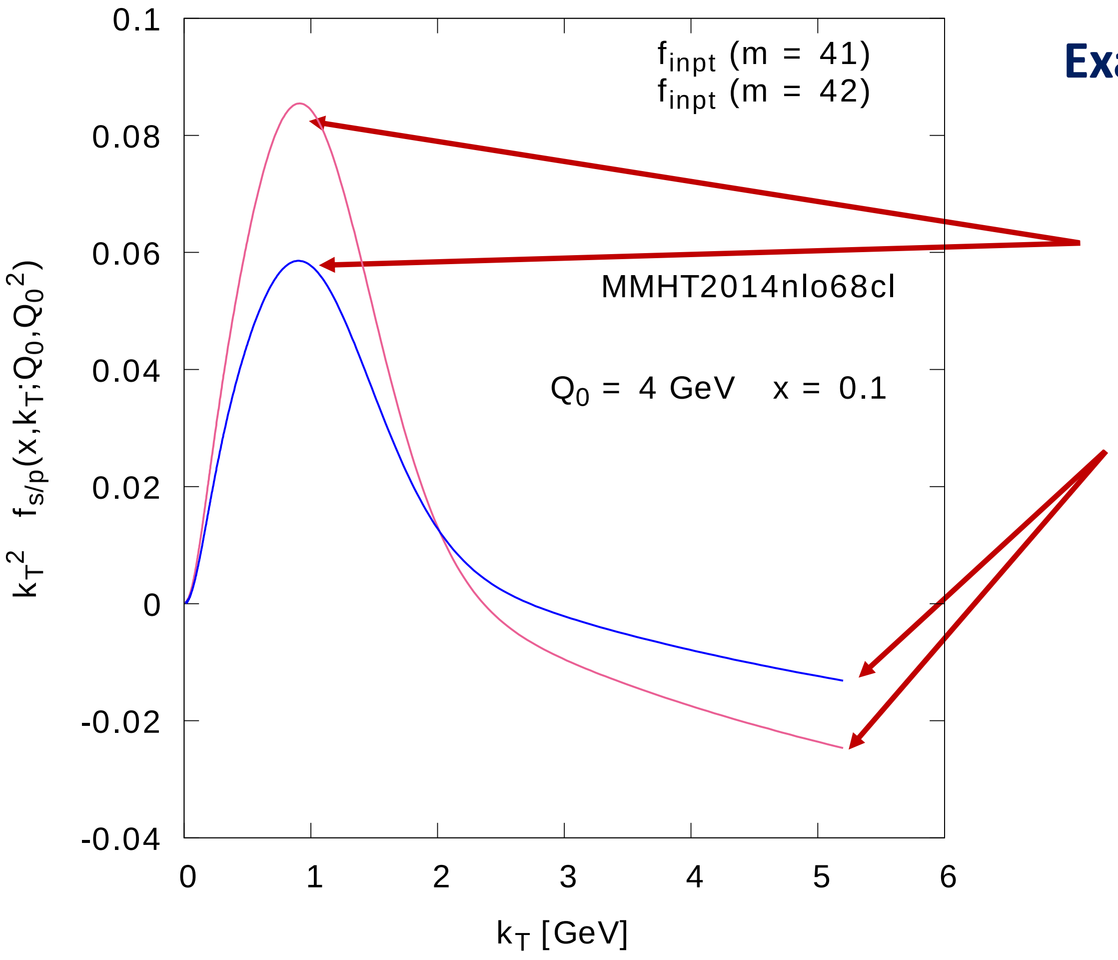
A postdiction of CDFII with E288 GAUSSIAN fit




A postdiction of CDFII with E288 SPECTATOR fit



# TMDs are affected by collinear distributions




**Example**: take two pdfs associated with the same flavor (s here) and compute the input TMD

Maybe unexpected **different small $k_T$ behavior** because of integral relation

Expected **different tails** because of the OPE expansion

Changing the integral **necessarily** changes the integrand

# Summary

- Consistent TMD parametrization for large TM at input scale
- Control over perturbative vs nonperturbative
- Quantifiable collinear effects at small $k_T$
- No need of $b_{max}$ or other auxiliary parameters
- Improved TM behavior in matching region (not today)

NEXT/SOON:

- More checks with data
- higher orders and polarized cases (Sivers)
- Incorporate NP calculations (lattice, EFT, …)

# Thank you

# Why are $b_*$ and $b_{max}$ used ?

$$f_{i/H}(x, b_T; \mu, \zeta) = \widetilde{C}_{ij}(x, b_T; \mu, \zeta) \otimes f_{j/H}(x; \mu) + \mathcal{O}(m b_T)$$

Powers of $\ln\left(\frac{\mu b_T}{2e^{-\gamma_E}}\right)$ = $\xrightarrow{b_T \to +\infty} +\infty$

$\xrightarrow{b_T \to 0} -\infty$

**LARGE $b_T$**: solved by arbitrary cutoff $b_{max}$

**SMALL $b_T$**: solved by choosing a different scale $\mu_{b_*}(b_T, b_{max})$

# Why $b_*$ and $b_{max}$?

$$f_{i/H}(x, b_T; \mu, \zeta) = \widetilde{C}_{ij}(x, b_T; \mu, \zeta) \otimes f_{j/H}(x; \mu) + \mathcal{O}(m b_T)$$

Powers of $\ln\left(\frac{\mu b_T}{2e^{-\gamma_E}}\right)$ = $\overset{b_T \to +\infty}{\longrightarrow} +\infty$, $\overset{b_T \to 0}{\longrightarrow} -\infty$

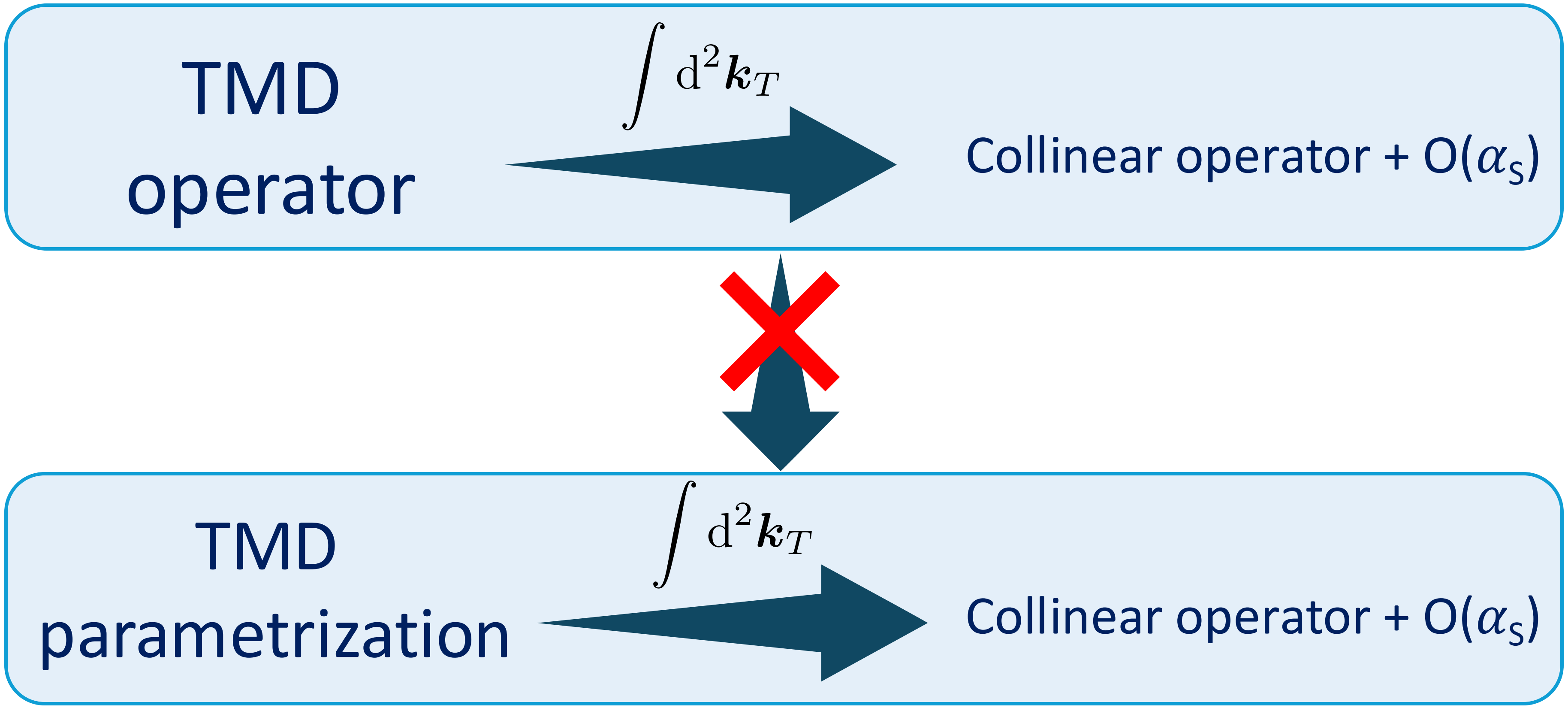
These problems are treated simultaneously in the stardard approach

BUT they are completely independent
and there is more to the story

# Why is this important?

- We can **quantitatively** and **conclusively** answer the question:

  How much collinear dependence do my TMD extractions carry?



# TMD from collinear factorization

$$\tilde{f}_i^{\mathrm{OPE}}(x, \boldsymbol{b}_{\mathrm{T}}; \mu, \zeta = \mu^2)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{2n} a_S^n L_b^k \widetilde{C}_{ij}^{(n,k)}(x) \otimes f_j(x; \mu)$$

$\overline{\mathrm{MS}}$ OPE coefficients

$\overline{\mathrm{MS}}$ PDF

QCD running coupling

$$L_b \equiv \ln\left(\frac{\mu b_T}{2e^{-\gamma_E}}\right)$$

# Conventional approach :

1

$$H \int \mathrm{d}^2\boldsymbol{k}_{1\mathrm{T}}\mathrm{d}^2\boldsymbol{k}_{2\mathrm{T}} f_{j/p}\left(x, \boldsymbol{k}_{1\mathrm{T}}; \mu, \sqrt{\zeta}\right) D_{h/j}\left(z, z\boldsymbol{k}_{2\mathrm{T}}; \mu, \sqrt{\zeta}\right) \delta^{(2)}\left(\boldsymbol{q}_\mathrm{T} + \boldsymbol{k}_{1\mathrm{T}} - \boldsymbol{k}_{2\mathrm{T}}\right)$$

Fourier Transform

$$H \int \frac{\mathrm{d}^2\boldsymbol{b}_\mathrm{T}}{(2\pi)^2} e^{-i\boldsymbol{b}_\mathrm{T}\cdot\boldsymbol{q}_\mathrm{T}} \tilde{f}_{j/p}\left(x, \boldsymbol{b}_\mathrm{T}; \mu, \sqrt{\zeta}\right) \tilde{D}_{h/j}\left(z, \boldsymbol{b}_\mathrm{T}; \mu, \sqrt{\zeta}\right)$$

# Solve evolution equations relating input scale with SIDIS scale

2

$$\frac{\partial \ln \tilde{f}_{j/p}\left(x, \boldsymbol{b}_{\mathrm{T}}; \mu, \sqrt{\zeta}\right)}{\partial \ln \sqrt{\zeta}} = \tilde{K}\left(\boldsymbol{b}_{\mathrm{T}}; \mu\right) \qquad \frac{\mathrm{d} \ln \tilde{f}_{j/p}\left(x, \boldsymbol{b}_{\mathrm{T}}; \mu, \sqrt{\zeta}\right)}{\mathrm{d} \ln \mu} = \gamma\left(\alpha_S(\mu); \mu/\sqrt{\zeta}\right)$$

$$\frac{\mathrm{d}\tilde{K}\left(\boldsymbol{b}_{\mathrm{T}}; \mu\right)}{\mathrm{d} \ln \mu} = -\gamma_K\left(\alpha_S(\mu)\right)$$

Same for FF

$$\mu = \sqrt{\zeta}$$
$$\mu_0 = \sqrt{\zeta_0}$$

$$\tilde{f}_{j/p}\left(x, \boldsymbol{b}_{\mathrm{T}}; \mu, \sqrt{\zeta}\right) = \tilde{f}_{j/p}\left(x, \boldsymbol{b}_{\mathrm{T}}; \mu_0, \sqrt{\zeta_0}\right) \times$$
$$\times \exp\left\{\int_{\mu_0}^{\mu} \frac{\mathrm{d}\mu'}{\mu'}\left[\gamma\left(\alpha_S(\mu'); 1\right) - \ln\left(\frac{\sqrt{\zeta}}{\mu'}\right)\gamma_K\left(\alpha_S(\mu')\right)\right] + \ln\left(\frac{\sqrt{\zeta}}{\sqrt{\zeta_0}}\tilde{K}\left(\boldsymbol{b}_{\mathrm{T}}; \mu_0\right)\right)\right\}$$

# Separate $b_T < b_{max}$ & $b_T > b_{max}$ regions with a $b_*$ prescription

3

$$\tilde{f}_{j/p}(x; \boldsymbol{b}_T; \mu_Q, Q) = \tilde{f}_{j/p}(x; \boldsymbol{b}_*; \mu_Q, Q) \underbrace{\frac{\tilde{f}_{j/p}(x; \boldsymbol{b}_T; \mu_Q, Q)}{\tilde{f}_{j/p}(x; \boldsymbol{b}_*; \mu_Q, Q)}}_{\exp\{-g_{j/p}(x, \boldsymbol{b}_T)\}}$$

Same for FF

Perturbatively calculable with fixed order collinear factorization

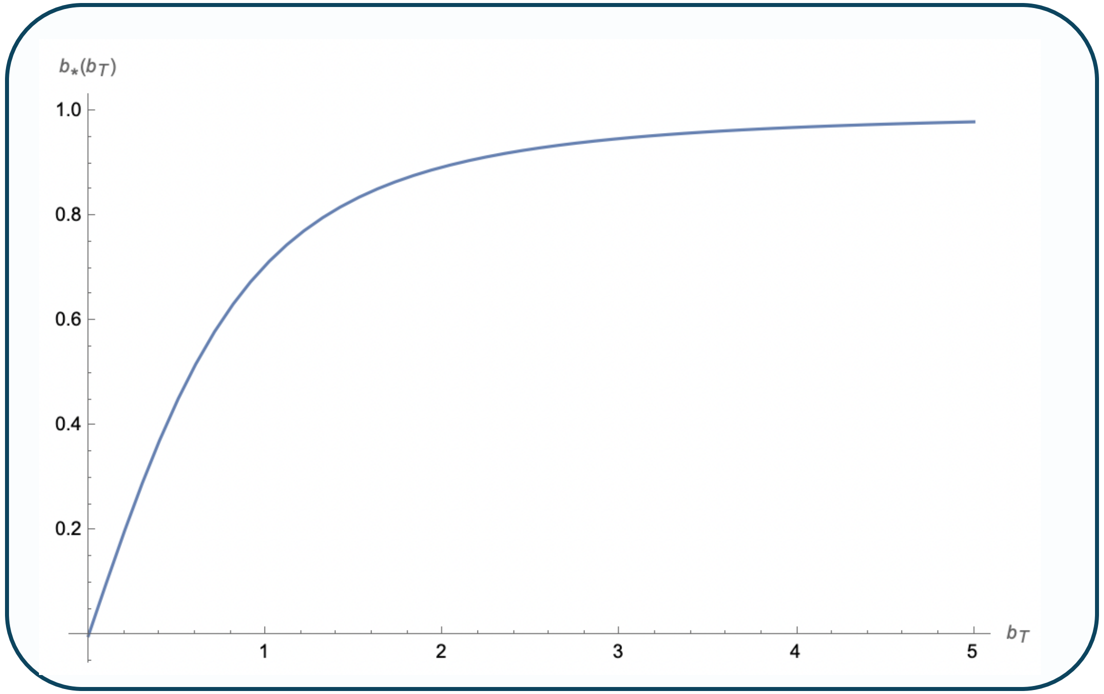
Nonperturbative

$$g_K(\boldsymbol{b}_T) \equiv \tilde{K}(\boldsymbol{b}_*; \mu) - \tilde{K}(\boldsymbol{b}_T; \mu)$$

# What are b* and bmax?

$$\boldsymbol{b}_*(b_T) = \begin{cases} \boldsymbol{b}_T, & b_T \ll b_{\max}, \\ \boldsymbol{b}_{\max}, & b_T \gg b_{\max} \end{cases}$$

$$\boldsymbol{b}_*(b_T) = \frac{\boldsymbol{b}_T}{\sqrt{1 + b_T^2/b_{\max}^2}}$$



# Choose ansatzes for g functions

4

$$g_{j/p}\left(x, \boldsymbol{b}_{\mathrm{T}}\right) = \frac{1}{4} M_F^2 b_{\mathrm{T}}^2$$

$$g_{h/j}\left(z, \boldsymbol{b}_{\mathrm{T}}\right) = \frac{1}{4z^2} M_D^2 b_{\mathrm{T}}^2$$

$$g_K\left(\boldsymbol{b}_{\mathrm{T}}\right) = \frac{g_2}{2M_K^2} \ln\left(1 + M_K^2 b_{\mathrm{T}}^2\right)$$

$$g_K\left(\boldsymbol{b}_{\mathrm{T}}\right) = \frac{1}{2} M_K^2 b_{\mathrm{T}}^2$$

# Collinear Evolution

Note : $\lim_{a_S \to 0} C_\Delta^c = 0$

$$\frac{\mathrm{d} f_i^c}{\mathrm{d}\ln\mu} \equiv 2P_{ij}^c \otimes f_j^c + p.s.$$

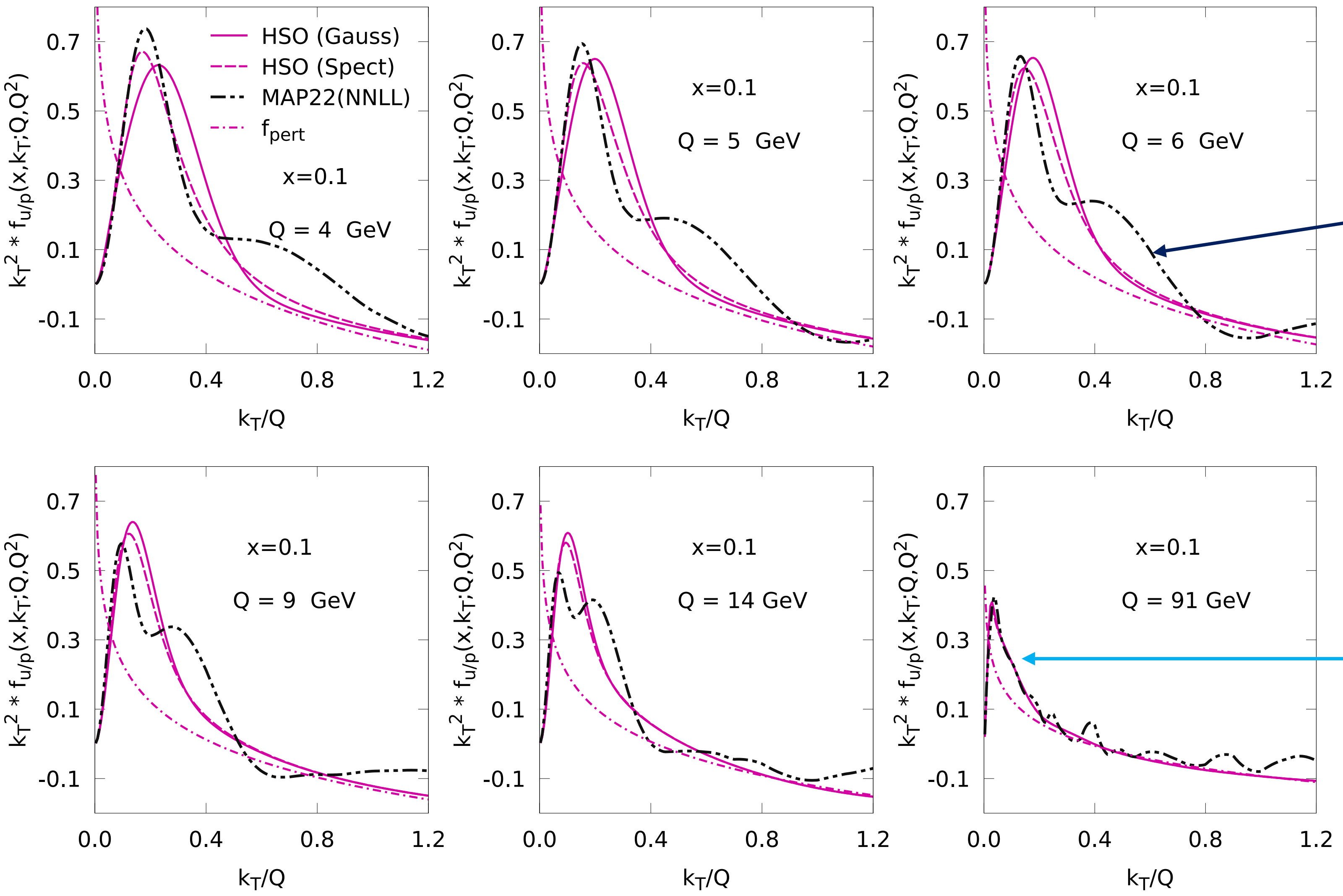
$$= 2P_{ij} \otimes f_j + C_{\Delta,ij}^c \otimes 2P_{jk} \otimes f_k + \frac{\mathrm{d}C_{\Delta,ij}^c}{\mathrm{d}\ln\mu} \otimes f_j + \frac{\mathrm{d}p.s.}{\mathrm{d}\ln\mu}$$

Usual evolution

Additional term (scheme change)

Power suppressed

# Comparison with MAP22




## Observations:

No tail matching for MAP

Different models can describe the small $k_T$ region at low Q

Model dependence washes out at large Q

How do we choose?

# Problems in the large kT-tail

$$\mathcal{H}_0\{\ln\left(\frac{\mu b}{C_1}\right)\}(k_T) = -\frac{1}{k_T^2}$$

$$\mathcal{H}_0\{\ln\left(\frac{\mu}{C_1}\sqrt{b^2 + b_{\text{min}}^2}\right)\}(k_T) = -\frac{b_{\text{min}}}{k_T}K_1\left(b_{\text{min}}k_T\right)$$

$$\mathcal{H}_0\{\ln^2\left(\frac{\mu b}{C_1}\right)\}(k_T) = -\frac{1}{k_T^2}\ln\left(\frac{\mu^2}{k_T^2}\right)$$

$$\mathcal{H}_0\{\ln^2\left(\frac{\mu}{C_1}\sqrt{b^2 + b_{\text{min}}^2}\right)\}(k_T) = -\frac{1}{k_T^2}\left[K_0\left(b_{\text{min}}k_T\right) + b_{\text{min}}k_T K_1\left(b_{\text{min}}k_T\right)\ln\left(\frac{b_{\text{min}}\mu^2}{2e^{-\gamma_E}k_T}\right)\right]$$