A road map to Small-x Physics & Parton Saturation at Colliders

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A few references

1) Iancu & Venugopalan (2003)
The Color Glass Condensate & High-energy QCD

2) Kovchegov & Levin (2012)
Textbook: QCD at high energies

3) Morreale & Salazar (2021)
Mining for Gluon Saturation at Colliders

History & motivation

Regge limit: $s \gg Q^2$, $x = Q^2/s$, fixed Q^2, $x \to 0$

Pre QCD
early 1960's

Regge theory based on S-matrix $\quad \sigma(s, t=0) \propto s^{\alpha(0)-1}$

* Froissart & Martin $\sigma \le \frac{\pi}{m_\pi^2} \ln^2(s/s_0)$ based on unitarity

CERN ISR 1973 : observe growth of pp cross-section

QCD :
1975-78 BFKL predicts $\alpha(0)-1 \simeq \frac{4 N_c \alpha_s}{\pi} \ln 2 \simeq 0.5$ for $\alpha_s \simeq 0.2$
hard pomeron intercept

1983-1986 GLR-MQ recombination effects (saturation)

$$\frac{\partial\, x g(x,Q^2)}{\partial \ln 1/x} = \underbrace{\frac{\alpha_s N_c}{\pi} \int \frac{dQ'^2}{Q'^2}\, x g(x, Q'^2)}_{\text{DGLAP}} \underbrace{- \frac{\alpha_s^2 \gamma}{Q^2 R^2} \left[x g(x, Q^2) \right]^2}_{\text{recombination}}$$

needed ← recombination
restore unitarity

1992 DL extracted $\alpha(0)-1 \simeq 0.08$ from pp data

soft pomeron (DL) $\simeq 0.08$ vs hard pomeron (BFKL) $\simeq 0.5$

1993-94 HERA measured DIS cross-section, dominance of gluon $xg(x) \sim x^{-\lambda}$

$\lambda \simeq 0.25$

1991-94 Nikolaev-Zakharov: dipole cross-section DIS

Mueller dipole cascade: reformulation of BFKL

unitarization, path towards calculate multiple scattering effects

1994 McLerran-Venugopalan: large-nucleus as dense classical field,

sources ρ and fields A, coherent scattering, gluon density

in heavy-ions (precursor of CGC), identify emergent scale Q_s

transverse density of partons

1996 Balitsky: Wilson line operator and OPE $\Rightarrow$ Balitsky hierarchy

1999 Kovchegov: Evolution of dipole @ large N_c

BK - equation

$$\frac{\partial N_{01}}{\partial \ln 1/x} = \frac{\bar{\alpha}_s}{2\pi} \int d^2x_2 \, \frac{x_{01}^2}{x_{02}^2 x_{12}^2} \Big[\overbrace{N_{02} + N_{12} - N_{01}}^{\text{BFKL}} - \underbrace{N_{02} N_{12}}_{\text{non-linear term}} \Big]$$

1997-2000 : JIMWLK

RG equation for $W[\rho]$ weight function of sources

in Hamiltonian formalism $\dfrac{\partial W[\rho]}{\partial \ln(1/x)} = H_{JIMWLK} W[\rho]$

2001 : BK & JIMWLK reconciled!

two faces of same coin : projectile vs target

2001 - beyond : NLO corrections to BFKL / BK / JIMWLK

Glasma as a transition between CGC & QGP

Correspondence with hadron structure TMDs & GPDs

Spin physics, entanglement, ...

CGC : theory of sources , fields & Wilson lines
and their RG structure rooted in QCD

Physics: multiple scattering (Wilson lines)
non-linear evolution (BK - JIMWLK)
emergence of saturation scale $Q_s(x, A)$
improvable : NLO corrections , beyond eikonal , ...

dipole picture + BK/JIMWLK
path towards unitarization can dynamically produce
transition between soft & hard intercepts!

multiparticle production beyond dipole : quadrupole , higher pts correlators

connection to heavy ions via initial condition & pre-thermalization

Phenomenology

HERA : total cross-section, diffraction, ...

RHIC & LHC :
pp/pA/AA: forward (multi) particle production
hadron, jets, photons, vector meson, energy correlators..
UPCs : photo-nuclear reactions (DIS like)

EIC : DIS off nuclei: enhancement of saturation through nuclei

Cosmic rays : super high-energy DIS !

Some recent connection to entanglement @ colliders

Outline

Lecture 1

1) Sources vs fields, MV model
2) Classical soln of Yang-Mills' eqs for a eikonal current
3) Effective CGC propagators and Wilson lines
4) Quark scattering and the dipole correlator
5) Two models for the dipole correlator

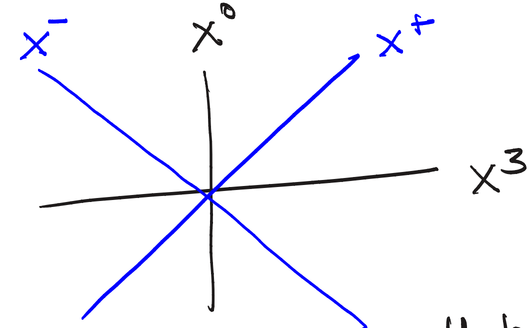
Lecture 2

6) DIS cross-section
7) RG evolution @ small-x : Balitsky-Kovchegov equation
8) Multiparticle production
9) Phenomenology

Sources & fields

Light-cone coordinates

$$x^{\pm} = \frac{x^0 \pm x^3}{\sqrt{2}} \, , \quad x_\perp$$



* Consider a nucleus moving in the + LC direction with large P^+

let's introduce scale $\Lambda^+ = x_0 P^+$ $(x_0 \ll 1)$ and separate modes

fast modes $k^+ > \Lambda^+$ (large-x partons)

dynamics is time dilated, and localized length contraction

treat as a recoil-less color density current J^μ

slow modes $k^+ < \Lambda^+$ (small-x partons)

dynamical modes treat as gauge field A^μ

Compute observable

$$\langle\langle \mathcal{O} \rangle\rangle = \int [\mathcal{D}J]\, W_{\Lambda^+}[J] \underbrace{\frac{\int^{\Lambda^+} \mathcal{D}A\mathcal{D}\psi\, e^{i(S_{QCD}[A,\psi] + \int A\cdot J)}\, \mathcal{O}[A,\psi]}{\int^{\Lambda^+} \mathcal{D}A\mathcal{D}\psi\, e^{i(S[A,\psi] + \int A\cdot J)}}}_{\langle \mathcal{O} \rangle_{\Lambda^+}[J]}$$

path integral for observable for given source configuration J

$$\langle\langle \mathcal{O} \rangle\rangle = \int [\mathcal{D}J]\, W_{\Lambda^+}[J] \langle \mathcal{O} \rangle_{\Lambda^+}[J]$$

for large current one can expand around classical solution

observable/kinematic dependent

$$\langle\langle \mathcal{O} \rangle\rangle = \int [\mathcal{D}J]\, W_{\Lambda^+}[J] \left(\mathcal{O}_{cl}[J] + \alpha_s \ln\left(\Lambda^+_{obs}/\Lambda^+\right) \mathcal{O}_{quantum} + \cdots \right)$$

For a nucleus moving relativistically $p^+ \gg m_p$

$$J^\mu_{(x)} = \delta^{\mu +} \rho(x^-, x_\perp)$$, strictly only true in non-abelian gauge theory when $A^- = 0$ due to current conservation

* $J^-, J_\perp$ components suppressed relative to J^+
* J^+ independent of x^+

MV model: gaussian functional for initial condition of W

$$W_{\Lambda_0}[\rho] = N e^{-\int d^2x_\perp dx^- \rho^a \rho^a / 2\mu^2}$$

$$\langle \rho^a_{(x)} \rho^b_{(y)} \rangle_{\Lambda_0} = \int [\mathcal{D}\rho] W_{\Lambda_0}[\rho] \, \rho^a_{(x)} \rho^b_{(y)} = \mu^2 \delta^{ab} \delta^{(2)}(x_\perp - y_\perp) \delta(x^- - y^-)$$

μ^2 color charge density $\propto A^{1/3}$

Classical Yang Mills for eikonal current

Simpler example: Maxwell's equations

$$\partial_\mu F^{\mu\nu} = J^\nu \quad , \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Eikonal current $\quad J^\nu(x) = \delta^{\nu +} \rho(x^-, x_\perp)$

Let's work in LC gauge $A^- = 0$

Since current is independent of x^+, look for soln's independent of x^+ $\Rightarrow F^{+-} = 0 \; , \; F^{j-} = 0$

$$\nu = - \;\Rightarrow\; \partial_+ \underbrace{F^{+-}}_{=0} + \partial_j \underbrace{F^{j-}}_{=0} = 0 \;\checkmark$$

$$\nu = \perp \;\Rightarrow\; \partial_+ \underbrace{F^{+j}}_{=0} + \partial_- \underbrace{F^{-j}}_{=0} + \partial_i F^{ij} = 0 \;\Rightarrow\; \partial_i F^{ij} = 0$$

$$-\nabla_\perp^2 A^j = 0$$

$$+ \text{ b.c.} \Rightarrow A^j = 0 \;\checkmark$$

$$\nu = + \Rightarrow \quad \partial_- F^{-+} + \partial_\perp F^{\perp +} = J^+ \qquad F^{\perp +} = \partial_\perp A^+$$

$$\partial_\perp F^{\perp +} = J^+$$

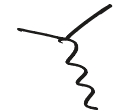
$$-\nabla_\perp^2 A^+ = \rho \quad \leftarrow \text{Poisson's equation}$$

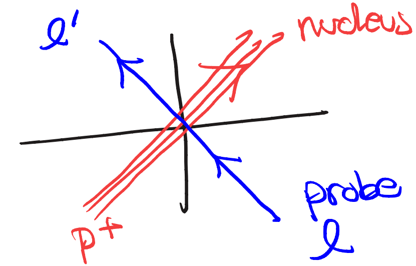
In momentum space $\quad \tilde{A}^+(k) = \dfrac{\tilde{\rho}(k)}{k_\perp^2}$

Exercise I: Show that Yang-Mills equations $[D_\mu, F^{\mu\nu}] = J^\nu$ same soln $A^\mu = \delta^{\mu +}\alpha$ with $-\nabla_\perp^2 \alpha = J^+$

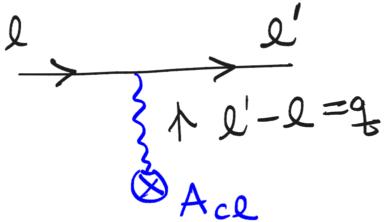
Exercise II: find soln in gauge $A^+ = 0$
hint: perform a gauge rotation

CGC effective vertex

For simplicity consider QED $\Rightarrow$ $-ie\gamma^\mu$ 



probe moving with $l^- \gg l_\perp$



$$\ldots(-ie)\gamma^\mu \ldots \tilde{A}_\mu(l'-l)$$

$$= -ie\gamma^- \tilde{A}^+(l'-l)$$

Recall
$$\boxed{a\cdot b = a^+b^- + a^-b^+ - a_\perp\cdot b_\perp}$$

$$\tilde{A}^+_{cl}(q) = \int d^4z\, e^{iq\cdot z} A^+_{cl}(z^-, z_\perp)$$

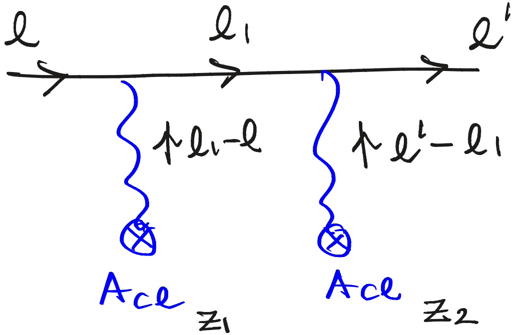
$$= 2\pi\,\delta(q^-)\int d^2z_\perp\, e^{-iq_\perp\cdot z_\perp}\int dz^-\, e^{iq^+z^-} A^+_{cl}(z^-, z_\perp)$$

$$V_1(l', l) = 2\pi\,\delta(l^- - l'^-)\gamma^-(-ie)\int d^2z_\perp\, e^{-i(l'_\perp - l_\perp)\cdot z_\perp}\int dz^-\, A^+_{cl}(z^-, z_\perp)\, e^{i(l'^+ - l^+)\cdot z^-}$$

drop phase

z^- support $\sim 1/p^+$
for external partons on-shell

so $l^+z^- \sim \dfrac{l_\perp^2}{l^- p^+} \sim \dfrac{l_\perp^2}{S} \ll 1$ similarly $l'^+z^- \ll 1$



l_1 : loop momentum

$$S_0(l_1) = \frac{\not{l}_1}{l_1^2 + i\varepsilon}$$

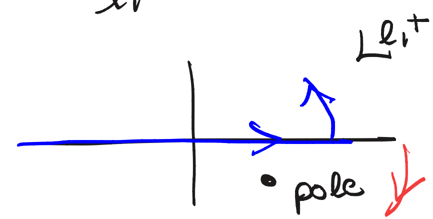
$$V_2(l', l) = \int \frac{d^4 l_1}{(2\pi)^4} V_1(l', l_1)\, S_0(l_1)\, V_1(l_1, l)$$

Dirac structure $\quad \gamma^- \not{l}_1 \gamma^- = 2 l_1^- \gamma^-$

l_1^- integration use delta function $\delta(l_1^- - l^-) \Rightarrow l_1^- = l^-$

l_1^+ integration $\displaystyle I = \int \frac{dl_1^+}{2\pi} \frac{e^{i l_1^+ (z_1^- - z_2^-)}}{\left[l_1^+ - \frac{l_{1\perp}^2 - i\varepsilon}{2 l_1^-}\right]}$ $\leftarrow$ recall $l_1^2 = 2 l_1^- l_1^+ - l_\perp^2$

$l_1^- > 0$



Two cases

- $z_1^- > z_2^-$ $\Rightarrow$ $i\, l_1^+ (z^- - z_2^-) < 0$ if $\mathrm{Im}(l_1^+) > 0$

  close contour upwards b/c contribution from semi-circle will be exp suppressed

  but there are no enclosed poles $\Rightarrow$ $I = 0$

- $z_2^- > z_1^-$ close contour downwards

  then we have a pole

$$I \sim e^{i\, l_{1\perp}^2 / 2 l_1^- (z_1^- - z_2^-)} \simeq 1 \quad \text{b/c} \quad \frac{l_{1\perp}^2}{S} \ll 1$$

$I \sim \Theta(z_2^- - z_1^-) \leftarrow$ ordering of scatterings

Carry out transverse integration

$$\int \frac{d^2 l_{1\perp}}{(2\pi)^2} \, e^{-i l_{1\perp}\cdot(z_{1\perp}-z_{2\perp})} = \delta^{(2)}(z_{1\perp}-z_{2\perp})$$

$\Rightarrow$

$$V_2(l',l) = 2\pi\,\delta(l^- - l'^-)\,\gamma^- \int d^2 z_\perp \, e^{-i q_\perp\cdot z_\perp} (-ie)^2 \underbrace{\int_{-\infty}^{\infty} dz_2^- \int_{-\infty}^{z_2^-} dz_1^- \, A^+_{cl}(z_1^-, z_\perp)\, A^+_{cl}(z_2^-, z_\perp)}_{\frac{1}{2} \int_{-\infty}^{\infty} dz_2^- A^+_{cl}(z_2^-, z_\perp) \int_{-\infty}^{\infty} dz_1^- A_{cl}(z_1^-, z_\perp)}$$

$$V_n(l',l) \qquad \frac{1}{n!} \prod_{i=1}^{n} (-ie) \int_{-\infty}^{\infty} dz_i^- \, A^+_{cl}(z_i^-, z_\perp)$$

$$\sum_{n=0}^{\infty} V_n \Rightarrow 2\pi\,\delta(l^- - l'^-)\,\gamma^- \int d^2 z_\perp \, e^{-i(l'_\perp - l_\perp)\cdot z_\perp} \, e^{-ie\int_{-\infty}^{\infty} A^+_{cl}(z^-, z_\perp)\, dz^-}$$

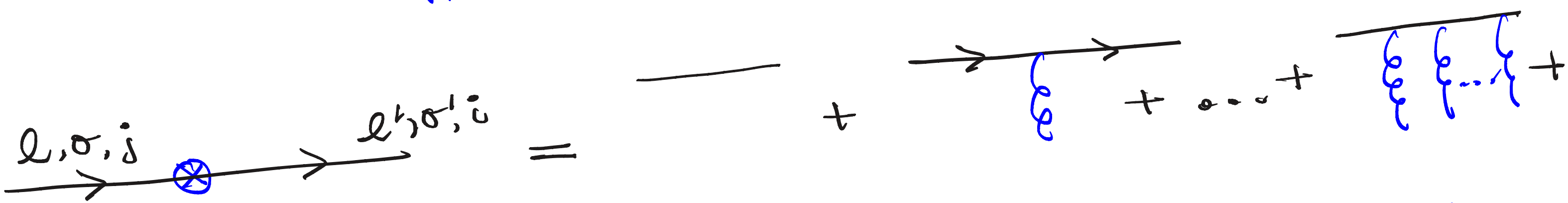
Same exercise can be carried out in QCD

Quark effective vertex

$$I_q(\ell',\ell) = 2\pi\,\delta(\ell^- - \ell'^-)\,\gamma^-_{\sigma'\sigma} \int d^2 z_\perp \, e^{-i(\ell'_\perp - \ell_\perp)\cdot z_\perp} \, V_{ij}(z_\perp)$$

$$V_{ij}(z_\perp) = \underline{P}\left[e^{-ig\int_{-\infty}^{\infty} A^{+,a}_{cl}(z^-, z_\perp)\, t^a_{ij}\, dz^-} \right], \quad t^a$$ fund representation of SU(3)

CGC effective vertex for quark



resums multiple eikonal scatterings

Gluon effective vertex

$$I_g(l', l) = 2\pi\, \delta(l^- - l'^-)\,(2l^-)\, g^{\mu\nu} \int d^2 z_\perp\, e^{-i(l'_\perp - l_\perp)\cdot z_\perp}\, U_{bc}(z_\perp)$$

$$U_{bc}(z_\perp) = \mathcal{P}\left[e^{ig\int_{-\infty}^{\infty} A^{+,a}_{cl}(z^-, z_\perp)\, T^a_{bc}\, dz^-} \right]$$

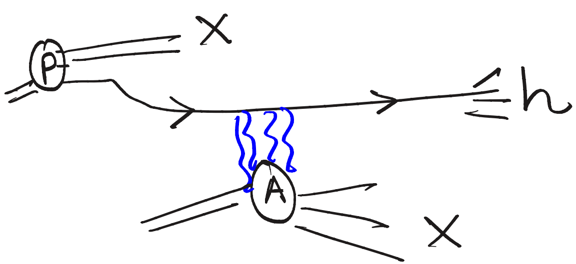
T^a adjoint representation of $SU(3)$

CGC effective vertex for gluon

resums multiple eikonal scatterings

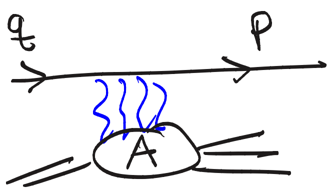
Quark scattering and the dipole

$pA \to h + X \leftarrow$ single inclusive hadron production



$$\frac{d\sigma^{pA\to h+X}}{d^2P_h\, dy_h} = x f_{i/p}(x) \otimes \frac{d\sigma^{iA\to i+X}}{d^2p\, dy} \otimes D_{i/h}(z)$$

Let's compute this objects



$q_\perp = 0$

$$S = \bar{u}(p,\sigma')\, I_q(p,q)\, u(q,\sigma)$$

$$S = 2\pi\delta(p^- - q^-)\, \bar{u}(p,\sigma')\gamma^- u(q,\sigma) \int d^2x_\perp\, e^{-ip_\perp\cdot x_\perp}\, V(x_\perp)$$

$$S = 2\pi\,\delta(p^- - q^-)\,\mathcal{M}$$

$$\mathcal{M} = \bar{u}(p,\sigma')\gamma^- u(q,\sigma) \int d^2x_\perp\, e^{-i p_\perp\cdot x_\perp}\, V(x_\perp)$$

$$\frac{1}{N_c}\frac{1}{2}\sum_{\sigma\sigma'} |\mathcal{M}|^2 = \underbrace{\mathrm{Tr}[\not{p}\,\gamma^-\not{q}\,\gamma^-]}_{\frac{1}{2}\times 8p^-q^-} \int d^2x_\perp d^2x_\perp'\, e^{-i p_\perp\cdot(x_\perp - x_\perp')} \underbrace{\frac{1}{N_c}\mathrm{Tr}[V(x_\perp)V^\dagger(x_\perp')]}_{S(x_\perp, x_\perp')}$$

color

Fourier transform

Accounting for flux factor:

$$\frac{d\sigma}{d^2p_\perp dy} = \frac{S_\perp}{(2\pi)^2}\left\langle \tilde{S}(p_\perp)\right\rangle_x$$

where

$$\tilde{S}(p_\perp) = \int d^2(x_\perp - x_\perp')\, e^{-i p_\perp\cdot(x_\perp - x_\perp')} \frac{1}{N_c}\mathrm{Tr}[V(x_\perp)V^\dagger(x_\perp')]$$

average over sources

$$x = \frac{p_\perp e^{-y}}{\sqrt{s}}$$

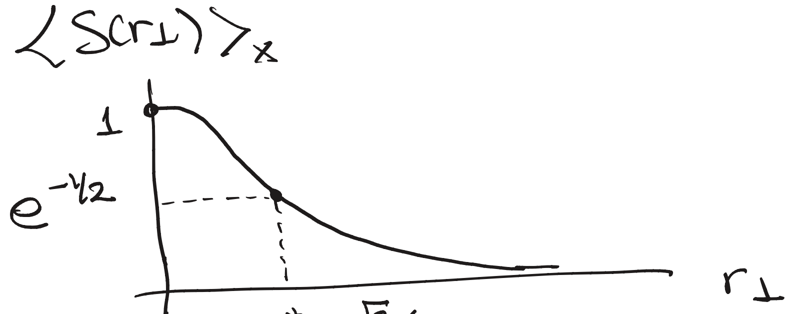
I assumed translational invariance S only depends $r_\perp = x_\perp - x_\perp'$

$S_\perp$ transverse area

What can we say about $\langle S(x_\perp, x'_\perp)\rangle_x$?

What happens when $r_\perp \to 0 \Rightarrow \langle S(r_\perp)\rangle_x = 1$ ← b/c Wilson lines are unitary matrices

$r_\perp \to \infty \quad \langle S(r_\perp)\rangle_x = 0$ ← decorrelation

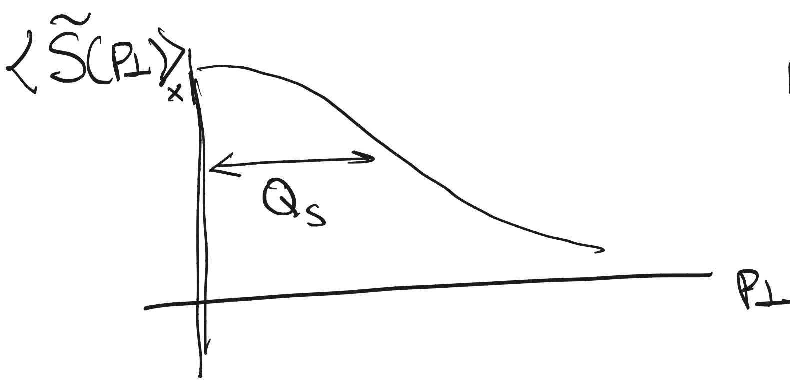


$r^* = \sqrt{2}/Q_s$ ← saturation scale

Two models for dipole

GBW : $\langle S(r_\perp)\rangle_x = e^{-\frac{1}{4}Q_s^2(x)\, r_\perp^2}$ $\qquad Q_s^2 = A^{1/3} Q_0^2 \left(\frac{x_0}{x}\right)^\lambda$

$\Rightarrow \langle \tilde{S}(p_\perp)\rangle_x \propto e^{-p_\perp^2/Q_s^2}$



multiple scattering

$\Rightarrow$ broadening controlled by saturation scale Q_s

Q_s grows with A and with smaller x !

MV model $\quad \rho^a(x_\perp) = \int dx^- \, \rho^a(x^-, x_\perp)$

$$\lambda^2 = \int dz^- \mu^2$$

$$\langle \rho^a(x_\perp) \rho^b(y_\perp) \rangle_{MV} = \lambda^2 \delta^{ab} \delta^{(2)}(x_\perp - y_\perp)$$

$$\langle \alpha^a(x_\perp) \alpha^b(y_\perp) \rangle_{MV} = \lambda^2 \delta^{ab} \int \frac{d^2 z_\perp}{(2\pi)^2} K_0(m|z_\perp - y_\perp|) K_0(m|z_\perp - x_\perp|)$$

introduced IR regulator $\frac{1}{k_\perp^2} \to \frac{1}{k_\perp^2 + m^2}$

$$\left\langle \frac{1}{N_c} \mathrm{Tr}[V(x_\perp) V^\dagger(y_\perp)] \right\rangle_{MV} = e^{-g^2 \frac{C_F}{2} \chi(x_\perp, y_\perp)}$$

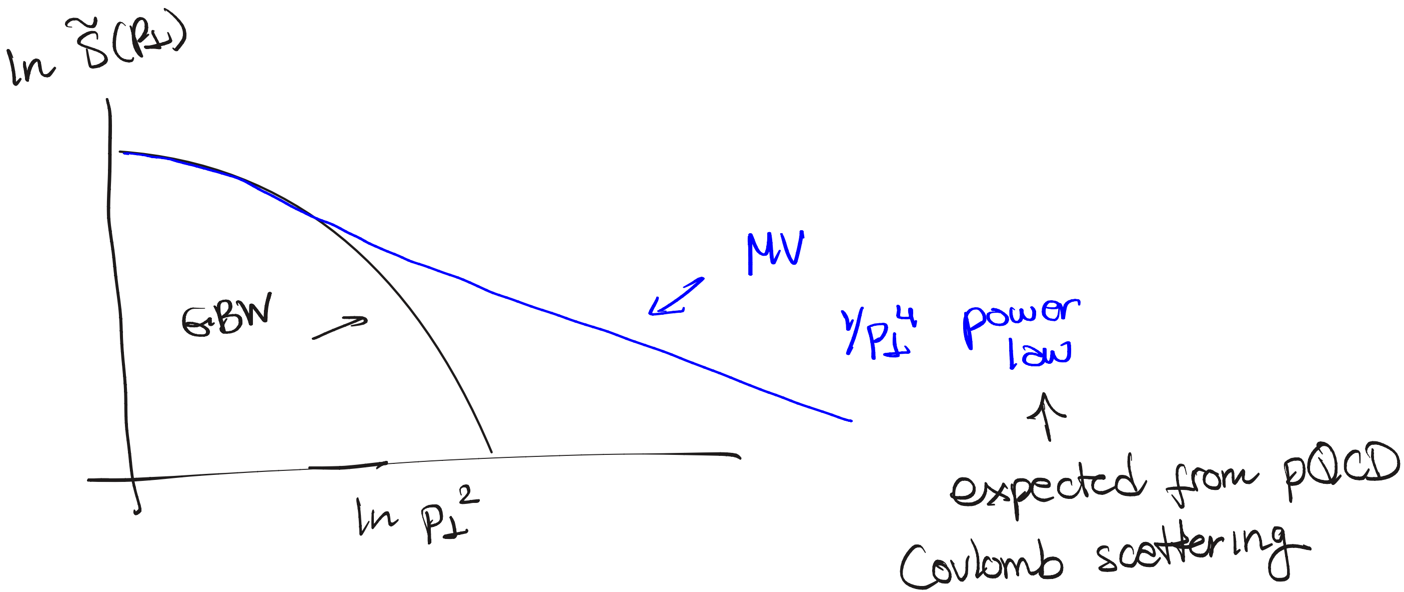
$$\chi(x_\perp, y_\perp) = \gamma(x_\perp, x_\perp) + \gamma(y_\perp, y_\perp) - 2\gamma(x_\perp, y_\perp), \quad \gamma(x_\perp, y_\perp)\delta^{ab} = \langle \alpha^a(x_\perp) \alpha^b(y_\perp) \rangle$$

$$\langle S(r_\perp) \rangle_{MV} = e^{-\frac{1}{4} Q_s^2 r_\perp^2 \ln(1/m r_\perp)} \leftarrow \text{MV model} \quad m r_\perp \ll 1$$

where $Q_s^2 \simeq \frac{g^2 C_F}{2\pi} \lambda^2$ recall $\lambda^2 \propto A^{1/3}$

nuclear enhancement of saturation scale

$$\langle \tilde{S}(p_\perp) \rangle = \begin{cases} e^{-p_\perp^2/Q_s^2} & p_\perp \lesssim Q_s \\ 1/p_\perp^4 & p_\perp \gg Q_s \end{cases}$$



Exercises Lecture I

① Solve YM eqs for eikonal current $J^{\mu}(x) = \delta^{\mu +}\rho(x^-, x_\perp)$

in $A^- = 0$ gauge

& then in $A^+ = 0$ gauge (perform gauge rot)

② Compute in detail quark & gluon CGC propagators

in $A^- = 0$ gauge, what happens to 4-gluon vertex?

③ Compute $\langle \alpha^a(x_\perp)\alpha^b(y_\perp)\rangle$ in the MV model

hint: find Greens function for $-\nabla_\perp^2 + m^2$

④ Compute $\frac{1}{N_c}\langle \mathrm{Tr}[V(x_\perp)V^\dagger(y_\perp)]\rangle$ in terms of $\langle \alpha^a(x_\perp)\alpha^b(y_\perp)\rangle$

⑤ Numerically evaluate Fourier transform of dipole in GBW & MV models

Recap lecture 1

1) large-x partons $J^+(x^-, x_\perp)$ small-x partons $A^\mu(x^-, x_\perp)$

2 solve classical YM $-\nabla_\perp^2 A^+_{cl} = J^+$

3) derived effective vertices in terms of Wilson lines

$$V(x_\perp) = \mathcal{P}\, e^{ig\int dz^- A^+_{cl}}$$

4) computed $q + A \to q + X$ scattering

$$\frac{d\sigma}{d^2p_\perp dy} \propto FT\left[\frac{1}{N_c}\langle Tr[V(x_\perp)V^\dagger(y_\perp)]\rangle_x\right]$$

5) MV model $\frac{1}{N_c}\langle Tr[V(x_\perp)V^\dagger(y_\perp)]\rangle_{x_0} = e^{-1/4\, Q_s^2 r^2 \ln(1/mr)}$

Today (Lecture 2)

1) DIS cross-section

2) Small-x evolution: Balitsky Kovchegov equation

3) Mutiparticle production [very briefly]

4) Phenomenology:

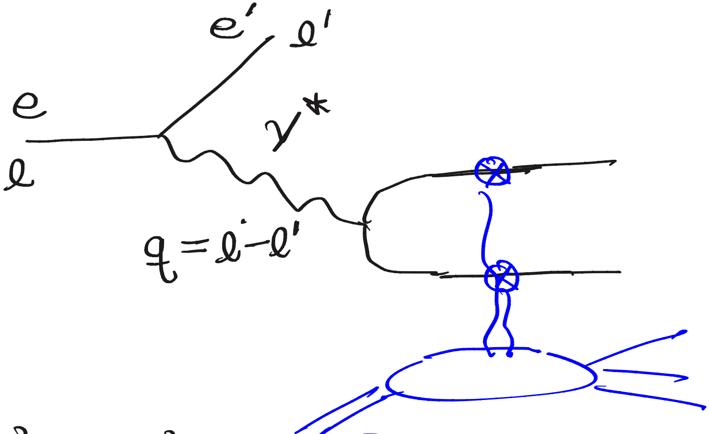
σ_r^{DIS} HERA

single inclusive hadron prod in pp/pA RHIC & LHC

two-particle prod in pp/pA RHIC

J/ψ photo-production UPCs @ RHIC & LHC

DIS cross-section



$$Q^2 = -q^2 > 0$$

$$x_{Bj} = \frac{Q^2}{2P\cdot q} \simeq \frac{Q^2}{W^2}$$

$$y = \frac{P\cdot q}{P\cdot l}$$

$$\frac{d\sigma}{dQ^2\,dy} = f_L(y, Q^2)\,\sigma_L(x_B, Q^2) + f_T(y, Q^2)\,\sigma_T(x_B, Q^2)$$

Let's compute $\gamma^* p \to X$ and use the optical theorem

$$\sigma_{tot}^{\gamma^* p} = \frac{\mathrm{Re}\left[\langle \mathcal{M}^{\gamma^* p \to \gamma^* p}\rangle\right]}{q^-}$$

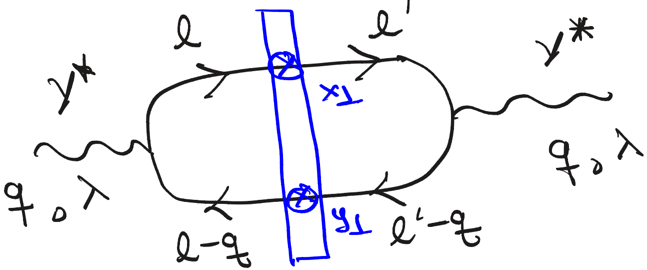
where

$$2\pi\,\delta(p_f^- - p_i^-)\langle \mathcal{M}\rangle = \langle S[s] - S[0]\rangle$$

Standard version opt thm: $\sigma_{tot} = \frac{1}{s}\,\mathrm{Im}\left[\mathcal{M}_{forward}\right]$

DIS cross-section

Frame $q_\perp = 0$



$$S^\lambda[8] = (-1) \int \frac{d^4\ell \, d^4\ell'}{(2\pi)^8}$$

$$\mathrm{Tr}\Big[S_0(\ell)(-ie \not{\epsilon}(q,\lambda)) S_0(\ell - q) T_q(\ell - q, \ell' - q') \, S_0(\ell' - q)(-ie \not{\epsilon}^*(q,\lambda)) S_0(\ell') T_q(\ell', \ell)\Big]$$

dipole $\frac{1}{N_c} \mathrm{Tr}[V(x_\perp) V^\dagger(y_\perp)]$

We can organize the result as:

$$\mathcal{M}^\lambda = 2q^- N_c \int d^2x_\perp \, d^2y_\perp \, [1 - S(x_\perp, y_\perp)] \, A^\lambda(x_\perp, y_\perp)$$

$$A^\lambda(x_\perp, y_\perp) = -\frac{(e e_f)^2}{2\pi} \int \frac{d^4\ell}{(2\pi)^3} \frac{d^4\ell'}{(2\pi)^3} \frac{(2q^-)\,\delta(\ell^- - \ell'^-)\, e^{i(\ell_\perp - \ell'_\perp)\cdot r_\perp} A^\lambda(\ell, \ell')}{[\ell^2 + i\epsilon][(\ell - q)^2 + i\epsilon][\ell'^2 + i\epsilon][(\ell' - q)^2 + i\epsilon]}$$

$$A^\lambda(\ell, \ell') = \frac{1}{(2q^-)^2} \mathrm{Tr}\left[\gamma^- \not{\ell} \, \not{\epsilon}(q,\lambda) \, (\not{\ell} - \not{q}) \, \gamma^- \, (\not{\ell}' - \not{q}) \, \not{\epsilon}^*(q,\lambda) \, \not{\ell}'\right]$$

DIS cross-section

Intergration over l'^- is trivial $\delta(l^- - l'^-)$

Dirac Algebra. Let's consider longitudinally polarized case

In light cone gauge $A^- = 0$ $\quad \varepsilon^\mu(q,\lambda) = \left(\frac{Q}{q^-}, 0, 0_\perp\right) \Rightarrow \not{\varepsilon}(q,\lambda=L) = \frac{Q}{q^-}\gamma^-$

$$\gamma^- \not{a} \gamma^- = \gamma^- \underbrace{\{\gamma^-, \not{a}\}}_{2a^-} + \underbrace{\gamma^-\gamma^-}_{0}\not{a}$$

$$A^{\lambda=L}(l,l') = \frac{1}{(2q^-)^2}\frac{Q^2}{(q^-)^2}\mathrm{Tr}\left[\gamma^- \not{l}\, \gamma^- (\not{l} - \not{q})\gamma^- (\not{l}' - \not{q})\gamma^- \not{l}'\right]$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$$A^{\lambda=T} = 8Q^2 \frac{l^-}{q^-}\frac{(l^- - q^-)}{q^-}\frac{(l^- - q^-)}{q^-}\frac{l^-}{q^-}$$

Let's define $z \equiv l^-/q^- \Rightarrow A^\lambda = 8\, z^2(1-z)^2 Q^2$

For transversely pol photon $\quad \varepsilon^\mu(q,\lambda) = (0, 0, \varepsilon_\perp^\lambda)$, $\varepsilon_\perp^\lambda = \frac{1}{\sqrt{2}}(1, i\lambda)$

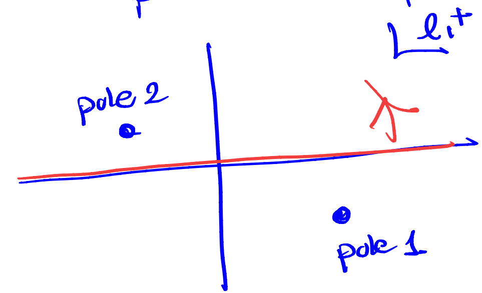
$$A^{\lambda=T} = 4\left[z^2 + (1-z)^2\right](l_\perp \cdot \varepsilon_\perp^\lambda)(l'_\perp \cdot \varepsilon_\perp^{\lambda *}) \quad \leftarrow \text{Exercise}$$

l^+ integration

$l^2 = 2l^+l^- - l_\perp^2$

$$I = \int \frac{dl^+}{2\pi} \frac{2q^-}{[l^2+i\varepsilon][(l-q)^2+i\varepsilon]} = \frac{-1}{z(1-z)} \frac{1}{2q^-} \int \frac{dl^+}{2\pi} \frac{1}{\underbrace{\left[l^+ - \frac{l_\perp^2 - i\varepsilon}{2q^- z}\right]}_{\text{pole 1}} \underbrace{\left[l^+ - \left(q^+ - \frac{l_\perp^2 - i\varepsilon}{2q^-(1-z)}\right)\right]}_{\text{pole 2}}}$$

$$I = \frac{-1}{z(1-z)} \frac{1}{2q^-} \frac{i\,\Theta(z)\Theta(1-z)}{\left[q^+ - \frac{l_\perp^2}{2q^- z(1-z)}\right]}$$



$$I = \frac{i\,\Theta(z)\Theta(1-z)}{[z(1-z)Q^2 + l_\perp^2]}$$, we used $Q^2 = -q^2 = -2q^+q^-$

$0 < z < 1$
else both poles sit same halfplane

Analogous result for l'^+ integration

$$A^\lambda = (ee_f)^2 \int_0^1 \frac{dz}{4\pi} \int \frac{d^2l_\perp\, d^2l'_\perp}{(2\pi)^4} \frac{e^{i(l_\perp - l'_\perp)\cdot r_\perp} A^\lambda}{[\bar{Q}^2 + l_\perp^2][\bar{Q}^2 + l'^2_\perp]}$$

Transverse integration

longitudinally pol photon $A^{\lambda=L}$ independent of $l_\perp$ & $l_\perp'$

$$\int \frac{d^2 l_\perp}{2\pi} \frac{e^{-i l_\perp \cdot r_\perp}}{[\bar{Q}^2 + l_\perp^2]} = K_0(\bar{Q} r_\perp) \; , \quad \text{identical structure } l_\perp' \text{ integral}$$

Assembling the result:

$$M^{\lambda=L} = \frac{\alpha_{em} e_f^2 N_c q^-}{\pi^2} \int d^2x_\perp d^2y_\perp \left[1 - \langle S(x_\perp, y_\perp)\rangle_x\right] \int_0^1 dz \, 4 z^2 (1-z)^2 Q^2 K_0^2(\bar{Q} r_\perp)$$

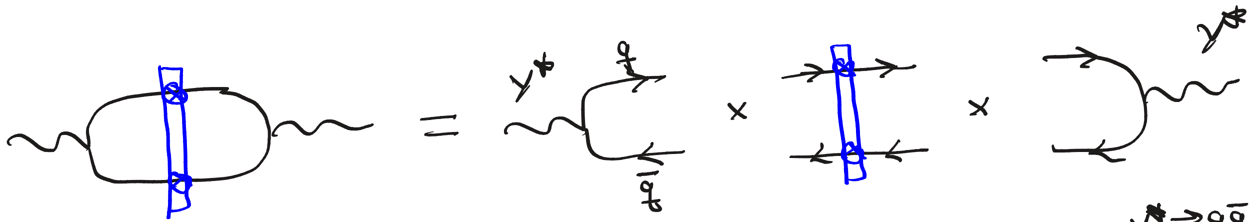
For transversely pol photon (exercise)

$$M^{\lambda=T} = \frac{\alpha_{em} e_f^2 N_c q^-}{\pi^2} \int d^2x_\perp d^2y_\perp \left[1 - \langle S(x_\perp, y_\perp)\rangle_x\right] \int_0^1 dz \left[z^2 + (1-z)^2\right] \bar{Q}^2 K_1^2(\bar{Q} r_\perp)$$

↑ averaged over 2 trans pol

$$r_\perp = x_\perp - y_\perp \qquad S(x_\perp, y_\perp) = \frac{1}{N_c} \mathrm{Tr}\left[V(x_\perp) V^\dagger(y_\perp)\right]$$

DIS cross-section : factorization

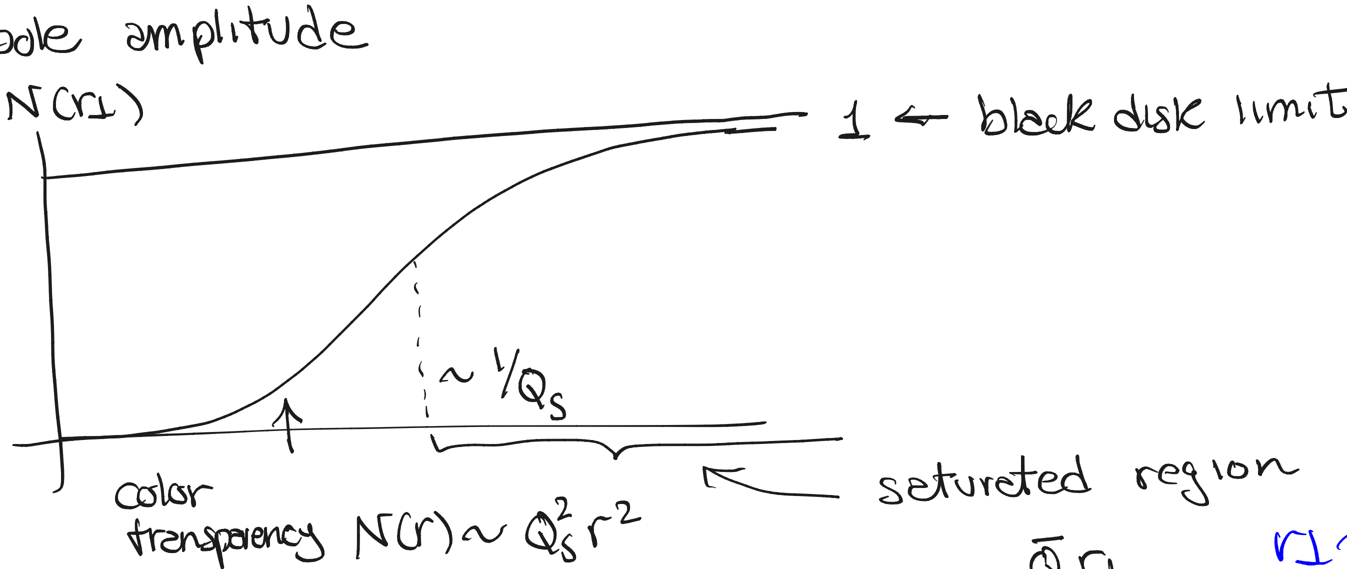


$$\psi^{\gamma^*\to q\bar{q}} \times [1-S] \times \left(\psi^{\gamma^*\to q\bar{q}}\right)^*$$

$$\left|\psi^{\gamma^*\to q\bar{q}}_{\lambda=L}\right|^2 = \frac{\alpha_{em}\, q^- e_f^2}{\pi^2}\, 4 z^2(1-z)^2 Q^2 K_0^2(\bar{Q} r_\perp)$$

$$\left|\psi^{\gamma^*\to q\bar{q}}_{\lambda=T}\right|^2 = \frac{\alpha_{em}\, q^- e_f^2}{\pi^2}\, [z^2+(1-z)^2]\bar{Q}^2 K_1^2(\bar{Q} r_\perp)$$

$$\langle N(r_\perp)\rangle_{MV} \equiv 1-\langle S(r_\perp)\rangle_{MV} = 1 - e^{-\frac{1}{4}Q_s^2 r^2 \ln(1/mr + e)}$$



$r_\perp \sim 1/Q$

Light-cone wavefunction $|\Psi_\lambda|^2 \sim e^{-\bar{Q} r_\perp}$

$Q \gg Q_s \Rightarrow$ one probes the color transparent regime

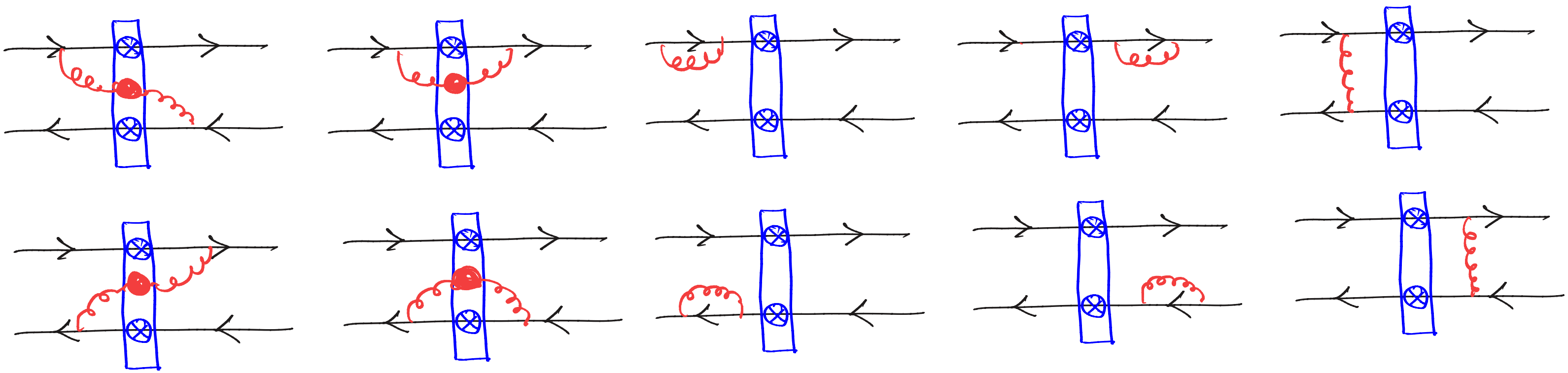
$\sigma \sim Q_s^2/Q^2$ (up to logs)

$Q \sim Q_s \Rightarrow$ one probes saturation region

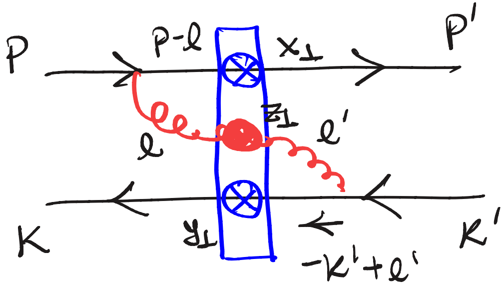
extreme limit $Q_s/Q \to \infty \Rightarrow \sigma =$ constant

Small-x evolution: Balitsky-Kovchegov

Consider a one-loop correction to the dipole



Let's focus on the phase space where radiated gluon has small longitudinal momentum ℓ^-, we will show this leads to a large logarithm!



$$S = \int \frac{d^4 l \, d^4 l'}{(2\pi)^8} G^{\nu\beta}(l') \, I_g^{\beta\alpha}(l', l) \, G^{\alpha\mu}(l)$$

$$\ldots I_q(p', p-l) \, S_0(\not{p}-\not{l}) \, ig\gamma^\mu t^a \ldots$$

$$\ldots I_q(-k, -k'+l') \, S_0(-\not{k}'+\not{l}') \, ig\gamma^\nu t^b \ldots$$

Trick $\quad \Pi^{\alpha\mu}(l) = -\sum_{\lambda=T} \varepsilon^\alpha(l,\lambda)\, \varepsilon^{*\mu}(l,\lambda) + \ldots$ off-shell piece

$$\Pi^{\nu\beta}(l')\, g^{\beta\alpha}\, \Pi^{\alpha\mu}(l) = \sum_\lambda \varepsilon^\nu(l',\lambda)\, \varepsilon^{*\mu}(l,\lambda)$$

Then : the fermion lines contract with polarization vectors

$$\varepsilon^\mu(l,\lambda) = \left(\frac{\varepsilon_\perp^\lambda \cdot l_\perp}{l^-}, 0, \varepsilon_\perp^\lambda\right)$$

Recall Dirac structure of I_q is γ^-

$$\ldots \Gamma_q(p', p-l)\, S_0(\not p - \not l)\, ig \not\epsilon^{*\mu}(l,\lambda)\, t^a \ldots$$

$$\overset{\text{Lorentz/Dirac}}{=} \gamma^-(\not p - \not l)\not\epsilon^{*\mu}(l,\lambda) \xrightarrow{l^- \to 0} \frac{\epsilon_\perp^{\lambda *}\cdot l_\perp}{l^-} 2p^- \gamma^- = 2p^- \frac{\epsilon_\perp^{\lambda *}\cdot l_\perp}{l^-}\gamma^-$$

$$\ldots \Gamma_q(-K, -K'+l')\, S_0(-\not K' + \not l')\, ig \not\epsilon(l',\lambda)\, t^b \ldots$$

$$\underset{z_g \to 0}{\overset{\text{Lorentz/Dirac}}{=}} 2K^- \frac{\epsilon_\perp^{\lambda}\cdot l'_\perp}{l^-}\gamma^-$$

Let's now examine l^+ integration

$$I = \int \frac{dl^+}{2\pi} \frac{2p^-}{[l^2+i\epsilon][(p-l)^2+i\epsilon]} = \frac{i\,\Theta(l^-)\Theta(p^- - l^-)}{\left[\frac{l^-}{p^-}\left(1-\frac{l^-}{p^-}\right)p^2 + \left(l_\perp - \frac{l^-}{p^-}p_\perp\right)^2\right]}$$

Similar for l'^+

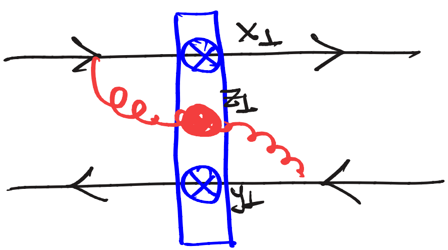
$$\xrightarrow[\frac{l^-}{p^-} \ll l_\perp^2/p^2]{} = \frac{i\,\Theta(l^-)\Theta(p^- - l^-)}{l_\perp^2}$$

Carrying out transverse integral (fourier transform)

$$\int \frac{d^2 l_\perp}{(2\pi)^2} \frac{l_\perp^i}{l_\perp^2} = \frac{i\, r_\perp^i}{2\pi\, r_\perp^2}$$

modulo perturbative part of dipole without radiation

$$S = \frac{\alpha_s}{\pi^2} \int \frac{dl^-}{l^-} \int d^2 z_\perp \frac{(x_\perp - z_\perp)\cdot(y_\perp - z_\perp)}{(x_\perp - z_\perp)^2 (y_\perp - z_\perp)^2} \quad V(x_\perp)\, t^a\, V^\dagger(y_\perp)\, t^b \;\; U^{ba}(z_\perp)$$



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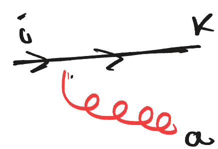
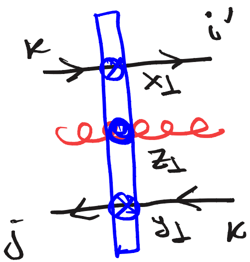
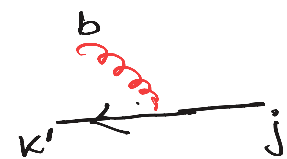






eikonal gluon emission

$$\frac{ig}{2\pi} (t^a)_{ki} \frac{(\vec{x}_\perp - \vec{z}_\perp)}{(x_\perp - z_\perp)^2}$$

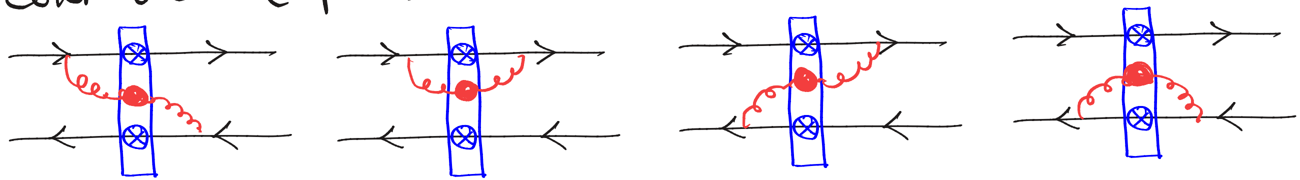
scattering

$$V_{i'k}(x_\perp)\, V^\dagger_{jk'}(y_\perp)\, U^{ba}(z_\perp)$$

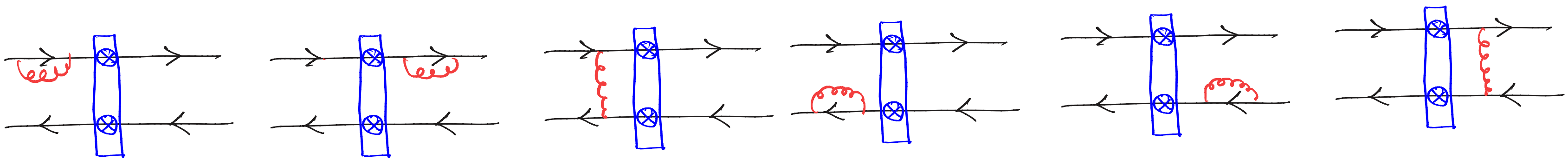
eikonal gluon absorption

$$\frac{-ig}{2\pi} (t^b)_{k'j'} \frac{(\vec{z}_\perp - \vec{y}_\perp)}{(z_\perp - y_\perp)^2}$$

Take color trace (dipole)



$$= \frac{\alpha_s}{\pi^2}\int\frac{dl^-}{l^-}\int d^2 z_\perp \left[\underbrace{\frac{2\,(x_\perp - z_\perp)\cdot(z_\perp - y_\perp)}{(x_\perp - z_\perp)^2 (z_\perp - y_\perp)^2} + \frac{1}{(x_\perp - z_\perp)^2} + \frac{1}{(y_\perp - z_\perp)^2}}_{\frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2 (z_\perp - y_\perp)^2}}\right] \mathrm{Tr}\left[V(x_\perp)\, t^a\, V^\dagger(y_\perp)\, t^b\, U^{ba}(z_\perp)\right]$$



$$= -\frac{\alpha_s}{\pi^2}\int\frac{dl^-}{l^-}\,\frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2 (z_\perp - y_\perp)^2}\; C_F\, \mathrm{Tr}\left[V(x_\perp)\, V^\dagger(y_\perp)\right]$$

Combining terms & using $t^a\, U^{ba}_{(z)} = V^\dagger(z)\, t^b\, V(z)$ and Fierz identity

$$\mathrm{Tr}\left[V(x_\perp)\, t^a\, V^\dagger(y_\perp)\, t^b\, U^{ba}(z_\perp)\right] - C_F\, \mathrm{Tr}\left[V(x_\perp) V^\dagger(y_\perp)\right]$$

$$= \frac{1}{2}\left[\mathrm{Tr}\left[V(x_\perp)V^\dagger(z_\perp)\right]\mathrm{Tr}\left[V(z_\perp)V^\dagger(y_\perp)\right] - N_c\, \mathrm{Tr}\left[V(x_\perp)V^\dagger(y_\perp)\right]\right]$$

The $\int \frac{d\ell^-}{\ell^-}$ produces a logarithmic divergence

but it is actually bounded by physical scales

Recall $\frac{\ell^-}{p^-} < \ell_\perp^2 / p^2 \sim \ell_\perp^2 / Q^2$ & $\ell_\perp / (x_0 S) < \frac{\ell^-}{p^-}$

virtuality of parent

equivalent to $\ell^+ < x_0 p^+$

modes above x_0 integrated out

$$\int_{(\ell_\perp^2/x_0 S)p^-}^{(\ell_\perp^2/Q^2)p^-} \frac{d\ell^-}{\ell^-} = \ln\left(x_0 S / Q^2\right) = \ln\left(x_0 / x\right)$$

$x = Q^2/S$

Dipole amplitude

$$N_{xy}(Y) = 1 - \frac{1}{N_c}\left\langle \mathrm{Tr}\left[V(x_\perp)V^\dagger(y_\perp)\right]\right\rangle_x \,, \quad Y = \ln\left(x_0/x\right)$$

Evolution

$$\frac{\partial N_{xy}(Y)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2}\int d^2z \frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2 (z_\perp - y_\perp)^2}\Big[N_{xz}(Y) + N_{zy}(Y) - N_{xy}(Y) \underbrace{- N_{xz}(Y) N_{zy}(Y)}_{\text{non-linear term}}\Big]$$

evolution absorbs corrections

$$\sim \alpha_s Y = \alpha_s \ln\left(x_0/x\right)$$

x_0 scale of initial condition

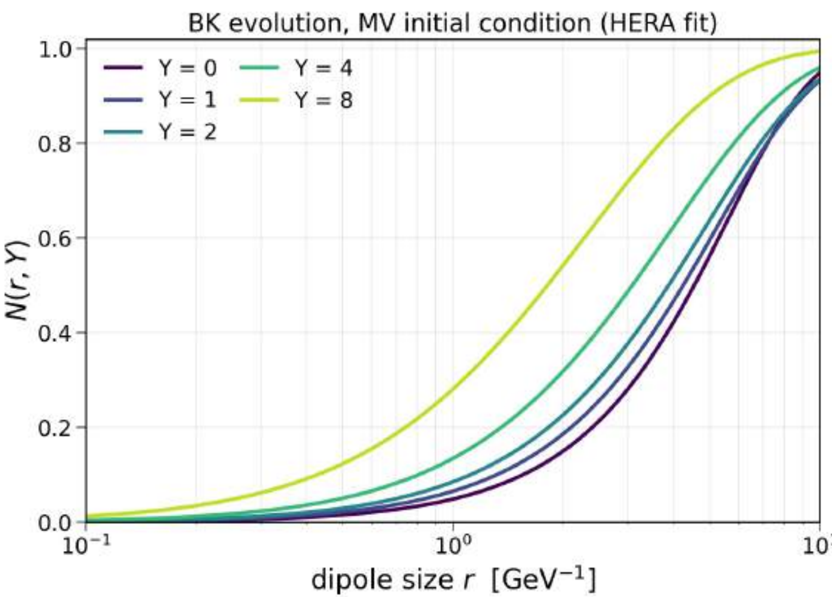
x physical scale set by scattering, in the case of DIS:

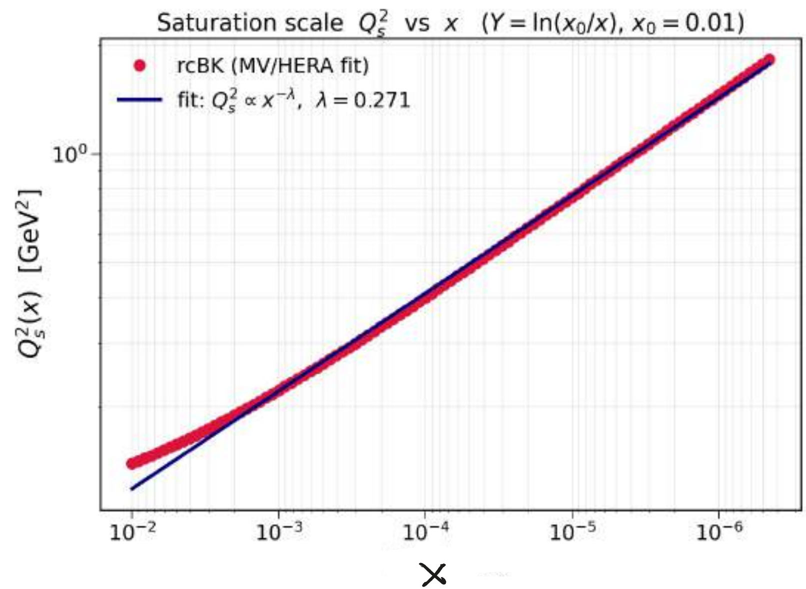
$$x = Q^2/s$$

Numerical solution to the BK equation

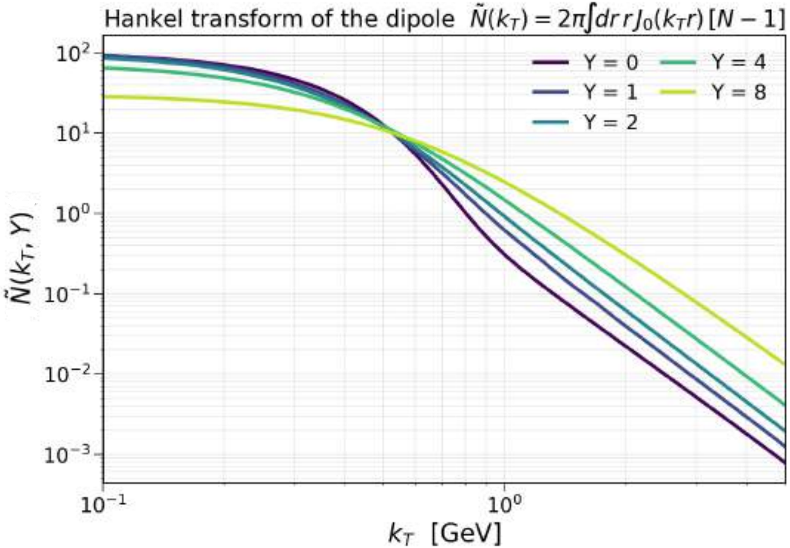
$$Y = \ln(x_0/x) \rightarrow x = x_0 e^{-Y}$$

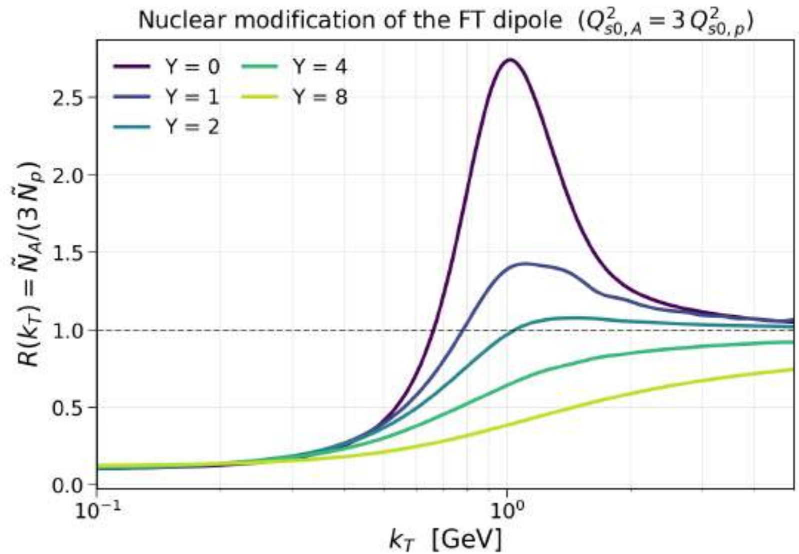
* running coupling following Balitsky prescription & $\mu^2 = C^2/r^2$, C^2 fitted to HERA



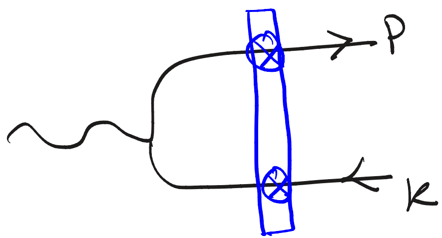


Numerical solution to the BK equation





Mutiparticle production: dijets in DIS



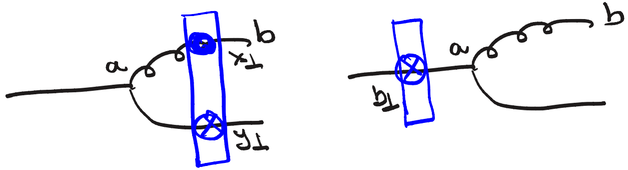
$$\mathcal{M} \sim \psi^{\gamma^* \to q\bar{q}} \otimes \left[V(x_\perp) V^\dagger(y_\perp) - \mathbb{1} \right]$$

$$\frac{d\sigma}{d^3p\, d^3k} \sim \int d^2x_\perp d^2y_\perp d^2x'_\perp d^2y'_\perp\, e^{-i p_\perp \cdot (x_\perp - x'_\perp)}\, e^{-i k_\perp \cdot (y_\perp - y'_\perp)}$$
$$\psi^{\gamma^* \to q\bar{q}}(x_\perp - y_\perp) \left[\psi^{\gamma^* \to q\bar{q}}(x'_\perp - y'_\perp) \right]^*$$
$$\left\langle \left[S^{(4)}(x_\perp, y_\perp, y'_\perp, x'_\perp) - S^{(2)}(x_\perp, y_\perp) - S^{(2)}(y'_\perp, x'_\perp) + 1 \right] \right\rangle_x$$

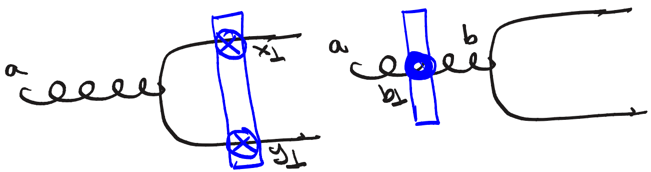
quadrupole $\quad S^{(4)}(x_\perp, y_\perp, y'_\perp, x'_\perp) = \frac{1}{N_c} \mathrm{Tr}\left[V(x_\perp) V^\dagger(y_\perp) V(y'_\perp) V^\dagger(x'_\perp) \right]$

dipole $\quad S^{(2)}(x_\perp, y_\perp) = \frac{1}{N_c} \mathrm{Tr}\left[V(x_\perp) V^\dagger(y_\perp) \right]$

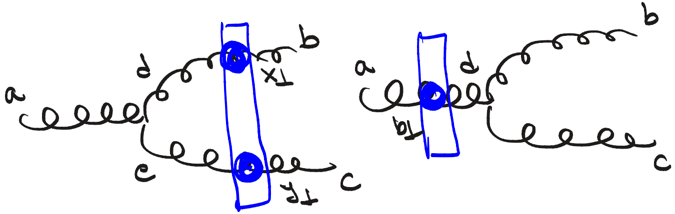
Mutiparticle production



$$\mathcal{M} \sim \psi^{q\to qg} \otimes \left[U^{ba}(y_\perp) V(x_\perp) t^a - t^b V(b_\perp) \right]$$



$$\mathcal{M} \sim \psi^{g\to q\bar{q}} \otimes \left[V(x_\perp) t^a V^\dagger(y_\perp) - t^b U^{ba}(b_\perp) \right]$$



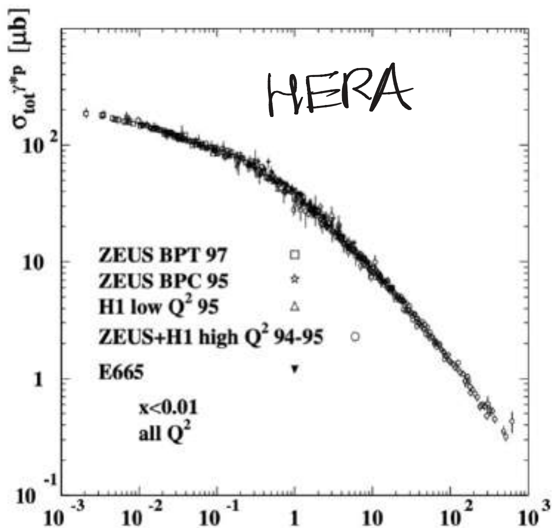
$$\mathcal{M} \sim \psi^{g\to gg} \otimes \left[U^{bd}(x_\perp) U^{ce}(y_\perp) f^{ade} - f^{dbc} U^{da}(b_\perp) \right]$$

cross-section will involve multipoint correlators of fundamental and adjoint Wilson lines, their evolution is dictated by JIMWLK equation (generalization of BK)

Reduced DIS cross-section

$$\sigma^{\gamma^* p \to X}(Q^2, x_{Bj})$$

$$\lambda_{eff} = \frac{d \ln \sigma (W^2)}{d \ln W^2}$$



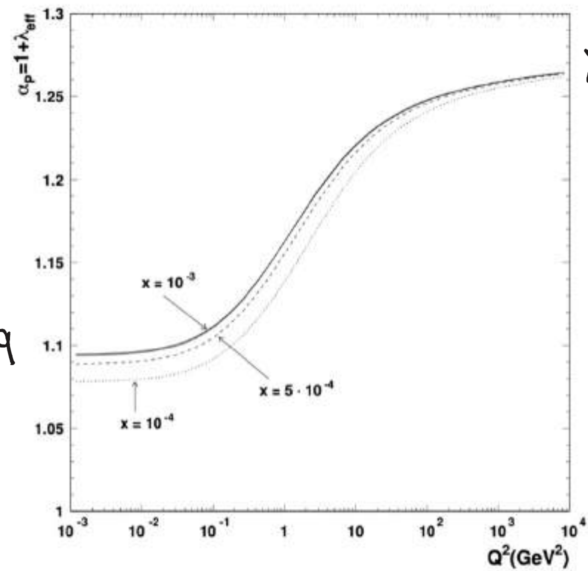
$$\tau = \frac{Q^2}{Q_s^2(x_{Bj})}$$

Stasto, Golec-Biernat, Kwiecinski (2000)

$1+\lambda_{eff}$

0.09

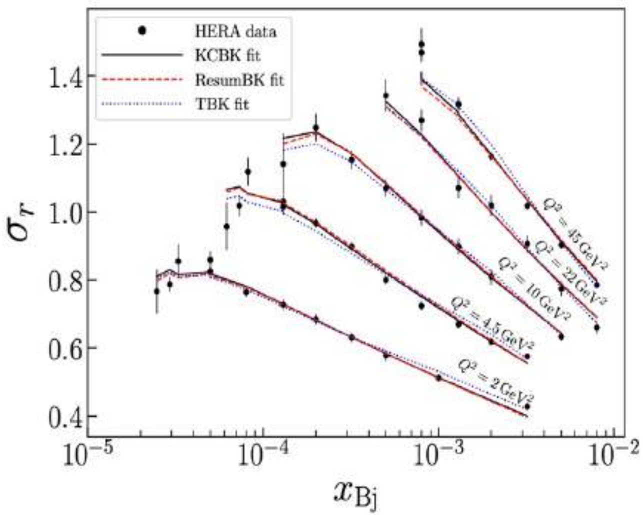
$\lambda_{eff} = 0.27$



Golec-Biernat, Wusthoff (1998)

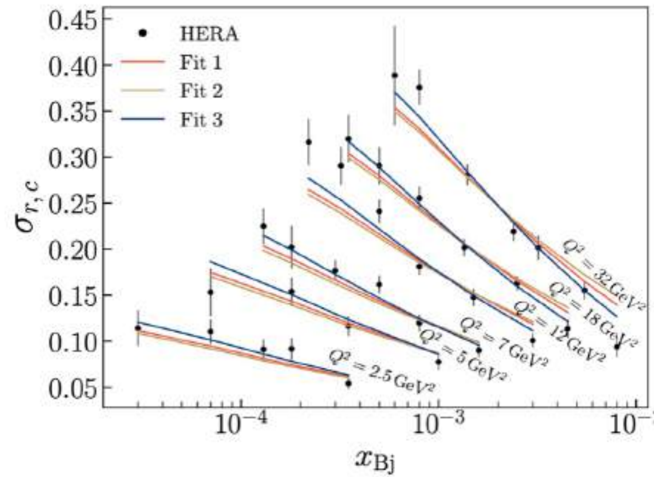
Reduced DIS cross-section

Total reduced cross-section



Beuf, Lappi, Hänninen, Mäntysaari (2020)

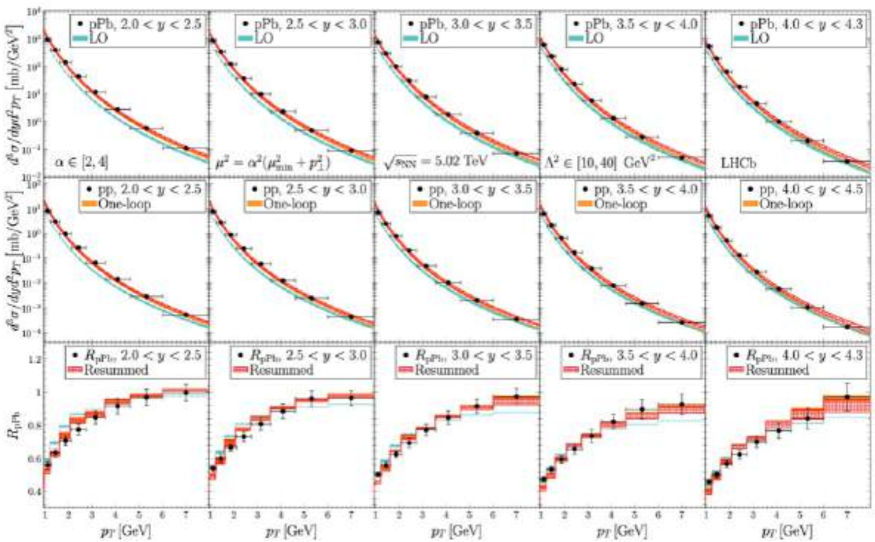
Charm reduced cross-section



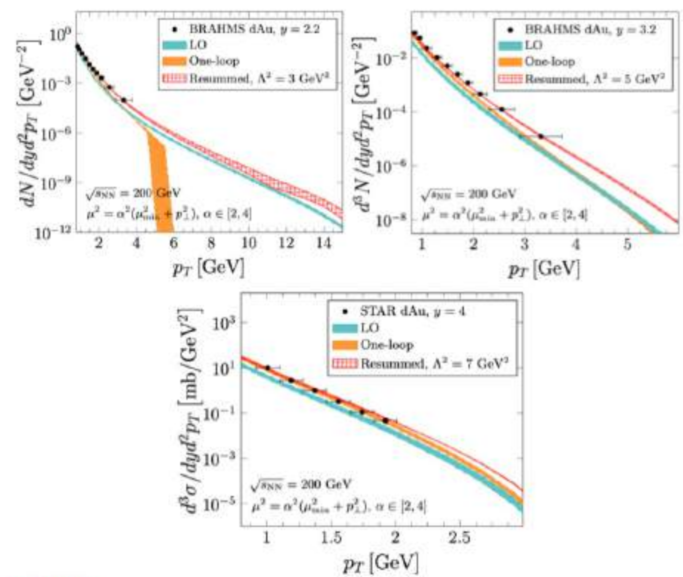
Hänninen, Mäntysaari, Paatelainen, Penttala (2023)

Single inclusive hadron proton-proton and proton-nucleus collisions

LHC



RHIC

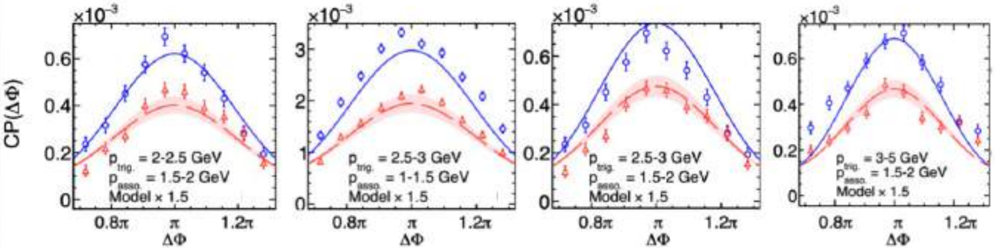


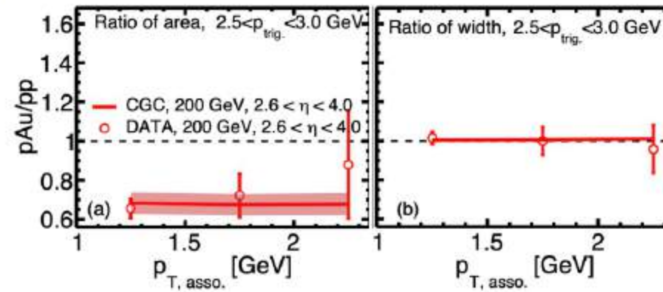
Shi, Wang, Wei, Xiao (2021)

Double inclusive particle production in pp/pA and ep/eA

Dihadron production in pp/pA

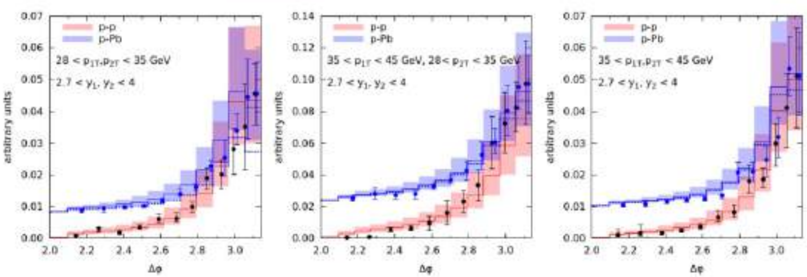
Caucal, Kang, Korcyl, Salazar, Schenke, Stebel, Venugopalan, Zhao (2025)
Data from STAR





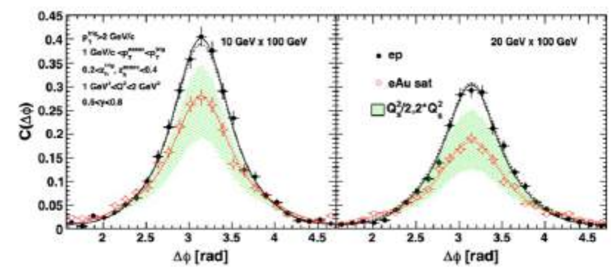
Dijet production in pp/pA

Van Hameren et al (2023) Data from ATLAS

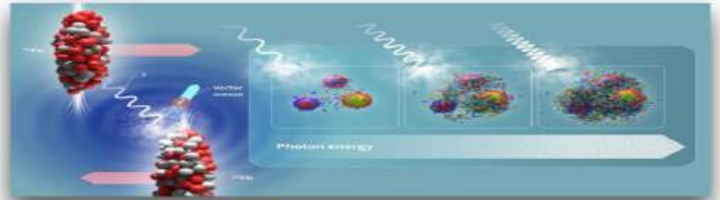


Di-hadron production in ep/eA

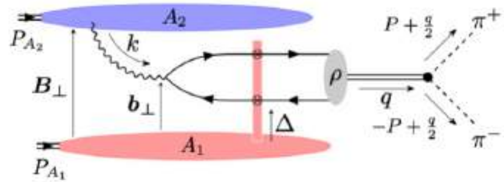
Aschenauer et al (2014)

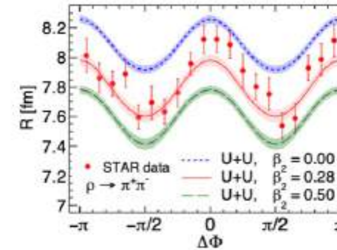


Wealth of data from UPCs: diffractive vector meson

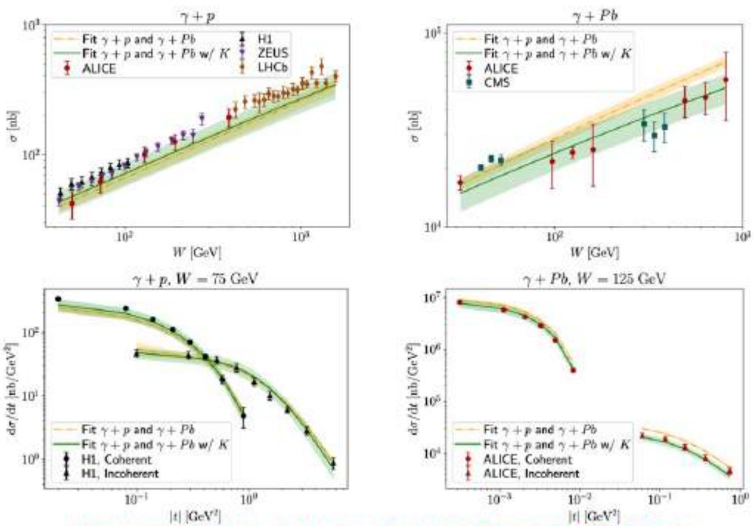


Differential measurements in azimuthal angle of decay -> access to nuclear structure



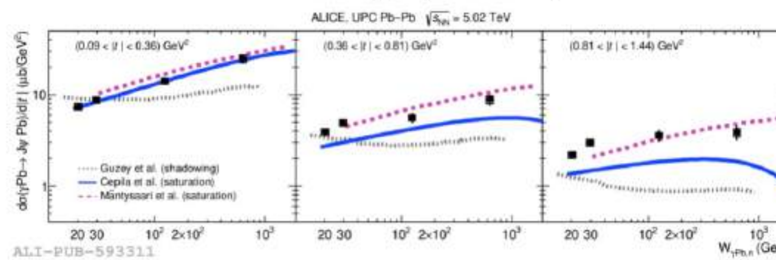


Global Bayesian coherent + incoherent J/ψ photo-production



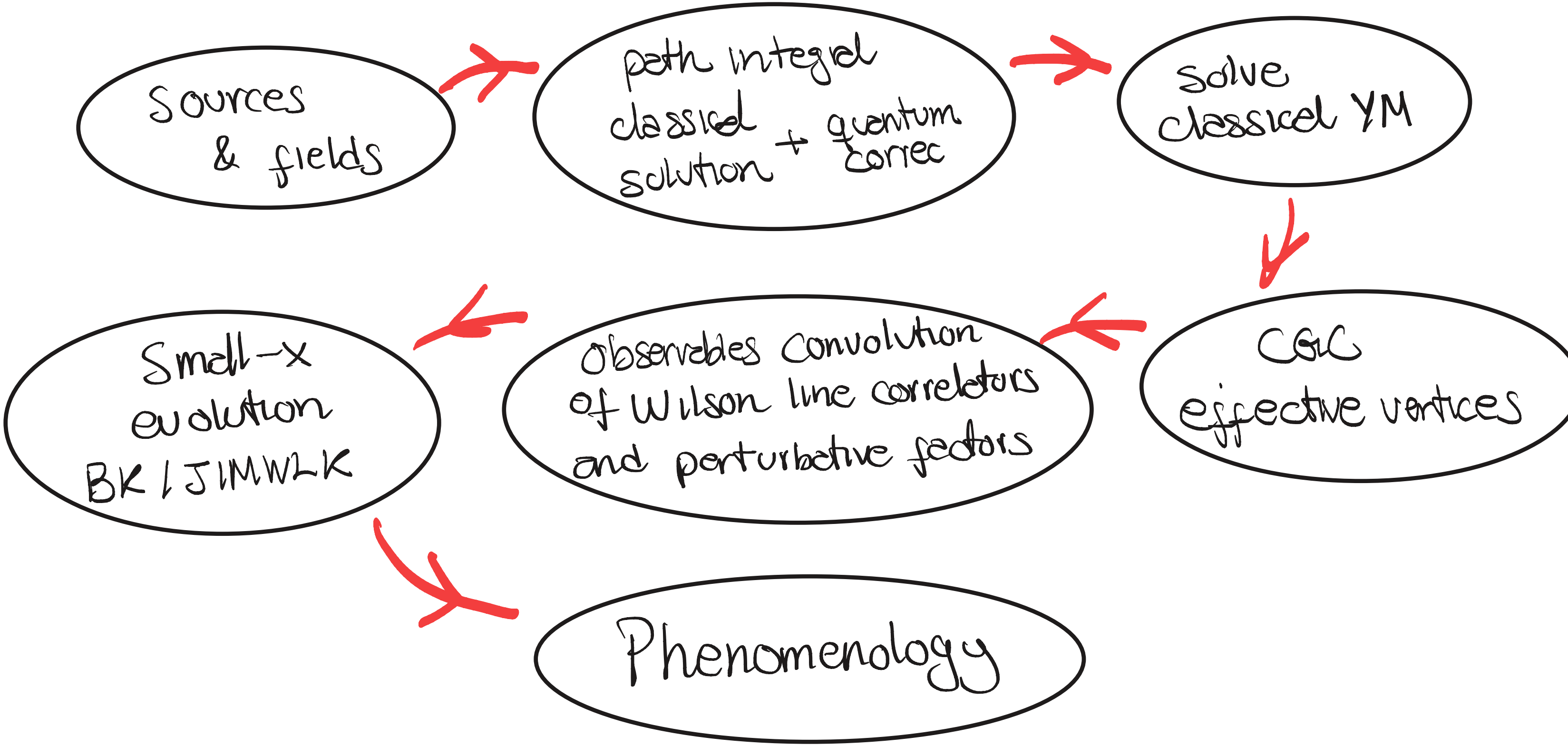
Mäntysaari, Roch, Salazar, Schenke, Shen, Zhao (2025)

Double-differential incoherent γA (QM 2025)



Mäntysaari, Salazar, Schenke (2022, 2024)
Data from ALICE (2025)

A roadmap to Small-x Physics & Parton Saturation



Exercises Lecture II

1. Derive the forward scattering $\gamma^* p \to \gamma^* p$ for both longitudinally & transversely polarized photon in detail filling the gaps in the notes of Lecture 2.

2. Complete the intermediate steps to derive the BK equation

3. One can obtain the BFKL equation by dropping the quadratic term $N_{xz} N_{yz}$. Investigate the solution to the BFKL equation

4. Show that the BK equation admits travelling wave solutions.

5. Compute the differential cross-section for quark-antiquark pair production in $\gamma^* p$ (dijet production @ leading order)

6. Correlation limit of dijet production in DIS

Let p & k be the momenta of the quark & antiquark.

Define momentum imbalance $K_\perp = p_\perp + k_\perp$ & relative momentum

$P_\perp = z_2 p_\perp - z_1 k_\perp$ where $z_1 = \frac{p^-}{q^-}$ & $z_2 = \frac{k^-}{q^-}$

Consider the limit $K_\perp \ll P_\perp$ in which the jets are produced back-to-back in the transverse plane.

Argue that in this limit one can perform a gradient expansion of the correlator around small $r_\perp = x_\perp - y_\perp$ & $r_\perp' = x_\perp' - y_\perp'$.

Show that in this limit the diff cross-section factorizes

$$d\sigma \sim H(z_1, z_2, P_\perp, Q)\, G_{WW}(K_\perp)$$