Finding the most compact configuration of neutron stars

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Abstract

We reproduce past solutions of the Tolman-Oppenheimer-Volkoff equations to estimate the maximum mass of a neutron star for two models of the equation of state (EoS), and discuss the future prospects of estimating a maximal-density constraint on the EoS. We use an EoS for a non-interacting ideal Fermi gas of pure neutrons, finding a maximum mass $M_{\mathrm{max}} \approx 0.71\,\mathrm{M}_{\odot}$ and a corresponding radius $R \approx 9.0$ km, agreeing with past studies. This procedure is repeated for an EoS modified to include the strong nuclear interaction. From this we find $M_{\mathrm{max}} \approx 2.22\,\mathrm{M}_{\odot}$ and $R \approx 10.0$ km, in agreement with past studies as well. The next steps of this project will be to assume the lower bound $M_{\mathrm{max}} \approx 2.0\,\mathrm{M}_{\odot}$ and find an EoS that satisfies this mass condition and minimizes sound speed c_s.

1 Introduction and background information

In this section, we provide some relevant background information on neutron stars and a brief motivation for the project.

1.1 Neutron stars from astronomical perspective

A neutron star is the remnant core of a star that stopped undergoing nuclear fusion, so radiation pressure no longer holds it up against gravitational collapse. Instead, it is in a stable hydrostatic equilibrium between the inward gravitational force and outward neutron degeneracy pressure and neutron-neutron repulsion via the strong nuclear interaction (Silbar & Reddy, 2004). Its radius is on the scale of $\sim$10 km and its mass is on the order of $\sim$1-2 $\mathrm{M}_\odot$. This reflects the extremely high density of a neutron star, $\sim 10^{15}\,\frac{\mathrm{g}}{\mathrm{cm}^3}$, an order of magnitude above nuclear saturation density, the density of an atomic nucleus. These values were determined from observations of pulsars, rapidly spinning neutron stars that emit beams of light along a specific direction in the rotating frame of each star. By measuring the timing of the light pulses intersecting Earth, astronomers were able to estimate the objects' small size (Carroll & Ostlie, 2017).

1.2 Neutron star formation mechanisms

Neutron stars form when the remnant core of a main sequence star undergoes gravitational collapse as its mass exceeds the Chandrasekhar limit of $\sim 1.4\,\mathrm{M}_\odot$. This is the limit beyond which electron degeneracy pressure can no longer support the star against gravitational collapse, so the next stable configuration is the one that characterizes neutron stars, where the star is held up by neutron degeneracy pressure (Carroll & Ostlie, 2017).

During gravitational collapse, protons and electrons combine to form neutrons and electron neutrinos, and the reverse reaction occurs because of beta decay. As the reverse reaction occurs, the proton and electron products are also repelled from each other due to degeneracy pressure, increasing the energy of the system slightly. The reactions proceed until reaching an equilibrium where the energy of the system is minimized and it would cost energy for either reaction to continue. Also, matter should be electrically neutral, which requires the charges of protons and electrons to cancel. The resulting matter is composed of mostly neutrons, with trace amounts of electrons and protons remaining (Silbar & Reddy, 2004).

In its stable state, the star is held up by the pressure created by neutron star matter. As we will show in the next section, neutron degeneracy pressure, which is a purely quantum mechanical effect that causes neutrons to repel, is not enough to support the heavy neutron stars. The bulk of the pressure originates from the nuclear interaction between neutrons.

Neutron degeneracy pressure arises from the Pauli exclusion principle for fermion wavefunctions. Neutrons, protons, and electrons are fermions, which means that they must have antisymmetric wavefunctions. Upon exchange of any two identical fermions, the wavefunction describing their superposition must be equal in magnitude and opposite in sign to the pre-exchange superposition wavefunction (Schroeder, 2000a).

1.3 Neutron star maximum mass

As matter accretes onto a neutron star, it will continue to exist in its stable configuration as long as hydrostatic equilibrium is maintained between gravity and neutron degeneracy pressure

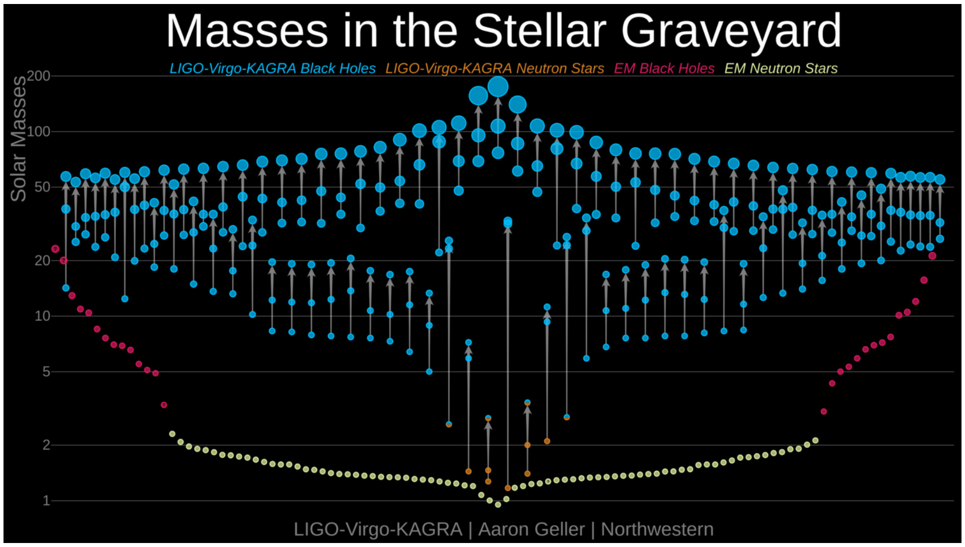

Figure 1: Masses of observed neutron stars and black holes. The mass gap lies between the EM Neutron Stars in yellow and the EM Black Holes in red. Figure adapted from (LIGO-Virgo-KAGRA, 2021).

and nuclear repulsion. Eventually when enough mass is accreted, the gravitational force exceeds degeneracy pressure and nuclear repulsion, and gravitational contraction occurs. This is a runaway mechanism because at this density, there is no longer any known force stronger than gravity, so it proceeds until the star collapses to become a black hole (Carroll & Ostlie, 2017).

Neutron stars have been observed at $\sim$1.4 - 2.0 $M_\odot$, and black holes have been observed at $\sim 5\,M_\odot$ and above. There is an observational "mass gap" between 2 and 5 $M_\odot$, in which there is an underabundance of neutron star and black hole observations, as shown in Fig. 1 (LIGO-Virgo-KAGRA, 2021).

This observational uncertainty of the maximum mass of a neutron star cannot be fully removed by theory, because the structure equation reflecting hydrostatic equilibrium depends directly on the strength of nuclear repulsion, which has large uncertainty. As such, the maximum mass of a neutron star is poorly constrained, and that is the primary motivation for studying the thermodynamic configurations of neutron stars.

1.4 Project overview

As alluded to in the previous section, the mass and radius of a neutron star can be predicted by solving a structure equation that encodes the hydrostatic equilibrium between inward gravitational force and the outward pressure of neutron star matter. The structure of a neutron star is governed by the neutron star's equation of state (EoS). There has been a plethora of EoSs from past research, but there is still significant uncertainty in the true EoS.

In this project (INTURN 24-1), we seek to find a constraint on the EoS corresponding to the most compact configuration of a neutron star. To do this, we have reproduced past studies of neutron star EoSs, focusing on the ideal Fermi gas model, which encodes neutron degeneracy pressure, and the strong nuclear interaction effect. The crucial next step in this study will be to make the assumption that the maximum neutron star mass is 2 $M_\odot$, a lower constraint according to astronomical observations. We will attempt to find the densest equation of state that predicts this assumed maximum mass.

2 Scientific results: reproducing the mass-radius relationship for various equations of state

In this section, we discuss the scientific results we have obtained during the program. We have reproduced the mass-radius relationship for various equations of state.

2.1 Neutron star structure equation

The mass and radius of a neutron star are found by solving the structure equations mentioned in the previous section. Tolman, Oppenheimer, and Volkoff derived these equations in Tolman (1939) and Oppenheimer & Volkoff (1939), so they are known as the Tolman-Oppenheimer-Volkoff (TOV) equations. They are two coupled first order differential equations derived from the condition for hydrostatic equilibrium between gravitational force and outward pressure:

$$\frac{dp}{dr} = -\frac{G\epsilon(r)\mathcal{M}(r)}{r^2}\left[1 + \frac{p(r)}{\epsilon(r)}\right]\left[1 + \frac{4\pi r^3 p(r)}{\mathcal{M}(r)}\right]\left[1 - \frac{2G\mathcal{M}(r)}{r}\right]^{-1}, \tag{1}$$

$$\frac{d\mathcal{M}}{dr} = 4\pi r^2 \epsilon(r)\,. \tag{2}$$

In this project, we use the natural unit system, in which $\hbar = c = 1$. This makes the gravitational constant $G = 6.72 \times 10^{-45}\,\mathrm{MeV}^{-2}$. The variables consist of r, the distance from the center of the star, p, pressure (distance dependent), ϵ, energy density (distance dependent), and $\mathcal{M}$, mass enclosed within the distance r from the center of the star. r and $\mathcal{M}$ are cumulative quantities and take on the values of neutron star radius R and mass M, respectively, at the surface.

The main expression outside of the brackets in Eq. (1) is derived from a free body diagram using Newtonian gravity, and the three correction factors in brackets are derived from general relativity (Silbar & Reddy, 2004). Eq. (2) is derived from mass density ρ integrated over the distance from the center of the neutron star, and we utilize the relativistic relation $\rho = \epsilon$ in natural units.

Intuitively, the initial conditions should be central pressure $p(0)$ and central cumulative mass $\mathcal{M}(0)$. However, there is an issue with using the values at $r = 0$ because this is the location of a singularity for the TOV equation. Therefore, practically, we need the initial conditions to instead be at some small but positive $r = r_{\mathrm{small}}$.

One has to specify by hand the value of central number density $n(r_{\mathrm{small}})$ or $p(r_{\mathrm{small}})$, which can be converted through the EoS, covered in section 2.2. Central cumulative mass $\mathcal{M}(r_{\mathrm{small}})$ is found by multiplying central mass density $\rho(r_{\mathrm{small}})$ by the volume of a small sphere with radius r_{small}.

To solve the TOV equation, we have coded a Python notebook.

The solution we want is $\mathcal{M}(r)$. Once $p(r) = 0$, it indicates that the surface $r = R$ has been reached, and $\mathcal{M}(r)$ flattens out around this distance. Fig. 2 shows example plots of $\mathcal{M}(r)$ and $p(r)$ for the central mass density $n(r = 0.1\,\mathrm{km}) = 7n_0$, where $n_0 \approx 0.16\,\mathrm{fm}^{-3}$ is nuclear saturation number density (Silbar & Reddy, 2004).

To find the total mass and the total radius of the neutron star, we simply terminate the solving algorithm at $p(r) = 0$ and pick out the values of $\mathcal{M}(R) = M$ and R here. We then convert these values to units of $\mathrm{M}_\odot$ and km, respectively.

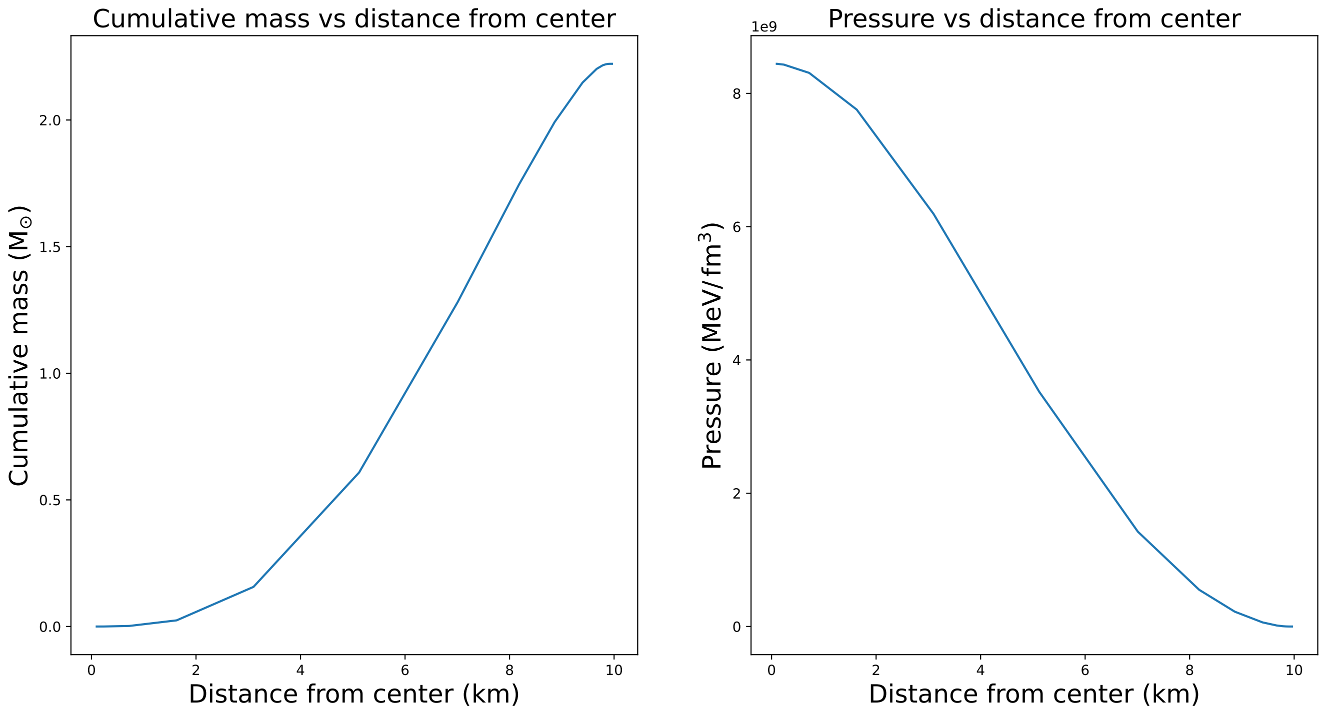

Figure 2: Plots of cumulative mass and pressure vs distance from the star's center. Note how both quantities flatten out near the star's surface ($R = r \approx 9.96\,\text{km}$). The mass of this neutron star is $\sim 2.22\,\text{M}_\odot$.

2.2 Equation of state of ideal Fermi gas

To solve the TOV equation, an equation of state is needed to convert between number density n, energy density ϵ, and pressure p. It is a relation between pressure and energy density that serves as a constraint on the thermodynamic properties of the matter in question. It is used in conjunction with the initial conditions $p(r_{\text{small}})$ and $\mathcal{M}(r_{\text{small}})$ to be an input for the TOV equation. Because the output of the TOV equation is a mass-radius relationship for any fixed EoS, there is a one-to-one correspondence between each EoS and each mass-radius relation. The EoS takes into account the components of pressure keeping the neutron star stable, including the nuclear interaction that is poorly constrained, so we have some freedom in choosing an EoS, and it is where the uncertainty in the maximum mass lies.

To reproduce previous studies of neutron stars held up only by neutron degeneracy pressure, not considering the nucleon interactions taking place, we find the EoS for the non-interacting ideal Fermi gas. This utilizes quantum statistics of a zero-temperature system of pure neutrons packed tightly enough that the Pauli exclusion principle determines the system's density. Rather than deriving an analytic function for pressure in terms of energy density, we derive formulae for energy density and pressure independently, and determine a range of discrete corresponding values for each.

Because it is convenient to specify a range of number densities n to compute values for, we try to find expressions with n as the independent variable.

For the Fermi momentum, which determines the energy density of the Fermi gas, we quote its expression from statistical mechanics (Schroeder, 2000a) (Silbar & Reddy, 2004):

$$k_F = \sqrt[3]{\frac{6\pi^2 n}{g}}\,, \tag{3}$$

where the number of spin states per energy state $g = 2$ for neutrons.

Also from statistical mechanics, we find the energy density by integrating the energy (Schroeder, 2000a) (Silbar & Reddy, 2004):

$$\epsilon = \frac{g}{2\pi^2}\int_0^{k_F} k^2\sqrt{k^2 + m_n^2}\,dk\,, \tag{4}$$

where k is momentum per particle and $m_n \approx 939.565\,\mathrm{MeV}$ is the mass per neutron. The square root term is the energy per particle, an expression from relativistic dynamics in natural units. The one other quantity we need before evaluating pressure is chemical potential (Schroeder, 2000b) (Silbar & Reddy, 2004):

$$\mu \equiv \frac{\partial\epsilon}{\partial n} = \sqrt{k_F^2 + m_n^2}\,. \tag{5}$$

Finally, a relation between pressure, energy density, chemical potential, and number density is derived from the thermodynamic identity (Schroeder, 2000b) (Silbar & Reddy, 2004).

$$p = -\epsilon + \mu n\,. \tag{6}$$

Using these relations, we specify a range of reasonable number densities and compute a value of pressure and energy density for each n.

Upon iterating over many central number densities and solving the TOV equation for a mass and radius for each $n(r_{\mathrm{small}})$, we plot the masses and radii to display the mass-radius relation. We used values of $n(r_{\mathrm{small}})$ between $0.1n_0$ and $100n_0$, obtaining the plot in Fig. 3 for the non-interacting Fermi gas model.

For this situation, the maximum mass is $\sim 0.71\,\mathrm{M}_\odot$ with a corresponding radius of ∼9.0 km. This is what we expect from the original work of Oppenheimer & Volkoff (1939).

We are missing the important nuclear interaction in the EoS, which supposedly will increase the maximum mass to $\sim 2\,\mathrm{M}_\odot$, so in the next section we account for this effect and study the result.

2.3 Including the nuclear interaction effect

The strong nuclear interaction causes neutrons to repel each other when they are extremely nearby, as in a neutron star (Silbar & Reddy, 2004). This extra force of repulsion is an important correction to the ideal Fermi gas model. We include it by modifying the formula for energy density. Instead of finding energy density from Fermi-Dirac statistics as in the Fermi gas case, we pull Eq. (11) for energy per neutron (excluding its rest mass) from Gandolfi et al. (2014):

$$E = a\left(\frac{n}{n_0}\right)^{\alpha} + b\left(\frac{n}{n_0}\right)^{\beta}\,. \tag{7}$$

Here, a, b, α, and β are parameters that we quote without going into the nuclear physics. Eq. (7) encodes both the ideal Fermi gas model and the strong nuclear effect. Then, the energy density can be found from Eq. (8):

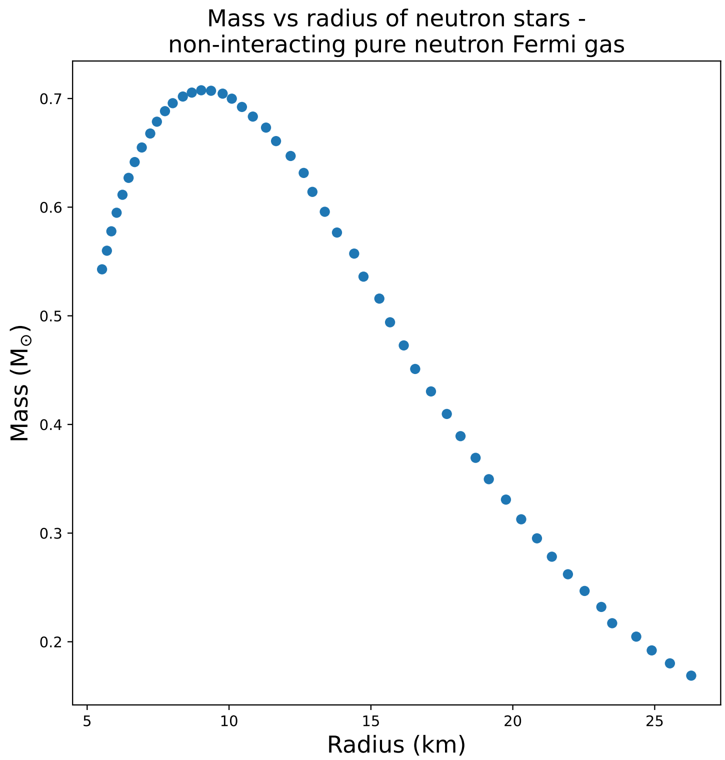

Figure 3: Mass-radius plot for non-interacting pure neutron Fermi gas

$$\epsilon = n\left(E + m_n\right)\ . \tag{8}$$

Plugging this in to the existing TOV solver code for a number density range of $0.5n_0$ to $10n_0$ yields a maximum mass of $\sim 2.22\,\mathrm{M}_\odot$ and a corresponding radius of ∼10.0 km, as seen in the mass-radius plot in Fig. 4.

This result roughly agrees with the analogous result from Silbar & Reddy (2004), in which the maximum mass was $\sim 2.3\,\mathrm{M}_\odot$ and the corresponding radius was ∼13.5 km.

3 Future prospects: Toward finding the most compact configuration

Having reproduced the results of past neutron star EoS studies to verify the validity of our TOV-solving program, we now seek to examine the lower bound of the EoS. We do this by making use of an equation relating the EoS to the sound speed c_s within the neutron star matter:

$$\frac{dp}{d\epsilon} = c_s^2\,. \tag{9}$$

This relation is expressed in natural units, where c_s can take on any values between 0 (indicating an EoS at the limit imposed by stability) and 1 (indicating an EoS at the limit imposed by causality, $c_s = c$). $c_s = 1$ imposes an upper bound on the EoS, while $c_s = 0$ imposes a lower bound, the end that we are interested in further constraining. The new condition that we will impose in this study is a maximum mass $M_{\max}$ of 2 $\mathrm{M}_\odot$. This is the lower bound of $M_{\max}$, corresponding to a stricter

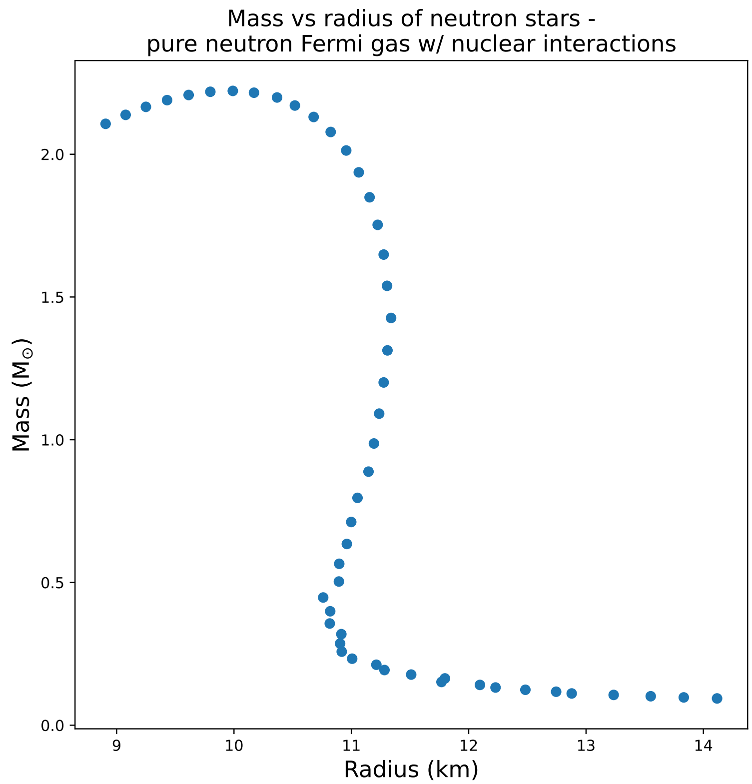

Figure 4: Mass-radius plot found from modifying the energy density to include the strong nuclear interaction

lower bound on the EoS. Drischler et al. (2021) does an excellent job of exploring the consequences of Eq. (9), and we will be drawing from their work in this next step of the project.

To estimate the EoS corresponding to a minimum c_s and $M_{\max} \approx 2.0\,\mathrm{M}_\odot$, we plan to vary the nuclear EoS above $n \sim$ (1-2)n_0 until the slope of the EoS approaches 0, while simultaneously requiring that the maximum mass reaches 2 $\mathrm{M}_\odot$. Below this density, the nuclear EoS can be fixed, e.g., by using Eq. (7). We may split the EoS above $n \sim$ (1-2)n_0 into a piece-wise function for different density ranges. At this point we will present an estimate of the lower bound of the neutron star EoS.

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