Summer Progress Report for INTURN Project: 24-6[1]

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Abstract

This report follows the plans and accomplishments made over the summer for INTURN project 24-6. This specific project goes over extreme equations of state within binary neutron star mergers. However in this report, we do not touch upon any interactivity with modeling or simulating BNS merger events. Instead this report is about the development of constructing an Equation of State for a singular compact star (either a White Dwarf or Neutron Star) and then using that EOS to then pass through a TOV Sovler to generate a Mass vs. Radius curve in units of Solar Masses and kilometers. To do this, Python was used to construct Equations of State based on generated tables of values for various parameters like fermi momentum and number densities.

Introduction

The work that has been done so far through this project has been both enlightening and a challenge. So far what has been done in this project has consisted of reading other papers on sound speeds and neutron stars, studying textbook material on equations of state, and even some lectures on Numerical Relativity which will be useful for the future directions we wish to take in this project. I want to proceed through this paper as less of a mundane status update but also as the thought process going through the project and some of the struggles or minor successes had along the way.

Modeling the Equation of State is one of the major components to working with the Structure Equations for compact stars. Stellar objects in general hold hydostatic equilibrium, which means that a star is able to withstand the force of gravity to prevent collapse by countering with internal pressure. In compacts stars, White Dwarfs and Neutron Stars, there is no more thermal pressure which withstands collapse but instead it is the nucleons themselves that resist collapse. This pressure is determined by the systems energy density which can be calculated by knowing the systems number density and fermi momenta. In a White Dwarf, this energy density comes from the massive nucleons present which the pressure which resists the collapsing of gravity comes from electrons, which is what is called electron degeneracy pressure in White Dwarfs.

In order to know about how the pressure changes in a star when modeling the entire object, we must know how the energy density changes as well which is what causes pressure changes. That is why the Equation of State is so important. It keeps track of the interplay between energy density and pressure so that stellar modelling and simulation can be performed. The Equation of State is also what allows us to work with the structure equations especially when general relativistic corrections are included. The structure equations then give us a full model of how mass and radius change due to internal factors like number density and central pressure.

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A Brief Mention on Units

It is easiest to pass everything through natural units, since all the units involved for equations will only be in MeV to some power or not. All while allowing unity amongst all fundamental constants.
$\hbar = c = k_b = 1$
Determining units and tracking them is tricky, especially when dealing with an alternate system of units like natural units for example. Working with them requires some good in depth understanding about how units translate across equations and systems. The upside to natural units and going between that and say cgs or SI is that all fundamental constants are taken out and simplifies calculations. It is interesting to see how then the final step for presenting data in this case is to figure out how to bring the units back to something that makes sense.

Equation of State and Theory

The construction of the equation of state for these compact stars involves finding properties about any fermions involved in the system. Each fermion has their own contributions towards the total energy density and pressure with a given fermi momentum or number density for each. Knowing either one of them will give the other because of how number density , n and fermi momentum p_F are related.

$$n = \frac{p_F}{3\pi^2} \tag{1}$$

This equation can be used to either find the fermi momentum using a given number density or vice verse. Regardless of this, it is the fermi momentum that is used to find its corresponding energy density and pressure. How exactly these two quantities are related comes from their thermodynamic relationship.
In a compact star, we are able to assume degeneracy because at the cores of these stars, the pressures are extremely high and therefore fermions have to take up the lowest energy states possible without violating Pauli's exclusion principle. The conditions of degeneracy are also in line with the conditions of fermions when at extremely low temperatures since they are also found in their lowest energy states.
Assuming that $T = 0$ in the first law of thermodynamics we can find that $P = \frac{\partial U}{\partial V} = n\mu - \epsilon$. So we see that pressure is the difference of the chemical potential (which is just the change in energy density with respect to number density) and energy density. This is safe for us to assume that the pressure of the system is based upon the energy density.
The equation for energy density and pressure for each fermion is

$$\epsilon(k) = \frac{\gamma}{2\pi^2} \int_0^k \sqrt{(k^2 + m^2)} k^2 dp \tag{2}$$

$$P(k) = \frac{1}{3} \frac{\gamma}{2\pi^2} \int_0^k \frac{k^2}{\sqrt{(k^2 + m^2)}} k^2 dp \tag{3}$$

These two equations are what produce our Equation of State (Shapiro & Teukolsky 1983). By modeling a range of difference fermi momenta we can generate a plot which can then be interpolated over or in an even more creative manner, a function can be generated in the form of a polytrope which will truly be the equation of state.

Generating Equations of State

This section is being treated on a case by case basis which will first look at the modeling process of White Dwarfs and then it will look into the process of modeling a full Neutron Star with all three fermions as individual contributors.

White Dwarf Equation of State

When coding the White Dwarfs EOS, I had started out by using cgs units instead of natural units in order to maintain a finer concept of units while progressing through the various equations and plotting. In the given paper, Neutron Stars for Undergraduates, it was suggested that the range of fermi momentum to be plotted would be between 0 and $2m_e$. This is mainly due in part to the relativistic parameter which is used in the equations of energy density and pressure where the integral is solved and instead uses this parameter,x, instead of explicitly using k and m

The energy density contribution by electrons in a white dwarf is similar to the equation above. The difference between them is that the former equation uses γ which is in place to denote the degeneracy of momentum states. However the entire constant set outside of the integral can be reduced to a single factor of $\epsilon - 0$ which is determined by some choice or in the convenience of the equation and its variables. In the case for White Dwarfs $\epsilon_0 = \frac{m_e^4 c^5}{\pi^2 \hbar^5}$. And the equations for energy density and pressure are (where $x = \frac{k_F}{m_e c}$)

$$\epsilon(x) = \frac{\epsilon_0}{8}[x(2x^2 + 1)(1 + x^2)^{\frac{1}{2}} - sinh^{-1}(x)] \tag{4}$$

$$P(x) = \frac{\epsilon_0}{72}[x(\frac{2x^2}{3} - 1)(1 + x^2)^{\frac{1}{2}} + sinh^{-1}(x)] \tag{5}$$

Although these are in cgs units and not in natural units as I had mentioned before, it still provides a legible curve that shows how pressure changes due to energy density. Thankfully these equations were derived from their integral form and can be used not just here in a pure electron gas but in the upcoming models that need it.(Silbar & Reddy 2004)

Neutron Star Equation of State

The Neutron Star Equation of State was one of the more trickier approaches when it comes to modeling an EOS. In the White Dwarf case, there was really only one fermion that was being accounted for, but for Neutron Stars and using the Fermi Gas Model all three fermions are included in the model. Because this is an Ideal Fermi Gas, none of these particles have nuclear interactions. Although this does restrict us from having a deeper more comprehensive understanding of the Neutron Stars core, we still retain other conditions seen in Neutron Stars.Mainly, the properties that are seen are charge neutrality and weak interaction equilibrium or β equilibrium.

Neutron Stars possess charge neutrality, which means that at the cores there is a zero net charge. That being said, we can assume that there must be equal numbers of protons and electrons within. $n_p = n_e$. And since there is proportional relationship between number densities and fermi momentum, they must also be equal to each other $k_{F,p} = k_{F,e}$.

Free neutrons within Neutron Stars will undergo a weak decay which causes a neutron to decay into a proton, electron, and an antineutrino. Because charge neutrality is a property of the neutron star, there must be an inverse reaction which takes place in the form of electron capturing. This

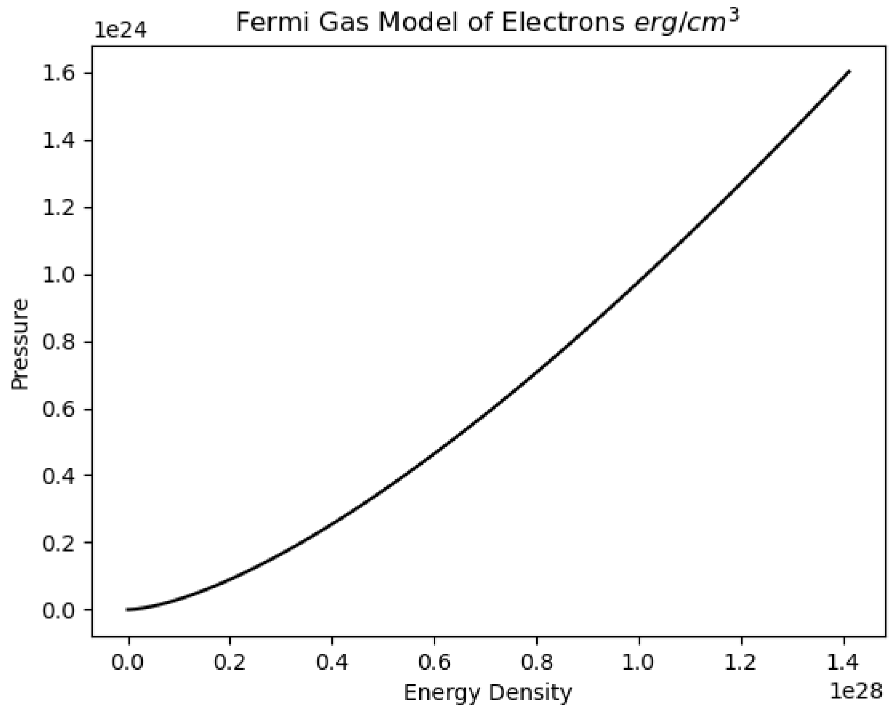

Fig. 1.— This is a plot of the energy density and pressure in an ideal Fermi Gas composed of just Electrons. Both of which have units of ergs$/cm^3$. The pressure here is due to electron degeneracy pressure and there is a higher amount of nucleons present which is what mainly contributes to the energy density.

reaction takes a proton and electron to produce a neutron and a neutrino.

$$n \longrightarrow p + e^- + \bar{\nu}_e \tag{6}$$

$$p + e \longrightarrow n + \nu_e \tag{7}$$

The reactions between electron capture and neutron decay exhibit beta equilibrium within a Neutron Star. To quantify this, we use a relationship between all three fermions chemical potentials to show that their interactions are in equilibrium.

$$\mu_n = \mu_p + \mu_e \tag{8}$$

Using this equality it is easy to find an equation that finds the fermi momentum of protons based on that of neutrons. When writing this code and determining the equation to use, any instance of $k_{F,e}$ is easily replaced with $k_{F,p}$ when used in the chemical equilibrium formula where, $\mu = \sqrt{k_F^2 + m^2}$.

The EOS constructed for Neutron Stars was initially started by two alternate processes. One was to manually input a range for the relativistic parameter like the ones in equation 4 and 5. The other, and more favorable method by me, is to use a range of number densities found from data tables

online specific for equations of state. It is a lot more fascinating and practical to use imported data when running this code for further use. However manually inputting a range for x_n was the best method for determining any error or numerical confusion.
Regardless of the method used to obtain the neutrons fermi momentum, the equations used to determine energy density and pressure, previously for white dwarfs, can also be used to find the individual energy densities and pressures of the fermions in the neutron star. These can all be summed to find the total contribution that each has towards the EOS of the Neutron Star which can be seen in Fig. 2.

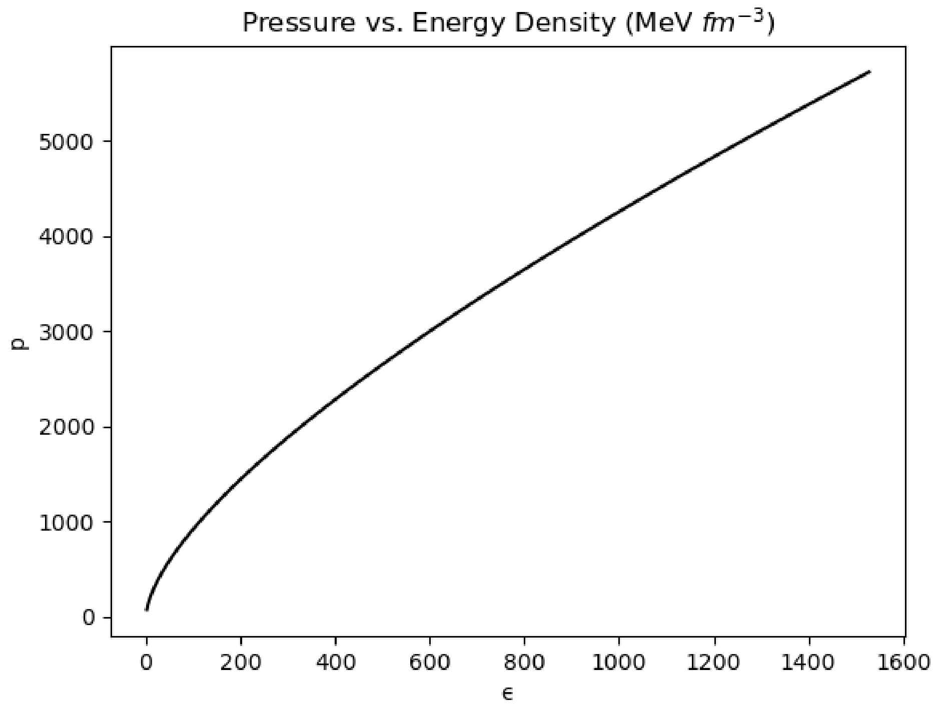

Fig. 2.— Equation of State, in natural units, found based upon the sum of individual fermion energy densities and pressures. The massive pressures here are due to degeneracy pressure just like in white dwarfs but this term increases very rapidly as opposed to energy density because of how compact Neutron Stars are in their cores.

TOV Solver

Constructing the TOV Solver code was both a simple process and a careful one much like the Equation of State code that had been written. First of all, this is when it was realized that the best way to handle two semi-large blocks of code was to separate them in classes. One for the construction of the EOS and the other to use that EOS to solve the TOV equation. The final output of which would be multiple curves that gives information about the Neutron Star being modelled. However, in this case what is cared about is not one specific Neutron Star but a whole range of them in order to find what the maximum masses are for a certain radius.

When doing this modelling, the first thing that comes to mind is where exactly we are drawing limits, or more physically, what the boundary conditions are. With pressure, it makes sense that once we see this value terminate at zero, then we have reached the edge of the star and we can break out of a loop and extract what the total mass is at that point and therefore what the maximum radius was for that star. The initial conditions were much more straightforward as we know that the mass at the very center of the star must begin at zero as we imagine that at this point we have not begun to collect anything about mass. And of course we are at a point where the radius is zero. As I have stated, pretty straightforward with this initial setup.

Structure Equations and the TOV Equation

Taking a step back to classic Newtonian mechanics, stars can in fact be modelled without considered anything higher level like general relativity for example. A star is held together in hydro-static equilibrium, meaning that a star pushes against its own gravity to avoid collapse by pressure either through thermonuclear processes or degeneracy pressure in cases where thermonuclear processes terminate completely.

$$\frac{dp}{dr} = -\frac{G\rho(r)M(r)}{r^2} \tag{9}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho(r) \tag{10}$$

These are the coupled differential equations for hydro-static equilibrium that use mass density to determine the changes of mass and pressure alongside mass itself in changes in pressure with respect to r.
It isn't too hard to assume that as you slowly move from the center of a star, something that is not recommended by any means, the mass density decreases by some amount and at some rate. This of course is determined by the equation of state, however the one that I had constructed uses energy density instead. This is a simple fix to include it into the classical newtonian equations since we can use the mass-energy equation in a way that shows the relationship between mass density and energy density $\epsilon \approx \rho c^2$.
Besides this minor change we could expect that the mass of a star will rise fast and begin to trait off at some asymptotic where the maximum mass is reached, and that the pressure will do the same but in a continuous decrease towards zero.

This all finally brings us to the TOV Equation. In White Dwarfs it is not absolutely necessary to use this form, so instead the classical Newtonian equations for hydro-static equilibrium can be used with ease. However, the main implication of the TOV is because in Neutron Stars, there is a significant effect of space-time due to the ratio because mass and the stars radius being large (or non-negligible)(Silbar & Reddy 2004). Because of this, general relativity must be taken into account and the GR corrections must be made, but only on the derivative for pressure.

$$\frac{dp}{dr} = -\frac{G\epsilon(r)M(r)}{c^2r^2}\left[1 + \frac{p(r)}{\epsilon(r)}\right]\left[1 + \frac{4\pi r^3 p(r)}{M(r)c^2}\right]\left[1 - \frac{2GM(r)}{c^2 r}\right]^{-1} \tag{11}$$

I went ahead and decided to turn the equation into a form that uses dimensionless values of pressure, energy density, and mass. Much in the same way as what was done in the construction

of the EOS and then plugging in constants back in later on. Its also a lot easier in this way since it can reduce the equation down to simpler terms.
To start, we take ϵ and p and divide them both by ϵ_0 which was defined previously. Instead of having this constant with units of $\frac{MeV}{fm^3}$, we convert these units to become $\frac{M_\odot}{km^3}$. This is a lot better because it makes these units make a lot more sense and digestible.

I will first establish two constants (keeping in mind here that c still is 1, and $M\odot$ is also considered to be equal to 1 since this is the unit of measurement).

$$R_0 = \frac{GM_\odot}{c^2} = 1.47\text{km} \tag{12}$$

$$\beta_\odot = \frac{4\pi}{M_\odot c^2}\epsilon_\odot \tag{13}$$

Here R_0 is equal to one half Schwartzchild radius of the sun. The constant $\beta_\odot$ is not too incredibly special, besides interestingly enough it has a unit of km^{-3} which is the same as number density, but it does help reduce the TOV equation into this much easier form alongside $R_\odot$.

$$\frac{dp}{dr} = -R_0\frac{\bar{\epsilon}\bar{M}}{r^2}\left[1 + \frac{\bar{p}}{\bar{\epsilon}}\right]\left[1 + \beta_\odot\frac{r^3\bar{p}}{\bar{M}}\right]\left[1 - \frac{2R_0\bar{M}}{r}\right]^{-1} \tag{14}$$

$$\frac{dM}{dr} = \beta_\odot r^2\bar{\epsilon} \tag{15}$$

Both of these equations are fed into a python class that is used to solve the TOV equation for a given pressure. This given pressure is set to be the central pressure at $r = 0$ where $M = 0$ initially. Within a stable neutron star, there are central energy densities that can range anywhere between one-half to around 10 times nuclear energy density according to preexisting literature(Glendenning 1997). Either by inspection or by creating a function based on values obtained, the corresponding pressure can be found in order to create a list of central pressures.

Polytropic Model

Using python to solve the TOV equation requires knowing what the energy density is from a given pressure. This is a simple way of explaining it but to do this in code, I use a method of curve fitting which creates an equation that can account for the wide range of densities within data.

The polytrope is a thermodynamic relationship which can be found in the ideal gas but in this context it is a relationship in astrophysics which shows how pressure depends on a given density. This makes sense considering that the pressure in thermodynamics involves the change of energy density with respect to number density.
In the Neutron Star EOS we can use a double polytrope and can find parameters that can fit the curve that was generated best.

$$\bar{\epsilon}(\bar{p}) = K_{NR}\bar{p}^{\frac{3}{5}} + K_R\bar{p} \tag{16}$$

Here we have two components, one corresponding to the non-relativistic regime where $\Gamma = \frac{3}{5}$, and the other term is set to a power of 1 because it is simple enough to give an analytic solution when solving for pressure. When this is done we find the constants, according to the data plotted.

$$\bar{\epsilon}(\bar{p}) = 1.977\bar{p}^{\frac{3}{5}} + 1.001\bar{p} \tag{17}$$

These constants could vary of course depending on the extent of the values plotted, but the power law still stays consistent.

TOV Solver

By inspection we can see that we will have a range of central pressures between 1 to 10^3 $\frac{\text{MeV}}{\text{fm}^3}$ (or 10^{-4} to 1, when dimensionless). Using this range, which is in agreement with the typical range stated for energy densities before (additionally, I can expand this range because of some of the central pressure values used in the Neutron Stars for Undergraduates paper)(Silbar & Reddy 2004), I can use a logspace command to create a range of values of central pressures to use when solving for the TOV equation. Creating this was an easy arrangement in my suitable python IDE, and so were the coupled differentials that can be easily utilized by the solveivp function thanks to scipy. However, the meticulous part had breached its way into my way.

What was a complicated process soon turned to be a very easy solution and insight into not just the equation but also the setup of initial conditions itself. One major note about the initial setup is the obvious, there is no way to make a computer divide by zero, so when it comes to starting at zero radius or zero mass, instead what needed to be done was to calculate the mass at a small incremental difference of radius from r = 0 to r = dr. This setup helped manage making sure that there were proper values for initial mass and radius that aren't explicitly zero but are the next step from both being equal to zero.

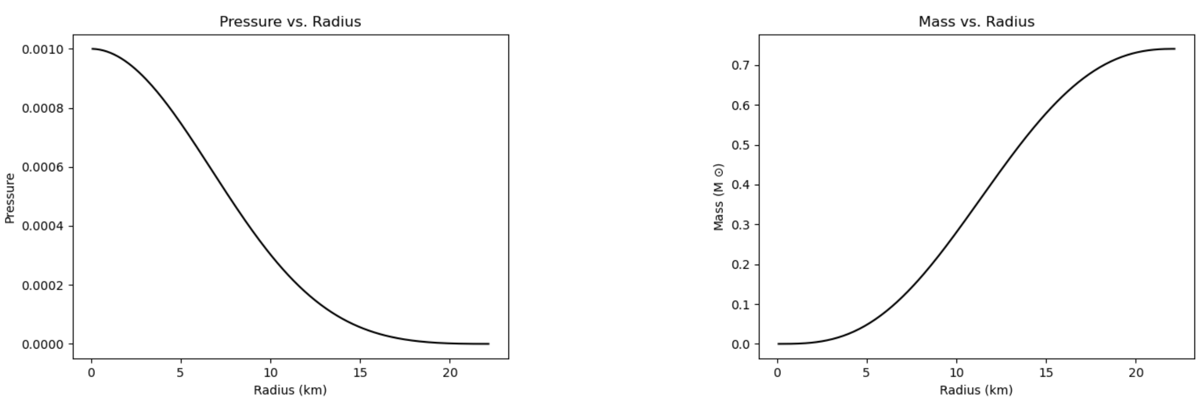

Fig. 3.— Pressure and Mass changing as a function of radius for a Neutron Star model with a central pressure, $\bar{p} = 10^{-3}$. These curves are, in a way almost mirror images of each other. We see that pressure decreases at a fast rate and mass does the opposite at a similar rate.

Now from this, it is possible now to see how the pressure and mass changes as a function of radius from a given central pressure. After running the script I had written that uses an EOS constructor and now a TOV Solver, I am able to obtain curves for pressure and mass, both as functions of radius. Not only this but a wide range of central pressures can be modeled, and all that needs to be extracted is the maximums of mass and radius for each. This is what has been done for an expanded range of central pressure to show the relationship between mass and radius for Neutron Stars shown in Figure 4.

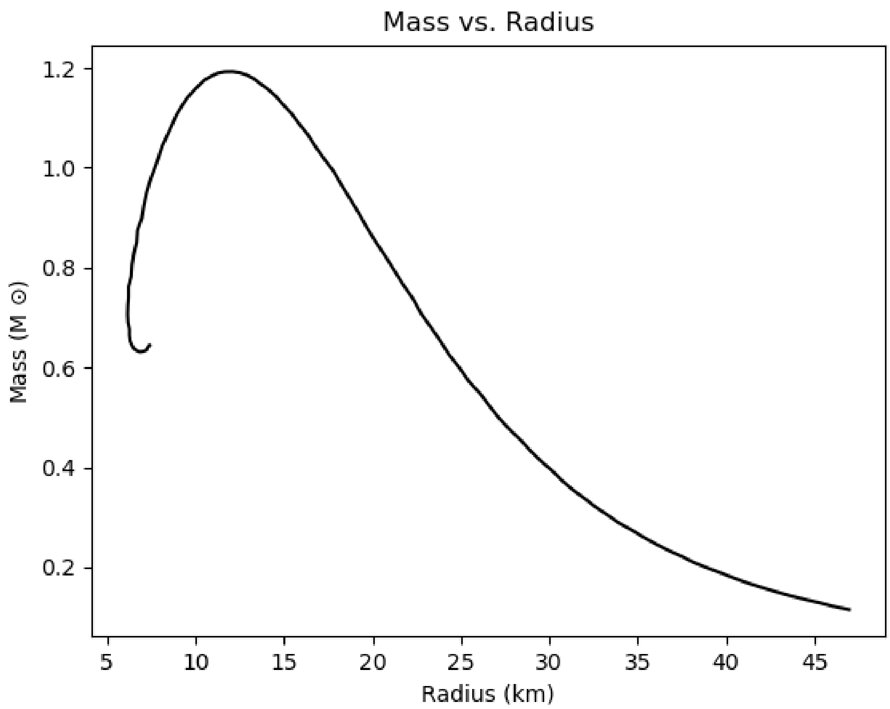

Fig. 4.— Maximum Mass and Radii for a range of central pressures ($10^{-6} < \bar{p}_c < 10^2$). There is a region near about 12 km where a maximum mass is reached. Beyond this, as radius decreases the mass then starts to decrease as well and soon initiates an in-spiral indicated by the small curl at the end of the curve on the left. What this means is gravitational collapse for a Neutron Star since it is deemed unstable beyond the point of the maximum mass.

The Future

Looking back on the progress of this seemingly simple part of the process shows great promise for what is to be done in the future as far as research and modelling Neutron Stars. What has been done here is for only single Neutron Stars, however in the future me and John wish to use HPC's not only to perform simulations that have more precision but also simulations that include the full extent of GR and in 3D space. This is especially the ambition for when it comes time to model Binary Neutron Star mergers, which will have to include GR, especially since what our interest in here is looking at sound speeds within Neutron Stars.

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