Mixed-Phase Strange Quark Stars

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Abstract

With the advent of LIGO in 2002, and with subsequent expansions and improvements, scientists can use gravitational waves to probe deeper into the universe than previously possible. As researchers cast a bigger net, they increase the chances of discovering objects with exotic compositions. This work characterizes one possible exotic object: mixed-phase, strange quark stars or nuggets. An equation of state is derived by modeling the object as a relativistic Fermi gas; this equation leads to solutions the Tolman-Oppenheimer-Volkov (TOV) Equations. More important, this investigation explores the methods for predicting characteristics such as mass and radii of hypothetical exotic objects in the universe.

1 Introduction

First observed over fifty years ago, neutron stars are often described as astrophysical laboratories. Their high density (∼1.5 solar masses and 10 km in diameter) presents opportunities to test theories at the frontiers of nuclear physics, particle physics, and astrophysics as researchers attempt to characterize exotic matter under extreme conditions. However, neutron stars are just one animal in a zoo of hypothetical exotic objects in the universe. As density increases within a neutron star, the quarks composing the protons and neutrons in a neutron star break out of their triplets. The resulting matter is a dense soup of quarks, gluons, and electrons, a quark star. The focus of this paper is a specific quark star composed of strange quarks and a mixed phase. This star and countless other theoretically constructed objects exist within the density range of neutron star to black hole. Their discovery by LIGO and ensuing study would provide further insight regarding exotic matter and new testing grounds for nuclear, particle, and astrophysical theories.

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2 Natural Units

Before modeling the composition of mixed-phase, strange quark stars, a brief discussion of natural units is in order. In the realm of nuclear and particle physics, the prevalence of Planck's constant $\hbar$ and the speed of light c can make calculations messy and cumbersome. In the interest of efficiency and ease of work, a unit system is adopted in which

$$\hbar = c = 1.$$

While expressions with $\hbar$ and c become easier to work with, vigilance is still required, for the units on other quantities change also within the system:

$$\text{Mass, Energy, Momentum} \Rightarrow MeV$$

$$\text{Force} \Rightarrow MeV^2$$

$$\text{Charge Density} \Rightarrow MeV^3$$

$$\text{Pressure, Energy Density} \Rightarrow MeV^4 \text{ or } \frac{MeV}{fm^3}$$

$$\text{Length, Time} \Rightarrow MeV^{-1}$$

Natural units are one of many systems used in various settings. For the remainder of this paper, natural units will be used.

3 Star Structure: Tolman-Oppenheimer-Volkov Equation

The approach to star structure here will be Newtonian. Consider a uniform sphere in equilibrium[1], and consider a small box within the sphere. The net forces acting on the box are given below in Figure 1:

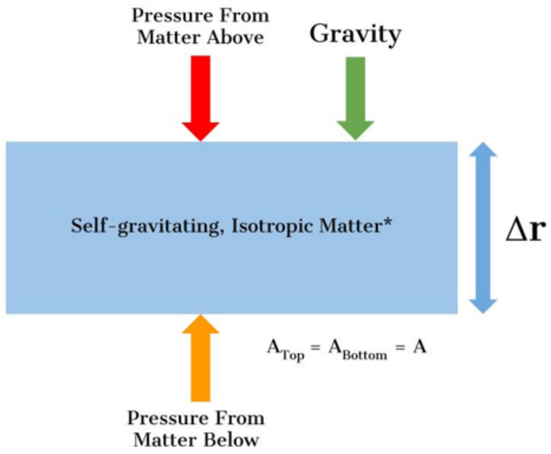


Figure 1: Forces acting within a uniform sphere in hydrostatic equilibrium.

The important thing to note is that the sum of forces in equilibrium must be zero. This fact, along with the fact that energy density and mass density are related by $\epsilon(r) = \rho(r)$, yields an equation for the rate of change of the pressure within the object as one moves out towards its surface:

$$\sum F = P(r)A - P(r+\Delta r)A - \frac{GM(r)\rho(r)A\Delta r}{r^2} = 0$$

$$\Delta P = P(r+\Delta r) - P(r) = -\frac{GM(r)\rho(r)}{r^2}\Delta r$$

$$\boxed{\frac{dP}{dr} = -\frac{GM(r)\epsilon(r)}{r^2}}$$

To find the rate of change of the mass moving away from the center, consider thin shells of the sphere instead of small boxes:

$$\Delta M = M(r+\Delta r) - M(r) = M_{shell} = V_{shell}\rho(r) = 4\pi r^2\epsilon(r)\Delta r$$

$$\boxed{\frac{dM}{dr} = 4\pi r^2\epsilon(r)}$$

These are the Newtonian TOV Equations. While these equations work in non-relativistic scenarios, the extreme pressure and energy density found in hypothetical objects requires general relativity. Solving Einstein's field equations for a uniform sphere yields the relativistic TOV Equations:

$$\boxed{\frac{dP}{dr} = -\frac{G\epsilon(r)M(r)}{r^2}\left[1+\frac{P(r)}{\epsilon(r)}\right]\left[1+\frac{4\pi r^3 P(r)}{M(r)}\right]\left[1-\frac{2GM(r)}{r}\right]^{-1}}$$

$$\boxed{\frac{dM}{dr} = 4\pi r^2\epsilon(r)}$$

Note that the first factor in the pressure equation is the Newtonian equation. The other factors are general relativistic corrections.

In both cases, the TOV and mass equations represent a coupled system with no analytical solution. Furthermore, there are more unknowns than equations. However, an equation of state for the material composing the object, specifically one relating pressure and energy density ($P(\epsilon)$), will allow for numerical solutions. The numerical solutions to the coupled system are possible masses and radii of the object. For this purpose, a 4th order Runge-Kutta Solver was constructed in C++, and a detailed description of the numerical method can be found in [1]. In short, a model of the matter composing the object leads to a solution of the TOV equation.

4 Object of Interest: Mixed-Phase, Strange Quark Nuggets

This project investigates a mixed-phase, strange quark star [2][3]. This object is a sphere composed of two phases (see figure 2): At the core is a dense, positively-charged, homogeneous soup of gluons and up, down, and strange quarks, with electrons moving throughout. The phase is positively-charged due in part to the mass of the strange quark (95 MeV) dwarfing those of the up (2.4 MeV) and down quarks (4.8 MeV). Outside the inner phase, densely packed electrons extend all the way to the surface. This phase is negatively-charged. The object is constructed to have a global net zero charge. Using these assumptions, one can find possible masses and radii of a mixed-phase, strange quark star.

5 Modeling the Object

5.1 Chemical Potential Parametrization

In physics problems, especially problems with phases in equilibrium, it is often helpful to figure out what is

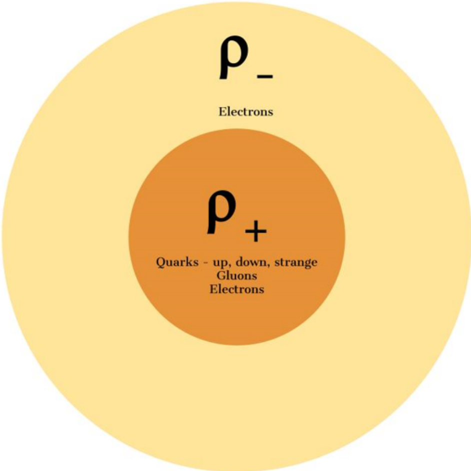


Figure 2: Mixed-phase, strange quark star composition. The inner phase exhibits a positive charge density, and the outer phase exhibits a negative charge density.

invariant or conserved with respect to some process. Once conserved quantities are identified, parameters associated with those quantities may be introduced, and the problem can be framed with respect to those parameters. That is the case here. Within the strange object, the main equilibrium process is:

$$d \leftrightarrows u + e^- + \nu_e$$

Moving to the right is beta decay, in which a down quark decays into an up quark and an electron (with mass 0.511 MeV). The extra energy is carried away by an antineutrino and kinetic energy of the products. Moving to the left is electron capture. In less dense stars, electron capture entails a proton and electron combining to form a neutron; however, the quarks in the strange star are not necessarily confined into protons and neutrons. Thus, up and down quarks are found in the equation instead. The neutrinos in this equilibrium may be disregarded, as they are not associated with quantities of interest.

There are two conserved quantities of interest in the equilibrium process above: One is baryon number, given by $\frac{1}{3}$ multiplied by the difference in number of quarks and number of anti-quarks.

$$B_i = \frac{1}{3}(n_{q_i} - n_{\bar{q}_i})$$

Protons and neutrons have baryon number 1, as they each consist of a trio of quarks, with different numbers up and down. A π^0 meson has baryon number zero, as it consists of a quark-anti-quark pair.

The other conserved quantity of interest is electric charge. Quarks possess a charge of some integer multiple of $\frac{1}{3}$, and electrons possess a charge of -1.

It is now possible to frame the problem around these two parameters. Associate with each particle involved a chemical potential energy μ_i (energy per particle), given by the sum of the product of baryon number B_i and baryon chemical potential μ_B (in energy per baryon) and the product of charge Q_i and charge chemical potential μ_e (energy per unit charge):

$$\mu_i = B_i\mu_B + Q_i\mu_e$$

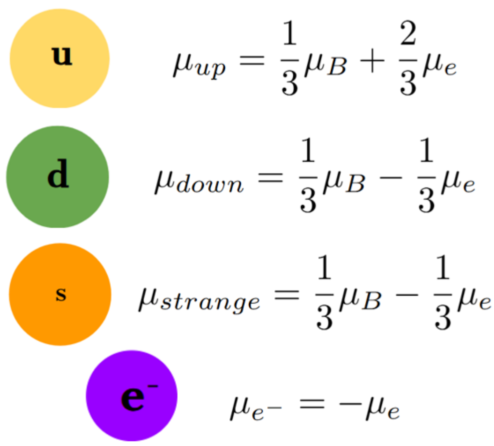


Figure 3: Parametrized chemical potentials of quarks and electrons. The particle energies are now in terms of baryon and charge chemical potential.

5.2 Relativistic Fermi Gas

The contents of the strange object will be modeled as a relativistic, zero-temperature Fermi gas. The zero-temperature model is applicable because the thermal pressure exerted by the quarks and electrons is negligible compared with the degeneracy pressure at temperatures below 10^{12} K. Neutron stars, which burn at around 10^8 K initially before cooling, fall well short of the mark. Furthermore, strange stars are denser than neutron stars, so the degeneracy pressure should be greater. It is therefore safe to use a zero-temperature model.

A relation between the momentum of each particle and the chemical potential energy parameters is useful in Fermi gas model calculations. The general version of the relativistic mass-energy relationship $E = mc^2$, with chemical potential energy μ_i taking the place of E, will provide just that:

$$E^2 = (pc)^2 + (mc^2)^2 = p^2 + m^2$$

$$\mu_i = \sqrt{p^2 + m_i^2}$$

$$\boxed{p_i = \sqrt{\mu_i^2 - m_i^2} = \sqrt{(B_i\mu_B + Q_i\mu_e)^2 - m_i^2}}$$

5.3 Reduction to One Parameter

Next, narrow down from two parameters to one parameter. The values of all quantities will then depend solely on the value of the baryon chemical potential μ_B. Use the assumption that the strange object is in a Gibbs Equilibrium; the pressure of the two phases must be equal. In the inner phase (see figure 2), pressure contributions come from the quarks and electrons, while in the outer phase, only electrons are present:

$$P_{inner} = P_{outer}$$

$$P_{e^-} + P_{quarks} = P_{e^-}$$

$$P_{quarks}(\mu_B, \mu_e) = \left(\sum_{i=u,d,s} P_{F_i} \right) - B = 0$$

In the above equation, B refers to the bag constant[4] in quantum chromodynamics (QCD)[5]. The bag constant is a value related to the bag model of quark confinement[6], and its presence in calculations of quark pressures is necessary. Further discussion of this concept will not occur; it is very interesting, but outside the scope of this project. Use for the bag constant a value of $B^{\frac{1}{4}} = 200\ MeV$[7]. Then, the sum of pressures being equal to zero is an equation one can numerically solve to find a function $\mu_e(\mu_B)$.

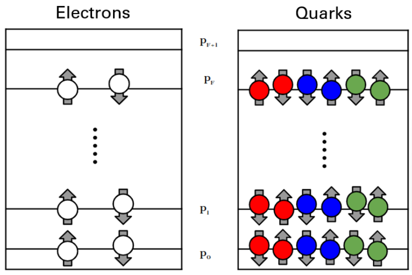
The pressure of each constituent comes from the Fermi gas model. The pressures are given by the equations below[8]. x_i has only been defined to make the equations more concise.

$$P_{F_i} = \frac{g_i}{6\pi^2} \int_0^{p_{F_i}} \frac{p^4}{\sqrt{p^2 + m_i^2}} dp$$

$$P_{F_i} = \frac{g_i m_i^4}{16\pi^2} \left[x_i(\frac{2}{3}x_i^2 - 1)\sqrt{1 + x_i^2} + \text{arcsinh}(x_i) \right]$$

$$x_i = \frac{p_{F_i}}{m_i} = \frac{\sqrt{(B_i\mu_B + Q_i\mu_e)^2 - m_i^2}}{m_i}$$

The factor g_i is the degeneracy of each particle at zero temperature[9]. The Pauli Exclusion Principle states no two particles can share the same quantum numbers. As the levels fill with electrons from the ground state to the Fermi level, the exclusion principle allows two particles at each energy, one spin up and one spin down. Quarks, however, have an additional quantum number given by color charge. Each quark can have color red, blue, or green, in addition to its spin. Thus, for each energy level, there are allowed six different quarks.



$$g_i = \begin{cases} 2, i = e^- \\ 6, i = u, d, s \end{cases}$$

Figure 4: Fermi Gas Degeneracy at T = 0 K.

Now, all of the necessary information is available to calculate the pressures. This yields the equation below:

$$\frac{8\pi^2 B}{3} = \sum_{i=u,d,s} m_i^4 \left[x_i(\frac{2}{3}x_i^2 - 1)\sqrt{1 + x_i^2} + \text{arcsinh}(x_i) \right]$$

The solution μ_e vs. μ_B may then be plotted:

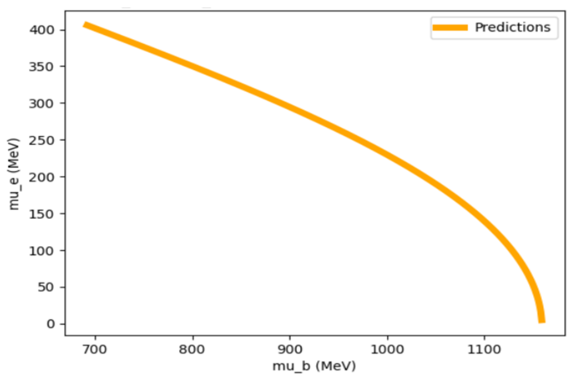


Figure 5: μ_e vs. μ_B for strange object matter.

Now, $\forall \mu_B \in [700, 1150]\ MeV$, $\exists$ an associated value of $\mu_e \in [0, 425]\ MeV$. There is only one parameter, and focus shifts to generating a curve relating pressure and energy density. Hurray!

5.4 Generating Pressures

Next, associate a pressure with each μ_B. This is as simple as finding the pressure of the electron phase (outer phase shell in figure 2), since the phases are in Gibbs Equilibrium and therefore maintain equal pressures:

$$P(\mu_B) = P_{F_{e^-}}(\mu_e)$$

$$P_{F_{e^-}} = \frac{m_{e^-}^4}{8\pi^2}\left[x_{e^-}(\frac{2}{3}x_{e^-}^2 - 1)\sqrt{1+x_{e^-}^2} + \text{arcsinh}(x_{e^-})\right]$$

With electrons, $\mu_e >> m_{e^-}$ and $x_{e^-} >> 1$, so appropriate approximations can be made:

$$\boxed{P_{F_{e^-}}(\mu_B) \approx \frac{m_{e^-}^4}{8\pi^2}\left[x_{e^-}(\frac{2}{3}x_{e^-}^2)(x_{e^-}) + \text{arcsinh}(x_{e^-})\right]}$$

5.5 Volume Fraction

The generation of energy densities is more complicated. First, the energy density of each phase must be found, which is easy enough with the Fermi gas model. The more difficult task is finding the fraction of the total volume that each phase inhabits. It is doable, however, by enforcing the global charge neutrality of the strange object and using a formula for the charge density of each phase.

The positive charge density of a phase is given by a derivative of pressure with respect to the charge chemical potential:

$$\rho = -\frac{\partial P}{\partial \mu_e}$$

Note that, when calculating the charge density of electrons, $\mu_e >> m_{e^-}$. With quarks, however, $\mu_e >> m_i$ can't generally be said. Use the variable x_{e^-} as defined above for conciseness:

$$x_{e^-} = \frac{p_{F_{e^-}}}{m_{e^-}} = \frac{\sqrt{\mu_e^2 - m_{e^-}^2}}{m_{e^-}} \approx \frac{\mu_e}{m_{e^-}}$$

The calculations follow as:

$$\rho_- = \frac{\partial P_{F_{e^-}}}{\partial \mu_e} \approx \frac{m_{e^-}^4}{12\pi^2}\frac{\partial}{\partial \mu_e}\left[x_{e^-}^4 + \frac{3}{2}\text{arcsinh}(x_{e^-})\right]$$

$$= \frac{m_{e^-}^4}{12\pi^2}\left[4\frac{\mu_e^3}{m_{e^-}^4} + \frac{3}{2}\frac{1}{\sqrt{1+\frac{\mu_e^2}{m_{e^-}^2}}}\right]$$

$$\boxed{\rho_- \approx \frac{m_{e^-}^4}{12\pi^2}\left(4\frac{\mu_e^3}{m_{e^-}^4}\right) = \frac{\mu_e^3}{3\pi^2}}$$

The same equation yields the charge density of the quark phase of the object:

$$\rho_+ = -\frac{\partial}{\partial \mu_e}\left[P_{F_{e^-}} + \sum_{i=u,d,s} P_{F_i}\right]$$

$$\boxed{\rho_+ \approx -\frac{\mu_e^3}{3\pi^2} - \sum_{n=0}^{3} c_n \mu_e^n}$$

The formula is extraordinarily messy, so c_n denotes coefficients, each of which is a function of baryon number B_i, charge number Q_i, masses m_i, and other variables. It is now feasible to find the volume fraction of the quark-electron mixed phase by enforcing global charge neutrality on the strange object:

$$f\rho_+ + \rho_- = 0,$$

where f is defined as

$$f = \frac{\text{Volume of Quark-Electron matter}}{\text{Total Volume}}.$$

Plugging in the formulas above for charge densities yields an expression for the volume fraction:

$$\boxed{f = -\frac{\rho_-}{\rho_+} = \frac{\frac{\mu_e^3}{3\pi^2}}{\frac{\mu_e^3}{3\pi^2} + \sum_{n=0}^{3} c_n \mu_e^n}}$$

5.6 Energy Densities

Almost there. Next, calculate the energy density of each phase using the Fermi gas model formula:

$$\epsilon_i = \frac{E_i}{V} = \frac{g_i}{2\pi^2}\int_0^{p_{F_i}} p^2\sqrt{p^2 + m_i^2}\,dp$$

$$= \frac{g_i m_i^3}{16\pi^2}\left[x_i(2x_i^2 + 1)\sqrt{1+x_i^2} - \text{arcsinh}(x_i)\right]$$

Recall that with electrons, $\mu_e >> m_{e^-}$, $x_{e^-} >> 1$. Also, the degeneracy factor remains the same as defined in Figure 4. Hence, the energy density of the electron phase and the mixed phase, respectively, are given by

$$\boxed{\epsilon_{e^-} \approx \frac{\mu_e^4}{4\pi^2 m_{e^-}} - \frac{\mu_e^3}{8\pi^2}\text{arcsinh}(\frac{\mu_e}{m_{e^-}})}$$

$$\boxed{\epsilon_{quarks} = \frac{3}{8\pi^2}\sum_{i=u,d,s} m_i^3 \left[x_i(2x_i^2+1)\sqrt{1+x_i^2} - \text{arcsinh}(x_i)\right]}$$

.
The total energy density of the strange object is then recovered using the volume fraction:

$$\epsilon_{total} = f\epsilon_{quarks} + \epsilon_{e^-}.$$

5.7 Equation of State

Associated with each μ_B now is a pressure and energy density. Plotting the ordered pairs (ϵ, P) using the parameter μ_B and finding a fit will allow for derivation of an equation of state for the strange object:

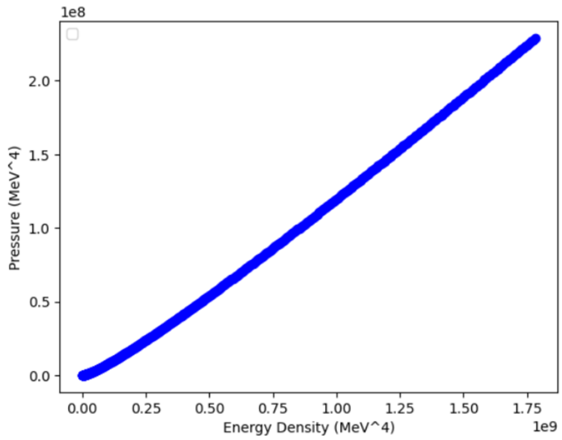


Figure 6: P vs. ϵ for mixed-phase, strange quark stars and nuggets. Exponential fit: $P(\epsilon) \approx 0.00632\epsilon^{1.142}$

The equation of state can now be used in solving the TOV and mass equations for mixed-phase, strange quark stars. This will allow for the finding of potential masses and radii of the hypothetical object based on the initial assumptions. Changing units to make the numbers more physical ($MeV^4 \to \frac{MeV}{fm^3}$), trying different fits for the equation of state, and repeating the process with different matter are all possible directions to move in from here.

6 Conclusion

In this project, methods for finding solutions to the TOV and mass equation system for a mixed-phase, strange quark star were discussed. A relativistic Fermi gas model aided in finding an equation of state for the object of interest, and programs in C++ and Python aided in procuring numerical solutions for all of the problems encountered. The project yielded an understanding of natural units and parametrization of problems as well as a tour of the wonderful world of extremely dense objects. Although most exotic objects cannot be characterized solely using a Fermi gas model, predictions for their masses and radii may be generated using a similar framework while employing more complicated physics. These predictions can then aid in the future detection and identification of these objects in LIGO probes. The value of parameters such as the Bag constant, the composition of the hypothetical object, and the model used to construct it may all be modified again and again to learn about different animals in the zoo, so these exotic objects may be identified and appreciated in the wild.

7 Acknowledgments

This was a valuable experience, one I truly enjoyed, and one I will remember for a long time. Thank you Dr. Reddy, thank you to the INT, and thank you to the NSF for this opportunity.

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