Neutrino Oscillations in Dense Media

Ellen Gates*
Department of Physics & Astronomy, University of Missouri - Columbia
(Dated: January 31, 2025)

Neutrinos are the second most abundant particle in the universe. They play an important role in many astrophysical processes, including in supernova explosions. It is thought that neutrinos play an important role in the formation of proto-neutron stars in the aftermath of a supernova, but a complete explanation for their role in this process does not exist. We set up a toy model of a supernova that explores the nature of neutrino oscillations in the aftermath of the explosion. Our results begin to give insight into the effect of a neutrino-dense environment on neutrino oscillations, showing a collective evolution of the flavor states.

I. INTRODUCTION

This project aims to study how neutrino oscillations are affected in an extremely neutrino-dense medium, like a supernova. We begin with a brief overview of neutrinos, neutrino oscillations, and supernovae, and then walk through the quantum kinetic equations we employed to describe neutrino flavor oscillations in a supernova environment. We also explain the preliminary results obtained from solving the quantum kinetic equations.

A. Neutrino Oscillations

Neutrinos are extremely light, neutral fermions that interact with other particles only via the weak force and gravity. They come in three flavor varieties: $|\nu_\alpha\rangle$, $\alpha = e, \mu, \tau$. Neutrinos were long thought to be massless, as the standard model predicts, but they have been observed to change, or oscillate, flavor as they travel through spacetime. That is, a neutrino of flavor α may later be observed to be in flavor state β. This is only possible if neutrinos have at least a very small mass and the different flavors have different masses. It has since been shown that neutrinos have three possible mass eigenstates: $|\nu_i\rangle, i = 1, 2, 3$, and each flavor is composed of a superposition of these mass eigenstates following the relation $|\nu_\alpha\rangle = \sum_i U^*_{\alpha i} |\nu_i\rangle$, where $U^*_{\alpha i}$ is the mixing angle matrix.

Looking at the time-independent Schrodinger equation for the mass eigenstates of the neutrino and, since neutrinos are extremely light, applying the relativistic approximation $p_i = \sqrt{E_i^2 - m_i^2} \approx E_i - \frac{m_i^2}{2E_i}$, we obtain the result

$$|\nu_i(t)\rangle \approx e^{-m_i^2/2E_i} |\nu_i\rangle$$

From this we get the probability of a neutrino oscillating from flavor α to flavor β:

$$\langle\nu_\beta|\nu_\alpha(t)\rangle \approx \sum_i U_{\beta i} U^*_{\alpha i} e^{-m_i^2 t/2E_i} \neq 0$$

We can now see that there is a nonzero probability that a neutrino will oscillate from flavor α to flavor β if it is composed of distinct mass eigenstates.

We can continue on in our derivation to find the probability of oscillation, say from the electron state to the muon state, and we obtain the result

$$P_{\nu_e \to \nu_\mu} = \sin^2(2\theta) \sin^2\left(\frac{\Delta m^2}{4|\bar{p}|}\right) \quad (1)$$

which gives

$$P_{\nu_e \to \nu_e} = 1 - \sin^2(2\theta) \sin^2\left(\frac{\Delta m^2}{4|\bar{p}|}\right) \quad (2)$$

For a comprehensive derivation, see Ref. [1].

B. Supernovae

Although neutrinos are extremely weakly interacting, they still play an important role in many astrophysical settings, including in core-collapsing supernovae (CCSNs). Neutrino heating is responsible for heating up the stellar matter that causes the initial explosion. Then, because of the process of electron capture ($p + e^- \to n + \nu_e$) which happens in abundance during a CCSN, ν_e are produced in large quantities (something like 10^{58} are emitted in the first few seconds of the explosion). The oscillations of these neutrinos are thought to govern both the creation of the proto-neutron star and the abundance of heavy metals created in the explosion [2].

Since a CCSN is such a neutrino-dense environment, neutrinos, which usually interact very little with baryonic matter, begin to self-interact. We must account for these self interactions in our models. No model thus far has been able to accurately recreate a SN environment. The work discussed in this paper begins to explore the interface between two commonly used models (discussed in Section II) in an attempt to create a model that bridges the knowledge gap between our observations of CCSNs and the role we believe neutrinos to play in this process.

* ebghht@umsystem.edu

II. QUANTUM KINETIC EQUATIONS

There are multiple approaches to studying neutrino oscillations in environments like SNs. One, the quantum many body approach, is detailed in Ref. [3]. Another approach uses quantum kinetic equations (QKEs) to describe neutrino oscillations. This approach is computationally far simpler, but there is some question as to whether it is as accurate at modeling oscillations as the quantum many body approach. One goal of this project was to compare our results with the results from a similar system that used the quantum many-body approach in order to determine the effectiveness of QKEs on a smaller scale.

A. Toy Model

Before deriving the QKEs, we must set up a toy model of a supernova environment. This model, in momentum space, is based on the model utilized in Ref. [3] and uses the following constraints:

$$\bar{p} = (p_x, p_y, p_z)$$

$$p_x = \frac{2\pi}{L} n_x$$

$$p_y = \frac{2\pi}{L} n_y$$

$$p_z = 0$$

$$n_{max} = 5$$

$$p_x > 0 \tag{3}$$

This creates a 2-dimensional grid of 35 discrete momentum modes, as seen in Fig. 1. A neutrino can occupy any of the 35 momentum bins, and, using natural units, the energy of the neutrino can be described by $E = |\bar{p}|$.

To further simplify our calculations, we will ignore antineutrinos and tau neutrinos. Considering only electron and muon neutrinos, we will place 6 ν_e and 4 ν_μ in 10 distinct momentum bins on the grid. The 6 ν_e were placed in the following bins:

$$\mathbf{p}_1 = (1, -4), \quad \mathbf{p}_6 = (3, -3), \quad \mathbf{p}_9 = (2, -2),$$
$$\mathbf{p}_{11} = (4, -2), \quad \mathbf{p}_{13} = (2, -1), \quad \mathbf{p}_{21} = (1, 1),$$

And the 4 ν_μ were placed in:

$$\mathbf{p}_{26} = (2, 2), \quad \mathbf{p}_{27} = (3, 2), \quad \mathbf{p}_{29} = (1, 3),$$
$$\mathbf{p}_{34} = (2, 4)$$

The system was then time evolved and we observed the effects of forward scattering and inelastic collisions on the neutrino oscillations.

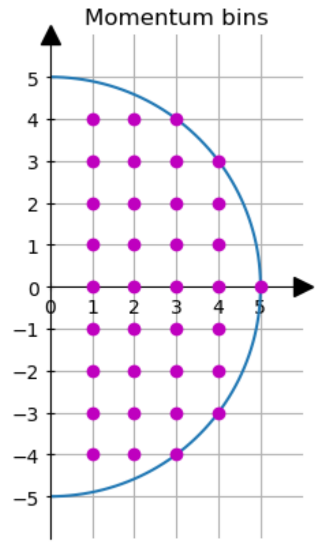

FIG. 1. Pink dots represent the 35 allowed momentum modes for the neutrinos, going out to $p_{max} = 5$ (blue line).

B. QKE Structure

The basic structure of a QKE that describes neutrino oscillations is as follows:

$$iDF = [H, F] + iC \tag{4}$$

Here, $[H, F]$ represents the coherent evolution of the vacuum mass and forward scattering. H is the Hamiltonian, which can be split into the vacuum mass and forward scattering Hamiltonian: $H = H_{vacuum} + H_{\nu\nu}$.

F is the density matrix

$$F = \begin{pmatrix} F_{LL} & F_{LR} \\ F_{RL} & F_{RR} \end{pmatrix} \tag{5}$$

In this case, we will only consider left-handed neutrinos, F_{LL}, as the standard model predicts. Because of this simplification, we can instead employ a density matrix in the flavor basis:

$$f_{n_p}(t) = \begin{pmatrix} f_{ee}(t) & f_{e\mu}(t) \\ f_{\mu e}(t) & f_{\mu\mu}(t) \end{pmatrix} \tag{6}$$

Then, since $f_{e\mu}(t) = f^*_{\mu e}(t)$ we can convert the density matrix to a polarization vector basis:

$$f_{n_p}(t) = \begin{pmatrix} f_1(t) & f_3(t) + if_4(t) \\ f_3(t) - if_4(t) & f_2(t) \end{pmatrix} \tag{7}$$

Note that the diagonal terms of this matrix represent the probability of being in the ν_e and ν_μ state, respectively, and are what we are most interested in. Our results will be expressed in this basis.

Finally, iC is the inelastic collision term. Also, because we are using a toy model with discrete momentum bins, we must discretize the QKEs, giving us functions that include the integers n_p and n_i. Below, we will further explore the form of each term in the QKEs.

1. Vacuum Mass Term

The vacuum mass term accounts for neutrino vacuum oscillations, without interaction from other neutrinos. It also takes into account the energy difference between mass eigenstates with the term $\Delta m^2 = m_2^2 - m_1^2$. The discretized vacuum mass term takes the form

$$\dot{f}_{n_p}(t) = -i\left[\frac{\Delta m^2}{4|\bar{p_n}|}R, f_{n_p}(t)\right] \tag{8}$$

where R is the mixing angle matrix

$$R = \begin{pmatrix} -\cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} \tag{9}$$

The vacuum mass term by itself can be solved analytically and we obtain the same probabilities derived in the quantum mechanical approach, shown in Eqs. 1 and 2.

2. Forward Scattering Term

Adding the forward scattering term allows for neutrino self interactions. It takes the form

$$\dot{f}_{n_p}(t) = -i\sqrt{2}\frac{G_F}{V}\sum_i \kappa_{ij}\left[f_{n_q}(t), f_{n_p}(t)\right] \tag{10}$$

G_F is the Fermi constant and describes the interaction via the weak force between neutrinos. $\sum_i \kappa_{ij}\left[f_{n_q}(t), f_{n_p}(t)\right]$ accounts for the interaction of the flavor states between neutrinos in different momentum bins, leading to collective flavor evolution. $\kappa_{ij} = 1 - \cos\theta_{ij}$ where

$$\cos\theta_{ij} = \frac{\bar{p_i}\cdot\bar{p_j}}{|\bar{p_i}||\bar{p_j}|}$$

and represents the angle between momentum bins p_i and p_j. κ_{ij} thus has a unique weight depending on the alignment of the momentum bins the neutrinos occupy.

3. Collision Term

The collision term takes the form

$$\begin{aligned} C(\vec{n_p}) = {} & \left(\frac{G_F}{V}\right)^2 V^{1/3}\sum_{\vec{n_2}}\sum_{\vec{n_3}}\sum_{\vec{n_4}}\delta_{\vec{n_p}+\vec{n_2},\vec{n_3}+\vec{n_4}}(1-\hat{n_p}\cdot\hat{n_2})(1-\hat{n_3}\cdot\hat{n_4}) \\ & \times\frac{|\vec{n_p}|+|\vec{n_2}|-|\vec{n_3}|}{\sqrt{(|\vec{n_p}|+|\vec{n_2}|-|\vec{n_3}|)^2-n_{4y}^2-n_{4z}^2}}\theta\left((|\vec{n_p}|+|\vec{n_2}|-|\vec{n_3}|)^2-n_{4y}^2-n_{4z}^2\right) \\ & \times\left(\bar{\delta}_{n_{4x},\sqrt{(|\vec{n_p}|+|\vec{n_2}|-|\vec{n_3}|)^2-n_{4y}^2-n_{4z}^2}}-\bar{\delta}_{n_{4x},-\sqrt{(|\vec{n_p}|+|\vec{n_2}|-|\vec{n_3}|)^2-n_{4y}^2-n_{4z}^2}}\right) \\ & \times\Big\{\Big[f_4(I-f_2)+\mathrm{Tr}(f_4(I-f_2))\Big]f_3(I-f_p)+(I-f_p)f_3\Big[(I-f_2)f_4+\mathrm{Tr}((I-f_2)f_4)\Big] \\ & \quad -\Big[(I-f_4)f_2+\mathrm{Tr}((I-f_4)f_2)\Big](I-f_3)f_p-f_p(I-f_3)\Big[f_2(I-f_4)+\mathrm{Tr}(f_2(I-f_4))\Big]\Big\} \end{aligned}$$

This term is responsible for allowing neutrinos to undergo inelastic collisions and switch momentum bins. Given a neutrino starting in momentum bin p that collides with one in momentum bin p_2, the two neutrinos will end up in bins p_3 and p_4 following the constraint $E(p) + E(p_2) = E(p_3) + E(p_4)$. In other words, the sum of the energy of the starting bins must be equal to the sum of the energy of the bins the neutrinos end up in to conserve momentum and energy.

Work related to the collision term is ongoing and, for the most part, will not be included in the "Results" section (Section III) of this paper.

C. A Note on Dimensional Analysis

In the interest of completeness, I will briefly walk through the dimensional analysis of these equations. As mentioned previously, we are using natural units, so we set $c = 1$ and $\hbar = 1$. That gives us time t in units of E^{-1} and momentum p in units of E.

Furthermore, we can set the energy

$$\sqrt{2}\frac{G_F}{V} = \mu$$

Putting all energies in terms of μ, and plugging in $\bar{p_n} = \frac{2\pi}{L}\sqrt{n_{ix}^2 + n_{iy}^2}$ we get

$$\frac{\Delta m^2}{4|\bar{p_n}|} \rightarrow \frac{\Delta m^2 L/\mu}{8\pi\sqrt{n_{ix}^2 + n_{iy}^2}}$$

We will set the constant $\frac{\Delta m^2 L/\mu}{8\pi} = 1$ because we are most interested in observing the regime where the energies μ and $\Delta m^2 L$ are on a similar scale. This yields the final form of the QKE:

$$\dot{f}_{n_p}(t) = -i\left[\frac{1}{\sqrt{n_{ix}^2 + n_{iy}^2}}R, f_{n_p}(t)\right] - i\sum_i \kappa_{ij}\left[f_{n_q}(t), f_{n_p}(t)\right] + C(n_p) \tag{11}$$

III. RESULTS

As explained in Section II, Subsection B, the results of the QKEs come in the form of a density matrix. In my results (see Fig. 2 for an example) the blue line represents the function $f_1(t)$, or $f_{ee}(t)$, or the probability of being in the ν_e state. The orange line, then, represents $f_2(t)$, $f_{\mu\mu}(t)$, or the probability of being in the ν_μ state. That is to say, the blue and orange lines represent the diagonal terms of the density matrix. Thus, the sum of these two probabilities is always one, as represented by the purple line.

A. Simplest Case

The simplest case we can model considers only the vacuum mass term (Eq. 8). As discussed previously, solving this QKE will yield the results from Eqs. 1 and 2, with the frequency of oscillations depending on the energy of the momentum bin the neutrino occupies.

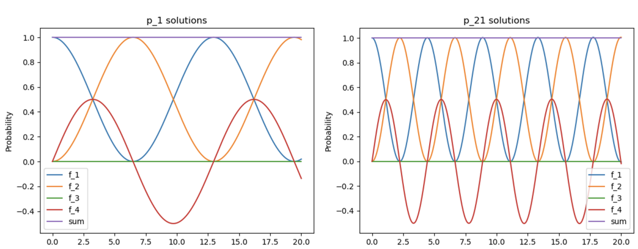

FIG. 2. Oscillation probability of neutrinos in momentum bins p_1 and p_{21}. Bin p_1 has the higher energy. Here, the red and green lines represent the off-diagonal terms and are not utilized in our results.

B. Coherent Evolution

Adding in the vacuum mass term, the neutrinos can interact with each other, thus affecting their oscillations. As a comparison with only the vacuum mass terms, we can look at the same two neutrinos in bins p_1 and p_{21}.

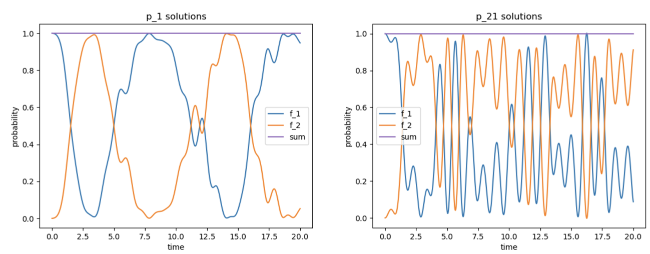

FIG. 3. Oscillation probabilities for neutrinos in momentum bins p_1 and p_{21} with the forward scattering term. Forward scattering interactions affect the oscillation of the neutrinos.

As we can see in Fig. 3, the oscillations are no longer perfectly sinusoidal. This is because the forward scattering term takes into account neutrino interactions that do not significantly change the particles' momenta, but do change the neutrinos' oscillations. This also creates collective oscillation among the neutrinos, where the different flavor states evolve together, as shown in Fig. 4, due to entanglement between the states.

C. Conclusions

These results, while still far from comprehensive, are promising. We can see that the QKEs are effective in accounting for both vacuum oscillations and forward scattering effects. The collision term, when added, will account for inelastic collisions between the neutrinos, allowing them to change their momenta. When this term

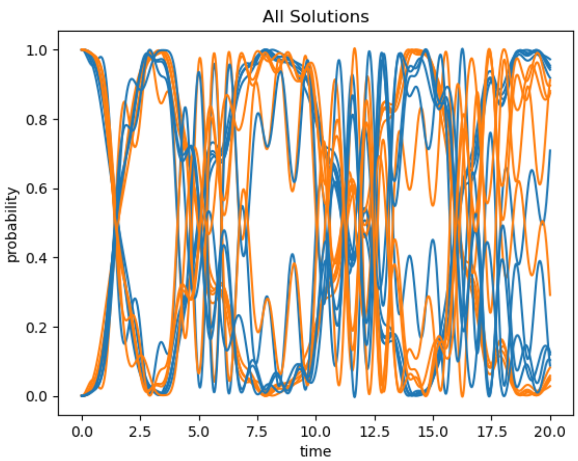

FIG. 4. graph of oscillation probabilities for all 10 neutrinos (blue representing neutrinos that begin in the ν_e state and orange representing those in the $\nu_\mu$state), showing that states that begin with the same probability (either 0 or 1) tend to evolve together.

is added, we expect to see even more collective evolution because of stronger interactions. Another effect we may see is flavor depolarization, which happens when neutrinos interact with each other so much, their flavor states become maximally entangled. The density matrix for a single neutrino in this state would have diagonal terms both equal to 1/2, meaning an observer is equally likely to find the neutrino in either flavor state.

Adding the collision term will also allow us to better compare our results to the quantum many body results from Ref. [3].

IV. ACKNOWLEDGEMENTS

I would like to thank Professor Vincenzo Cirigliano for his guidance on this project, as well as the National Science Foundation and University of Washington's Physics REU for supporting this research.

[1] B. Kayser. Neutrino mass, mixing, and oscillation. In *Flavor Physics for the Millennium*. World Scientific, 2001.
[2] V. Cirigliano, M. Paris, and S. Shalgar. Collective neutrino oscillations with the halo effect in single-angle approximation. *Journal of Cosmology and Astroparticle Physics*, 2018(11):019, 2018.
[3] V. Cirigliano, S. Sen, and Y. Yamauchi. Neutrino many-body flavor evolution: The full hamiltonian. *Physical Review D*, 110(12), 2024.