DRAFT VERSION JANUARY 29, 2025
Typeset using LaTeX **twocolumn** style in AASTeX631

Neutrino Many-Body Systems

ARIELLE FROMMER
ADVISORS: DR. VINCENZO CIRIGLIANO AND DR. YUKARI YAMAUCHI[1]

[1] *University of Washington Institute for Nuclear Theory*

ABSTRACT

Neutrinos are nearly massless particles in the Standard Model with the unique ability to oscillate mass, or flavor. As they are produced in many different environments, such as in solar radiation, supernovae, and nucleosynthesis, determining how they behave in many body systems is crucial to understanding astronomical phenomena. We can set up systems of many neutrinos, initializing specific momentum and flavor configurations and letting the system time-evolve. Building off of the work of Cirigliano et al. (2024), we use the formalism of second quantization to construct a Hamiltonian, considering the kinetic energy terms and interaction potentials, and using creation and annihilation operators to describe flavor-changing oscillations. We specifically compare the behavior of flavorful systems with flavorless systems. We examine the relationship between the kinetic and potential energy terms in a one-flavor model with 10 momentum modes, analyzing how different energy scales influence the system's dynamics. We also find that flavorless systems do not equilibrate or thermalize as expected. The behavior of neutrino many-body systems can illuminate how these particles interact and time-evolve, and this work opens future avenues of research into the thermalization behavior of one-flavor systems.

1. INTRODUCTION

Neutrinos are tiny, nearly massless Standard Model particles that only interact via the weak force. Neutrinos can occur in three flavors – electron, muon, and tau – corresponding to the leptonic interaction that produces them. However, neutrinos created as a specific flavor may be observed at a later time with a different flavor. This peculiar phenomenon is known as flavor oscillation, and it occurs because neutrinos exist as a superposition of energy-mass eigenstates whose probability amplitudes oscillate with time. Depending on the probability amplitudes of the different mass eigenstates, a neutrino may be observed with a different mass and therefore a different flavor.

Despite their weak interactions, neutrinos may be the key to understanding crucial astrophysical phenomena. Neutrinos are produced in stellar nucleosynthesis, supernovae, and black holes. They are also emitted by our own Sun, and may have played a role in Big Bang nucleosynthesis. Understanding how many neutrinos interact and oscillate in a given system may shed light on their evolution and provide key insights into their role in astrophysics.

1.1. *Formalism and Second Quantization*

Neutrino many-body systems can be modelled and understood through formalism known as second quantization. Specifically, we write our system of neutrinos in terms of an occupation number basis – where each bin represents a given momentum and flavor state that the neutrino could occupy. Because neutrinos are fermions and cannot occupy the same state, each occupation number will be either 0 or 1.

The occupation number basis enables us to use creation and annihilation operators to create or destroy a neutrino in a given-many body system. For example, in a system of 2 flavors and 4 momentum bins, the occupation number basis is:

$$|e_1, \mu_1, e_2, \mu_2, e_3, \mu_3, e_4, \mu_4\rangle$$

If we have an electron neutrino in p_1 and a muon neutrino in p_3, the occupation number basis is:

$$|1,0,0,0,0,1,0,0\rangle$$

If we want to change change the momentum of the first neutrino to p_2, and its flavor to a muon, we could do it this way:

$$a^\dagger_{\mu_2} a_{e_2} \, |1,0,0,0,0,1,0,0\rangle = |0,0,0,1,0,1,0,0 >$$

The Hamiltonian, or total energy of the system, can be expressed in terms of these creation and annihilation operators.

1.2. *Constructing the Hamiltonian*

In a many-body system of neutrinos, there are several different energies to consider as a part of the Hamiltonian. The vacuum oscillation – the kinetic energy term – measures the

energy of a single neutrino, and includes creation and annihilation operators that change the neutrino flavor. The kinetic energy term of Hamiltonian for a single-flavor system is as follows:

$$H_{kin} = \sum_p |p| a_p^\dagger a_p$$

The interaction potential accounts for interaction energy between two neutrinos:

$$H_{\upsilon\upsilon} = \frac{G_F}{\sqrt{2}} \sum_{p,p',q,q'} \delta(p+q-p'-q') \times a_{p'}^\dagger a_{q'}^\dagger a_p^\dagger a_q^\dagger \times g(p',p,q',q)$$

1.3. *Number Operators*

Number operators express the behavior of different states as time evolves. Each bin (ie. a muon neutrino with momentum p_1, an electron neutrino with momentum p_3) has a number operator that represents the expectation value for that bin, or the probability that we find a neutrino in that bin. N_i^+ operators count the likelihood of finding an electron plus the likelihood of finding a muon in a given momentum state: $N_i^+ = N_i^e + N_i^\mu$. Conversely, $N_i^- = N_i^e - N_i^\mu$.

2. MODELING

2.1. *Two-Flavor Case*

The toy model I worked with for the two-flavor case contained 4 momentum modes and 2 neutrinos. In order to constrain the momentum and reduce the complexity of the Hamiltonian, we set up a momentum conservation relation: $p_1 + p_2 = p_3 + p_4$. We set the initial state to be one electron neutrino with momentum p_1 and one electron neutrino with momentum p_2. We compared the number operators of the initial occupied bins, or the probability amplitude of (p_1, e) and (p_2, e).

Next, we considered the number operator N_i^+ and found that $N_1^+ = N_2^+$, $N_3^+ = N_4^+$, and $N_1^+ + N_2^+ = N_3^+ + N_4^+$. These relations are consistent with the momentum conservation that was set at the beginning of the model and with our understanding of number operators. The states are written as:

$|e_1, \mu_1, e_2, \mu_2, e_3, \mu_3, e_4, \mu_4\rangle$ where the subscript indicates the momentum. Here are the states that time evolve:

$C_1 |1,0,1,0,0,0,0,0\rangle$ (initial state)
$C_2 |1,0,0,1,0,0,0,0\rangle$
$C_3 |0,1,1,0,0,0,0,0\rangle$

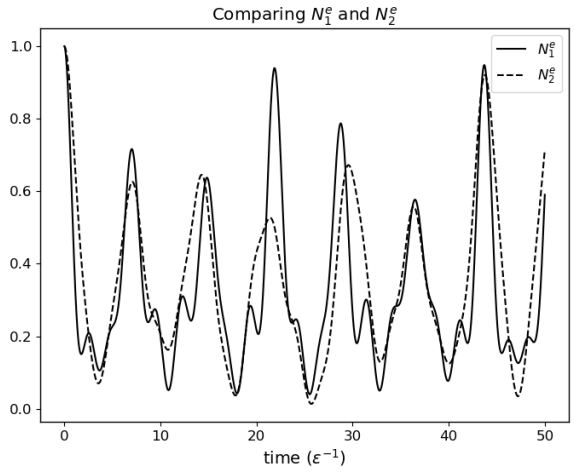

Figure 1. N_1^e and N_2^e appear to oscillate together, which is consistent with our momentum conservation relation that says two neutrinos will either be have momenta p_1 and $p2$ or p_3 and $p4$. The slight variation occurs because of the flavor oscillation.

$C_4 |0,1,0,1,0,0,0,0\rangle$
$C_5 |0,0,0,0,1,0,1,0\rangle$
$C_6 |0,0,0,0,1,0,0,1\rangle$
$C_7 |0,0,0,0,0,1,1,0\rangle$
$C_8 |0,0,0,0,0,1,0,1\rangle$

Given these amplitudes, the number operators should be equal to:

$$N_1^+ = N_1^e + N_1^\mu = |C_1|^2 + |C_2|^2 + |C_3|^2 + |C_3|^2$$
$$N_2^+ = N_2^e + N_2^\mu = |C_1|^2 + |C_3|^2 + |C_2|^2 + |C_4|^2$$
$$N_3^+ = N_3^e + N_3^\mu = |C_5|^2 + |C_6|^2 + |C_7|^2 + |C_8|^2$$
$$N_4^+ = N_4^e + N_4^\mu = |C_5|^2 + |C_7|^2 + |C_6|^2 + |C_8|^2$$

These derived relationships are consistent with the graphical model.

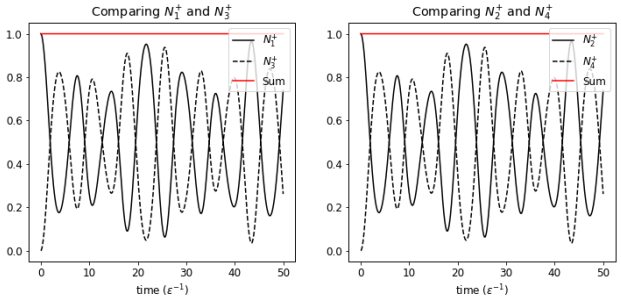

Figure 2. While the oscillation amplitude varies throughout, the number operators all sum to 1 as expected.

The time-evolution of these 8 states was also plotted.

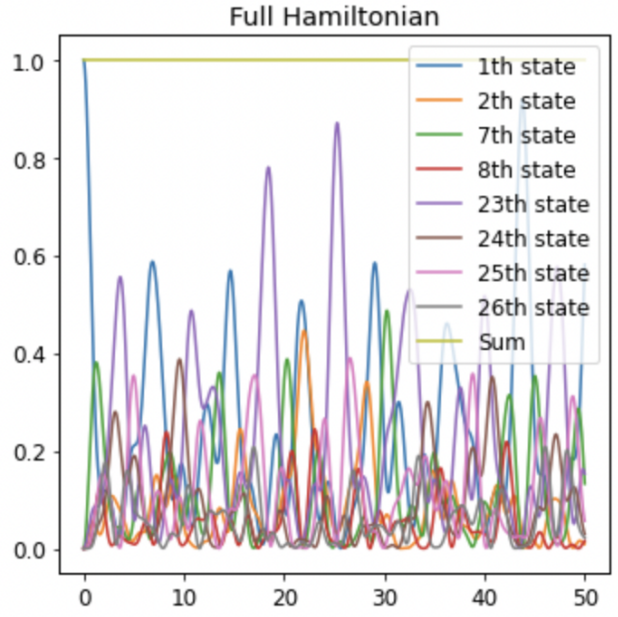

Figure 3. We plotted the amplitudes over time of the 8 states that do time-evolve.

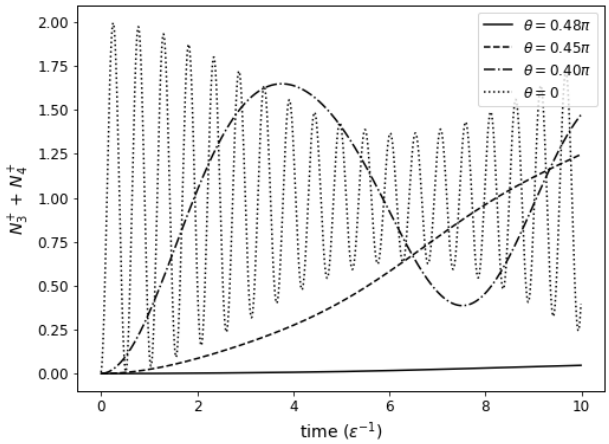

Figure 4. We reconstructed a plot from Cirigliano et al. 2024.

Then, we mainly explored the one-flavor case and examined momentum modes.

2.2. *One-Flavor Case: 4 Momentum Modes*

Moving onto the one flavor case, we first modeled the same system of 4 momentum modes and 2 neutrinos with the same momentum relation, but excluding flavor. We began in the same initial state with two electrons of momentum p_1 and p_2. Because we have taken out flavor, the amplitudes of each state now oscillate between p_1 and p_2 and p_3 and p_4 without variation. The states are written as: $|e_1, e_2, e_3, e_4\rangle$.

Here are the states that time-evolve in this system:

$C_1 |1,1,0,0\rangle$ (initial state)
$C_2 |0,0,1,1\rangle$

When we look at the number operator relations, this uniformity also emerges. We no longer consider N_i^+ and N_i^- in a flavorless case, since $N_i = N_i^e$. The number operators are as follows:

$N_1 = |C_1|^2 = N_2$
$N_3 = |C_2|^2 = N_4$

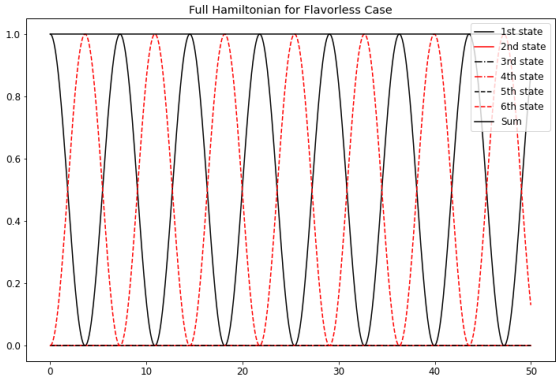

Figure 5. We plotted the amplitudes over time of the 2 states that do time-evolve.

2.3. *One-Flavor Case: 10 Momentum Modes*

Next, we model a case with 10 momentum modes and 4 neutrinos. For both models, we began with the neutrinos in p_1, p_2, p_4, and p_5. We tried 2 different momentum relations. The states are written as: $|e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\rangle$. For the first model, we set these momentum relations:

$p_1 + p_2 = p_3 + p_4$
$p_4 + p_5 = p_6 + p_7$
$p_7 + p_8 = p_9 + p_1 0$

Given the initial state, here are the states that time-evolve in this model:

$C_1 |1,1,0,1,1,0,0,0,0,0\rangle$ (initial state)
$C_2 |1,1,0,0,0,1,1,0,0,0\rangle$
$C_3 |0,0,1,1,0,1,1,0,0,0\rangle$

Given these amplitudes, the relevant number operators are:

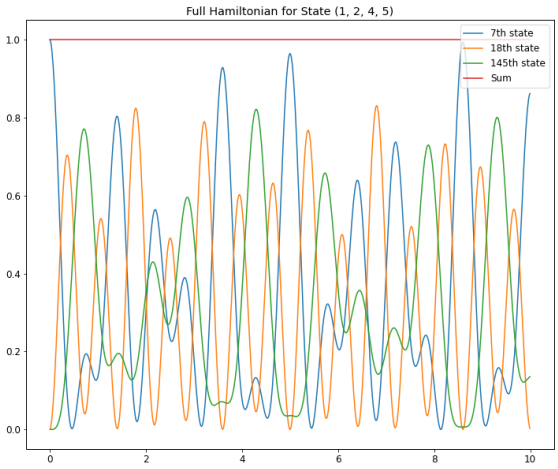

Figure 6. We plotted the amplitudes over time of the 3 states that do time-evolve.

$$N_1 = |C_1|^2 + |C_2|^2 = N_2$$
$$N_3 = |C_3|^2 = N_4$$
$$N_1 = 1 - N_3$$
$$N_4 = |C_1|^2 + |C_3|^2 \; N_6 = |C_2|^2 + |C_3|^2 = N_7$$
$$N_5 = |C_1|^2$$
$$N_6 = 1 - N_5$$

Note how N_4 is not equal nor inverse to another number operator, because it straddles two momentum relations.

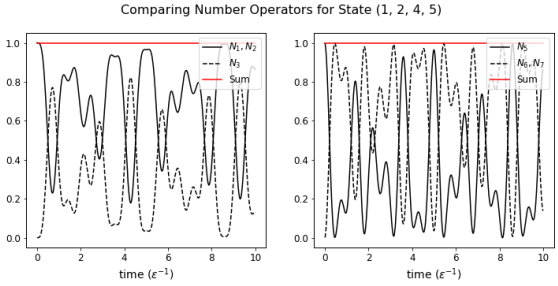

Figure 7. We plotted the number operators of the states that time-evolve.

These derived relationships are consistent with the graphical model.

We modeled a similar setup of 10 momentum modes and 4 neutrinos, but with different momentum relations. For this model, we set the following momentum relations:

$$p_1 + p_2 = p_3 + p_4$$
$$p_1 + p_2 = p_5 + p_6$$
$$p_3 + p_5 = p_7 + p_8$$

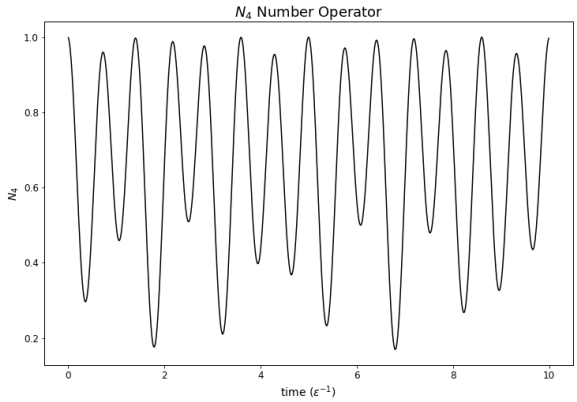

Figure 8. We plotted the N_4, which has a different amplitude than the other number operators because it straddles 2 momentum relations.

$$p_7 + p_8 = p_9 + p_{10}$$

We initialized the model in an initial state of $|0,0,0,0,0,0,1,1,1,1\rangle$. Based on the momentum relations, the system can evolve into the following states:

$C_1\,|0,0,0,0,0,0,1,1,1,1\rangle$ (initial state, 209th state)
$C_2\,|0,0,1,0,1,0,0,0,1,1\rangle$ (164th state)
$C_3\,|0,0,1,0,1,0,1,1,0,0\rangle$ (159th state)

When we graph the Hamiltonian over 30,000 time steps, we notice that the probability amplitude of the initial state remains high, and the states do not "talk to each other" as much, so to speak, as they did in the other model.

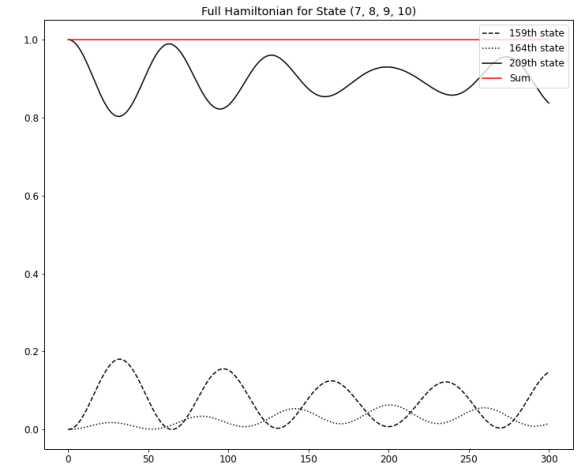

Figure 9. We plotted the Hamiltonian of the system and found that it was most likely in its initial state.

When we plotted the number operators, a similar pattern emerges. The expectation values of N_9 and N_10 remain highest, as did those of N_7 and N_8, which make sense since the two states that their respective bins are occupied both have higher amplitudes (C_1 and C_3) than the remaining state (C_2).

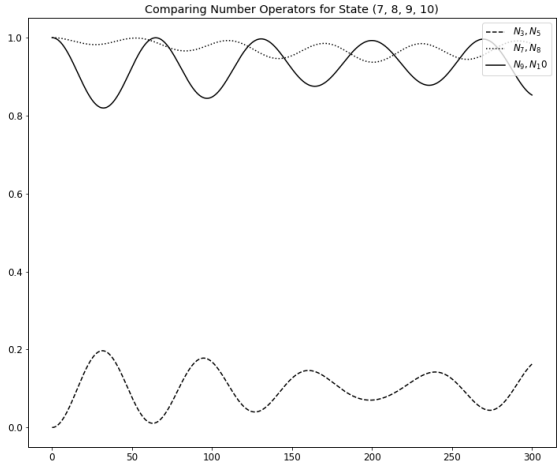

Figure 10. We plotted the number operators N_3 and N_5, N_7 and N_8, and N_9 and N_{10}.

The reason that the states with higher energies do not mix as much comes down to the values of the kinetic energy and the potential energy in this system. In any given two-level system, the Hamiltonian is given as:

$$\begin{bmatrix} \Delta & \epsilon \\ \epsilon & -\Delta \end{bmatrix}$$

Δ is the kinetic energy difference in the system, and ϵ is the g-factor value, which is related to the potential energy. In the limit of $\Delta \to \infty$, the following eigenvalues and eigenvectors emerge for a two-level system:

$$\frac{1}{\sqrt{2}}\left(\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\right)$$

We can see that the amplitudes of each state are the same, so they will oscillate with equal and opposite probability. If we look at the same setup in the limit of $\epsilon << \Delta$, then different probabability amplitudes emerge:

$$\frac{\begin{bmatrix}1\\0\end{bmatrix}+\frac{\epsilon}{2\Delta}\begin{bmatrix}0\\1\end{bmatrix}}{\sqrt{1+\frac{\epsilon^2}{4\Delta^2}}}$$

In our example, we found $\Delta = 0.00601331$ and $\epsilon = -0.0019435$ between states p_3, p_5, p_7, and p_8. The ratio of potential to kinetic energy is thus quite small, making the probability amplitude on the interaction states much less than the amplitude on the initial state.

2.4. *Equilibration*

Finally, we can consider the system's behavior when we implement kinetic energy conservation, or $\Delta \to 0$, thus correcting for this effect. If kinetic energy is conserved, we expect all states to have the same kinetic energy, which means that they should all become equally likely. This relation is represented by the following equation, which defines the number operators based on equally weighted probabilities:

$$\langle N \rangle = \sum_i w_i \langle \Psi_i | N | \Psi_i \rangle$$

When the number operators become equally likely, we say that the system is equilibrating, or thermalizing, in a microcanonical ensemble. In Cirigliano et al. (2024), two-flavor systems of many neutrinos were shown to equilibrate. However, in one-flavor systems, the neutrinos were found not to equilibrate.

We tested a system with 6 neutrinos and 20 momentum modes, comparing the equilibration when we initalized the system with one flavor (e,e,e,e,e,e), two flavors (e,e,e,μ,μ,μ) and three flavors (e,e,μ,μ,τ,τ). We observed the evolution of each number operator. In addition, we calculated the equilibrated, equally weighted number operators in the microcanonical ensemble and observed how the number operators approach these values.

The system equilibrated fastest in the three-flavor case, as all the number operators approached their microcanonical ensemble values. The system equilibrated in the two-flavor case as well, though not as frequently. Interestingly, the system did not equilibrate at all in the one flavor case. Other configurations were attempted in the one-flavor case, though none led to equilibration.

The reasons that the one-flavor system does not equilibrate are not immediately clear. Equilibration may relate to the energy level spacings of neutrinos, which could be different across flavorless and flavorful systems. In future work, we hope to investigate the lack of equilibration in a flavorless system.

3. ACKNOWLEDGEMENTS

I would like to thank Dr. Yukari Yamauchi and Dr. Vincenzo Cirigliano for their help and guidance on this project,

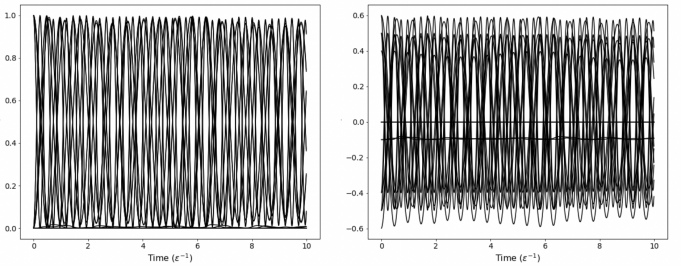

Figure 11. The one-flavor system does not equilibrate as expected. The left plot displays time-evolution of number operators, while the right plot compares the number operators to their microcanonical values.

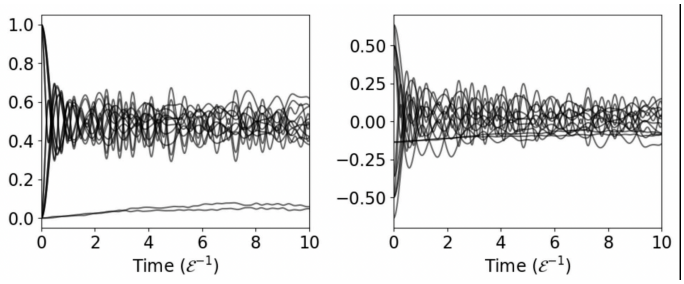

Figure 12. The two-flavor system does equilibrate as expected. The left plot displays time-evolution of number operators, while the right plot compares the number operators to their microcanonical values.

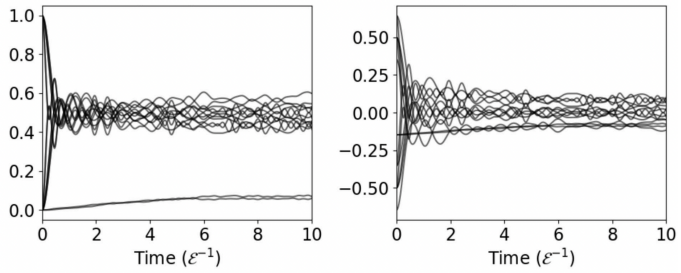

Figure 13. The three-flavor system equilibrates the quickest of all the systems we modeled. The left plot displays time-evolution of number operators, while the right plot compares the number operators to their microcanonical values.

and to the National Science Foundation and University of Washington REU for supporting my research.

REFERENCES

Cirigliano, V., Sen, S., & Yamauchi, Y. 2024, Phys. Rev. D, 110, 123028