Study of the Light Front Quantum Harmonic Oscillator in Tilted Coordinates

Anne Lashbrook
Department of Physics, Saint Louis University, St. Louis, MO 63103, USA

Gerald A. Miller and Adam Freese
Department of Physics, University of Washington, Seattle, WA 98195, USA
(Dated: August 23, 2023)

The light front formalism as applied in tilted coordinates is studied for the quantum harmonic oscillator. An attempt is made to find the wavefunction for this system analytically, but numerical techniques prove necessary. The wavefunction for the system is then found by breaking the Hamiltonian in two parts and treating the latter part as a perturbation, and the first steps are made in calculating the energy eigenvalues for the quantum harmonic oscillator in tilted coordinates numerically.

I. INTRODUCTION

Theoretical nuclear physics aims to provide a sound description of systems of hadrons and nuclei. Ideally, these descriptions would come directly from quantum chromodynamics. However, oftentimes approximation methods prove necessary. The path integrals encountered from quantum chromodynamics in describing the physics within the atomic nucleus are analytically difficult. Oftentimes, analytic or perturbation solutions for low-energy QCD are difficult or impossible to obtain due to the nonlinear nature of the strong force and the large coupling constant. To alleviate these obstacles, nuclear physicists have introduced multiple computational and theoretical techniques. Of particular interest in this study is the light front formalism.

This study begins by introducing the light front formalism and its application to tilted coordinates. The Hamiltonian operator is then formulated for the quantum harmonic oscillator in tilted coordinates, and the process of finding the wavefunction for this system is discussed. Initially, attempts to find the wavefunction analytically are made, but numerical techniques prove necessary in approximating the energy eigenvalues for this system.

A. The Light Front Formalism

Introduced by Paul Dirac in his 1949 paper Ref [1], the light front formalism employs an alternative synchronization convention to provide a different description of the physics within the atomic nucleus. Rather than using the Einstein synchronization convention, the light front formalism utilizes light front synchronization. The differences between these two synchronization conventions are visualized in Figure 1.

It is important to note that the time for the signal to complete its full journey is the same under both conventions. Ultimately, the physical predictions of a relativistic theory are independent of the synchronization convention applied. The use of a particular convention is arbitrary,

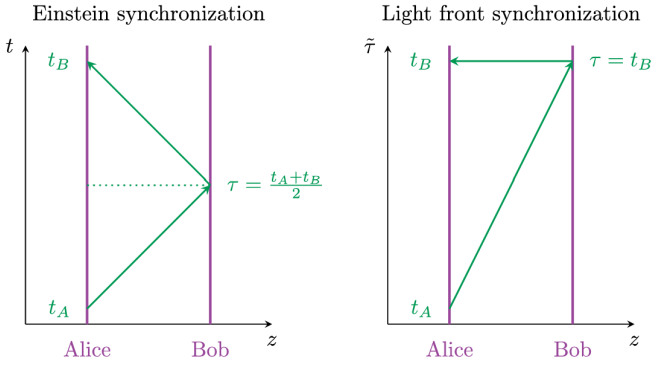

FIG. 1. Visualization of Einstein versus light front synchronization. Image Credit: Adam Freese. Under the Einstein synchronization convention, a signal traveling between Alice and Bob takes the same amount of time regardless of the direction of travel. Thus, the time Bob receives Alice's signal is the average of time A, the time Alice sent the signal, and time B, the time Alice received Bob's signal. Under light front synchronization, a signal can travel instantly int he negative z-direction. Therefore, the time Bob sends the signal and the time Alice receives it are both marked as time B.

but using new conventions may be more likely to solve certain problems.

Freese and Miller acknowledge some concerns regarding the usage of the light front formalism in Ref [2]. One being that densities calculated in the light front may describe physical systems moving at the speed of light thus distorting those calculations. Another concern is that the negative components of four-vectors and tensors have no clear physical meaning in the light front. To alleviate these concerns, tilted coordinates apply light front time but standard, Cartesian spatial coordinates.

B. Tilted Coordinates

Tilted light front coordinates are originally defined by Blunden, Burkardt, and Miller in Ref [3] and are ex-

pressed in Eqn 1.

$$\begin{aligned} \tilde{\tau} = \tilde{x_0} \equiv x^+ &= t + z \\ \tilde{x}^0 &\equiv x \\ \tilde{y}^0 &\equiv y \\ \tilde{z}^0 &\equiv z \end{aligned} \tag{1}$$

The tilde above tilted coordinates distinguishes them from standard, Cartesian coordinates. The question now is how to employ tilted coordinates in describing physical systems.

To apply tilted coordinates to quantum-mechanical systems, we construct the suitable Hamiltonian operator from the mass-shell relation for tilted coordinates. The mass-shell relation for Cartesian coordinates is the usual:

$$E = \sqrt{m^2 + \mathbf{p}^2} \tag{2}$$

This relation looks different under tilted coordinates:

$$\tilde{E} = \frac{m^2 + \tilde{\mathbf{p}}^2}{2\tilde{p_z}} \tag{3}$$

The derivation of the mass-shell relation for tilted coordinates is given in detail by Freese and Miller in Ref [2]. Interestingly, for tilted light front coordinates, we have a rational expression for the mass-shell relation. Having a rational expression rather than irrational could prove promising in solving problems of interest in QCD.

With an expression for the mass-shell relation for tilted coordinates, we construct the Hamiltonian operator:

$$\hat{H} = \frac{m^2 + \tilde{\mathbf{p}}^2}{2\tilde{p_z}} + V \tag{4}$$

This allows us to solve for the wavefunction for physical systems under tilted coordinates using the Schrödinger equation.

II. FINDING THE WAVEFUNCTION

In understanding the light front formalism as applied in tilted coordinates and its possible applications, it is important to test its Hamiltonian in simple theories. In this context, this project applies the light front formalism to the quantum harmonic oscillator. The quantum harmonic oscillator is a relatively simple quantum mechanical system, and it is one of the few for which analytical solutions are known making it a great system with which to explore the properties of tilted coordinates.

With the suitable Hamiltonian expressed in Eqn 4, we want to solve the Schrödinger equation for this system. First attempts look to find the wavefunction analytically. If analytic solutions can be found, that will directly yield the energy eigenvalues for this system, the states where the wavefunction does not change over time.

A. Analytical Attempt

Substituting in the operator identities for the Hamiltonian and momentum operators, the Schrödinger equation we want to solve is:

$$iE\psi = \frac{m^2 - \frac{\partial^2}{\partial \mathbf{x}^2}}{2\tilde{p_z}}\psi + \frac{1}{2}m\omega^2\mathbf{x}^2\psi \tag{5}$$

To solve this partial differential equation, we apply separation of variables. However, an interesting difficulty arises in trying to separate out the Schrödinger equation. The Hamiltonian operator mixes together the orthogonal and z-components of the momentum operator. This causes the equation to be inseparable in terms of x, y, and z. Thus, a coordinate transformation is necessary:

$$\begin{aligned} u &= x(\frac{\tilde{p_z}}{m})^{1/4} \\ v &= y(\frac{\tilde{p_z}}{m})^{1/4} \end{aligned} \tag{6}$$

Using this coordinate transformation, the Schrödinger equation is now separable in terms of u, v, and $\tilde{p_z}$. Proceeding with separation of variables, solutions for the u and v components of the wavefunction follow the form of Weber polynomials:

$$\begin{aligned} \psi_u'' + (\lambda_u - m^2\omega^2u^2)\psi_u = 0 \\ \psi_v'' + (\lambda - \lambda_u - m^2\omega^2v^2)\psi_v = 0 \end{aligned}$$

where $\lambda = \lambda_u + \lambda_v$. However, the ordinary differential equation for the p_z component of the wavefunction has a complicated form:

$$\begin{aligned} \psi_{p_z}''(-p_z m^2\omega^2) + \psi_{p_z}'(m^2\omega^2) \\ +\psi_{p_z}(m^2{p_z}^2 + 2iEm^2{p_z}^2 + m^3 \\ -\frac{5m^2\omega^2}{4p_z} + \lambda {p_z}^{1/2}m^{1/2}) = 0 \end{aligned}$$

This ordinary differential equation proves to be a major obstacle in analytically finding a wavefunction for the quantum harmonic oscillator in tilted coordinates. There are some boundary conditions we could try applying such as having the wavefunction be finite at the origin and vanish as p_z goes to infinity. Still, due to the complicated form for this p_z-component of the wavefunction, pursuing numerical techniques may prove to be more fruitful.

B. Numerical Technique

The numerical technique applied to obtain the wavefunction involves breaking the Hamiltonian operator into two parts, H_0 and V:

$$\begin{aligned} H_0 &= \frac{p_\perp^2 + m^2 + p_z^2}{2p_z} + \frac{m\omega^2 r^2}{2} \\ V &= \frac{m\omega^2 z^2}{2} \end{aligned} \tag{7}$$

where $p_\perp$ represents the momentum operator in the x and y directions and $r^2 = x^2 + y^2$. In redefining p_z as $p_n = \frac{(2n+1)\pi}{2L}$ where n takes on integer values, this breaks the original Hamiltonian operator into a radial part and a longitudinal part along the z-axis. In doing so, the H_0 Hamiltonian can be treated as a modified planar harmonic oscillator which has solutions that can be found analytically The obstacle to overcome in solving the Schrödinger equation with H_0 as the Hamiltonian will be accounting for the additional m^2 and p_z^2 terms. Having broken apart the original Hamiltonian operator, we solve the Schrödinger equation with H_0 and then account for the V part of the Hamiltonian as a small adjustment to the wavefunction. Ultimately, we obtain a wavefunction that represents the complete Hamiltonian.

a. Planar Quantum Harmonic Oscillator: The H_0 Hamiltonian is treated as a modified planar quantum harmonic oscillator which has known solutions. The solutions for the planar quantum harmonic oscillator in cylindrical coordinates consist of Laguerre polynomials which solve the following ordinary differential equation:

$$x(L_n^\alpha)'' + (\alpha + 1 - x)(L_n^\alpha)' + nL_n^\alpha = 0.$$

In solving the Schrödinger equation with H_0 as the Hamiltonian operator we expect to see solutions containing Laguerre polynomials.

The following paragraph goes in detail describing how the radial wavefunction is obtained for this system. In defining $x = \sqrt{p_n m\omega^2} r^2$, the Schrödinger equation for the radial part of the wavefunction becomes:

$$xR'' + R' + \frac{2Ep_n - m^2 - p_n^2 - \sqrt{p_n m\omega^2}x}{2\sqrt{p_n m\omega^2}R} = 0 \quad (8)$$

where R(x) is the radial wavefunction. Letting $R = e^{-x/2}H(x)$, equation 8 becomes:

$$xH'' + (1 - x)H' + \mathcal{E}_n H = 0 \quad (9)$$

where $\mathcal{E}_n = \frac{2Ep_n - m^2 - p_n^2}{4\sqrt{p_n m\omega^2}} - \frac{1}{2}$. This now follows the form of ordinary differential equations for which Laguerre polynomials are solutions. The radial wavefunction is now written:

$$R = e^{\frac{-x}{2}}H(x) = ke^{\frac{-\sqrt{p_n m\omega^2}r^2}{2}} L_{\mathcal{E}_n}(\sqrt{p_n m\omega^2}r^2) \quad (10)$$

where k is a normalization constant. To account for the influence of the V Hamiltonian operator, we multiply the radial function by an exponential function:

$$\psi(r, z) = ke^{\frac{-\sqrt{p_n m\omega^2}r^2}{2}} L_{\mathcal{E}_n}(\sqrt{p_n m\omega^2}r^2)\frac{e^{ip_n z}}{\sqrt{2L}} \quad (11)$$

where L binds integration along the z-axis. Altogether, Eqn 11 expresses the wavefunction for the light front quantum harmonic oscillator in tilted, cylindrical coordinates.

b. Normalization The wavefunction is normalized in the longitudinal direction by dividing by $\sqrt{2L}$. To normalize the wavefunction radially, we construct the normalization integral:

$$\int_0^\infty re^{\frac{-ar^2}{2}} e^{\frac{-br^2}{2}} L_l(ar^2)L_{l'}(br^2)\, dr \quad (12)$$

where $a = \sqrt{p_n m\omega^2}$, $b = \sqrt{p_{n'} m\omega^2}$, $l = \mathcal{E}_n$, and $l' = \mathcal{E}_{n'}$. The solution to integrals of this form is not found in tables of known integrals containing Laguerre polynomials, so the normalization constant for the radial part of the wavefunction is calculated individually for each matrix element of the constructed Hamiltonian matrix.

III. CONSTRUCTING THE HAMILTONIAN MATRIX

Now having a wavefunction for the light front quantum harmonic oscillator in tilted coordinates, we construct the Hamiltonian matrix. From this Hamiltonian matrix, we can obtain approximate energy eigenvalues for this system. Since the numerical technique breaks the Hamiltonian operator into two parts, the Hamiltonian matrix is the sum of two matrices: one for H_0 and another for V. The matrix for H_0 is relatively simple to construct since analytical solutions to the Schrödinger equation exist with H_0 as the Hamiltonian operator. Therefore, the H_0 matrix is diagonal with the energy eigenvalues of the radial wavefunction going down the diagonal.

$$E_{n,l} = \frac{(l + \frac{1}{2})4\sqrt{p_n m\omega^2} + m^2 + p_n^2}{2p_n} \quad (13)$$

$$\begin{bmatrix} E_{0,0} & 0 & 0 & \cdots \\ 0 & E_{1,0} & 0 & \cdots \\ 0 & 0 & E_{0,1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It is important to note that a mapping convention is used to fill out the matrix elements since the energy eigenvalues depend on two integer values: n and l. The mapping convention follows the following pattern.

n, l	Matrix Row/Column Number
0, 0	0
1, 0	1
0, 1	2
1, 1	3
2, 0	4
0, 2	5
2, 1	6
1, 2	7
2, 2	8
⋮	⋮

The V matrix takes more effort to construct. Using the wavefunction in Eqn 11 as the basis, we calculate each individual matrix element for a finite number of matrix dimensions. First, the radial component of the wavefunction is normalized for the row/column numbers in question using Eqn 12. For known values n and l, the integral in Eqn 12 can be solved. Then, the inner product of the V Hamiltonian operator is taken with the wavefunction:

$$\langle \psi_{n,l} \, |V| \, \psi_{n',l'} \rangle$$

$$\int_0^\infty \int_{-L}^{L} kre^{\frac{-\sqrt{p_n m\omega^2} r^2}{2}} L_l(\sqrt{p_n m\omega^2} r^2) \frac{e^{ip_n z}}{\sqrt{2L}} (\frac{1}{2} m\omega^2 z^2)$$
$$e^{\frac{-\sqrt{p_{n'} m\omega^2} r^2}{2}} L_{l'}(\sqrt{p_{n'} m\omega^2} r^2) \frac{e^{ip_{n'} z}}{\sqrt{2L}} \, dz \, dr$$

The potential matrix is not diagonal, and there is no general formula for matrix elements like in the case for the H_0 matrix. Therefore, the V matrix is finite for however far out calculations are carried. As the dimensions of the V matrix approach infinity, this method yields exact energy eigenvalues. This is important to note regarding the numerical technique being used. The value L confines the wavefunction in the longitudinal direction, along the z-axis. This confinement forces the wavefunction to go to zero as it approaches -L and +L in the z-direction. Therefore, for large and small values L, this adjustment for the V part of the Hamiltonian operator breaks down. The following is the four-by-four example of the potential matrix:

$$\begin{bmatrix} \frac{1}{6}L^2m\omega^2 & -\frac{L^2m\omega^2}{\pi^2} & 0 & -\frac{L^2m\omega^2}{\pi^2} \\ -\frac{L^2m\omega^2}{\pi^2} & \frac{1}{6}L^2m\omega^2 & -\frac{L^2m\omega^2}{\pi^2} & 0 \\ 0 & -\frac{L^2m\omega^2}{\pi^2} & \frac{1}{6}L^2m\omega^2 & -\frac{L^2m\omega^2}{\pi^2} \\ -\frac{L^2m\omega^2}{\pi^2} & 0 & -\frac{L^2m\omega^2}{\pi^2} & \frac{1}{6}L^2m\omega^2 \end{bmatrix}$$

The Hamiltonian matrix is also calculated symbolically for the nine-by-nine case as expressed in Figure 2. It is important to note the dependence on the square of the bounds of integration in the longitudinal direction, L^2. This motivates analysis into the L-dependence of the energy eigenvalues calculated using this numerical technique. Do energy states exist where there is minimal dependence on L?

To approximate the energy eigenvalues of the light front quantum harmonic oscillator in tilted coordinates, we sum the H_0 and V matrices for increasing matrix size. The summed matrix is diagonalized symbolically for the four-by-four and nine-by-nine cases, but numerical methods are necessary for increasing dimensions. The eigenvalue dependence on L is visualized graphically for the four-by-four and nine-by-nine matrix cases in Figures 2 and 3.

[h!]

Ultimately, we want to find regions where the wavefunction depends on L minimally since L is an arbitrary

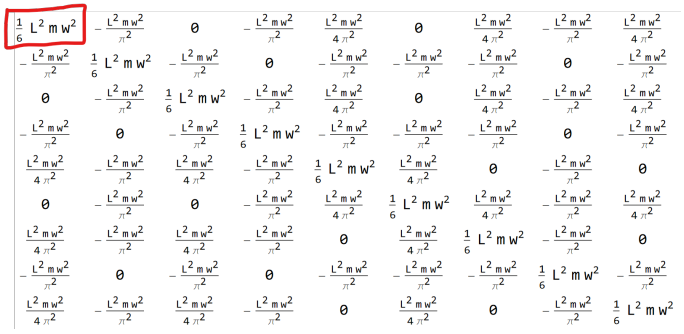

FIG. 2. Symbolic depiction of the 9x9 V matrix. The ground state (n = n' = 0, l = l' = 0) is boxed in red.

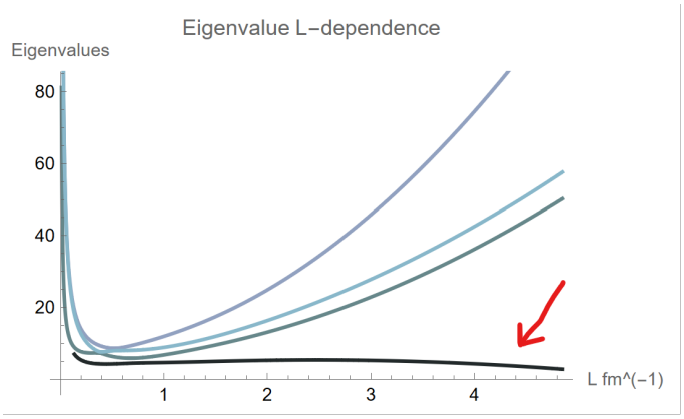

FIG. 3. Eigenvalue Dependence on L for the 4x4 Hamiltonian matrix. The darkest line indicated by the red arrow represents the ground state while lighter colors go towards higher energy states.

value introduced in confining the wavefunction. This region may be found in the ground state. Interestingly, however, the ground state eigenvalue, the darkest color, goes towards negative infinity for large values L while all other eigenvalues tend to positive infinity as L increases. Further study into the trend of the energy eigenvalues as the Hamiltonian matrix approximation increase in dimensions is necessary to clarify this result.

Further, substituting in values $m = 1.7fm^{-1}$, $\omega = 2.5fm^{-1}$, and $L = 7 * \frac{1}{\sqrt{m\omega}}$ into the diagonalized Hamiltonians for the four-by-four and nine-by-nine cases yields the following approximate eigenvalues expressed in Figures 4 and 5. These are common values used on the scale of nuclear physics research. We do not necessarily expect these values to tend towards those of the quantum harmonic oscillator in standard, Cartesian coordinates as the dimensions of the Hamiltonian matrix increase.

IV. DISCUSSION AND CONCLUSION

In diagonalizing both the four-by-four and nine-by-nine Hamiltonian matrices, we find that they both con-

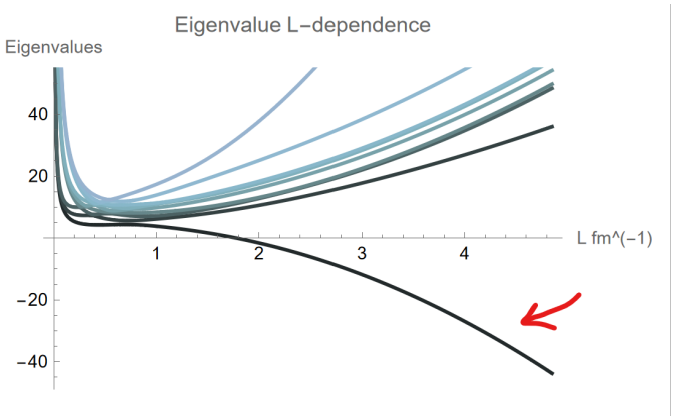

FIG. 4. Eigenvalue Dependence on L for the 9x9 Hamiltonian matrix. The darkest line indicated by the red arrow represents the ground state while lighter colors go towards higher energy states.

$$\begin{pmatrix} 4.98815 & 0 & 0 & 0 \\ 0 & 27.7053 & 0 & 0 \\ 0 & 0 & 33.2027 & 0 \\ 0 & 0 & 0 & 56.1879 \end{pmatrix}$$

FIG. 5. Diagonalized 4x4 Hamiltonian matrix with $m = 1.7fm^{-1}$, $\omega = 2.5fm^{-1}$, and $L = 7 * \frac{1}{\sqrt{m\omega}}$.

tain a dependence on L^2 and this will most likely continue for larger Hamiltonian matrix dimensions. L was an arbitrary length scale value we introduced with which to confine the wavefunction longitudinally. Therefore, for large or small values L, the energy eigenvalue approximations break down which is evident in Figures 2 and 3. Again, it is interesting to note the trend of the ground state eigenvalues to tend towards negative infinity for increasing L. Further study would involve numerically calculating potential matrices of increasing dimension to see how this trend develops. If the dependence on L for any eigenvalue minimizes, that would be a good energy state to analyze further in approximating the wavefunction for the light front quantum harmonic oscillator in tilted coordinates.

Beyond applying tilted light front coordinates to the quantum harmonic oscillator, it is necessary to test them for multiple quantum mechanical systems to understand their properties. Following this study of the quantum harmonic oscillator, the application of tilted coordinates to other potentials and boundary conditions will help illuminate their properties. Another system of interest is applying tilted coordinates to a linear potential.

$$\begin{pmatrix} -17.1916 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 21.1723 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 26.6064 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 27.3079 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 31.2571 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 33.5688 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 34.2141 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 44.3899 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 90.3621 \end{pmatrix}$$

FIG. 6. Diagonalized 9x9 Hamiltonian matrix with $m = 1.7fm^{-1}$, $\omega = 2.5fm^{-1}$, and $L = 7 * \frac{1}{\sqrt{m\omega}}$.

V. SUMMARY

In summary, we make an attempt to analytically determine the wavefunction for the light front quantum harmonic oscillator in tilted coordinates. Numerical methods prove necessary, so we solve for the wavefunction by breaking the Hamiltonian operator into H_0 and V. The wavefunction for H_0 is found analytically and modified to account for V. The H_0 and V matrices are constructed, summed, and diagonalized for the four-by-four and nine-by-nine potential matrix cases. The dependence on the bounds of integration, -L to L, is discussed.

ACKNOWLEDGEMENTS

This work was conducted as part of the University of Washington Physics REU program supported by NSF Grant No. 2243362. Thank you to the University of Washington Physics Department and the Institute for Nuclear Theory for hosting this REU program. Thank you to Dr. Gerald A. Miller and Dr. Adam Freese for mentoring me this summer.

[1] P. A. M. Dirac, Forms of relativistic dynamics, Rev. Mod. Phys. **21**, 392 (1949).

[2] A. Freese and G. A. Miller, Light front synchronization and rest frame densities of the proton: Electromagnetic densities, Physical Review D **107**, 10.1103/physrevd.107.074036 (2023).

[3] P. G. Blunden, M. Burkardt, and G. A. Miller, Light front nuclear physics: Toy models, static sources, and tilted light front coordinates, Physical Review C **61**, 10.1103/physrevc.61.025206 (2000).