A Minimal Equation of State for Neutron Stars

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We construct the minimal equation of state that can account for the key features of the mass-radius curve for typical neutron stars in the mass range $1.3 - 2\ M_{\odot}$ where $M_{\odot} = 1.98 \times 10^{33}$ g is the mass of the sun. We identify the correlation between neutron star masses and radii and the parameters of the simplified EoS. The minimal EoS consists of a realistic model of the crust, and two distinct linear segments characterized by a constant speed of sound. The first segment describes the EoS at intermediate density and is characterized by the pressure at baryon density $n_B \simeq 2\ n_{\rm sat}$ where $n_{\rm sat} \simeq 0.16$ fm^{-3} is the nuclear saturation density. The slope of the second segment is varied over the range allowed by the principle of causality, which requires the speed of sound to be less than the speed of light, and the condition that the EOS produces massive neutron stars with a mass $> 2\mathrm{M}_{\odot}$. This simplified EOS can reproduce the key feature of the neutron mass-radius relation, and we find that the radius at 1.4 $\mathrm{M}_{\odot}$ is sensitive to pressure at $n_B \simeq 2n_{\rm sat}$ but relatively insensitive to the speed of sound. The maximum mass was sensitive to the energy density at which the pressure increases rapidly and the high-density speed of sound.

I. INTRODUCTION

When a massive star reaches the end of its life, its iron core implodes and triggers a supernova explosion [10]. This can leave behind the initial mass of the neutron star (NS). The entire NS is roughly the density of the nucleus of an atom ($10^{14}\ g/cm^3$), while matter that is normally interacted with on an every-day basis is on the order of 1 g/cm^3. At this nuclear density, protons and electrons are close enough that β^+ decay occurs often, resulting in neutron-rich matter. When the neutrons are so dense, their wave functions overlap. Since neutrons are fermions, they obey the Pauli exculsion principle which states that there can only be one neutron per quantum state. Neutrons can have spin up and spin down, meaning there can be two neutrons per energy state. Since NSs are cold objects, their thermal energy is negligible, so the Pauli exclusion principle has a large effect on the energy of the system. The neutrons are forced into high energy states, causing an outward pressure, called degeneracy pressure, that counteracts gravity.

As neutrons reach higher and higher energy levels, it can be more energetically favorable for them to decay: $n \rightarrow p + e^- + \bar{\nu}$, so not all protons and electrons are crushed into neutrons during the core collapse. In a NS, the rates of neutron decay and electron capture occur at equal rates, meaning the star is in chemical equilibrium. Deeper in a NS, other particles, like muons, can stably exist because their rest mass energies are lower than the Fermi energy of the neutrons. [7]

Currently, the accepted structure for the interior of a NS is divided into 5 sections: the outer crust, the neutron drip line, the inner crust, the outer core, and the inner core. The outer crust is a few hundred meters thick with densities of $10^4 - 10^{11} g\ cm^{-3}$, and is made of neutron-rich nuclei surrounded by a gas of free electrons. The neutron drip line is the point when neutrons are no longer bound to the nucleus. The inner crust, also called the free neutron regime has densities of $10^{11} - 10^{14} g\ cm^{-3}$. The outer core is a liquid of mainly neutrons but has some electrons, protons, and muons. The structure of the inner core is unknown, but the presence of hyperonic matter (matter made of strange quarks) has been theorized.

Their interior structure cannot currently be directly measured. Instead, relationships between density and pressure, called equations of state (EoS), are created using what is known about how matter interacts. An EoS, along with the structure equations (3 and 5), is used to create a model of a NS. The results of the models are compared to observations of the masses and radii of known NSs.

II. THE STRUCTURE EQUATIONS

Using purely Newtonian mechanics, the structure equations can be derived, providing a starting point for creating an EoS. To begin deriving the structure equations, take a box with cross-sectional area A and thickness dr. The pressure on the bottom face is $p(r) = F(r)/A$, while the pressure on the top face is $p(r+dr) = F(r+dr)/A$. The force due to gravity is

$$F = \frac{GMm}{r^2}. \tag{1}$$

Combining these equations with Einstein's $E = mc^2$, the following equations are derived:

$$\frac{dp}{dr} = -\frac{G\rho(r)M(r)}{r^2} = -\frac{G\epsilon(r)M(r)}{c^2r^2} \tag{2}$$

$$\frac{dM}{dr} = 4\pi r^2\rho(r) = \frac{4\pi r^2\epsilon(r)}{c^2} \tag{3}$$

$$M(r) = 4\pi \int_0^r r'^2 dr' \rho(r') = 4\pi \int_0^r r'^2 dr' \epsilon(r')/c^2 \quad (4)$$

Here, $M(r)$ is the mass contained within a given radius. ρ is the mass density and ϵ is the energy density. The switch from ρ to ϵ is important because it allows energy from particle interactions to be accounted for. ϵ and p have the same units: $ergs/cm^3$. Equations 2 and 3 can be solved by integrating from $r = 0$ to a maximum value, R, where the pressure reaches 0. This system requires two initial conditions: pressure, p_0, when $r = 0$ and $M(0) = 0$. The Newtonian equations work for modeling some stars, like small white dwarfs, but relativistic corrections are required to appropriately model NSs. Adding relativistic corrections to Equation 2 results in the Tolman-Oppenheimer-Volkov (TOV) equation [8][9].

$$\frac{dp}{dr} = -\frac{G\epsilon(r)M(r)}{c^2r^2} \times \left[1 + \frac{p(r)}{\epsilon(r)}\right]\left[1 + \frac{4\pi r^3 p(r)}{M(r)c^2}\right]\left[1 - \frac{2GM(r)}{c^2 r}\right]^{-1} \quad (5)$$

The first two additional terms are special relativity corrections and the third term is a general relativity correction. All of these corrections are greater than 1, seeming to "increase" the effect of gravity, resulting in smaller maximum masses and radii that can be achieved. Substituting in dimensionless quantities makes computation significantly easier. To do so, we define $\bar{M} = \frac{M}{M_\odot}$, $\bar{R} = \frac{R}{R_0}$ where $R_0 = \frac{GM_\odot}{c^2}$, $\bar{\epsilon} = \frac{\epsilon}{\epsilon_s}$ where $\epsilon_s = \frac{M_\odot c^2}{(4/3)\pi R_0}$, and $\bar{p} = \frac{p}{\epsilon_s}$. Substituting these dimensionless variables into equations 3 and 5 gives

$$\frac{d\bar{M}}{d\bar{r}} = 3\bar{r}^2\bar{\epsilon} \quad (6)$$

and

$$\frac{d\bar{p}}{d\bar{r}} = -\frac{\bar{\epsilon}\bar{M}}{2\bar{r}^2}\left[1 + \frac{\bar{p}}{\bar{\epsilon}}\right]\left[1 + \frac{3\bar{r}^3\bar{p}}{\bar{M}}\right]\left[1 - \frac{2\bar{M}}{\bar{r}}\right]. \quad (7)$$

If the system of equations is being solved with a differential equation solver, the second term in brackets will pose a problem with division by 0 because the mass contained within the starting radius is 0. To address this, the mass term can be expanded to $(4/3)\pi\bar{r}^3\bar{\rho}$. When working in Python, the differential equations were instead solved manually.

III. SOLVING THE COUPLED ODES

These differential equations cannot be solved analytically and instead must be done numerically. To find the maximum mass and radius for a given pressure, the differential equation is solved out to farther and farther radii until the pressure becomes negative. The last radius is then recorded and that value is passed into the mass equation to get the total mass of the star. This process can be repeated with several initial pressures to generate a mass-radius curve for a given EOS. Multiple approaches can be used to generate these curves.

A. Mathematica

To solve the differential equation for a given starting pressure, use a "Do" loop. This loop will iterate on the variable rf, the end point that the differential equation solver will go to. It is set to range from 1km to 40km in 100m increments. When working with Mathematica, an equation very similar to 7 was used, but the radius was not dimensionless; it was kept in kilometers. The "Check" is in place because NDSolve was reaching the maximum number of steps (which was set to 20,000). This caused the loop to not recognize that the pressure had gone negative because the exception was returned, so the returned radius of the NS was larger than it should've been. A sample of the code is included below. pReal and mReal can both be plotted versus r to see how pressure and mass change throughout a star. To generate a mass-radius plot, another "Do" loop is needed to iterate through multiple values of central pressure, in this case represented by pbar0R.

```
Do[Check[
  solution =
   NDSolve[{pbar'[x] == rhsGR[x],
     mbar'[x] == rhs2[x],
     pbar[r0] == pbar0R,
     mbar[r0] == 0}, {pbar, mbar},
     {x, r0, 40}, MaxSteps -> 20000],
     Break[], NDSolve::mxst];
 pReal = pbar /. First[fullsol];
 mReal = mbar /. First[fullsol];
 test = Re[pReal[rf]];
 If[test < 0, Break[],
 {rMax = rf, mMax = Re[mReal[rf]]}],
 {rf, 1, 40,0.1}]
```

When Equation 7 is solved, the second term in parentheses causes problems for NDSolve due to division by 0 when the radius is 0. To address this, the mass term must be expanded at very small radii.

$$M \approx \frac{4\pi r^3 \epsilon}{3c^2} \quad (8)$$

B. Python

Scipy does have a differential equation solver, but it struggled with the small numbers that were necessary to solve these ODEs. Instead, a while loop was used, and the equations were manually stepped through. Each differential equation was defined as a function, and the parameters were stepped through, increasing R by dr each step. The end values of R and M were then added to a

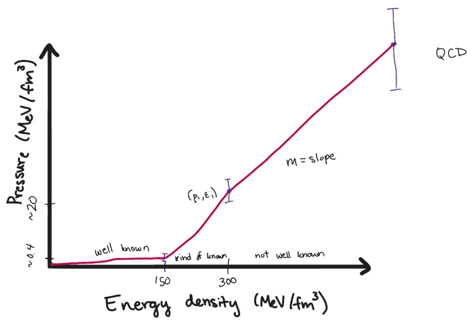

FIG. 1. A simplified drawing of regions of an EoS. The error bars at the end of each section show how well understood that segment of the EoS is. At extreme densities, the EoS becomes well known again because it can be calculated using quantum chromodynamics (QCD).

list. This loop was nested inside another while loop that iterated through a range of pressures, allowing a mass-radius curve to be plotted.

```
while P>0 and R<40:
    R += dr
    M += dmdr(P,R) * dr
    P += dpdr(P,M,R) * dr
```

IV. CONSTRUCTING AN EOS

A simplified model of an EoS can be constructed using three distinct sections, as shown in Figure 1. At low energy densities, below nuclear saturation energy density, about 150 MeV/fm^3, the pressure changes almost negligibly. This portion of the equation of state describes the behavior in the crust of a neutron star and is well known. The second section of this equation of state describes the NS at intermediate density, and is less well known. This is characterized by the pressure at baryon density $n_B \approx 2n_{sat}$ where $n_{sat} \approx 0.16 fm^{-3}$. The pressure in the outer core region is thought to be around 20 MeV/fm^3 [3]. This corresponds to an energy density of around 300 MeV/fm^3. The third segment of the EoS is not well known, so there are many different theories used to represent this section.

There are many approaches to constructing an EoS. One of the simplest is to treat a NS as a Fermi gas of neutrons. In this model, each energy level can only be occupied by two neutrons, one with spin $+\frac{1}{2}$, and one with spin $-\frac{1}{2}$. At the non-relativistic and relativistic approximations, this model can be approximated as a polytrope, $p = k\epsilon^{\gamma}$, where γ is 5/3 and 4/3 respectively [7][6]. When $\gamma = 4/3$, the maximum mass is independent of central pressure. To create an EoS for an arbitrarily relativistic Fermi gas, a linear combination of both polytropes can be used.

The Fermi gas model can then be improved by adjusting to account for the presence of protons and neutrons. This is necessary because free neutrons are not stable. Instead, it is energetically favorable for them to decay. If there are some protons present in the NS, they will also obey the Pauli exclusion principle, and if they reach a high enough energy level, the neutron will not decay. This shrinks the maximum mass that a NS can reach. The model can further be improved by including nuclear interactions. The total energy per particle can be written as

$$E = (n.\alpha) = E(n, 0) + \alpha^2 S(n) \tag{9}$$

where $\alpha = \frac{N-Z}{A}$ and $S(u)$ is defined as

$$S(u) = (2^{2/3} - 1)\frac{3}{5}\left\langle E_F^0 \right\rangle (u^{2/3} - F(u)) + S_0 F(u). \tag{10}$$

Another improvement to this EoS can be made by ensuring that causality is satisfied by respecting the conditions set in Equation V. More detail on this process can be found in the paper by Silbar and Reddy [7].

Another approach to constructing an EoS is creating a piecewise function to model the behavior of matter in different sections of the NS. One example of this is done by Macher and Schaffner-Bielich [4]. They use four parts for their EoS: crust, a hadronic phase for the outer core, a mixed phase of both hadronic and deconfined quark matter, and finally pure quark matter. For the crust, they use the BPS [2] and NV[5] EoS. For their hadronic phase, they used a polytropic EoS, and for the final two phases, they used linear EoSs.

One of the EoS models for a NS is a piecewise function using the BPS-NV [2][5] model for the crust and the APR [1] model for the core of the star. The EoS and resulting mass-radius plot using these models are shown in Figure 3. We hoped to use our simple linear fit EoS to extract the mass of a NS matching the results given by the complex EoS.

V. LINEAR EOS

In all EoS models, there is a period where pressure increases slowly as a function of energy density, then after a point, ϵ_0, the pressure begins increasing rapidly. We are considering this point to be around nuclear saturation energy denisty, 150 MeV. A linear EoS is the simplest EoS that can be used to model this. When p $\propto$ ϵ, the pressure only monotonically approaches 0, but never becomes negative, resulting in an infinite radius for a NS with any given central pressure. If the EoS is of the form

$$p = \left\{ \begin{array}{ll} 0 & \text{if } \epsilon < \epsilon_0 \\ c_s^2 & \text{if } \epsilon \geq \epsilon_0 \end{array} \right\}, \tag{11}$$

the pressure will become negative if $\epsilon_0 > 0$, allowing a mass-radius plot to be created, as shown in Figure 2. The slope of this segment is constrained by the principle

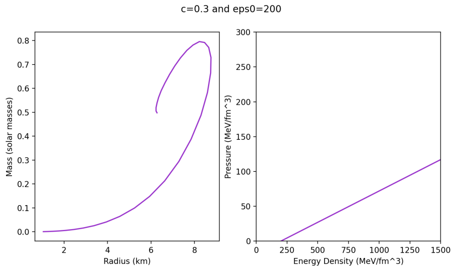

FIG. 2. **Left:** A mass-radius plot of a linear EoS where $c_s^2 = 0.09$ and $\epsilon_0 = 200$ MeV. **Right:** A plot of the EoS, showing that the x-intercept and slope match the constants that it was assigned.

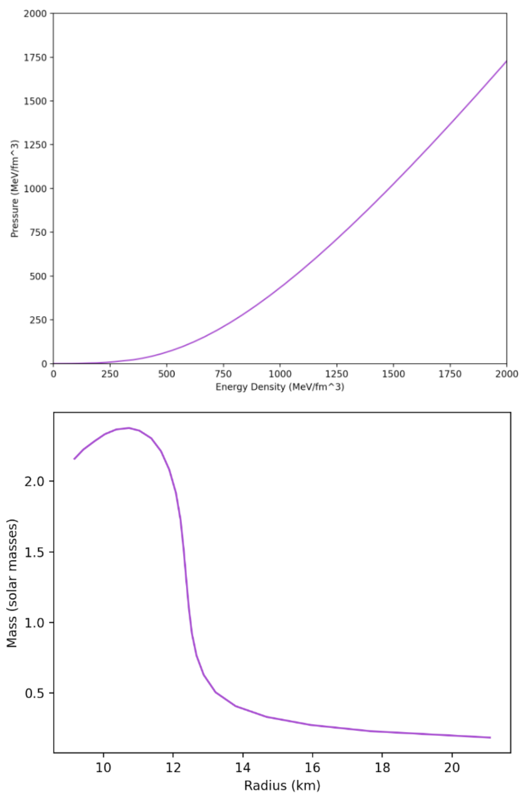

FIG. 3. Mass-radius curves generated using a linear EoS. The slope, c_s^2, is held constant at 0.3. The x-intercept, ϵ_0, is varied, and its values are displayed in the legend.

of causality. This requires the speed of sound to be less than the speed of light, following the relation

$$\left(\frac{c_s}{c}\right)^2 = \frac{dp}{d\epsilon}. \tag{12}$$

A. Pure Linear EoS

Using the pure linear EoS described in Section V, we analyzed how the maximum mass and radius at 1.4 $M_\odot$ changed as c_s^2 and ϵ_0 were altered. The general trend can be seen in Figure 4. When ϵ_0 is increased, both the mass and radius decrease. When c_s^2 is increased, the mass and radius increase. These patterns were investigated more thoroughly by creating "iso-c" curves for both M and $R_{1.4M_\odot}$. These curves were constructed by holding c_s^2 constant and changing ϵ_0, then fitting a line to all the points. This process was repeated for multiple values of c_s^2.

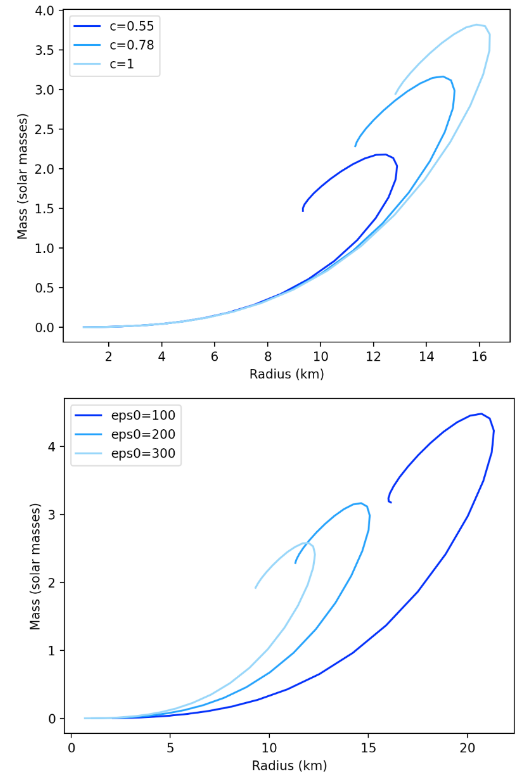

FIG. 4. Mass-radius curves generated using a linear EoS. **Top:** The x-intercept, ϵ_0, is held constant at 200 MeV. The slope, c_s^2, is varied, and the values of c_s are displayed in the legend. **Bottom:** The slope is held constant at 0.61. The x-intercept, is varied, and its values are displayed in the legend.

To fit the curves for maximum mass, we used a line with the form

$$M = aB^{-1/2}, \tag{13}$$

as shown in Figure 5. As c_s increased, so did the fit value of a. Fitting the curves for radius followed a slightly different procedure. To get the radius at 1.4 $M_\odot$, either interpolation or the point nearest to 1.4 $M_\odot$ can be used. If the second method is used, it is important to get the largest radius allowed at 1.4 $M_\odot$ because there can be multiple radii for one mass and it is important to make sure the maximum mass $\geq$ 1.4 $M_\odot$. The curves for the radius are shown in Figure 6. They were fit with an equation of the form

$$R = aB^{-1/3} + d. \tag{14}$$

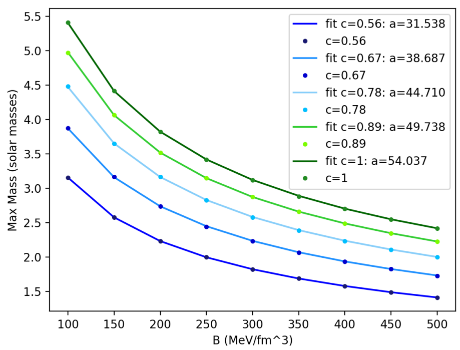

FIG. 5. The "iso-c" curves for the maximum mass. Each curve is fit using an equation proportional to $1/B^{1/2}$. The resulting fit parameters and the value of c_s are shown in the legend.

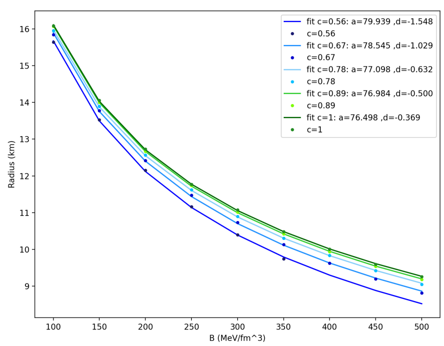

FIG. 6. The "iso-c" curves for the radius at 1.4 $M_\odot$. Each curve is fit using an equation of the form $1/B^{1/3} + f$. The resulting fit parameters and the value of c_s are shown in the legend.

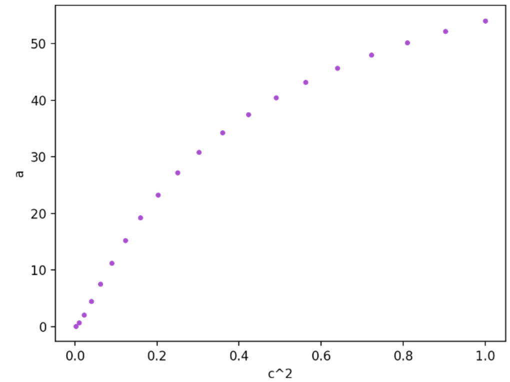

FIG. 7. A plot of the a values used in the fits for the mass "iso-c" curves. At small c_s^2 values, there is non-linear behavior, but around $c_s^2 = 0.4$, the points look nearly linear.

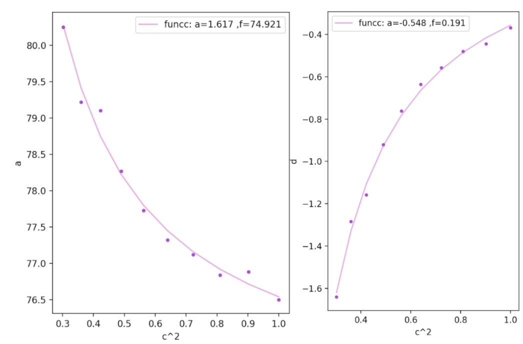

FIG. 8. **Left:** A plot of the a values used in the fits for the radius "iso-c" curves. **Right:** A plot of the d values used in the fits for the radius "iso-c" curves.

These fits allowed us to figure out what the maximum mass and R$_1$.4$M_\odot$ would be for any value of ϵ_0 for a given value of c_s once we had one value. We wanted to be able to predict the mass and radius just given ϵ_0 and c_s without an additional data point. To do this, we attempted to create fits of the fit constants.

There was no obvious function to use to fit the fit constants for the mass "iso-c" curves. For both the fit constants in Equation 14, we used a fit of the form

$$\text{const} = \frac{a}{c_s^2} + f. \tag{15}$$

The error in these fits is due to the issues with calculating R$_{1.4M_\odot}$ discussed above.

The mass-radius plots generated by the pure linear EoS did not match the high density part of Figure 3 well; it was only able to be a good approximation around one point, so we had to choose if we wanted it to match the maximum mass or R$_{1.4M_\odot}$.

B. Linear EoS with Crust

To have a better fit while still keeping the EoS simple, we added the crust EoS from BPS-NV [2][5]. This meant that for pressures up to $3.63172009e-7M_\odot/km^3$, or around 0.405 MeV/fm^3, the pressure is non-zero and slowly increasing. This switched the tail from the left to the right, giving a similar shape to the mass-radius curve in Figure 3. Figure 9 shows that adding in the crust does not make a large difference at high central pressures, which aligns with what we expect. At very low central pressures, there is a large difference between the equations of state, and that also aligns with our expectations because that is where the equations of state differ.

C. Piecewise Linear EoS with Crust

Neither of these models matched what is known about the equation of state very well. Looking back at Figure 1, it makes more sense to use the crust and two diatinct

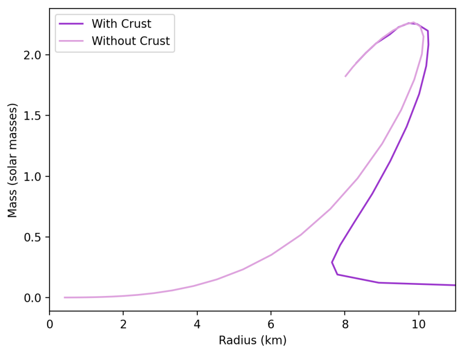

FIG. 9. A comparison of the mass-radius plot with and without the crust included in the EoS. Without the crust, the tail goes to the left, while with the crust, the tail goes to the right. At high densities, close to the maximum mass, the two curves are very similar.

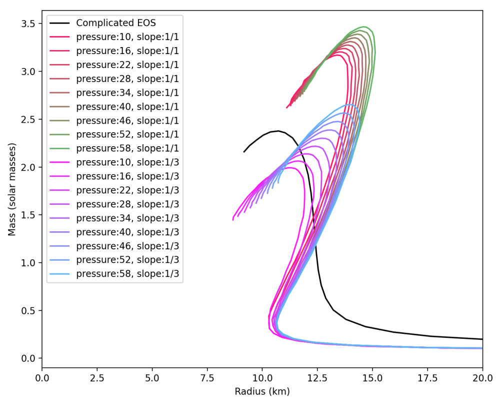

FIG. 10. The mass-radius plots of the EoS model compared to the mass-radius plot creating using the BPS-NV and APR EoSs. Each group has the same slope; those with lower maximum masses have a slope of 1/3, while the other group has a slope of 1.

linear segments. The parameters we used for the model were changing the pressure at ϵ_1 (300 MeV/fm^3) and the slope after that point. Changing these two parameters allowed us to investigate how sensitive the maximum mass and the radius at 1.4 $M_\odot$ are to that section of the EoS. The maximum value of the slope must still be less than 1 to preserve causality, as shown in Equation 12, and the curves where $c_s^2 = 1$ are the maxima for this EoS. The mass-radius plots using this EoS are shown in Figure 10.

VI. DISCUSSION AND CONCLUSION

We investigated how important the inner core EoS is for determining the maximum mass of a NS. To do this, we used a simple linear fit to closely match the APR [1] EoS at high densities and approximated the low-density EoS as either 0 or by using the BPR-NV [2][5] crust equa-

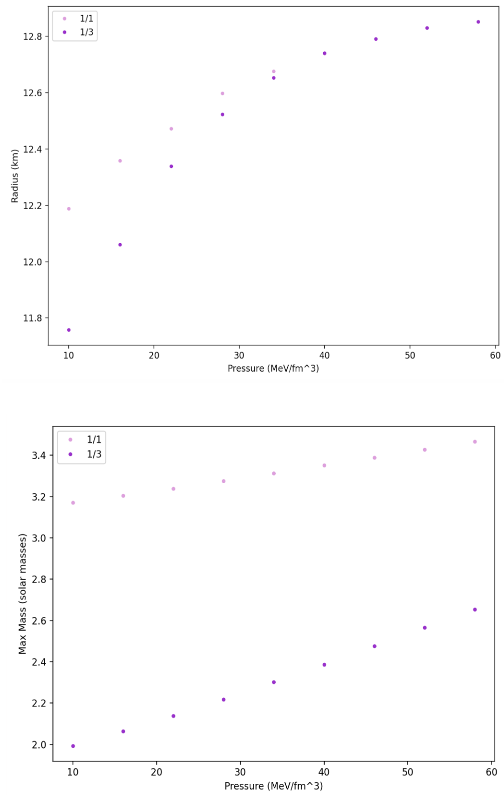

FIG. 11. **Top:** As the pressure increases at ϵ_1, the two curves converge, so the slope of that section does not have a large impact. This is likely because ϵ is low enough at the Chandrasekhar limit that not much of that portion of the equation of state is relevant. **Bottom:** The greater the slope is, the less that the pressure at ϵ_1 matters for the maximum mass. The slope does have a large impact on the maximum mass.

tion. Neither simplified EoS fit the curve well for the entire range, but both could be fit well near a specific area, often chosen to be either maximum mass or the radius at $1.4M_\odot$. The trends from this extremely simplified model suggest that the outer core EoS is very important for creating the mass-radius plots for a NS.

Further conclusions could be drawn from the other simplified model that used two linear segments. This simplified EoS with the crust and two distinct linear sections is adequate to accommodate the gross features observed on the Mass-Radius plot. The radius at 1.4 $M_\odot$ was mostly depending on the pressure at $2n_{sat}$, with the slope having little impact, especially at large p values. The maximum mass was dependant on slope and pressure. The slope had a more significant impact than pressure, and ss slope increased, the pressure at $2n_{sat}$ had less of an effect.

If the radius at 1.4 $M_\odot$ can only be measured within 10%, the slope of the second distinct linear section does have a measurable impact on the NS. The error bars on the measurements would be roughly 1.2 km, and the differences between the two slopes never exceeded 0.4 km, so all reasonable EoS would be well within the error bars.

In Figure 10, the radius at 1.4 $M_\odot$ is always within the error bars, so the simplified EoS would indistinguishable from the complex one given that measurment precision.

If the mass of the NS can also only be measured to an accuracy of 10%, the error bars would be around 0.2-0.3 km. Both the slope and pressure at ϵ_1 have effects on the maximum mass that exceed the error bounds, making the simplified EoS distinguishable from the more complex one given that measuring ability. Some of the simplified EoSs result in maximum masses within the error bounds, but the majority do not. If a high slope is chosen, the pressure at ϵ_1 has much less of an impact on maximum mass. At low slopes, like 0.3, changing the pressure at ϵ_1 changes the mass by an amount close to the hypothetical measurement uncertainty. When measuring mass, it is possible to distinguish between the complex and simplified EoSs.

With more precise measurements of NS masses and radii, more outer core EoSs can be ruled out. As the radii of low-mass NSs are measured more precisely, the pressure around $2n_{sat}$ can be further constrained. As the upper limit on the mass of a NS is better determined, the slope can be more constrained, limiting the speed of sound in inner regions of the NS, revealing more information about the structure. As observations continue to advance in this "Golden Age" of neutron star astrophysics, constraints on the EoS should greatly increase.

VII. ACKNOWLEDGEMENTS

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