Bounds on the Strength of Order and Disorder Parameters in Quantum Spin Chains with Finite Abelian Symmetries
REU Final Presentation

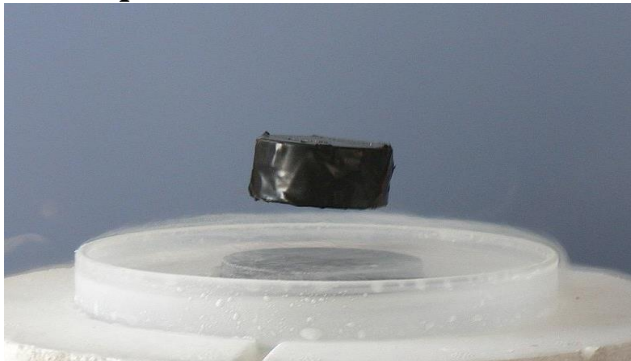**Maxim Zelenko[1]**

Mentor: Lukasz Fidkowski[2]

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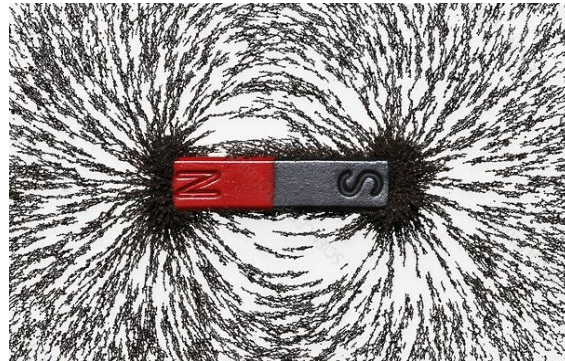# Motivation: Condensed Matter Physics

- **An objective of condensed matter:** to study emergent phenomena in quantum many-body systems.
- Examples of emergent phenomena:
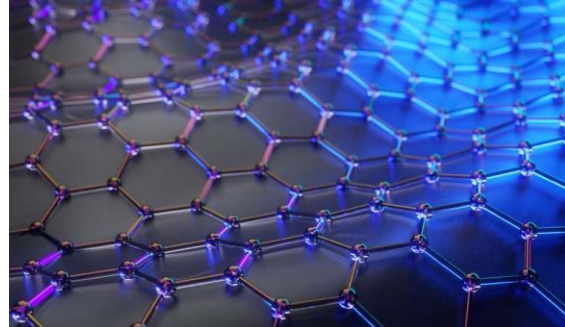 - Superconductors
 - Ferromagnets and antiferromagnets
  - Superfluids
  - Graphene

Motivation: 1D Systems Studied in Condensed Matter

- Serve as playgrounds for studying emergent phenomena
- Examples:
  - Thin wires
  - Optical lattices (top right)
  - Carbon nanotubes (two pictures below)
  - 1D arrays of interacting quantum dots, vortices, or other confined quantum systems
- **These serve as inspirations for instructive toy models, such as quantum spin chains (the subject of this talk!)**

a Optical lattice

b Real crystal
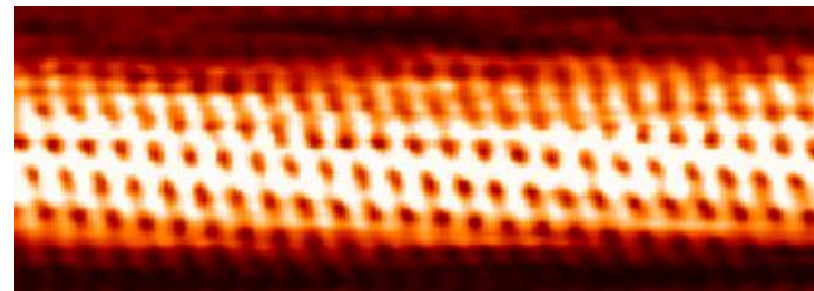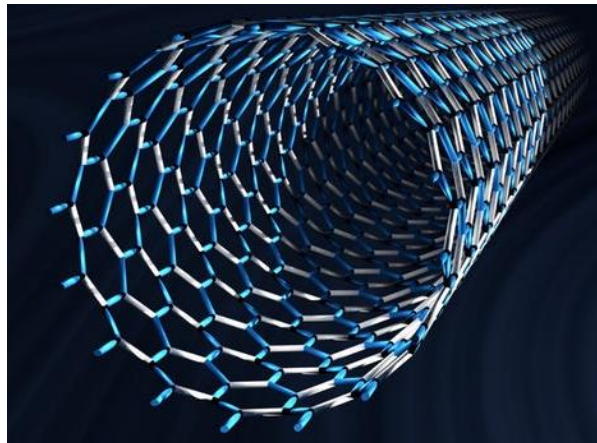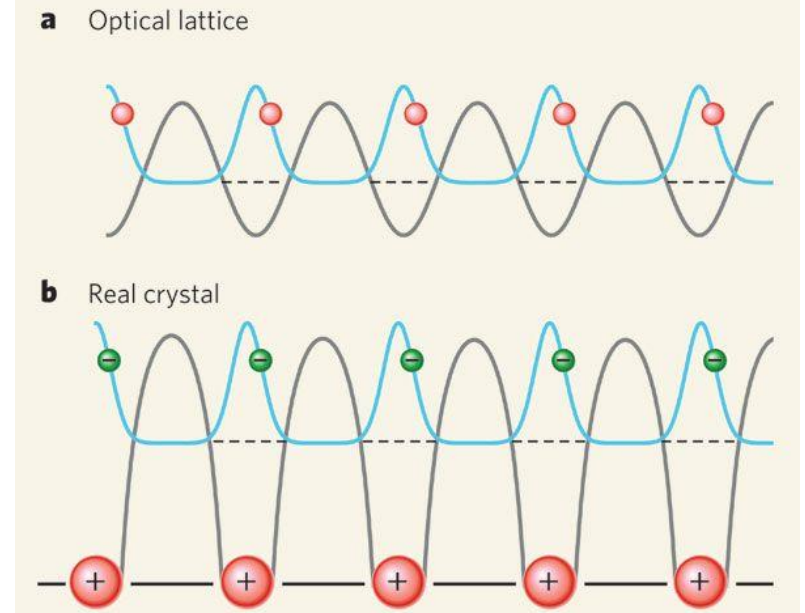
Spin Chains: a Physical Origin

- Consider some electronic material:

b Real crystal
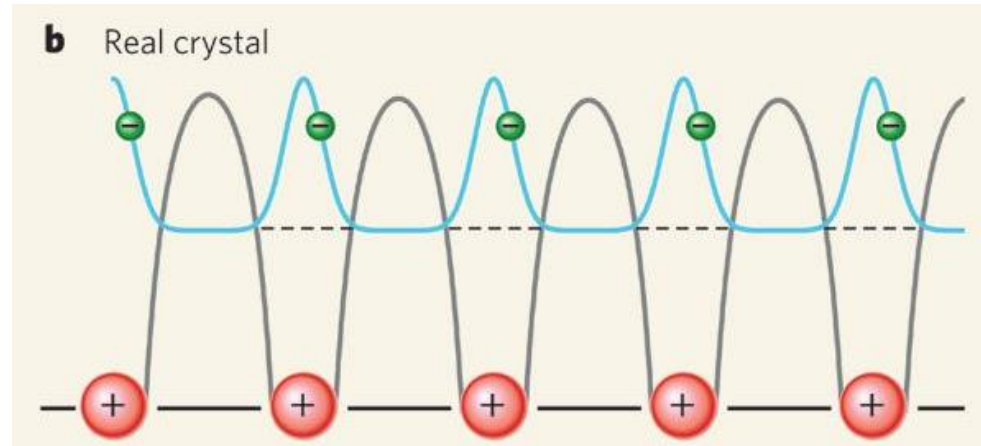
- Can model the system's dynamics and interactions by the following Hamiltonian:

$$\hat{H} = \hat{K}_e + \hat{V}_{ee} + \hat{V}_{ei} + \hat{K}_i + \hat{V}_{ii}$$

| Kinetic energy term for valence electrons | Coulombic/electrostatic repulsion term for the valence electrons | Electron-ion electrostatic attraction term. | Kinetic term for ions | Electrostatic repulsion term for ions |

Spin Chains: a Physical Origin

- Simplifying assumptions to make the Hamiltonian more tractable:

1. Electron-ion attraction term V_{ei} is a **spatially-periodic lattice potential** experienced by the valence electrons (see figure). Also, ignore lattice distortions.

2. This periodic potential V_{ei} is an array of deep quantum wells (**each well corresponds to an ion site**), so that there is one valence electron localized at each well/ion/lattice site.

a Optical lattice

b Real crystal
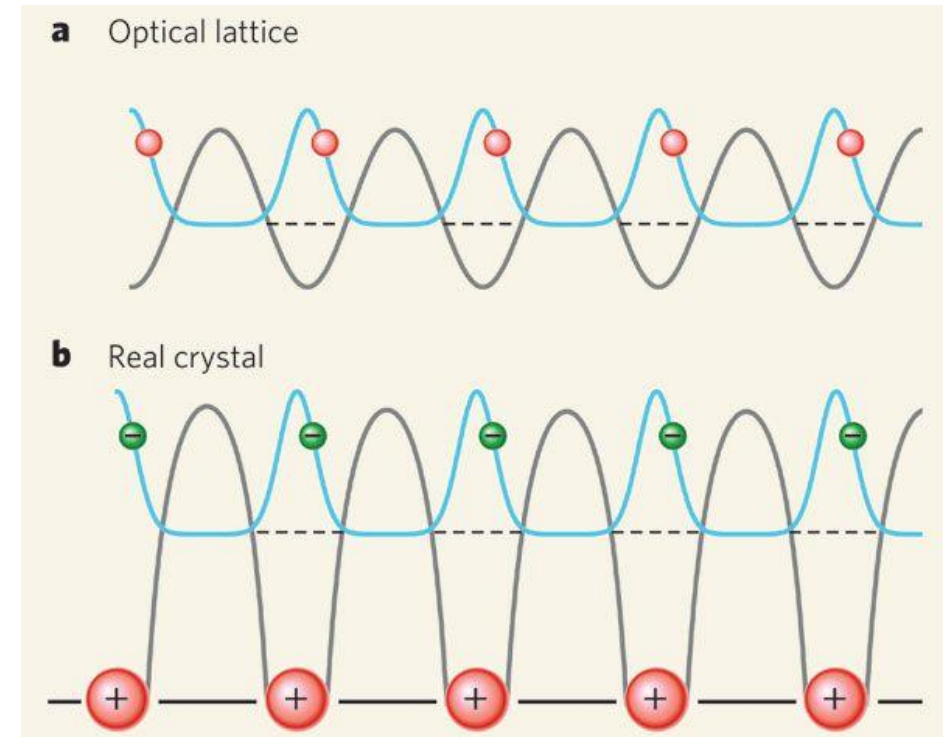
- Such assumptions result in a Hamiltonian telling us that the **spins of nearest-neighbor electrons either have a tendency to align or anti-align, thereby explaining ferromagnetism and anti-ferromagnetism.**

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Spin Chain Hilbert Space: The Rigorous Definition

- **Definition:** A **spin chain Hilbert space** is a Hilbert space that is equal to the tensor product of the finite-dimensional inner-product (or Hilbert) spaces corresponding to each site.

Each lattice site corresponds to a finite-dimensional Hilbert space
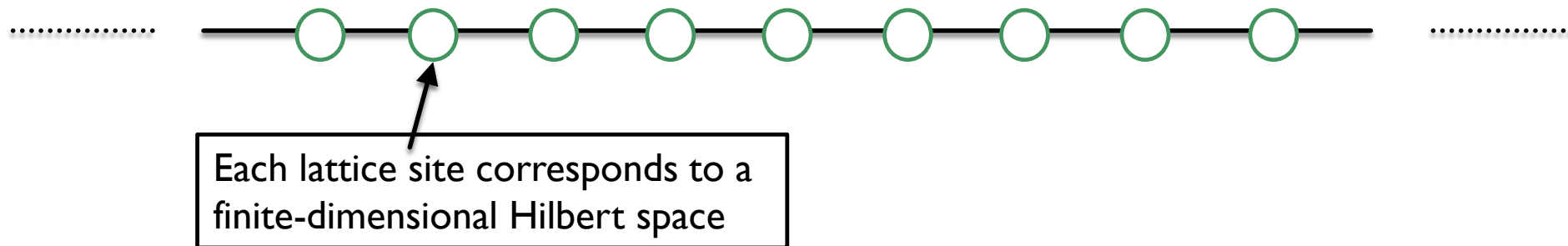
- To keep things simple, all the finite-dimensional Hilbert spaces of the lattice sites have the same dimension, which we denote by d.

- **Intuition for tensor product:** If each site contains a spin-1/2 particle like the electron, then $d=2$, so if the total number of sites is L, the dimension of the whole spin chain Hilbert space is 2^L, such that a general state in spin chain is a superposition of 2^L possible configuration states.

Spin Chains: Ising Model

- Consider the following Hamiltonian, called the **1D Ising Hamiltonian** and defined on a spin chain with *infinitely-many lattice sites*

$$H = -\sum_i \sigma_i^z \sigma_{i+1}^z$$

- Right away, we notice that this Hamiltonian only includes spin degrees of freedom:
  - σ_i^z is the operator that corresponds to the measurement of the **z-component of the spin of the electron at the ith lattice site.**
- In the basis of spin up and spin down states for the z-component of spin, the **Pauli operators are represented by the following matrices:**

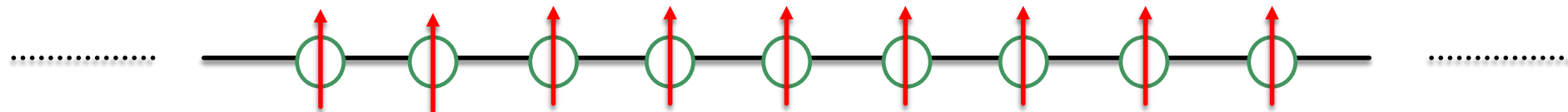$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
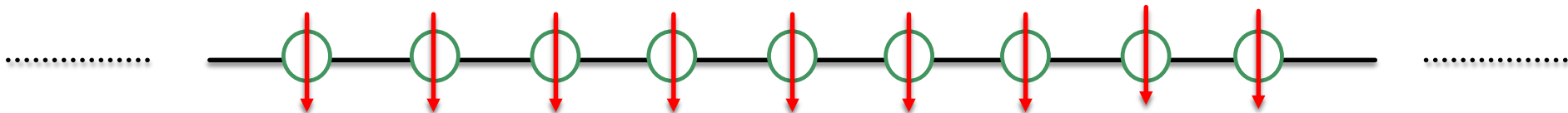
Ising Model

$$H = -\sum_i \sigma_i^z \sigma_{i+1}^z$$

- The ground state of the 1D Ising Hamiltonian is *two-fold degenerate*, i.e. the ground state subspace is two-dimensional and is spanned by two states:
  - All spin up in z-orientation

  - All spin down in z-orientation

Transverse Field Ising Model

- But suppose we introduce a *transverse* **magnetic field** in the x-direction. Then we add another term to the Ising Hamiltonian, leading us to the **transverse field Ising model:**

$$H = -\sum_i \sigma_i^z \sigma_{i+1}^z - B \sum_i \sigma_i^x$$

- |B| is the strength of the magnetic field, and as we increase this strength, the spins go from being spin up or down with a z-orientation to being more and more oriented in the x-direction of the magnetic field.

Basis of ground states when B = - ∞

Basis of ground states when B = 0

Basis of ground states when B = ∞
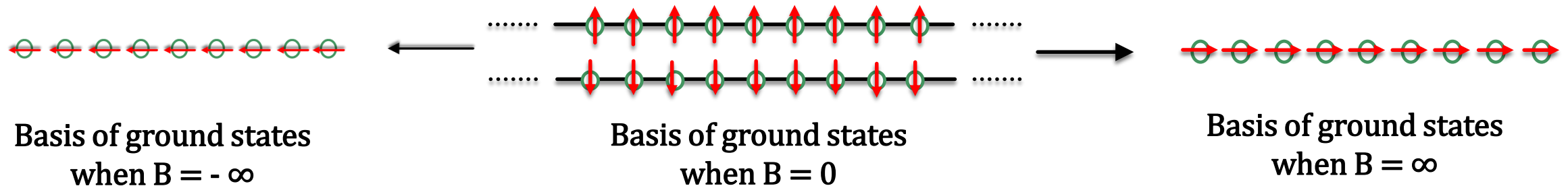
Transverse Field Ising Model Phase Diagram

- Hence, the two-fold spin-up-and-down degeneracy of the ground state should disappear when |B| is large enough, causing all the spins to be oriented toward the magnetic field.

- The loss of ground state degeneracy happens at |B| =1 and corresponds to the transition from the ferromagnetic phase to the paramagnetic phase (as shown in the phase diagram below)

Ferromagnetic
phase

-1   0   1   B

Critical points

Paramagnetic
phases
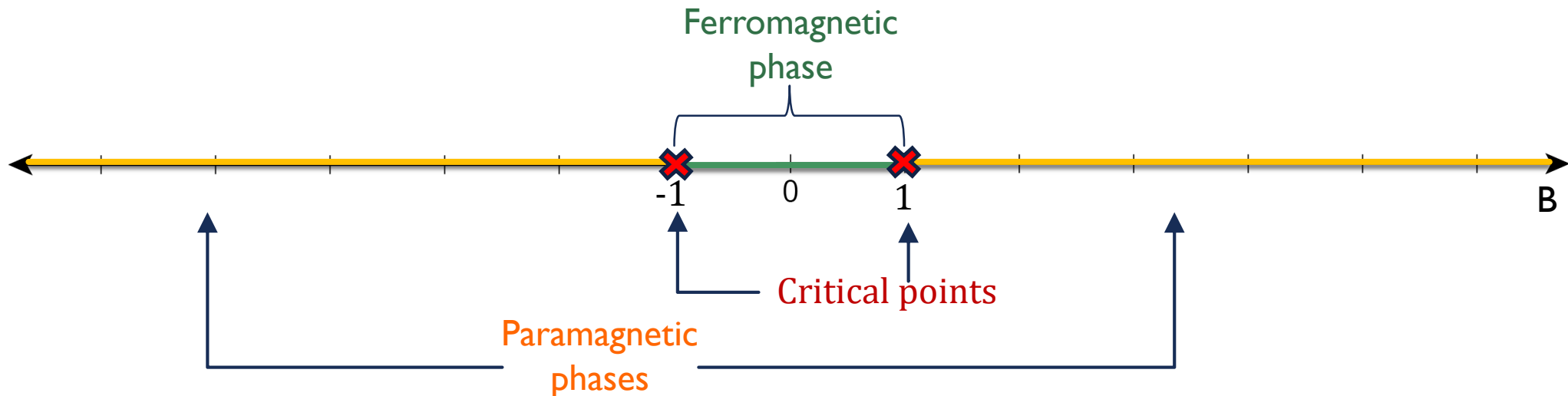
But how can we know where the critical points occur? Magnetization

- **Magnetization**: In electrodynamics, a pseudovector field that represents the density of the magnetic dipole moment, i.e. evaluates **magnetic dipole per unit volume** at every position.

- Spin can be thought of as a measurement proportional to the magnetic dipole moment, so σ_i^z also corresponds to measurement of the z-component of the magnetic dipole moment at the ith lattice site.

- Since our system is discrete, can treat "per lattice site" as "per unit volume," so that the **measurement by** σ_i^z **essentially means the measurement of the z-component of magnetization.**

- **Net magnetization**: Average magnetization over the whole system.

Observations about Net Magnetization in Transverse Ising

- **One can make two key observations about the ground states $|\Omega\rangle$ of the transverse field Ising model:**

  1. For any ith lattice site, $\lim_{j \to \infty} \langle \Omega | \sigma_i^z \sigma_j^z | \Omega \rangle \neq 0$ for |B| < 1 and = 0 otherwise.

  2. $\lim_{|i-j| \to \infty} \langle \Omega | \prod_{k=i+1}^{j} \sigma_k^x | \Omega \rangle \neq 0$ for |B| > 1 and = 0 otherwise.

- #1 informs us that for |B| < 1, the ground states have a correlation between a measurements of the z-component of magnetization at any ith lattice site and an analogous measurement at a distance infinitely far away.

- This signifies that there must be a net magnetization of the spins in the z-direction *at least* in the case that |B|<1.

- #2 means that for |B| > 1, spin flips applied to a ground state at almost every subinterval of the spin chain **must result in a state that overlaps with (i.e. not orthogonal to) that ground state.**

The Two Observations Combine into a Full Picture

- With spin flips most often not resulting in states orthogonal to $|\Omega\rangle$ for $|B| > 1$, we can deduce that the net magnetization is zero for $|B| > 1$.

- **This makes the ferromagnetic phase**, which is characterized by nonzero net magnetization, be **exactly $|B| < 1$.**

- In addition, this also **makes $|B| > 1$ the paramagnetic phase**, since we see zero net magnetization and the spins like to orient in the direction of the transverse magnetic field as $|B| \rightarrow$ infinity.
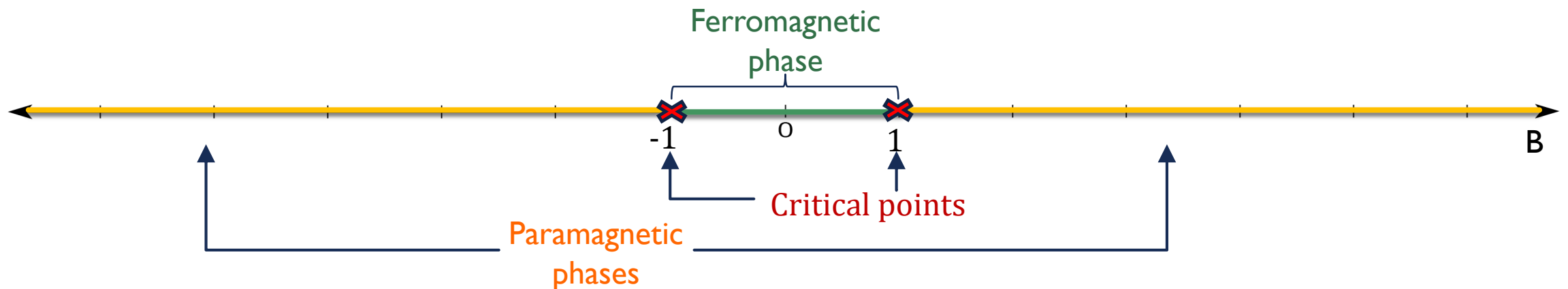
Can such analysis be generalized to any spin-chain model?

- We were lucky that in the Ising model, we could analyze net magnetization to identify the collective behavior of spins under different magnetic regimes.

- But magnetization is a very specific kind of observable and it would be nice to be able to identify phases for other more complicated Hamiltonians defined over the spin-chain.

- This provides a motivation for generalizing the measurement procedure via order and disorder parameters.
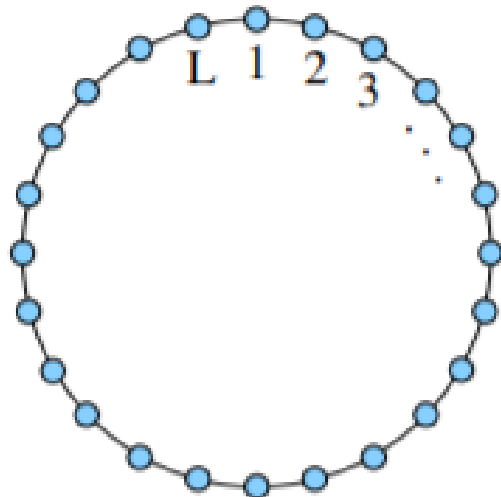
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Spin Chains: Heads-Up

- We will focus our attention on two types of spin chains:
    - **Infinite:**

    - **Finite spin chains with L spins arranged in a circle** (thereby periodic boundary conditions):

Hamiltonians: Heads-Up

- Our Hamiltonians of interest correspond to nearest neighbor interactions that have a bounded norm:

$$H = \sum_{i=1}^{L} H_{i,i+1},$$

such that

$$\text{supp}(H_{i,i+1}) \subset \{i, i+1\} \text{ and } \|H_{i,i+1}\| \leq 1.$$

- Assume that Hamiltonian H is symmetric with respect to some **Abelian group G of finite order** n.

- A representation of G as a direct product of cyclic groups is: $G = \langle S^{(1)} \rangle \times \langle S^{(2)} \rangle \times ... \times \langle S^{(m)} \rangle$

- Further assume that **G is an onsite symmetry**, i.e. can write for any generator: $S^{(j)} = \prod_{i=1}^{L} S_i^{(j)}$, where $\text{supp}(S_i^{(j)}) = \{i\}$.

- **Generic assumption:** Trivial subspace (i.e. eigenspace of G with eigenvalue 1) has a non-degenerate lowest energy state.

Definition of Order Parameter

Definition : *A collection of operators* $\{O_i : i \in X\}$ *with* $X \subset 1, 2, ..., L$ *is a called a* (δ, ℓ, k) *order parameter for state* $|\psi\rangle$ *if and only if*

1. O_i *transforms under a nontrivial irreducible representation of* G *such that generator* $S^{(k)}$ *is not in the kernel of the representation.*

2. O_i *is supported on* $[i - \ell, i + \ell]$.

3. $|\langle\psi| O_i^\dagger O_j |\psi\rangle \geq \delta$ *for all* $i, j \in X$ *with* $|i - j| \geq 2\ell$.

4. $\|O_i\| \leq 1$.

- **Please note:** This is a generalization of Michael Levin's definition provided in:
M. Levin, *Constraints on order and disorder parameters in quantum spin chains*, arXiv:1903.09028
https://arxiv.org/pdf/1903.09028.pdf
- While Levin's definition was just for $\mathbb{Z}_2$ symmetry, mine is for all finite abelian groups.

Definition of Disorder Parameter

Definition : *A collection of operators $\bigcup_{g \in G-\{1\}} \{O_i^{(g)} : i \in X\}$ with $X \subset 1, 2, ..., L$ is a called a (δ, ℓ) disorder parameter for state $|\psi\rangle$ if and only if*

1. $O_i^{(g)}$ *transforms under some irreducible representation of G (can be trivial or nontrivial).*

2. $O_i^{(g)}$ *is supported on $[i - \ell, i + \ell]$.*

3. $|\langle \psi | O_i^{(g)\dagger} O_j^{(g)} \prod_{p=i+1}^{j} g_p |\psi\rangle \geq \delta$ *for all $i, j \in X$ with $|i - j| \geq 2\ell$.*

4. $\left\| O_i^{(g)} \right\| \leq 1.$

- **Please note:** This is a generalization of Michael Levin's definition provided in:
M. Levin, *Constraints on order and disorder parameters in quantum spin chains*, arXiv:1903.09028
- While Levin's definition was just for $\mathbb{Z}_2$ symmetry, mine is for all finite abelian groups.

One of the main results by Michael Levin that I generalized to all finite Abelian groups

Theorem : Let $|\psi\rangle$ be an eigenstate of the Abelian symmetry group G of order n. For any given $\delta \in [0,1]$ and every pair of disjoint intervals I_1 and I_2, the state $|\psi\rangle$ is either $\delta/(n - |G/\langle S^{(j)}\rangle|)$ weakly-ordered on I_1, I_2 with respect to generator $S^{(j)}$ or $(1-\delta)/n$ weakly-disordered on the complementary intervals J_1, J_2.

- Note: This result is for the finite circular spin-chains.

- Using general results like the one above, I aim to prove some theorems regarding how the strengths δ of order and disorder parameters are bounded from below for spin chains with any Abelian symmetries.

- We are seeking to prove results that are analogous to those in Michael Levin's paper:
M. Levin, *Constraints on order and disorder parameters in quantum spin chains*, arXiv:1903.09028

- Most important goal: To relate Levin's constraints on order and disorder parameters to the classification of phases of the spin chain using group cohomologies $H^2(G, U(1))$

Acknowledgements

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- National Science Foundation and the University of Washington Physics Department

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1. A. Altland and B. Simons, Condensed Matter Field Theory (Cambridge University Press, Cambridge, 2010).
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- https://quantum-solutions.com/wp-content/uploads/2018/11/Webp.net-resizeimage.png

- https://en.wikipedia.org/wiki/Pauli_matrices

Spin Chains: a Physical Origin

- Consider two valence electrons inhabiting neighboring quantum wells. Due to tunneling, their wavefunctions overlap. The overall wavefunction for the two electrons must be anti-symmetric.

(a)

(b)

- The two electrons electrostatically repel, so it would be more energetically favorable if the *spatial part* of their overall wavefunction was anti-symmetric.

(c)

- Such spatial anti-symmetry enforces a symmetric alignment of the electron spins, so **there is an energetic benefit for nearest-neighbor spins to align.**
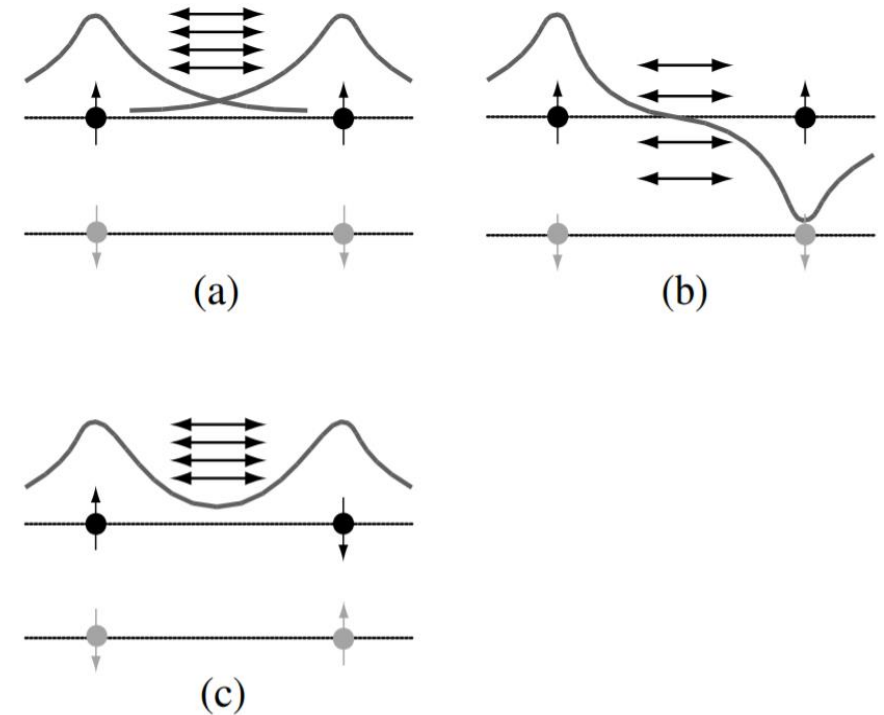
General Topological Picture of Phases

A real vector space formed by Hermitian operators acting on some Hilbert space. For our purposes, the Hilbert space is vector space of all possible states in some quantum system.

A subspace formed by Hermitian operators that can be physically realized as Hamiltonians describing localized interactions.

Disconnected subset consisting of local Hamiltonians that are gapped.
Gapped means there is a nonzero difference between energies of the ground state and first excited state **and** the ground state is nondegenerate.

Suppose we wanted to consider systems with a certain kind of symmetry

A real vector space formed by Hermitian operators.

Subspace of Hamiltonians that describe localized interactions.

Hamiltonians obeying the symmetry