General Constraints on Order and Disorder Parameters in Quantum Spin Chains with Finite Abelian Symmetries

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Abstract

In a paper called "Constraints on order and disorder parameters in quantum spin chains," Michael Levin derived general constraints on order and disorder parameters in Ising symmetric spin chains. Levin's main result in his paper was a theorem showing that in a circular spin chain, any Hamiltonian that has a non-degenerate ground state and is gapped, translationally invariant, and Ising symmetric must at least have either a nonzero order parameter or a nonzero disorder parameter. In the process of proving the theorem, he proved a lemma that made a general statement about correlation and symmetry defect properties of any state in a circular spin chain. These properties namely had to do with notions of order and disorder that are weaker than long-range order and disorder. In this report, we prove an extension of the lemma to all finite Abelian symmetries. Based on this generalization, we discuss some possible implications regarding how Levin's theorem could be generalized to arbitrary finite Abelian symmetries as well.

1 Introduction

One of the objectives of condensed matter physics is to study emergent phenomena in quantum many-body systems. Often, systems differing in microscopic detail, such as different interaction potentials and chemical makeup, display some common collective behavior. For this reason, condensed matter physicists are interested in the unifying principles of these common emergent behaviors rather than in peculiar details of very specific systems [1].

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Some examples of condensed-matter systems with interesting emergent phenomena include superconductors, superfluids, ferromagnets, anti-ferromagnets, and graphene.

This paper explores some theoretical properties of quantum spin chains. Though spin chains are toy models, they are inspired by real physical examples of 1D condensed matter systems, such as thin wires, optical lattices, carbon nanotubes, and one-dimensional arrays of quantum dots, vortices, and other confined quantum systems. Another important physical origin of the spin chains is the positive ion lattice of some typical electronic material like a metal or an insulating solid [1].

In this report, we provide an extension of the theoretical results in Ref. [2], a recent paper by Michael Levin. While Levin derived general constraints on order and disorder parameters in Ising symmetric spin chains, we discuss how some of these constraints could be extended to spin chains with arbitrary finite Abelian symmetries. In section 2, we rigorously define the spin chain Hilbert space and indicate the properties of Hamiltonians we are interested in. In section 3, we generalize Ref. [2]'s definitions of weak order and weak disorder to any finite Abelian symmetry. In section 4, we exploit these definitions to formulate and prove a finite-Abelian extension of Lemma 1 from Ref. [2]. This lemma was originally a statement that if one restricts an Ising symmetric state in the circular spin chain to some arbitrary pair of disjoint intervals, the state must either be weakly ordered or weakly disordered. Here, we generalize this lemma by proving an analogous claim for any state with a finite Abelian symmetry. Along with two other lemmas, Levin used Lemma 1 to prove Theorems 1 and 2, the main results of his paper. Theorem 1 claimed that in a circular spin chain, any Hamiltonian that has a non-degenerate ground state and is gapped, translationally invariant, and Ising symmetric must at least have either a nonzero order parameter or a nonzero disorder parameter. Theorem 2 is a weaker statement but for Hamiltonians that are not required to be translationally invariant. Based on our generalization of Lemma 1, we suggest how Theorems 1 and 2 from Ref. [2] could be generalized in section 5.

2 Setup of the Problem

We consider finite one-dimensional spin chains consisting of L spin sites forming a circular lattice. The spin at each site forms a d-dimensional vector space, where d is a finite nonzero integer. In other words, our system's state space is a d^L dimensional Hilbert space $\mathcal{H}$ that is a tensor product of all the d-dimensional vector spaces of the individual lattice sites. We label the L sites by 1, 2, 3,..., L to make $\{1, 2, ..., L\}$ denote the set of all lattice sites (see Figure 1(a) in Ref [2]). Our Hamiltonians of interest correspond to nearest neighbor interactions that have a bounded operator supnorm:

$$H = \sum_{i=1}^{L} H_{i,i+1}, \tag{1}$$

such that

$$\text{supp}(H_{i,i+1}) \subset \{i, i+1\} \quad \text{and} \quad \|H_{i,i+1}\| \leq 1.$$

We assume that H is symmetric with respect to some Abelian group G of finite order n. That is, there exists a faithful unitary operator representation R of G on $\mathcal{H}$ such that H

commutes with every element of the image $R(G)$. For convenience, we will treat the image $R(G)$ as the group G itself, so $\forall g \in G$, $R(g)$ will be denoted by g whenever there is no danger of ambiguity.

Just as in Ref [2], we further assume that the Abelian symmetry of the Hamiltonian is *on-site*. To clarify, the symmetry G is on-site if and only if the representation R mentioned above is such that $\forall g \in G$, the operator $R(g)$, which we simply denote by g, is of the form

$$g = \prod_{i=1}^{L} g_i, \tag{2}$$

where supp(g_i)=$\{i\}$.

3 Generalized Definitions of Weak Order and Disorder

Let us begin by introducing some notation: for each subset $X \subset \{1, 2, ..., L\}$, let G_X be the action of group G restricted to the linear subspace of $\mathcal{H}$ corresponding to X, and $\forall g \in G$, let g_X denote the element of G_X corresponding to g, i.e.

$$g_X = \prod_{i \in X} g_i.$$

Definition 1 *Let I_1 and I_2 be disjoint intervals. A state $|\psi\rangle$ is δ weakly-ordered on I_1, I_2 with respect to $g \in G - \{1\}$ if and only if there exists an operator B such that*

1. *B is supported on $I_1 \cup I_2$.*
2. *B transforms under the trivial representation of the group G, i.e. $\forall h \in G$, $h^\dagger B h = B$.*
3. *B transforms under a nontrivial irreducible representation ρ of G_{I_2}, such that g_{I_2} is not in the kernel of ρ. In other words, $\rho(g_{I_2}) \neq 1$ and $\forall h \in G$,*

$$h_{I_2}^\dagger B h_{I_2} = \rho(h_{I_2}) B.$$

4. *$|\langle\psi| B |\psi\rangle| \geq \delta$.*
5. *$\|B\| \leq 1$.*

A direct consequence of conditions 2 and 3 is that $\forall h \in G$, $h_{I_1}^\dagger B h_{I_1} = \overline{\rho(h_{I_2})} B$, so another way to state these two conditions is that B transforms under some nontrivial irreducible representations of G_{I_1} and G_{I_2} that are oppositely charged (i.e. that are complex conjugates of each other), such that B also transforms non-trivially under the respective cyclic subgroups $\langle g_{I_1}\rangle$ and $\langle g_{I_2}\rangle$. Let us further note that given that G is Abelian, all of its irreducible representations are one-dimensional and are thereby equivalent to their own characters. Therefore, any $\rho(h_{I_2})$ must be an $O(h)^{th}$ root of unity, where $O(h)$ is the order of group element h. Regarding element g, $\rho(g_{I_2})$ must be a *nontrivial* $O(g)^{th}$ root of unity, so $\exists k \in \{1, 2, ..., O(g) - 1\}$ s.t.

$$g_{I_2}^\dagger B g_{I_2} = e^{i\frac{2\pi k}{O(g)}} B. \tag{3}$$

Definition 2 *Let I_1 and I_2 be disjoint intervals. A state $|\psi\rangle$ is δ weakly-disordered on I_1, I_2 if and only if $\forall g \in G - \{1\}$, there exists an operator $C^{(g)}$ such that*

1. *$C^{(g)}$ is supported on $I_1 \cup I_2$.*
2. *$C^{(g)}$ transforms under the trivial representation of the group G, i.e. $\forall h \in G$,*

$$h^\dagger C^{(g)} h = C^{(g)}.$$

3. *$C^{(g)}$ transforms under some irreducible representation ρ' (can be trivial or nontrivial) of the group G_{I_2}, i.e. $\forall h \in G$,*

$$h_{I_2}^\dagger C^{(g)} h_{I_2} = \rho'(h_{I_2}) C^{(g)}.$$

4. *$|\langle\psi| C^{(g)} g_J |\psi\rangle| \geq \delta$, where J is an interval between I_1 and I_2.*
5. *$\left\|C^{(g)}\right\| \leq 1$.*

Again, a direct consequence of conditions 2 and 3 is that $\forall h \in G,\ h_{I_1}^\dagger C^{(g)} h_{I_1} = \overline{\rho'(h_{I_2})} C^{(g)}$. Furthermore, B from the definition of weak order and $C^{(g)}$ from the definition of weak disorder both satisfy the property that $\forall h \in G$, $\exists k \in \{0, 1, 2, ..., O(h) - 1\}$ s.t.

$$h_{I_2}^\dagger D h_{I_2} = e^{i\frac{2\pi k}{O(h)}} D \text{ and } h_{I_1}^\dagger D h_{I_1} = e^{-i\frac{2\pi k}{O(h)}} D, \tag{4}$$

where $D = B$ or $C^{(g)}$.

4 Proof of a Generalization of Lemma 1 from Ref. [2]

Theorem 3 (Generalized Lemma 1) *Let $|\psi\rangle$ be an eigenstate of the Abelian symmetry group G of order n. For any given $\delta \in [0, 1]$ and every pair of disjoint intervals I_1 and I_2 separated by a distance of at least one lattice site, the state $|\psi\rangle$ is either $\delta/(n - |G/\langle g\rangle|)$ weakly-ordered on I_1, I_2 with respect to some $g \in G - \{1\}$ or $(1 - \delta)/n$ weakly-disordered on the complementary intervals J_1, J_2.*

Proof. Just like Michael Levin in his proof of Lemma 1, we use the Fuchs-van de Graaf inequality [3]. A consequence of this inequality is that for any two states $|\psi\rangle$ and $|\psi'\rangle$ and any subset $X \subset \{1, 2, ..., L\}$, we have

$$\max_{\text{supp}(A)\subset X} \frac{1}{2} |\langle\psi| A |\psi\rangle - \langle\psi'| A |\psi'\rangle| + \max_{\text{supp}(U)\subset X^c} |\langle\psi| U |\psi'\rangle| \geq 1, \tag{5}$$

where operators A satisfy $\|A\| \leq 1$ and operators U are unitary. Let $|\psi'\rangle = g_{I_2} |\psi\rangle$ and $X = I_1 \cup I_2$. Substituting this into Eq. (5) yields

$$\max_{\text{supp}(A)\subset I_1 \cup I_2} \frac{1}{2} |\langle\psi| (A - g_{I_2}^\dagger A g_{I_2}) |\psi\rangle| + \max_{\text{supp}(U)\subset J_1 \cup J_2} |\langle\psi| U g_{I_2} |\psi\rangle| \geq 1. \tag{6}$$

It is clear from Eq. (6) that for any given $\delta \in [0, 1]$, either

(i) $\exists g \in G - \{1\}$ s.t. the first term in Eq. (6) is greater than or equal to δ.

(ii) $\forall g \in G - \{1\}$, the second term in Eq. (6) is greater than or equal to $1 - \delta$.

We claim that in case (i), $|\psi\rangle$ is $\delta/(n - |G/\langle g\rangle|)$ weakly-ordered on I_1, I_2 with respect to g, while in case (ii), it is $(1-\delta)/n$ weakly-disordered on J_1, J_2.

Case (i): We define $F := A_* - g_{I_2}^\dagger A_* g_{I_2}$, where A_* is a choice of operator A that maximizes the first term in Eq. (6), and let

$$B_\rho := \frac{1}{2n^2} \sum_{a\in G} a^\dagger \left(\sum_{b\in G} \overline{\rho(b_{I_2})} b_{I_2}^\dagger F b_{I_2} \right) a, \tag{7}$$

where ρ is any irreducible representation of G.

Since G is a finite Abelian group, it can can be written as a direct product of finite cyclic groups:

$$G = \langle S^{(1)}\rangle \times \langle S^{(2)}\rangle \times ... \times \langle S^{(m)}\rangle,$$

where $S^{(1)}$, $S^{(2)}$,..., $S^{(m)}$ are a certain choice of generators of G. To simplify notation, let $\omega_u := O(S^{(u)})$. Then, for any irreducible representation ρ, $\rho(S_{I_2}^{(u)})$ must be an ω_u^{th} root of unity, i.e. $\exists k_{\rho u} \in \{0, 1, 2, ...\omega_u - 1\}$ s.t.

$$\rho(S_{I_2}^{(u)}) = \exp\left[i\frac{2\pi k_{\rho u}}{\omega_u}\right].$$

Based on this, we can rewrite Eq. (7) as

$$B_\rho = \frac{1}{2n^2}\sum_{a\in G} a^\dagger \left(\sum_{v_m=0}^{\omega_m-1} e^{-i\frac{2\pi k_{\rho m} v_m}{\omega_m}} S_{I_2}^{(m)^{-v_m}} \right.$$
$$\left.\left(... \left(\sum_{v_2=0}^{\omega_2-1} e^{-i\frac{2\pi k_{\rho 2} v_2}{\omega_2}} S_{I_2}^{(2)^{-v_2}} \left(\sum_{v_1=0}^{\omega_1-1} e^{-i\frac{2\pi k_{\rho 1} v_1}{\omega_1}} S_{I_2}^{(1)^{-v_1}} F S_{I_2}^{(1)^{v_1}} \right) S_{I_2}^{(2)^{v_2}} \right) ... \right) S_{I_2}^{(m)^{v_m}} \right) a, \tag{8}$$

Since for any integer $\omega > 1$, the sum of all ω^{th} roots of unity is always zero, we can see from Eq. (8) that the sum of B_ρ over all irreducible representations of G is

$$\sum_\rho B_\rho = \frac{1}{2n}\sum_{a\in G} a^\dagger F a, \tag{9}$$

Notice that the projection of F onto the space of operators that transforms under the trivial representation of $\langle g_{I_2}\rangle$ equals to zero, that is,

$$\sum_{v=0}^{O(g)-1} g_{I_2}^{\dagger\ v} F g_{I_2}{}^v = 0, \tag{10}$$

because inserting the expression for F in terms of A_* into this sum results in a telescoping sum. An immediate consequence of Eq. (10) is that $\forall\rho$ s.t. $\rho(g_{I_2}) = 1$, we have that $B_\rho = 0$. Thus, if we let N be the set of all irreducible representations of G_{I_2} s.t. $\rho(g_{I_2}) \neq 1$, then

$$\sum_\rho B_\rho = \sum_{\rho\in N} B_\rho. \tag{11}$$

Combining Eqs. (9) and (11) yields

$$\sum_{\rho\in N} B_\rho = \frac{1}{2n}\sum_{a\in G} a^\dagger F a, \tag{12}$$

Since $\forall a \in G$, $|\psi\rangle$ is an eigenstate of a (the eigenvalue of which must be an $O(a)^{th}$ root of unity), it is clear that the right-hand side of Eq. (12) has the same absolute value of the expectation value with respect to $|\psi\rangle$ as $F/2$, so

$$\left|\sum_{\rho\in N} \langle\psi| B_\rho |\psi\rangle\right| = \frac{1}{2}|\langle\psi| F |\psi\rangle| = \max_{\text{supp}(A)\subset I_1\cup I_2} \frac{1}{2}|\langle\psi| (A - g_{I_2}^\dagger A g_{I_2}) |\psi\rangle| \geq \delta. \tag{13}$$

Hence, $\exists\tilde{\rho} \in N$ s.t $|\langle\psi| B_{\tilde{\rho}} |\psi\rangle| \geq \delta/\text{card}(N)$. It is easy to check that $B_{\tilde{\rho}}$ satisfies conditions 1, 2, 3, and 5 met by operator B in Definition 1 of weak order. To complete the proof of our claim for case (i), all we have left is to find $\text{card}(N)$.

To construct an irreducible representation of G_{I_2}, one must map g_{I_2} to one of the $O(g)^{th}$ roots of unity. Regardless of which root g_{I_2} gets mapped, the number of possible images that other elements of G_{I_2} can have under an irreducible representation is the same. This implies that the number of irreducible representations in which g gets mapped to 1 must be $n/O(g)$, which equals to the order of the quotient group $G/\langle g\rangle$. Thus, $\text{card}(N) = n - |G/\langle g\rangle|$, so $|\langle\psi| B_{\tilde{\rho}} |\psi\rangle| \geq \delta/(n - |G/\langle g\rangle|)$.

In all, we found an operator $B_{\tilde{\rho}}$ that satisfies the conditions for B in Definition 1 with δ replaced by $\delta/(n - |G/\langle g\rangle|)$, making $|\psi\rangle$ qualify as a $\delta/(n - |G/\langle g\rangle|)$ weakly-ordered state on I_1, I_2.

Case (ii): For all $g \in G - \{1\}$ and all irreducible representations ρ of G_{J_2}, define

$$C_\rho^{(g)} := \frac{1}{n^2}\sum_{a\in G} a^\dagger \left(\sum_{b\in G} \overline{\rho(b_{J_2})} b_{J_2}^\dagger U_*^{(g)} b_{J_2}\right) a,$$

where $U_*^{(g)}$ is a choice of unitary operator U that maximizes the second term in Eq. (6). By the same reasoning as for Eq. (9), we can obtain

$$\sum_{\rho} C_\rho^{(g)} = \frac{1}{n}\sum_{a\in G} a^\dagger U_*^{(g)} a, \tag{14}$$

Again, we use the fact that $|\psi\rangle$ is an eigenstate of G to obtain from Eq. (14) that

$$\left|\sum_{\rho} \langle\psi| C_\rho^{(g)} g_{I_2} |\psi\rangle\right| = |\langle\psi| U_*^{(g)} g_{I_2} |\psi\rangle| = \max_{\text{supp}(U)\subset J_1\cup J_2} |\langle\psi| U g_{I_2} |\psi\rangle| \geq 1 - \delta. \tag{15}$$

Therefore, there exists an irreducible representation $\tilde{\rho}$ s.t $|\langle\psi| C_{\tilde{\rho}}^{(g)} g_{I_2} |\psi\rangle| \geq (1-\delta)/n$. In addition to this property, one can check that $C_{\tilde{\rho}}^{(g)}$ satisfies the other defining properties that make $|\psi\rangle$ qualify as a $(1-\delta)/n$ weakly-disordered state on J_1, J_2. This completes the proof of our theorem.

5 Discussion of Future Work

To precisely figure out the finite-Abelian extensions of Theorems 1 and 2 in Ref. [2], we are analyzing how Levin's Lemmas 2 and 3 generalize to any finite Abelian symmetry. Currently, we suspect that Lemma 3 could be easy to generalize. At least, we can see how a version of a famous result by Hastings [4] that the proof of Lemma 3 relies on can easily be generalized by replacing the projection of states onto the even subspace of $\mathcal{H}$ by a projection onto the trivial subspace of $\mathcal{H}$ (i.e. eigenspace with eigenvalue 1) of an Abelian symmetry. Regarding Lemma 2, we still need some time to draw some conclusions regarding its generalization.

Theorems 1 and 2 rely on rigorous definitions of order and disorder parameters that are specific to Ising symmetric spin chains. Levin provides these definitions in Ref [2]. Based on how weak order and disorder have been generalized, we find it most intuitive for the finite-Abelian extensions of order and disorder parameters to be as follows:

Definition 4 *Let $g \in G - \{1\}$. A collection of operators $\{O_i : i \in X\}$ with $X \subset \{1, 2, ..., L\}$ is called a (δ, ℓ, g) order parameter for a state $|\psi\rangle$ if and only if*

1. *O_i transforms under a nontrivial irreducible representation of G such that g is not in the kernel of the representation.*
2. *O_i is supported on $[i - \ell, i + \ell]$.*
3. *$|\langle\psi| O_i^\dagger O_j |\psi\rangle| \geq \delta$ for all $i, j \in X$ with $|i - j| \geq 2\ell$.*
4. *$\|O_i\| \leq 1$.*

The set X is called the domain of definition of the order parameter.

Definition 5 *A collection of operators $\bigcup_{g \in G-\{1\}} \{O_i^{(g)} : i \in X\}$ with $X \subset \{1, 2, ..., L\}$ is called a (δ, ℓ) disorder parameter for a state $|\psi\rangle$ if and only if*

1. *$O_i^{(g)}$ transforms under some irreducible representation of G (can be trivial or nontrivial).*
2. *$O_i^{(g)}$ is supported on $[i - \ell, i + \ell]$.*
3. *$|\langle\psi| O_i^{(g)\dagger} O_j^{(g)} \prod_{p=i+1}^{j} g_p |\psi\rangle| \geq \delta$ for all $i, j \in X$ with $|i - j| \geq 2\ell$.*
4. *$\left\| O_i^{(g)} \right\| \leq 1$.*

The set X is called the domain of definition of the disorder parameter.

Relying on these definitions, we hope to generalize Theorem 1 into a claim of the following form: in a circular spin chain, for any gapped, translationally invariant, and G-symmetric Hamiltonian that has a non-degenerate lowest energy state $|\Omega\rangle$ in the trivial subspace, the state $|\Omega\rangle$ must either have a (δ, ℓ, g) order parameter for some $g \in G - \{1\}$ or a (δ, ℓ) disorder parameter defined over the whole spin chain (i.e. $X = \{1, 2, ..., L\}$) with δ equal to some fixed positive number less than 1 and ℓ somehow bounded from above. Additionally, we hope to achieve a similar finite-Abelian extension of Theorem 2.

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