Light Front Wavefunction in Quasi-Elastic Electron Deuteron Scattering

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Abstract

Recent interest in short range correlations of bound nucleons gave rise to our interest in the light front treatement of quasi-elastic electron deuteron scattering. The end goal of this research project is to examine the difference in cross sections between tensor polarized and unpolarized deuteron. This paper explores the simple case of scalar electron-deuteron scattering to determine whether the kinematic approximation given in the section II is appropriate for handling this system or if higher order corrections must be made. A conclusion is not met here but discussion of further research to continue this investigation is provided.

I. Introduction

Deuteron, being the only bound state of two nucleons, offers both experimentalists and theorists a simple system for exploring nucleon-nucleon interactions. Deuteron has the the especially useful property of existing in a superposition of the $L = 0$ and $L = 2$ states. While the tensor nuclear force commutes with the total angular momentum J, it does not commute with either the orbital or spin angular momentum of the composite system. This offers experimentalists a chance to use magnetic field interactions with deuteron to alter the amplitude of angular of the deuteron in the $L = 0$ and $L = 2$ states. As the $L = 2$ state has constituents which are more localized near the center of mass, this compression, along with examining the difference in quasi-elastic cross section, offers researchers a chance to probe these short range correlations. A letter of intent was recently submitted and approved for this very purpose, JLab LOI12-14-002.

In this paper, we attempt to model the deuteron-electron system with scalar particles in an effort to determine the error of a particularly useful approximation, given in [1], can be used to determine the scattering cross of these quasi-elastic scattering processes. Error is plotted and discussion of results and further research is given. Those unfamiliar with the light front techniques should refer to [1].

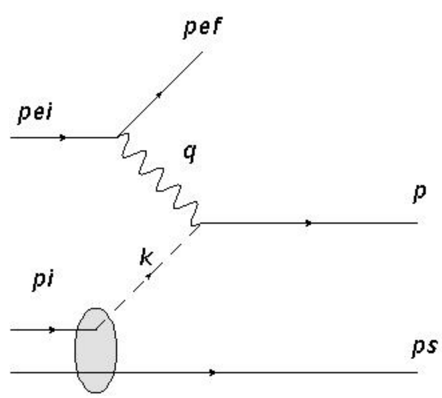

Figure 1: *Feynman Diagram of Scattering Process. Denote the mass of the deuteron M and the mass of a nucleon m. The four momentum of the virtual photon is* $q = (\nu, 0, 0, -\sqrt{Q^2 + \nu^2})$.

II. Wavefunction

I. Light Front Wave Function

In his review of light front quantization [2], Miller makes the following simplification,

$$\sum_f \int \frac{d^3 p_s}{E_s} d^4 p \delta(p^2 - m^2) \delta^4(p_i + q - p - p_s) |\langle f(p)|J(q)|i\rangle|^2 \tag{1}$$

$$= \sum_f \int \frac{d^3 p_s}{E_s} \delta((p_i + q - p_s)^2 - m^2) |\langle f(p)|J(q)|i\rangle|^2 \tag{2}$$

$$(p_i + q - p_s)^2 - m^2 \approx -Q^2 + q^- k^+. \tag{3}$$

$$= \sum_f \int \frac{d^2 p_s dp_s^+}{p^+} \delta(-Q^2 + q^- k^+) |\langle f(p)|J(q)|i\rangle|^2 \tag{4}$$

$$p_s^+ = k^+ \tag{5}$$

$$= \int \frac{d^2 k dk^+}{k^+} \delta(-Q^2 + q^- k^+) \sum_f |\langle f(p)|J(q)|i\rangle|^2 \tag{6}$$

$$= \int \frac{d^2 k dk^+}{k^+} \delta(-Q^2 + q^- k^+) \sum_f \langle i|J(q)|f(p)\rangle \langle f(p)|J(q)|i\rangle \tag{7}$$

$$= \int \frac{d^2 k dk^+}{k^+} \delta(-Q^2 + q^- k^+) \langle i|J^\dagger(q) J(q)|i\rangle \tag{8}$$

$$J(q) = b_{k^+=mx}(\vec{k}) \tag{9}$$

$$= \int \frac{d^2 k dk^+}{k^+} \delta(-Q^2 + q^- k^+) \langle i|b^\dagger_{k^+=mx}(\vec{k}) b_{k^+=mx}(\vec{k})|i\rangle \tag{10}$$

$$= \int \langle i|b^\dagger_{k^+=mx}(k) b_{k^+=mx}(k)|i\rangle d^2 k_\perp \tag{11}$$

This approximation is valid where the largest two contributions to the energy of the system are the rest masses and the energy of the incoming virtual photon (See appendix for derivation and discussion of light front approximation). Note that in equation (5), we have chosen the reference from to be that of the bound state.

The error of this approximation in the limits where: the binding energy's contribution to the total energy of the system is negligible: the magnitude of the momenta of the constituent particles in the rest frame of the deuteron is also negligible in comparison to the energy of the system: the Bjorken approximation is appropriate (see appendix), is given below in terms of the kinematic values of the virtual photon as well as the Bjorken variable $x = \frac{Q^2}{2m\nu}$. The percent error is plotted in figure 2.

$$Error(Q^2, \nu)_{lim} = \frac{MQ^2}{4\nu(Q^2 - M\nu)}. \tag{12}$$

$$Error(x, \nu)_{lim} = \frac{Mx}{4\nu(x-1)}. \tag{13}$$

This simplification grants two insights. First, that for scalar particle interactions, the mod-squared of the composite, momentum space, light front wave function of this the total system can be approximated with the mod-squared of the transfer matrix elements. Second, that we can simplify the calculations of the cross section as below.

$$\sigma = \frac{1}{32\pi^2 M |\vec{p_{ei}}|} \int \langle i(k_\perp)|b^\dagger_{k^+=mx}(k) b_{k^+=mx}(k)|i(k_\perp)\rangle d^2 k_\perp \tag{14}$$

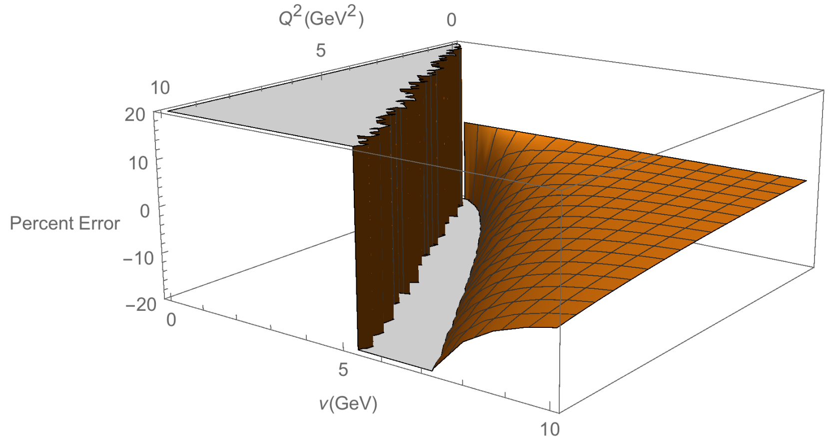

Figure 2: *Plot of the percent error of light front approximation.*

$$\sigma = \frac{1}{16\pi^2 M|\vec{p}_{ei}|}\int \langle i(k_\perp)|i(k_\perp)\rangle d^2k_\perp \tag{15}$$

Using the Feynman Rules for the scalar particle model given in [1], the transfer matrix elements given below in terms of the deuteron's rest frame kinematics.

$$\mathcal{M} = \frac{g}{(p_{ei}-p_{ef})^2}\frac{g}{M(M-2E_s)} \tag{16}$$

The photon propogator represents the mod-squared of the momenum space, light front wavefunction of the electron, while the nucleon propogator represents the mod-squared of the momentum space light front wave function of the deuteron in its rest frame, defined up to some complex phase. In terms of the momentum of the proton in the deuteron's rest frame, the momentum space deuteron wave function is given by

$$\psi_{LF}(|\vec{p}_s|) = \frac{g}{M(M-2\sqrt{\vec{k}^2+m^2})}. \tag{17}$$

For later convenience in calculating the cross section, the momentum of this expression can be transformed to the light front variables as follows (See appendix for derivation of transformation).

$$f(\vec{k}^2+m^2) = f(\frac{\vec{k}_\perp^2+m^2}{4x(1-x)}). \tag{18}$$

In light front variables we then have

$$\psi_{LF}(|\vec{k}|) = \frac{g}{M(M-\sqrt{\frac{\vec{k}_\perp^2+m^2}{x(1-x)}})} \tag{19}$$

II. Bethe-Salpeter Wave Function

The Lippman-Schwinger equation for a two-particle transition matrix is [3]

$$T = K + KGT, \tag{20}$$

where K is the irreducible two-particle scattering kernal and G is the completely disconnected two-particle propogator. A pole in the T matrix corresponds to a two-particle bound state. Investigation of the pole's residue gives an equation for the bound state vertex Γ

$$\Gamma = KG\Gamma. \tag{21}$$

We can identify the wavefunction Ψ as $G\Gamma$, we then recover the following equation

$$\Phi = GK\Phi. \tag{22}$$

Using the notation of [4], we find

$$\Psi(k, p_i) = G(k + \frac{p_i}{2}, k - \frac{p_i}{2}) \int \frac{d^4k'}{(2\pi)^4} iK(k, k', p_i)\Psi(k', p_i). \tag{23}$$

The bound-state amplitude is then given by

$$\Psi(k_1, p_i) = \Phi(k_1 - \frac{p_i}{2}, p_i). \tag{24}$$

Using this formalism, Miller et. al. showed that the light front wave function for a two-particle bound state is given by

$$\psi_{BS}(k, p_i) = \frac{k^+(p_i^+ - k^+)}{\pi p_i^+} \int_{-\infty}^{\infty} dk^- \Psi(k, p_i). \tag{25}$$

Miller shows that for the toy model

$$\Psi(k, p_i) = \frac{-ig}{(k^2 - m^2 + i\epsilon)((p_i - k)^2 - m^2 + i\epsilon)} \tag{26}$$

Rewriting this in light front coordinates,

$$\Psi(k, p_i) = \frac{-ig}{k^+(p_i - k)^+} \frac{1}{k^- - \frac{\vec{k}_\perp^2 + m^2}{k^+} + \frac{i\epsilon}{k^+}} \frac{1}{p_i^- - k^- - \frac{(\vec{p}_i - \vec{k}_\perp)^2 + m^2}{p_i^+ - k^+} + \frac{i\epsilon}{p_i^+ - k^+}} \tag{27}$$

Using the Residue Theorem and taking the contour integration in the upper half plane, the deuteron rest frame wave function is found to be the following.

$$\psi(k_\perp) = -g \frac{\Theta(x(1-x))}{M^2 - \frac{|\vec{k}_\perp|^2 + m^2}{x(1-x)}} \tag{28}$$

III. Cross Section

I. Exact Cross Section

An expression for scalar particle interaction cross section is derived here. As seen in section II, the expression for the differential cross section of a scalar particle interaction is given by the following.

$$\sigma = \frac{1}{32\pi^2 M|\vec{p}_{ei}|} \int \frac{d^3p_s}{E_s} d^4p_s \delta^4(p_i + q - p_s - p)|\mathcal{M}|^2 \tag{29}$$

$$= \frac{1}{64\pi^2 M|\vec{p}_{ei}|} \int \frac{d^3p_s}{E_s} \frac{d^3p}{E} \delta^4(p_i + q - p_s - p)|\mathcal{M}|^2 \tag{30}$$

$$= \frac{1}{64\pi^2 M|\vec{p}_{ei}|} \int \frac{d^3p_s}{E_s(p_s)E'(p_s)} \delta^4(M + \nu - E_s(p_s) - E'(p_s))|\mathcal{M}(p_s)|^2. \tag{31}$$

$$E'(p_s) = \sqrt{(\vec{q} - \vec{p}_s)^2 + m^2} \tag{32}$$

For simplification, denote the argument of the delta function $f(p_s)$.

$$f(p_s) = M + \nu - \sqrt{\vec{p_s}^2 + m^2} - \sqrt{(\vec{q} - \vec{p_s})^2 + m^2} \tag{33}$$

Due to the quadratic nature of this function in p_s, there are two values of the momentum for which the argument is zero. Denote these momenta p_1 and p_2. The delta function must then be redefined as follows.

$$\delta(f(p_s)) = \frac{\delta(p_s - p_1)}{|f'(p_1)|} + \frac{\delta(p_s - p_2)}{|f'(p_2)|} \tag{34}$$

Integrating over the spectator momentum, this expression simplifies as follows

$$\sigma = \frac{g^4}{64\pi^2 Q^2 M^3 |\vec{p_{ei}}|} \sum_{j=1}^{2} \frac{\Theta(p_j)}{E_s(p_j)E'(p_j)} \frac{p_j^2}{(M - 2E_s(p_j))^2} \frac{1}{|f'(p_j)|}. \tag{35}$$

By numerically evaluating the expressions for the zeros for all values of positive Q^2, ν and $\theta \epsilon (0, \pi)$, we find that one zero is negative in this region.

$$\sigma = \frac{g^4}{32\pi Q^4 M^3 |\vec{p_{ei}}|} \int \frac{\Theta(p_1)}{E_s(p_1)E'(p_1)} \frac{p_1^2}{(M - 2E_s(p_1))^2} \frac{d\theta}{|f'(p_1)|} \tag{36}$$

II. Matrix Element and Bethe-Salpeter Cross Section

As Miller showed in his simplification, the cross section in the kinematic limit given can be calculated as follows.

$$d\sigma = \frac{1}{16\pi^2 M^3 |\vec{p_{ei}}|} \int \langle i|i\rangle d^2k_\perp \tag{37}$$

Here the cross sections for the matrix element determined wave function as well as the Bethe-Salpeter wave function are determined.

Matrix element wave function

$$d\sigma = \frac{g^4}{16\pi^2 M^3 |\vec{p_{ei}}| Q^4} \int \frac{1}{(M - \sqrt{\frac{\vec{k}_\perp^2 + m^2}{x(1-x)}})^2} d^2k_\perp \tag{38}$$

$$\sigma = \frac{g^4}{8\pi M^3 |\vec{p_{ei}}| Q^4} \int \frac{k_\perp}{(M - \sqrt{\frac{\vec{k}_\perp^2 + m^2}{x(1-x)}})^2} dk_\perp \tag{39}$$

The integration over the momentum diverges but we can regulate it with a cutoff momentum λ. It's believed that this cutoff is necessary due to the neglect of the form factors of the calculation.

Bethe-Salpeter Wavefunction

$$d\sigma = \frac{1}{18\pi^2 M |\vec{p_{ei}}|} \int |\frac{g}{Q^2} \frac{g}{M^2 - \frac{|\vec{k}_\perp|^2 + m^2}{x(1-x)}}|^2 dk_\perp^2 \tag{40}$$

$$d\sigma = \frac{1}{8\pi M |\vec{p_{ei}}|} \int |\frac{g}{Q^2} \frac{g}{M^2 - \frac{|\vec{k}_\perp|^2 + m^2}{x(1-x)}}|^2 k_\perp dk_\perp \tag{41}$$

$$d\sigma = \frac{g^4}{8\pi M Q^4 |\vec{p_{ei}}|} \int \frac{k_\perp dk_\perp}{(M^2 - \frac{|\vec{k}_\perp|^2 + m^2}{x(1-x)})^2} \tag{42}$$

$$\sigma = \frac{g^4}{8\pi M Q^4 |\vec{p_{ei}}|} \frac{Q^4 \left(Q^2 - 4m\nu\right)^2}{32 m^2 \nu^2 \left(16\nu^2 m^4 + M^2 \left(Q^4 - 4m\nu Q^2\right)\right)} \tag{43}$$

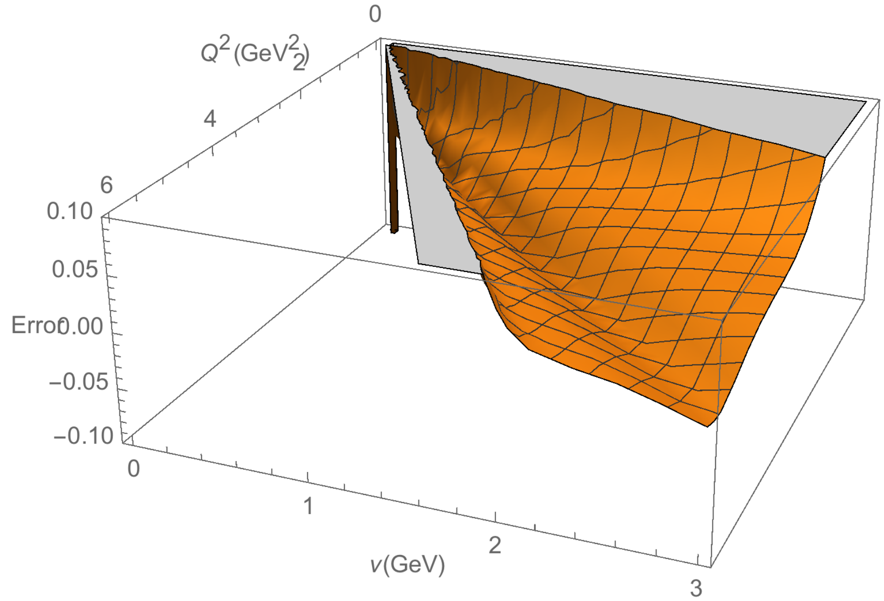

Figure 3: *Light front vs Matrix Element Error*

III. Cross Section Error

Bethe-Salpeter vs Matrix Element Wavefunction
This is the least computationally intensive calculation due to the nature of the theta function in the expression for the exact cross section. While this calculation does not give us information about the error of the light front cross section, it provides interesting information about the difference between the Bethe-Salpeter and Matrix Element methods. Examination of figure three will show that there is strong correlation between these two methods for $x = 1$.

Bethe-Salpeter vs Exact Cross section
As the Bethe-Salpeter is the exact initial wave function of the deuteron, this error calculation tells us the most about the validity of the light front approximation. We notice however that for the majority of the value of Q^2 and ν that the error goes to 1. This is indicating that the Bethe-Salpeter cross section is underestimating the cross section heavily in this region.

Exact vs Matrix Element Cross Section
Unfortunately due to the time constraint and because of the large processing power required to plot this error, the this error could not be determined in time but is currently being processed.

IV. Appendix

I. Equal Time Rest Frame Momentum to Light Front Coordinate Momentum

The purpose of this subsection is to derive a transformation between the equal-time rest frame nucleon momentum and light front momentum.

$$x = \frac{k^+}{P^+} = \frac{E(k) + k_3}{2E(k)} \tag{44}$$

$$= \frac{\sqrt{k_\perp^2 + k_3^2 + m^2} + k_3}{2\sqrt{k_\perp^2 + k_3^2 + m^2}} \tag{45}$$

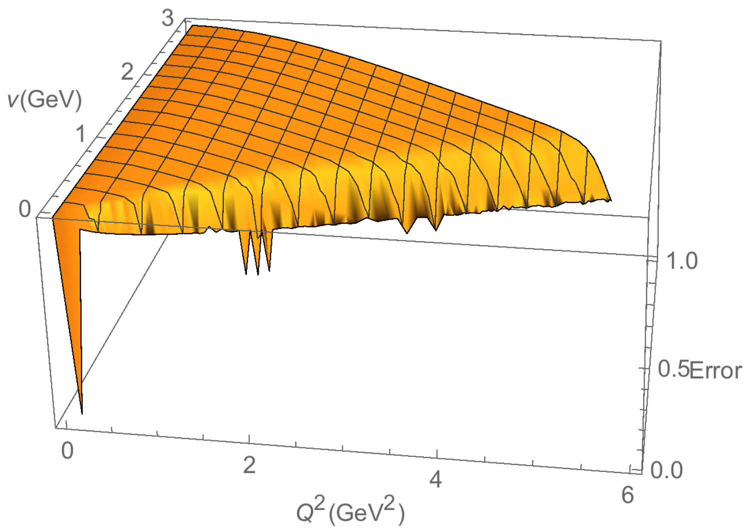

Figure 4: *Exact vs Bethe-Salpeter cross section error*

$$k_3 = (x - \frac{1}{2})\sqrt{\frac{\vec{k}_\perp^2 + m^2}{x(1-x)}} \tag{46}$$

$$\vec{k}^2 + m^2 = \frac{\vec{k}_\perp^2 + m^2}{4x(1-x)} \tag{47}$$

Note that there is a subtlety in the x chosen. We have chosen $x = \frac{k^+}{P^+}$ however, for deep inelastic scattering processes x is typically chosen so that $x = \frac{Q^2}{2m\nu}$. If we plug in $k^+ = mx$, the constraint given in equation 11, we find that these variables differ by a value of two.

II. Kinematic Approximation

As discussed in section II, Miller simplified the

$$\sum_f \int \frac{d^3 p_s}{E_s} d^4 p \delta(p^2 - m^2) \delta^4(p_i + q - p - p_s) |\langle f(p)|J(q)|i\rangle|^2 \tag{48}$$

$$\approx \int \langle i|b^\dagger_{k^+=mx}(k) b_{k^+=mx}(k)|i\rangle d^2 k_\perp \tag{49}$$

in the kinematic limit

$$(p_i + q - p_s)^2 - m^2 \approx -Q^2 + q^- k^+. \tag{50}$$

This section discusses the physical interpretation of this approximation by examining the neglected and providing an approximation scheme for which this error is small. An expression for the error of this approximation is also derived.

$$(p_i + q - p_s)^2 - m^2 = -2E_f \nu - 2E_f M + M^2 + 2\nu M - 2p_{fz}\sqrt{\nu^2 + Q^2} - Q^2 \tag{51}$$

$$-Q^2 + q^- k^+ = -\left(\nu + \sqrt{\nu^2 + Q^2}\right)(Ef - M + p_{fz}) - Q^2 \tag{52}$$

In the approximation where the binding energy and spectator momentum is small in comparison the the energy of the system, these expressions simplify as follows

$$(p_i + q - p_s)^2 - m^2 \approx -Q^2 + M\nu \tag{53}$$

$$-Q^2 + q^- k^+ \approx \frac{1}{2}\left(M\left(\nu + \sqrt{\nu^2 + Q^2}\right) - 2Q^2\right) \tag{54}$$

and under the Bjorken Limit

$$-Q^2 + q^- k^+ \approx -Q^2 + \frac{M^2}{4} x + M\nu \tag{55}$$

$$-Q^2 + q^- k^+ \approx -Q^2 + M\nu \tag{56}$$

where I have chosen the deep inelastic scattering Bjorken variable $x = \frac{Q^2}{2m\nu} \approx \frac{Q^2}{M\nu}$.

The approximation used by Miller can be seen to be a result of neglecting binding energy, spectator momentum and taking the Bjorken limit. The $\frac{M^2 x}{4}$ term however offers value in the calculation of the error of this approximation as well as offers to be the first higher order correction to the light front approximation used. The error is found using this term as well as equation (54) to be the following.

$$Error(Q^2, \nu)_{lim} = \frac{MQ^2}{4\nu(Q^2 - M\nu)}. \tag{57}$$

$$Error(x, \nu)_{lim} = \frac{Mx}{4\nu(x - 1)}. \tag{58}$$

Higher order expression for this is

$$(p_i + q - p_s)^2 - m^2 \approx -Q^2 + q^- k^+ - M\frac{Q^2}{4\nu}. \tag{59}$$

If we set these expressions equal to zero, we find the new

$$k^+ = \frac{Q^2(4\nu + M)}{2\left(4\nu^2 + Q^2\right)}. \tag{60}$$

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