Effective Theory Interaction for a Quantum Gas in the Unitary Limit

Allison Fink
Advisor: Wick Haxton
Institute for Nuclear Theory REU Program,
The University of Washington
August 26, 2009

Objective

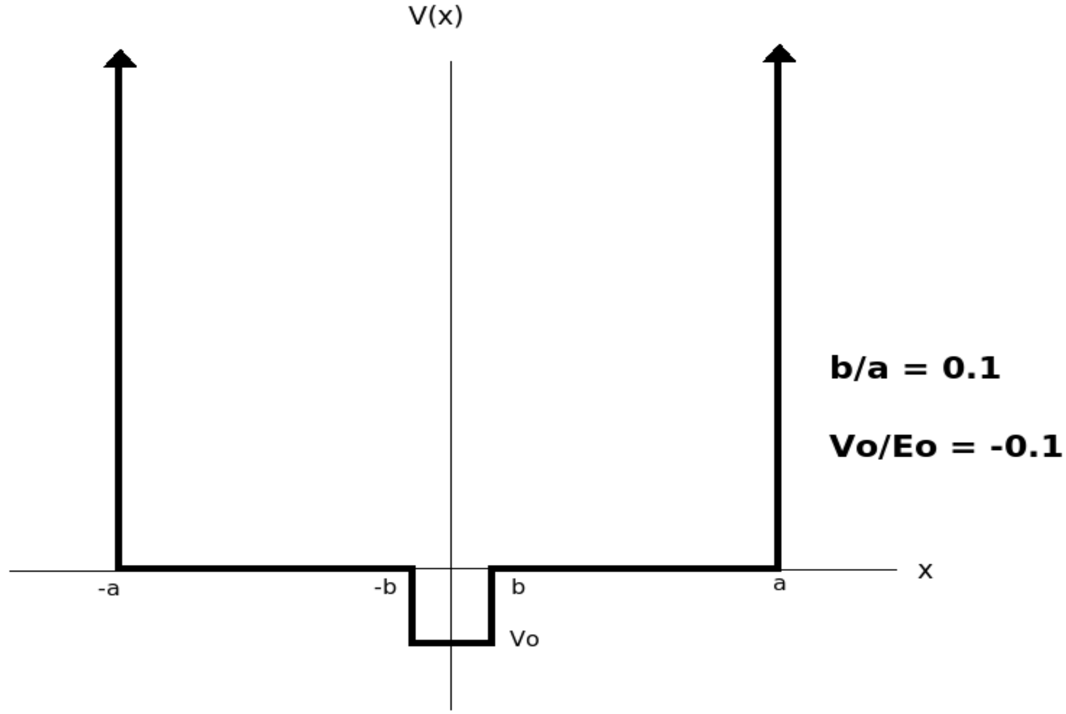
Determine the form of an effective theory interaction for a two particle system of a quantum gas in the unitary limit.

Result will be useful for describing very weakly bound particles in high potentials.

Effective Theories

- Low energy models for potential energy interactions involving high energy physics that we can't see.
- A wave function of low energy cannot be sensitive to a potential acting over a range much smaller than its wavelength, so the detailed structure of the potential can be treated as a perturbation.
- Create an approximation to the potential valid at low energies by fixing the parameters of the perturbing potential to agree exactly with the lowest energy measurements.

Example



$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

$\hat{H}_0$ Infinite square well Hamiltonian: leaves out short range potential

$\hat{H}_1$ Adds in an unknown short range potential V(x) as a correction

Must determine an approximate expression for V(x) consistent with low energy data.

Delta Function Approximations

$$E_N^{(1)} = \int_{-\infty}^{\infty} \mid \Psi_N^{(0)}(x) \mid^2 V(x) dx$$

$\mid \Psi_N^{(0)}(x) \mid^2$ smooth, $V(x)$ sharp: can expand smooth wave function in a Taylor series:

$$E_N^{(1)} = \int_{-\infty}^{\infty} \left(\mid \Psi_N^{(0)}(x) \mid^2 |_{x=0} V(x) + x \frac{d}{dx} \mid \Psi_N^{(0)}(x) \mid^2 |_{x=0} V(x) + \ldots \right) dx$$

Therefore express $V(x)$ as

$$V(x) = \sum_{n=0}^{\infty} a_n \frac{d^n}{dx^n} \delta(x)$$

$$a_0 = \int_{-\infty}^{\infty} V(x) dx \qquad a_1 = \int_{-\infty}^{\infty} x V(x) dx \qquad a_2 = \int_{-\infty}^{\infty} \frac{1}{2} x^2 V(x) dx$$

Simple Application of an Effective Theory for Perturbed Well

Keeping the first two nonzero terms:

$$V(x) = a_0\delta(x) + a_2 \frac{d^2}{dx^2}\delta(r)$$

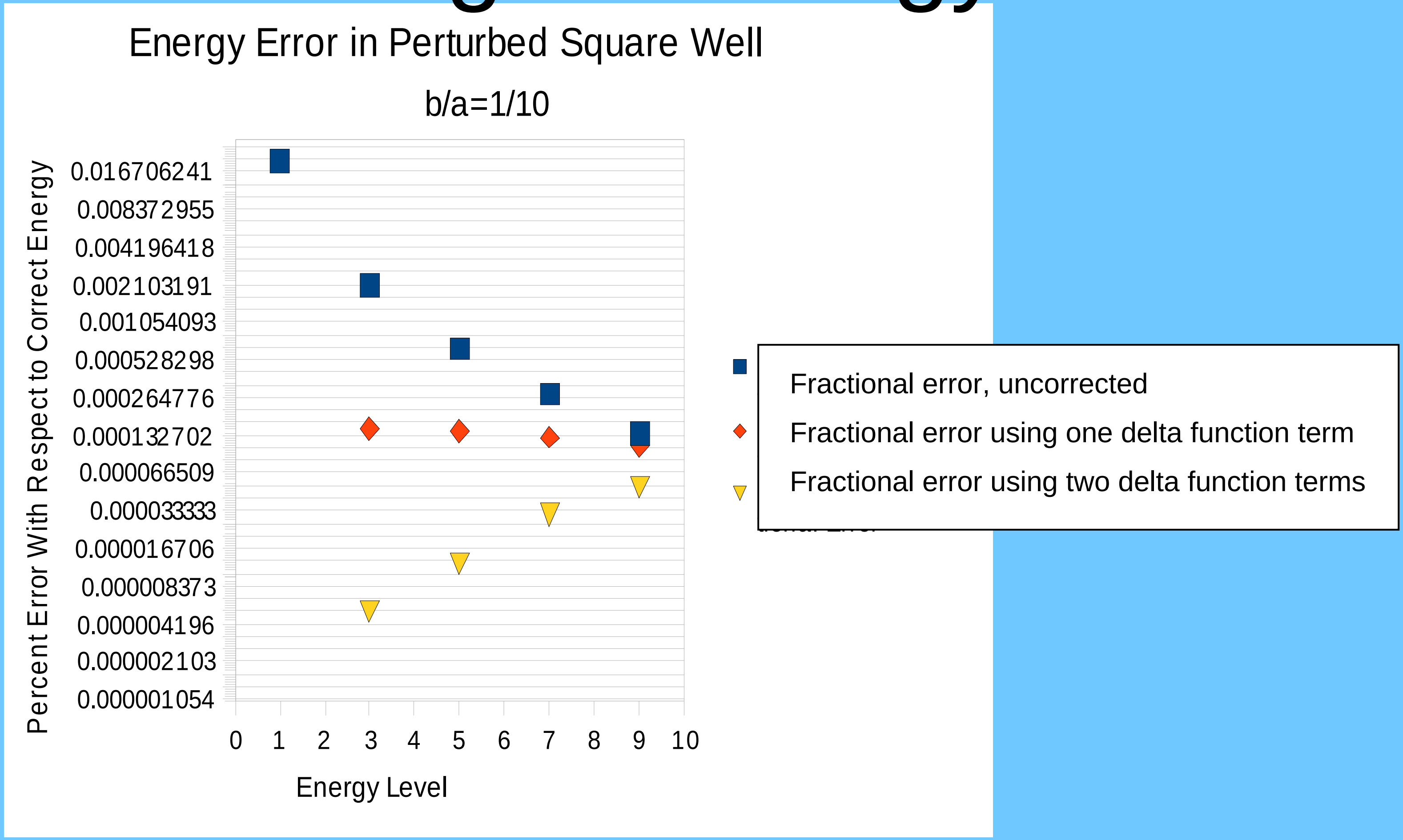
- Determine the coefficients by solving

$$E_1 - E_1^{(0)} = a_0 \mid \Psi_1^{(0)}(x) \mid^2|_{x=0} + a_2 \frac{d^2}{dx^2} \mid \Psi_1^{(0)}(x) \mid^2|_{x=0} = 0$$

$$E_2 - E_2^{(0)} = a_0 \mid \Psi_2^{(0)}(x) \mid^2|_{x=0} + a_2 \frac{d^2}{dx^2} \mid \Psi_2^{(0)}(x) \mid^2|_{x=0} = 0$$

- Apply the resulting $V(x)$ to higher order $\Psi_N^{(0)}(x)$ to determine shifts $E_N^{(1)}$.

Results of applying this effective theory to determine higher energy states



Error goes down as more terms are added within this range

For higher energies not shown, wave function more oscillatory; can “see” structure of short range interaction; higher order terms in the Taylor expansion become more important; effective theory will break down.

Main Task

Motivation: nuclear physics problems with very sharp short range potential, very small binding energy

My project explored: 2 particle system in free space

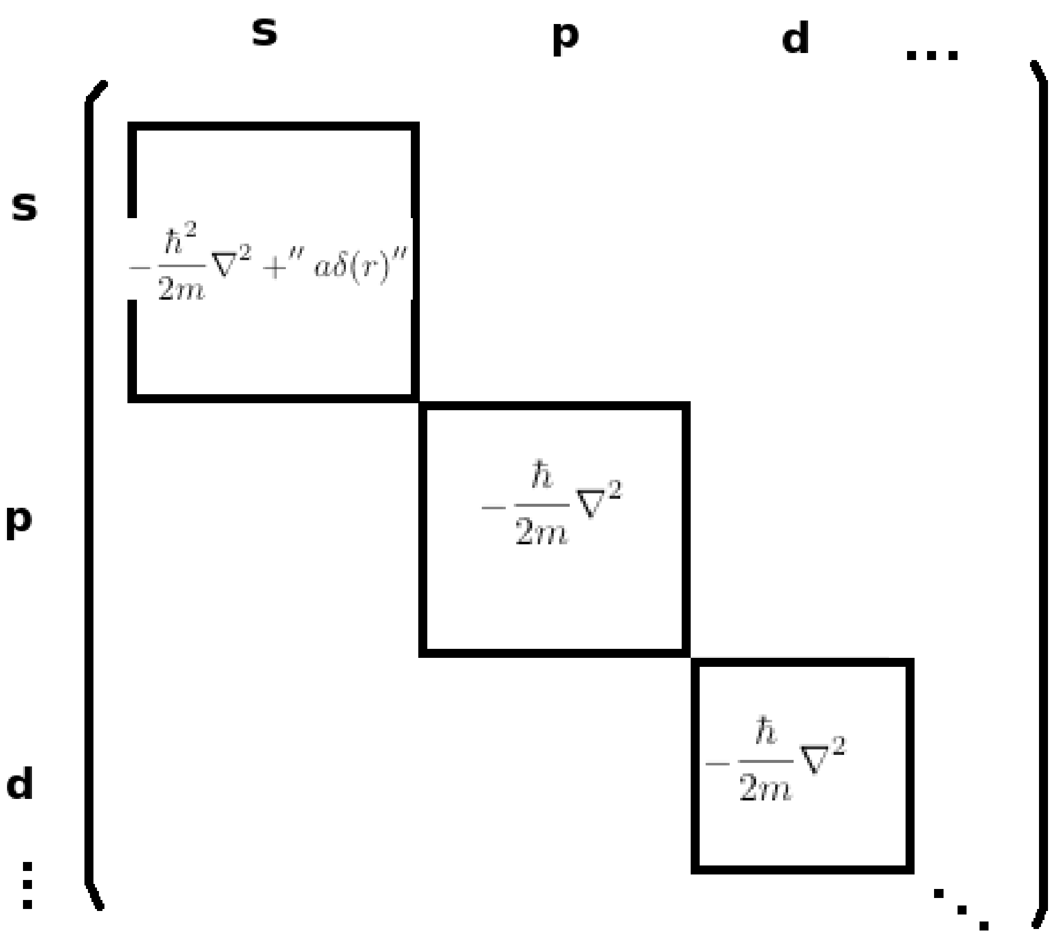
interaction as a "fuzzy" delta function

In free space, $E_{binding} \rightarrow 0$ (unitary limit)

$$\hat{H}_{unitary} = -\frac{\hbar^2}{2m}\nabla^2 +'' a\delta(r)''$$

Can only use a finite number of states in Hamiltonian; must determine the form of an effective theory interaction that accounts for those left out according to basis used.

Matrix Representation for $\hat{H}_{unitary}$



$$|\Psi\rangle = |\Psi_s\rangle + |\Psi_p\rangle + |\Psi_d\rangle + \ldots$$

$$\hat{H}_{unitary} = \hat{H}_s + \hat{H}_p + \hat{H}_d + \ldots$$

$$\hat{H}_{unitary}\,|\Psi\rangle = 0$$

Determine a_Λ from condition $\hat{H}_s\,|\Psi_s\rangle = 0$

Setup For Problem

$$\hat{H}_s = -\frac{\hbar^2}{2m}\nabla^2 +'' a_\Lambda \delta(r)''$$

$$''\delta(r)'' = \sum_{N'=0}^{\Lambda} \sum_{N=0}^{\Lambda} | N\rangle\langle N | \delta(r) | N'\rangle\langle N' |$$

- Find a_Λ such that $\hat{H}_s | \Psi_s\rangle = 0$
- Use a reduced basis including s states of principle quantum number $N = 0$ to $N = \Lambda$

Set up an effective Hamiltonian taking into account reduced basis

Bloch-Horowitz Equation

$$\hat{P} = \sum_{N=0}^{\Lambda} \mid N\rangle\langle N \mid \qquad \hat{Q} = \sum_{N=\Lambda+1}^{\infty} \mid N\rangle\langle N \mid \qquad \hat{P} + \hat{Q} = 1$$

$$\hat{P}\left(\hat{H} + \hat{H}\hat{Q}\frac{1}{E - \hat{Q}\hat{H}}\hat{Q}\hat{H}\right)\hat{P} \mid \Psi\rangle = E\hat{P} \mid \Psi\rangle$$

- Gives $\hat{H}_{eff} = \hat{H} + \hat{H}\hat{Q}\frac{1}{E - \hat{Q}\hat{H}}\hat{Q}\hat{H}$ producing same energy E as complete Hamiltonian

- For $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 +'' a_\Lambda \delta(r)''$ correction is zero for

all $\langle \Psi_{N'} \mid \hat{H} \mid \Psi_N \rangle$ except $\langle \Lambda \mid \hat{H} \mid \Lambda \rangle$

$$\begin{pmatrix} \hbar\omega \left(\frac{1}{2} \left(\frac{1}{2} + \mathbf{L2} \right) + a \sqrt{\frac{\text{Gamma}\left[\frac{1}{2}+\text{L2}\right]^2}{(\text{L2}\,!)^2}} \right) & \hbar\omega \left(\frac{1}{2} \sqrt{\mathbf{L2} \left(\frac{1}{2} + \mathbf{L2} \right)} + a \sqrt{\frac{\text{Gamma}\left[\frac{1}{2}+\text{L2}\right] \text{Gamma}\left[\frac{3}{2}+\text{L2}\right]}{(-1+\text{L2})\,!\,\text{L2}\,!}} \right) \\ \hbar\omega \left(\frac{1}{2} \sqrt{\mathbf{L2} \left(\frac{1}{2} + \mathbf{L2} \right)} + a \sqrt{\frac{\text{Gamma}\left[\frac{1}{2}+\text{L2}\right] \text{Gamma}\left[\frac{3}{2}+\text{L2}\right]}{(-1+\text{L2})\,!\,\text{L2}\,!}} \right) & \hbar\omega \left(\frac{1}{2} \left(-\frac{1}{2} + 2\,\mathbf{L2} \right) + a \sqrt{\frac{\text{Gamma}\left[\frac{1}{2}+\text{L2}\right]^2}{((-1+\text{L2})\,!)^2}} \right) \\ a\,\hbar\omega \sqrt{\frac{\text{Gamma}\left[-\frac{1}{2}+\text{L2}\right] \text{Gamma}\left[\frac{3}{2}+\text{L2}\right]}{(-2+\text{L2})\,!\,\text{L2}\,!}} & \hbar\omega \left(\frac{1}{2} \sqrt{(-1 + \mathbf{L2}) \left(-\frac{1}{2} + \mathbf{L2} \right)} + a \sqrt{\frac{\text{Gamma}\left[-\frac{1}{2}+\text{L2}\right] \text{Gamma}\left[\frac{1}{2}+\text{L2}\right]}{(-2+\text{L2})\,!\,(-1+\text{L2})\,!}} \right) \end{pmatrix}$$

Result:

$$a_\Lambda = \frac{1}{\sum_{j=0}^{[\frac{\Lambda}{4}]} \frac{\Gamma(\frac{\Lambda}{2}-2j+\frac{1}{2})}{(\frac{\Lambda}{2}-2j)!}}$$

where $[\frac{\Lambda}{4}]$=the greatest integer less than or equal to $\frac{\Lambda}{4}$.

Solution greatly simplifies problem for particles in trap:

$$\hat{H}\hat{Q}\frac{1}{E-\hat{Q}\hat{H}}\hat{Q}\hat{H} = 0$$

for all elements in the space-no Bloch Horowitz correction needed

Only need solve $\hat{H}_s \mid \Psi_s\rangle = E \mid \Psi_s\rangle$

can apply a_Λ to problems with more bodies.

Next Step

Establish how the effective interaction changes if we reduce the basis by one element and change $\hat{H}_{eff}(\Lambda)$ $\hat{H}_{eff}(\Lambda-2)$

By setting $\hat{P} = \sum_{N=0}^{\Lambda-2} | N\rangle\langle N |$ $\hat{Q} = | \Lambda\rangle\langle\Lambda |$

Bloch Horowitz Equation gives the correction to top corner matrix element $\langle\Lambda-2 | \hat{H}_{eff}(\Lambda) | \Lambda-2\rangle$ producing $\langle\Lambda-2 | \hat{H}_{eff}(\Lambda-2) | \Lambda-2\rangle$

Equation will give $a_{\Lambda-2}$ in terms of a_{Λ}.

Acknowledgements

Wick Haxton for advising me

Doron Gazit for his help

The directors of the REU program

My physics professors at Bryn Mawr

NSF for providing this research opportunity

I've really enjoyed being here. Thank you all.