Effective Theories and Quantum Field Theory

Or How I Spent My Summer

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Abstract

My work falls into two main categories of study: effective theories in quantum mechanics and quantum field theory (including some application of the techniques of effective theory). In particular, I studied the use of effective theories in potential models, including necessary corrections to operators in the effective theory and comparisons of different effective theories. In quantum field theory, I studied the derivation of Feynman rules from a Lagrangian, dimensional regularization, renormalization, and spontaneous symmetry breaking.

1 Introduction

Effective field theories have played a significant role in the development of the Standard Model of particle physics, from the Fermi point model of the weak interactions to the practice of "integrating out" heavy particles. My goal for the summer was, therefore, to learn how to apply effective theory techniques in quantum field theory and to understand current work in nuclear physics using effective field theories. I studied the application of effective theories to Schrödinger quantum mechanics to gain a clear understanding of the physics of effective theories as one part of my project. For a second part, I studied various subjects in quantum field theory with an emphasis on the derivation

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of Feynman rules and renormalization using dimensional regularization. I then applied what I learned of quantum field theory to studying spontaneous symmetry breaking, an effective field theory for a two scalar boson theory, and understanding recent work on nucleon-nucleon scattering using effective field theories.

2 Effective Theories

Effective theories operate on the principle that physical quantities can be given by a power series in the energy of the system in question; that is, we can write a measurable quantity (or the corresponding operator) as

$$Q = c_0 + c_1 \frac{E}{\Lambda} + c_2 \left(\frac{E}{\Lambda}\right)^2 + \cdots ,$$

where the coefficients are either constants or operators and Λ is some energy scale. If Λ is much higher than the energies of interest, then an effective theory in which Q is defined as the first one or two non-zero terms of the power series gives accurate results (the errors are on the order of the first non-zero term left out of the effective theory). Also, because high order terms give significant effects for $E \sim \Lambda$, the effective theory breaks down at about Λ, indicating that some new physical effect occurs at that energy scale.

By eliminating all but a few terms of the power series for Q, an effective theory allows calculations that might otherwise be impossible or exceedingly difficult. For example, Newtonian mechanics, an effective theory with truncation after leading order, is much simpler to use at low velocities (corresponding to energies low compared to mc^2) than special relativity. In another case, the high energy theory might be unknown, but a finite number of coefficients in an effective theory can be calculated from experimental results. Furthermore, the coefficients in the effective theory can be used to predict Λ, the energy scale at which some unknown physical effect will take place.

Below are two examples of effective theories. The first is an example in which the high-energy theory is completely known, so the effective theory is derived by a perturbative expansion of the full theory. In the second, the effective theory is derived from the results of simulated experiments.

2.1 A Matrix Hamiltonian Effective Theory

Consider a 3×3 matrix Hamiltonian

$$\mathcal{H} = \begin{bmatrix} m_1 & g & \lambda_1 \\ g & m_2 & \lambda_2 \\ \lambda_1 & \lambda_2 & M \end{bmatrix}, \; m_1, m_2 << M.^1 \tag{1}$$

We assume the conditions necessary for the couplings g, λ_1, and λ_2 to be perturbatively small, so the eigenvalues are (to first order) m_1, m_2, and M. Further, the energy eigenstates are slightly rotated in the 3D Hilbert space from the standard basis states. If we take the standard basis to be the observed states, we find an oscillation of states, similar to either entanglement of states or a simplified neutrino oscillation model.

Because M is a large energy scale, we can develop an effective with only two states (in either the observed or eigenstate basis) that is a truncated power series in $\frac{\lambda_1}{M}$ and $\frac{\lambda_2}{M}$ (the coupling constants here represent the energy at which we are working). One such effective theory uses the effective Hamiltonian

$$\mathcal{H}_{eff} = \begin{bmatrix} m_1 & g \\ g & m_2 \end{bmatrix}. \tag{2}$$

We see that this effective Hamiltonian (2) agrees with the corresponding matrix elements of (1).

To 0^{th} order in $\frac{1}{M}$, this effective Hamiltonian predicts the matrix elements of operators (such as the propagator). However, if we want agreement to higher order with the full theory, then we need either to correct (2) or include correction terms explicitly in our operators (or, for some operators, do both). These correction terms are dependent on the effective Hamiltonian used as well as the basis in which the Hamiltonians are written. The propagator, for example, is exactly the same for the effective and full theories in the energy eigenbasis, but includes a correction term (up to second order in $\frac{1}{M}$) of

$$U_{correction}(t) = \frac{1}{M^2} e^{-iMt} \begin{bmatrix} \lambda_1^2 & \lambda_1\lambda_2 \\ \lambda_1\lambda_2 & \lambda_2^2 \end{bmatrix}$$

in the basis of observed states. This correction term arises from the fact that the eigenstates are related to the observed states by a 2D rotation in the effective theory as compared to a 3D rotation in the full theory.

[1] Please note that I use the particle physics units in which $\hbar = c = 1$ throughout this paper.

2.2 A Potential-Well Effective Theory

In this model, which follows closely the ideas of [Lepage], we consider a potential in the Schrödinger equation that approaches a Yukawa potential at long distances but is non-perturbatively different at a short range. To model the low energy (long-distance) behavior of the full theory, we develop an effective potential with a high-energy scale corresponding to a wavelength of a. To 0^{th} order in the expansion, we can write the effective potential as

$$V_0(r) = -\frac{e^{-\frac{r}{r_0}}}{r}\,\mathrm{erf}(\frac{r}{\sqrt{2}a}),$$

which incorporates the correct long-range behavior. In this example, however, we calculate quantities such as the binding energies and scattering phase shifts directly from Shrödinger's equation, so we reduce the error in our predictions by adding an additional correction term to our potential:

$$V_{eff}(r) = V_0(r) + ca^2\delta_a(r), \qquad (3)$$

where δ_a is a "smeared out delta function," that is, a function that is nearly zero outside of $r \leq a$ and is about 1 inside $r \leq a$. Any form for this delta function will give valid low energy results; an effective theory is not sensitive at low energies to its high energy behavior (or else the only valid "effective" theory would have to be the full theory. Quite a lot of physics would be impossible if that were the case).

Then we choose the coefficient c so that the lowest scattering energy phase shifts match for the true potential and the effective potential 3. (We match the two theories at the lowest energy available because the error should behave as the first term in the truncation of the full theory, a power law.) See figure 1 for an example effective potential. With our effective theory now fixed, we calculate predictive binding energies and scattering phase shifts. In this case, because I also calculated data for the full theory numerically, I studied the errors in the predictions and found that the errors do indeed follow a power law form and were very reasonable for low energies (see figures 2 and 3). The bound state wave functions also agreed at long distances (low energies) for the two potentials. At high energies (corresponding to wavelengths of less than a), however, the theories behave radically differently; the effective and full potentials need not have the same number of bound states. As stated above, the low energy results are the same for different forms of the correction term δ_a, but the high energy results may be different.

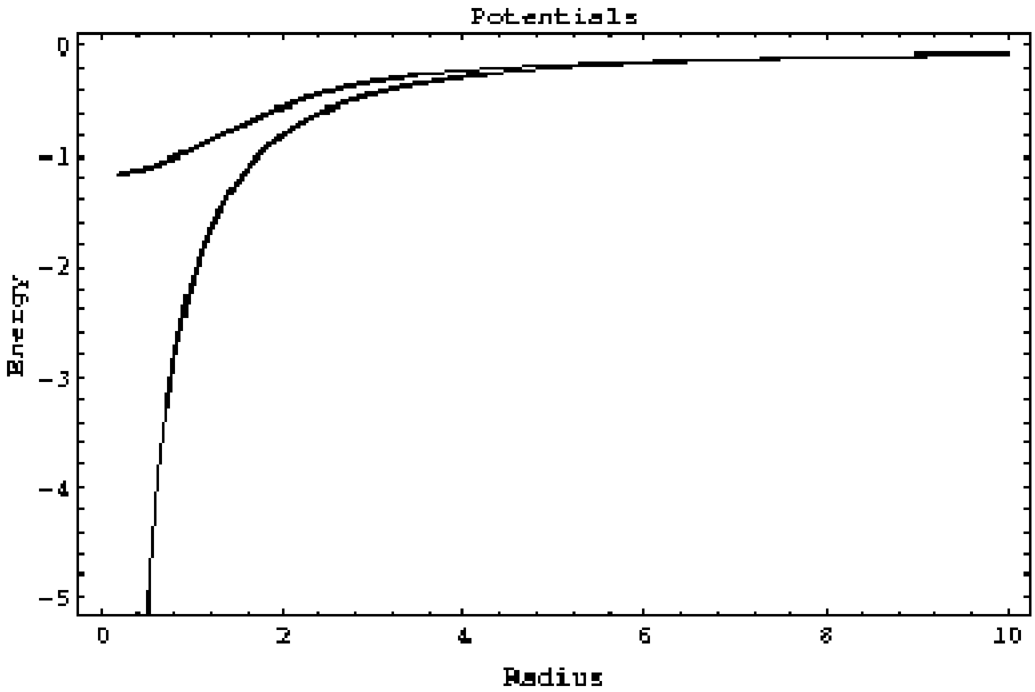

Figure 1: One choice of the effective potential and the true potential. The true potential is singular at the origin (and steeper than a Yukawa potential), and the effective potential is finite.

To predict the expectation value of an operator using the effective theory, we cannot simply use the operator acting in the effective theory. First, the Hilbert spaces of the full and effective theories are quite different, so we must include a coefficient for the effective theory operator to take into account renormalization effects. Further, as in the example above, we include a correction term to raise the order of the error in the prediction. Thus, for a generic operator O, we have the formula

$$O_{pred} = \alpha O_{eff} + \beta a^n \delta_a,$$

where the δ_a term is the first order correction corresponding to contributions from all wavelengths shorter than a and a^n is included to make β dimension-

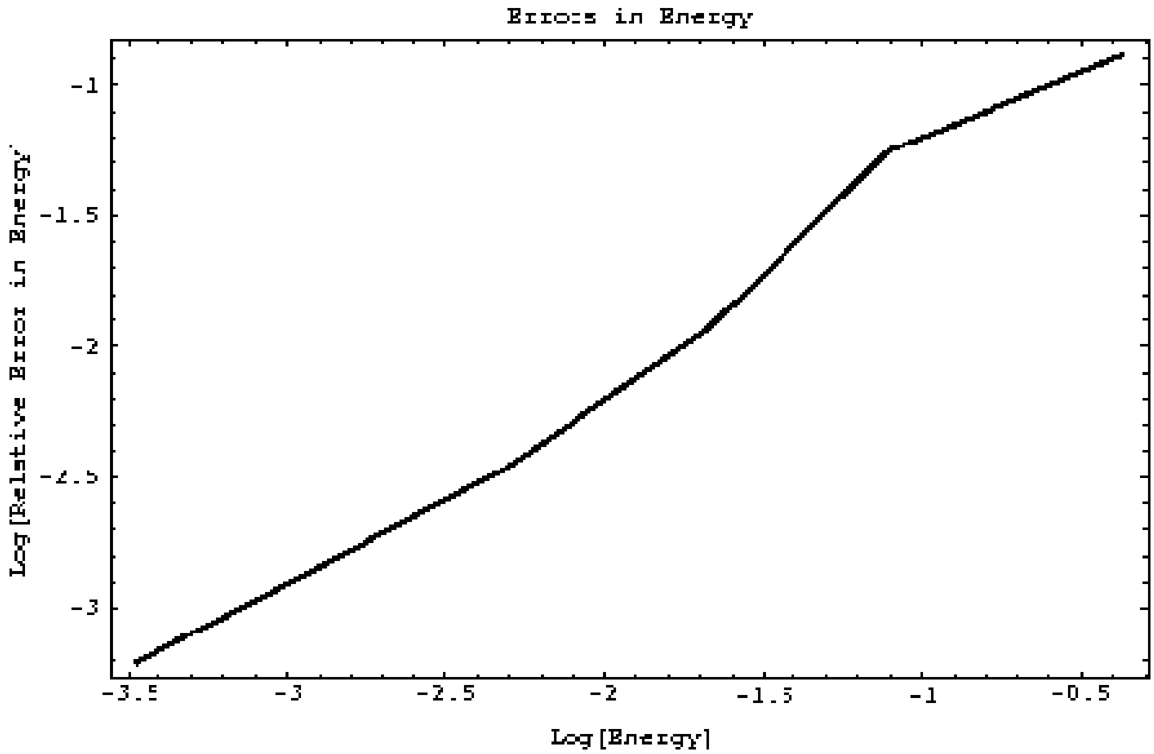

Figure 2: A log-log plot of the relative error in the energy eigenvalues between the effective and full theories.

less. We need two low energy data points to fix both coefficients.

2.3 Other Effective Theories

I also either worked out or researched several other effective theories (including contemporary applications of effective field theory), one of which I will discuss below. In researching effective theories, I read the review articles [Manohar], [Kaplan], and [Georgi], as well as [Lepage].

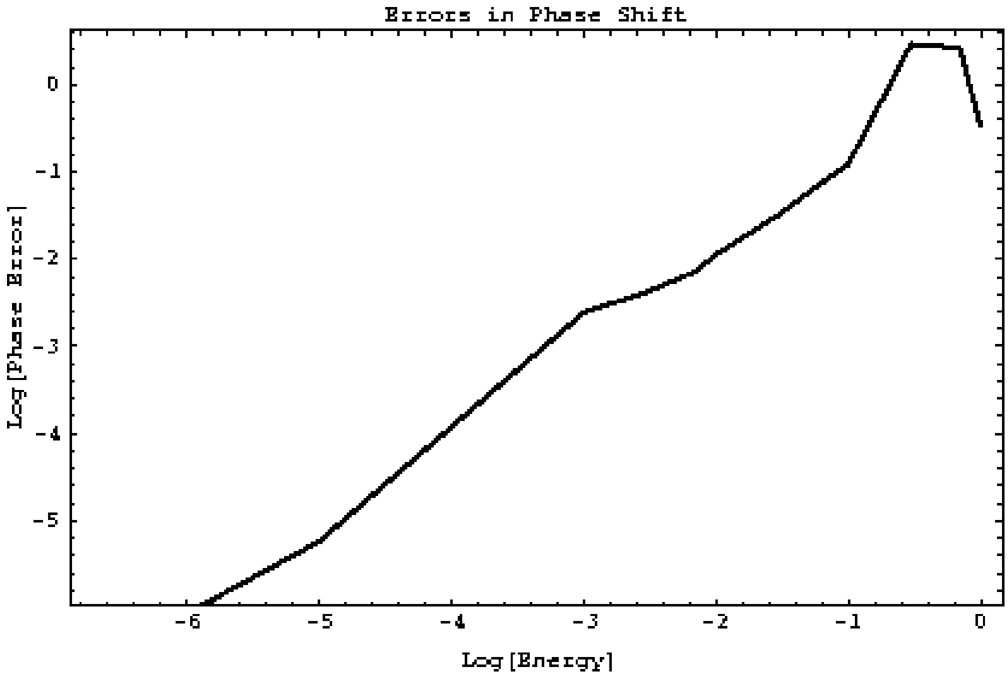

Figure 3: A log-log plot in the error between the scattering phase shifts in the effective and full theories as a function of the scattering energy.

3 Quantum Field Theory

3.1 Studying

Quantum field theories explain physics at the smallest understood distance scales; relativistic field theories take the place of Schrödinger quantum mechanics at subatomic levels. Quantum fields have successfully predicted results including the gyromagnetic ratio of the electron, cross-sections for particle scattering, decay rates of fundamental particles, and some aspects of big bang cosmology, and the mathematical formalism of field theories has further demonstrated the relationship of particle interactions to underlying symmetries of nature and explained the masses of some particles within that framework. This summer, I studied several aspects of quantum field theories, ranging from the derivation of Feynman rules to spontaneous symmetry breaking.

I began working from general principles at the beginning of the summer following [Ramond], through the derivation of Feynman rules and renormalization for a ϕ^4 theory from a path integral formalism. I also studied the derivation of Feynman rules from a more Hamiltonian formalism in [Peskin & Schroeder], [Veltman], and [Bjorken & Drell], mainly using the ϕ^4 theory as a concrete example. I also learned dimensional regularization techniques and several renormalization prescriptions (including the mass-independent $\overline{\mathrm{MS}}$ scheme), particularly from [Peskin & Schroeder]. After exhausting the ϕ^4 theory, I also did some sample renormalization calculations in QED (making occasional reference to [Peskin & Schroeder] and [Carlson]).

The final subject I studied in quantum field theory was gauge theory; I read [Quigg]. I began by learning about Noether currents and general properties of symmetries and then studied how the introduction of a new particle (in all the cases I studied, a vector boson) or particles can create a local gauge symmetry (a symmetry in which a transformation can vary from point to point in space-time and still leave the system invariant) in a Lagrangian. Particularly, I looked at U(1) gauge theories and SU(2) gauge theories for doublets and triplets. After that preparation, I worked through spontaneous symmetry breaking and the Higgs mechanism for U(1) and SU(2) triplet theories as well as the SU(2) $\times$ U(1) symmetry of the electroweak force. Those processes are responsible both for hiding the symmetry of the Lagrangian and for giving the gauge particles (the particles introduced to create the gauge symmetry), such as the $W^{\pm}$ particles, mass.

3.2 An Effective Field Theory Calculation

As a tie-in to my effective theory studies, I designed an effective field theory for a theory with two massive scalar particles, ϕ_1 and ϕ_2, where the ϕ_2 has a much greater mass. The Lagrangian for this theory was

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1\partial^\mu\phi_1 - m_1^2\phi_1^2) + \frac{1}{2}(\partial_\mu\phi_2\partial^\mu\phi_2 - m_2^2\phi_2^2) - \frac{\lambda_1}{4!}\phi_1^4 - \frac{\lambda_2}{4!}\phi_2^4 - \frac{\lambda_3}{(2!)^2}\phi_1^2\phi_2^2. \tag{4}$$

I used two different renormalization prescriptions to study this theory, the $\overline{\mathrm{MS}}$ scheme and a scheme in which the scattering amplitude was fixed at an (unphysical) space-like momentum.

In the effective theory, we only consider ϕ_1 particles with energies insufficient to excite the ϕ_2 particle field. Then, by doing pertubative matching to the full theory for $\phi_1\phi_1 \to \phi_1\phi_1$ scattering (that is, power-series-expanding the amplitude) in both renormalization schemes, I replaced two ϕ_2 exchange interactions with an effective vertex

$$i\alpha\,(s+t+u),\ \alpha = \frac{\lambda_3^2}{6\cdot 32\pi^2 m_2^2},$$

where s, t, and u are the Mandelstam variables corresponding to the momentum squared flowing through the vertex. Thus, we have an effective Lagrangian of

$$\mathcal{L}_{eff} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) - \frac{\lambda}{4!}\phi^4 + \frac{\alpha}{(2!)^2}\phi^2\partial_\mu\partial^\mu\phi^2, \tag{5}$$

where the subscript 1 has been dropped for simplicity (the ϕ^2 particle does not exist in the effective theory, and its mass is the energy scale of new physics). We then match the scattering amplitude (related to the cross-section) at a low (center-of-mass frame) scattering energy for the effective theory to that for the full theory, including correction terms appropriate to the renormalization scheme, and then calculate scattering amplitudes for a range of scattering energies. We again find that the errors between the full and effective theories follow a power-law form (see 4).

3.3 Nucleon-Nucleon Scattering

Finally, I read and studied the article by [Kaplan et. al.] on effective field theories for nucleon-nucleon scattering. The paper reviewed and modified "power-counting" arguments for determining the order of terms in the energy power series expansion given above for the case of non-relativistic nucleon-nucleon scattering and developed several different effective theories for the scattering process. First, effective theories without pions were discussed, including potential models and Feynman amplitude expansions, and the scattering phase shifts for the $^1S_0$ partial wave were computed. The paper found that the potential models were effective theory expansions in an energy scale of about 35 MeV instead of the pion mass $m_\pi \sim 140$MeV (for effective theories without pions) because of an unusually long scattering length. Even

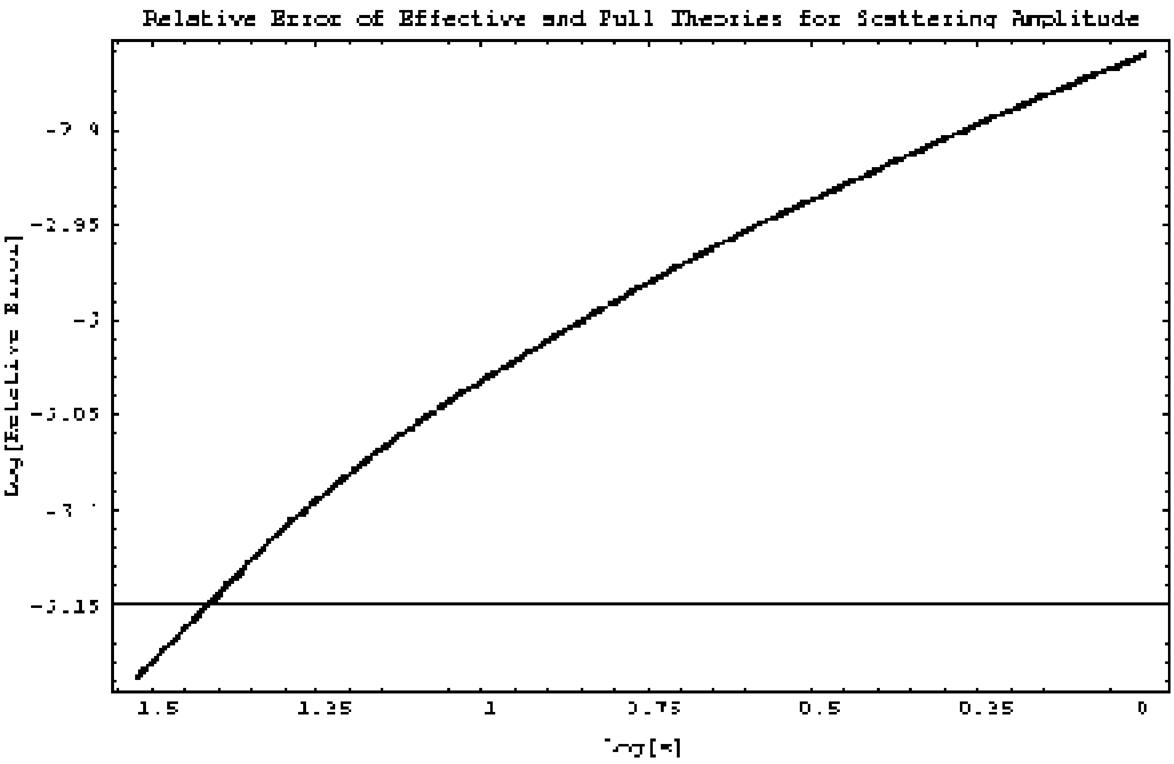

Figure 4: The relative error of the Feynman amplitude for $\phi_1\phi_1 \rightarrow \phi_1\phi_1$ scattering in the effective and full theories using $\overline{\mathrm{MS}}$ renormalization. The variable s is the square of sum of the incoming 4-momenta in any reference frame (it is Lorentz invariant).

including pion-exchange terms did not extend the range of validity of the potential model effective theories enough to be of much use. However, using the fact that

$$|\vec{p}| \cot \delta(\vec{p}) = i|\vec{p}| + \frac{4\Pi}{M}\frac{1}{\mathcal{A}},$$

where $\mathcal{A}$ is the Feynman amplitude for the scattering process, the authors derived a power series expansion for $|\vec{p}| \cot \delta(\vec{p})$ in terms of the effective Feynman amplitude. That effective theory agreed well with other models for the expected range of energies (that is, until about m_π without single pion exchange and farther with one-pion exchange).

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