Th2a-1

7/11/2013

Plan for Renormalization Group 2:

(1) Recap of some important points from the slides shown yesterday (using W2a-2 and 2b plus the exercises)

(2) Local projection (visualizing potential changes in coordinate space) and flow to universal potentials (where does unresolved physics go?)

(3) Alternative generators — Wegner and block diagonal ($V_{low\,k}$)

(4) Operators and many-body contributions

(5) Perturbativeness — Weinberg eigenvalues

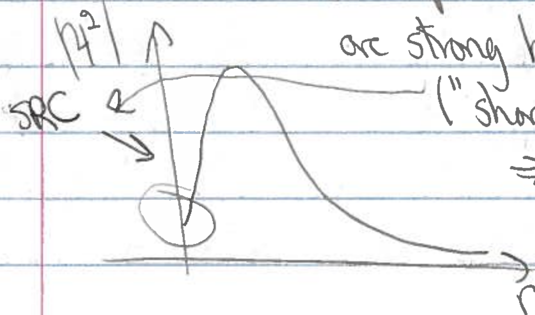
(6) Computational aspects

7/10/2013

- Recap of important points from the slides

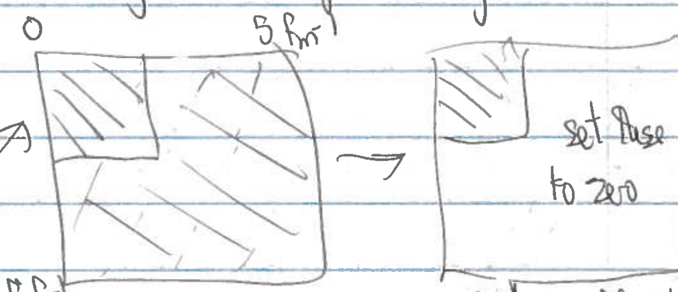
• Nuclei would be at low resolution based on Fermi momentum in large nucleus.

- recall from exercises $\rho = \frac{2}{3\pi^2}k_f^3$ (for protons and neutrons)
  $\Rightarrow k_f = (3\pi^2\rho/2)^{1/3}$ (see next page)
  and density of heavy nuclei about constant $\Rightarrow k_f \approx 1.1-1.3\ \text{fm}^{-1}$
- So typical relative momentum of $\approx 1\ \text{fm}^{-1}$ (≈ 200 MeV) in a large nucleus. Even less in light nuclei
- But if the potential has a repulsive core, then there are strong high momentum components ("short-range correlations")



$\Rightarrow$ · slow-down convergence of many-body nuclei.
· e.g. matrices get too big.

• low pass filter fails even for low energy.



Why? Because of quantum mechanics. $T = V + V\frac{1}{E-H_0}V + \dots$

If strong off-diagonal coupling $\Rightarrow$ can't ignore.

$$\Rightarrow \underbrace{\langle k|T|k\rangle}_{\text{low momentum}} = \langle k|V|k\rangle + \frac{2}{\pi}\int k'^2 dk' \frac{\langle k|V|k'\rangle\langle k'|V|k\rangle}{(k^2-k'^2)/m} + \dots$$

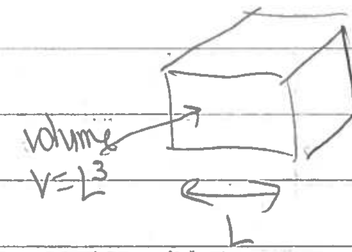
Solution? Unitary transformation to decouple! Use RG to do it.

7/10/2013

WQa-2b

[Aside: deriving $\rho = \frac{2}{3\pi^2} k_F^3$ ...

- For a non-interacting Fermi gas, imagine putting N particles in a box of side L $\Rightarrow \rho = \frac{N}{L^3}$



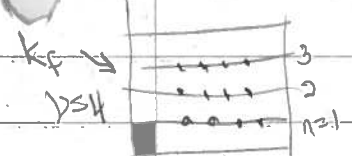
' when in doubt for uniform system, ~~put in box and take V→∞ limit at end~~

- Let ν be the spin-isospin degeneracy:

  $\nu = 2$ for neutrons only (spin up, spin down)

  $\nu = 4$ for symmetric matter $(\uparrow, \downarrow, n, p)$

- Apply periodic boundary conditions (pbc) $\Rightarrow$ discrete momentum levels:



Then $N = \sum_{n}^{n(k_F)} \nu \cdot 1$ ← ν in each state

But pbc: $e^{ik_n(x+L)} = e^{ik_n L}$ in each dimension

$\Rightarrow k_n L = 2\pi n \quad n = 0, 1, 2, 3, \ldots$ are allowed

$\Rightarrow n = \frac{kL}{2\pi}$ or $\Delta n = 1 = \frac{L}{2\pi}\Delta k$ in each dimension

in large V limit $\Rightarrow \sum_n \rightarrow \left(\frac{L}{2\pi}\right)^3 \int d^3k = \frac{V}{(2\pi)^3}\int d^3k$

$\Rightarrow N = \frac{V}{(2\pi)^3}\int^{k_F} d^3k \, \nu$

↖ volume of sphere in large V limit

or $\boxed{\frac{N}{V} = \rho = \frac{1}{8\pi^3}\cdot\frac{4}{3}\pi k_F^3 \cdot \nu = \frac{\nu k_F^3}{6\pi^2}}$

]

Th2a-2

7/11/2013

Wednesday exercises review

1(c) Why would we want to repeat nuclear structure calculations for different values of the SRG λ (or s).

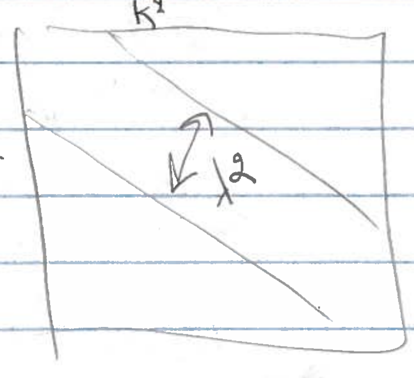
do 2, 3, first then return

- Observables are supposed to be unchanged
  - $\Rightarrow$ test if a quantity is an observable (example, clear demonstration that D-state probability in the deuteron is not)
  - $\Rightarrow$ determine the scale dependence of a quantity
- Test for errors
- Test approximations
  - We will see this particularly in considering many-body potentials and other operators.

2. General equation is $\frac{dH_s}{ds} = [\eta(s), H_s] = [[G_s, H_s], H_s]$
   - T_{rel} (or T) doesn't change by construction
   - What if we used $T = T_{cm} + T_{rel}$ for G_s instead of T_{rel}?
     (answer: $[T_{cm}, H_s] = 0$, so no difference!)
   - other choices for G_s?

3. $\langle k|\frac{dV_s}{ds}|k'\rangle \equiv \frac{dV_s(k,k')}{ds} = -(k^2-k'^2)^2 V_s(k,k') + \frac{2}{\pi}\int_0^\infty q^2 dq\,(k^2+k'^2-2q^2)V_s(k,q)V_s(q,k')$

   - If $-(k^2-k'^2)^2 V_s(k,k')$ dominates then $V_s(k,k') \doteq V_{s=0}(k,k')\,e^{-\left(\frac{k^2-k'^2}{\lambda^2}\right)^2}$ $\quad k \neq k'$ (must be sufficiently off diagonal)



- Look at slides

Th2a-3

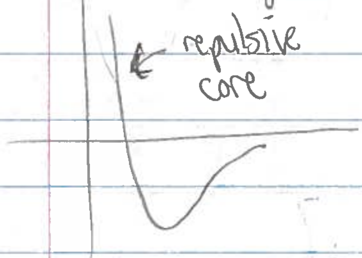
7/11/2013

Visualizing the softening of NN interactions

• In momentum space we associate softening of a potential with the decreased coupling between high and low momentum:

$$\langle k_{high} | V | k_{low} \rangle \to 0$$

• But what does this do to our picture of potentials having strong short-range repulsion?



• Visualizing is not so easy, because the potential becomes non-local, so it is a functional of r and r'

• note that the $-(\vec{k}^2-\vec{k}'^2)^2 V_s(\vec{k},\vec{k}')$ term in the SRG equation (not partial wave projected) can be written using

$$\vec{k}^2-\vec{k}'^2 = (\vec{k}+\vec{k}')\cdot(\vec{k}-\vec{k}') \to \vec{p}\cdot\vec{q}$$

as an explicit function of total momentum $\vec{p} = (\vec{k}+\vec{k}')$ and not just momentum transfer $\vec{q} = \vec{k}-\vec{k}' \Rightarrow$ nonlocal

Why do partial waves not mix? (except for already coupled) ⇒ insert expressions, only k^2, k'^2, q^2 dependence from SRG

• Plan: use a local projection

· The high-momentum tails of low-energy wave functions are suppressed by RG evolution which implies the wave function variation over short distances is small. So in the non-local Schrödinger equation:

treat as constant over range of V nonlocality

$$-\frac{1}{2\mu}\nabla^2\psi(\vec{r}) + \int d^3r'\, V(\vec{r},\vec{r}\,')\psi(\vec{r}\,') = E\psi(\vec{r})$$

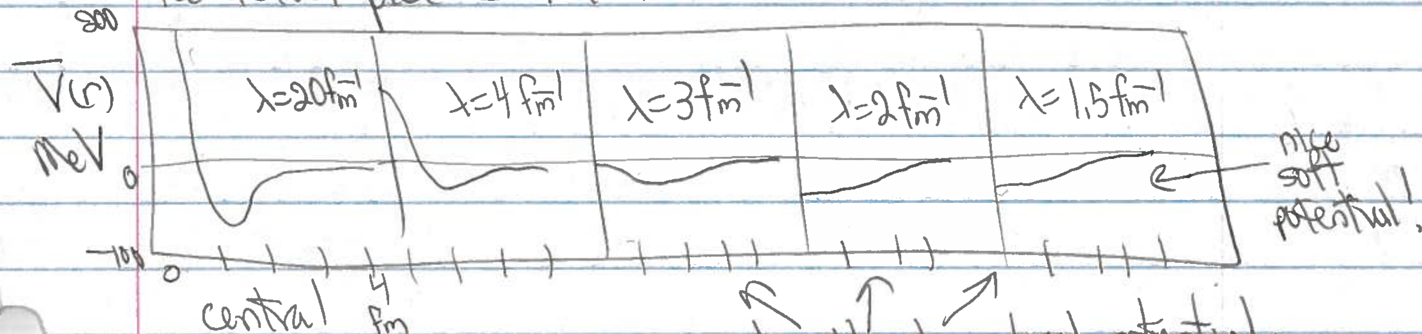
$$\to -\frac{1}{2\mu}\nabla^2\psi(\vec{r}) + \psi(\vec{r})\int d^3r'\, V(\vec{r},\vec{r}\,') \approx E\psi(\vec{r})$$

7/11/2013

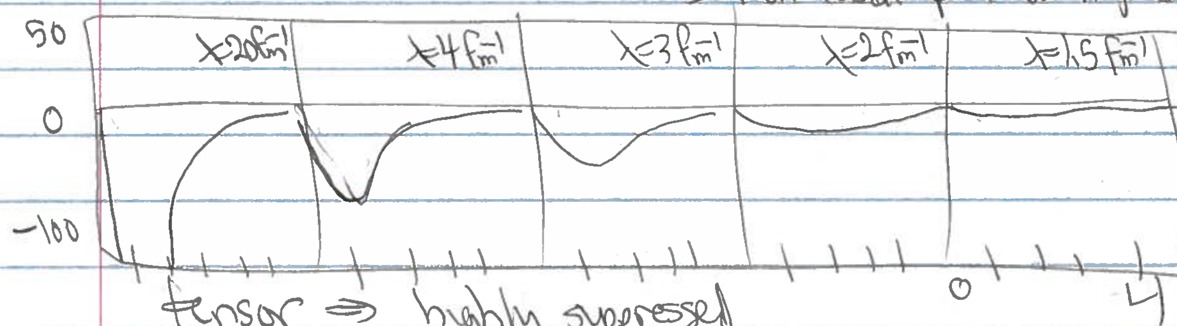
$\Rightarrow$ define $\overline{V}_\lambda(r) = \int d^3r' \, V_\lambda(\vec{r}, \vec{r}')$ as the local projection.

- Kyle Wendt has developed this idea further, to apply beyond S-waves (which is all that survives the angular integral).

- We'll sketch the result for the AV18 potential and look at the actual pictures later.

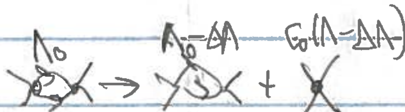


purely attractive local potential, so phase shift must fail to change sign
$\Rightarrow$ non-local part at higher momentum



tensor $\Rightarrow$ highly suppressed
- D-state probability changes greatly
- But asymptotic D-S ratio unchanged!
- What about quadrupole moment?

- Different potentials evolve to same in both momentum rep. (at momentum below λ and in local projection).
- Where do you expect high energy contributions to go? cf.



$\Rightarrow$ see slides (same thing here!)

Th2a-5

7/11/2013

- Wegner choice for the flow-equation generator
  $\Rightarrow$ use the diagonal of H_s in whatever basis you are in. E.g., $H_d = T_{rel} + V_s(k,k)$ ← diagonal

- Let's consider the general case with $H_{ii} \equiv e_i$ where we are labeling the basis elements $|i\rangle, |j\rangle, \ldots$
  - Note that these could be plane waves, harmonic oscillators, 2 particle or more, ...

$$\langle i|\frac{dH_s}{ds}|j\rangle = \frac{dH_{ij}}{ds} = \langle i|[[H_d,H_s],H_s]|j\rangle$$

$$= \langle i| \overset{ⓐ}{H_d H_s H_s} - \overset{ⓑ}{H_s H_d H_s} - \overset{ⓒ}{H_s H_d H_s} + \overset{ⓓ}{H_s H_s H_d} |j\rangle$$

insert $\sum_k |k\rangle\langle k|$ between H_s's and use $H_d|j\rangle = e_j|j\rangle$, etc.

$$\Rightarrow \frac{dH_{ij}}{ds} = \sum_k (\overset{ⓐ}{e_i} - \overset{ⓑ}{e_k} - \overset{ⓒ}{e_k} + \overset{ⓓ}{e_j}) H_{ik} H_{kj}$$ ← simple matrix multiplication!

$i=j \Rightarrow dH_{ii}/ds = 2\sum_k (e_i - e_k)|H_{ik}|^2$

We want to ask: what can we say about $\frac{d}{ds}\sum_{i\neq j} |H_{ij}|^2$?
- This is the sum of the squares of the off-diagonal ($i\neq j$) parts. Does it decrease?

- The full sum is $\sum_{ij} |H_{ij}|^2 = \sum_{ij} H_{ij} H_{ji} = \mathrm{Tr}\, H_s^2$ = constant! The trace is invariant.

$$\text{So } \frac{d}{ds}\sum_{i\neq j} |H_{ij}|^2 = -\frac{d}{ds}\sum_i |H_{ii}|^2 = -2\sum_i \underset{e_i}{H_{ii}} \underset{\text{from equation}}{\frac{dH_{ii}}{ds}} = -4\sum_{i\neq k} e_i(e_i - e_k)$$

$$= -2\sum_{i\neq k} (e_i - e_k)^2 |H_{ik}|^2 \leq 0 !$$

$\Rightarrow$ except for degeneracies, off-diagonal elements decrease.

Th 2a-6

7/11/2013

- The use of T_{rel} instead of $H_d = T_{rel} + H_{ii}$ is ok for nuclear physics, at least in the momentum basis, because
  $T_{rel} \gg (V_s)_{ij}$ so $H_d \approx T_{rel}$

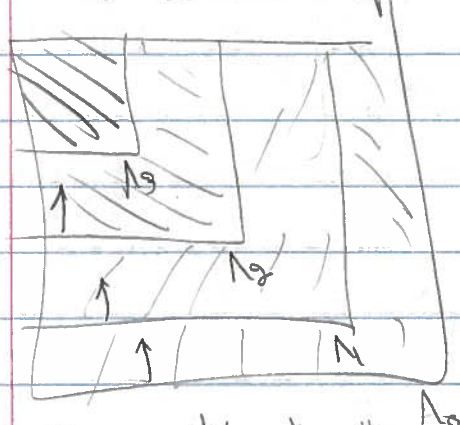
85
Schlund 2

- It can fail, though. $\Rightarrow$ see Wendt et al. with large Λ leading order forces.

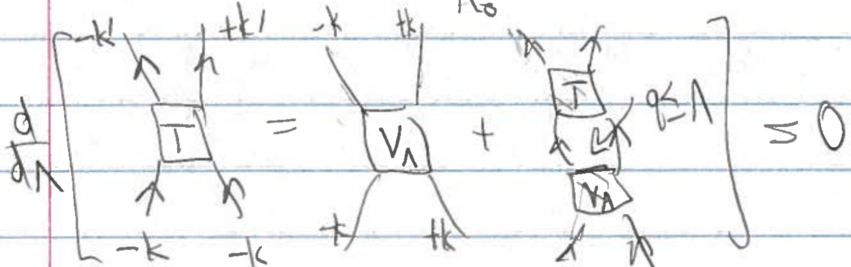
  $\Rightarrow$ good example of decoupling.

7/11/2013

• $V_{\text{low }k}$ RG equation — Bogner, Kuo, Schwenk (2001)



• based on requiring the half-off shell T matrix to be invariant with a change in cutoff on the sum over intermediate states.



initial $\quad T(k',k;k^2) = V_{NN}^{(k',k)} + \frac{2}{\pi}\mathcal{P}\int_0^{\Lambda_\infty} \frac{V_{NN}(k',p)\,T(p,k;k^2)}{k^2-p^2}\, p^2\,dp$

principal value so real

half-on-shell because $k;k^2$ but $p \neq k$

for all $k,k'<\Lambda$ { with cutoff $\quad = V^{\Lambda}_{\text{low }k}(k',k) + \frac{2}{\pi}\mathcal{P}\int_0^{\Lambda} \frac{V^{\Lambda}_{\text{low }k}(k',p)\,T(p,k;k^2)}{k^2-p^2}$

Take $\frac{dT}{d\Lambda}=0 \Rightarrow \frac{d}{d\Lambda}V^{\Lambda}_{\text{low }k}(k',k) = \frac{2}{\pi}\frac{V^{\Lambda}_{\text{low }k}(k',\Lambda)\,T(\Lambda,k;\Lambda^2)}{1-(k/\Lambda)^2}$ ← derivation is not immediate, (see Bogner et al.)

cf. partial wave SRG equation (with $G_s = T_{rel}$)

$$\frac{d}{d\lambda}V_\lambda(k,k') = \left(-\frac{4}{\lambda^5}\right)(k^2-k'^2)^2\,V_\lambda(k,k') + \frac{2}{\pi}\int_0^\infty (k^2+k'^2-2q^2)\,V_\lambda(k,q)\,V_\lambda(q,k')\,q^2\,dq$$

$\boxed{\text{from } \frac{d}{ds} = \frac{d}{d\lambda}\left(\frac{d\lambda}{ds}\right) \text{ and } s=1/\lambda^4}$

• Compare rhs: T matrix for $V^{\Lambda}_{\text{low }k}$ but just potential for SRG

⇒ SRG much easier for $A>2$ (otherwise need T matrix in all channels).

Thda-8

7/11/2013

· Can we get a $V_{low\,k}$-like potential from the SRG flow equation by an appropriate choice of G_s? Yes!

· Use $\frac{dH_s}{ds} = [[G_s, H_s], H_s]$ with $G_s(\Lambda) = \left(\begin{array}{c|c} PH_sP & 0 \\ \hline 0 & QH_sQ \end{array}\right)\Lambda$

· Choose Λ and then G_s is the running Hamiltonian with the off-diagonal blocks defined by Λ set equal to zero.

· P and Q are projection operators, $P+Q=1$.

· Proof that this does what we want [Gubankova et al.]

· A measure of off-diagonal coupling is QH_sP so this is the part that does the coupling. (upper, lower)

$$\Rightarrow \sum_n \langle \psi_n | (QH_sP)^\dagger (QH_sP) | \psi_n \rangle = \mathrm{Tr}[PH_sQH_sP] \geq 0 \quad (\text{since } Q^2=Q,\ P^2=P)$$

$[G_s, H_s] = [PH_sP + QH_sQ, H_s]$

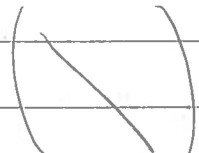
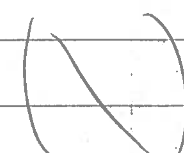
· Now check how this changes with s using $\frac{d}{ds}H_s = [\eta_s, H_s]$

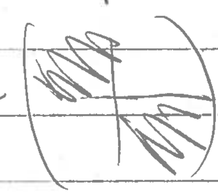
$$\frac{d}{ds}\mathrm{Tr}[PH_sQH_sP] = \mathrm{Tr}[P\eta_sQ(QH_sQH_sP - QH_sPH_sP)]$$

$$+ \mathrm{Tr}[(PH_sPH_sQ - PH_sQH_sQ)Q\eta_sP]$$

you are invited to prove it! $\Rightarrow$ $= -2\,\mathrm{Tr}[(Q\eta_sP)^\dagger(Q\eta_sP)] \leq 0$

$\Rightarrow$ The off-diagonal QH_sP block will decrease (or not increase) as s increases.

· Two examples: $G_s = T_r$  $\rightarrow$ goes to 

$G_s =$  $\Rightarrow$ goes to .

Does it always evolve to the pattern of G_s? See pictures!

7/11/2003

SRG Flow of Operators

- We'll have more to say about operators in a future lecture; just some basic ideas here.

- When we transform $H_s = U(s) H_{s=0} U^\dagger(s)$, the wave functions also get transformed: $|\psi_n^{s}\rangle = U(s)|\psi_n^{s=0}\rangle$, so that energies are unchanged $E_n = \langle\psi_n^{s=0}|H_{s=0}|\psi_n^{s=0}\rangle = \langle\psi_n^{s}|H_s|\psi_n^{s}\rangle$
- So this means that any operator O must be transformed:

$$O_s = U(s) O U(s)^\dagger$$

- We can calculate this directly by constructing $U(s)$ and $U(s)^\dagger$. We do this by first evolving H to H_s, then finding all the eigenstates $|\psi_n^s\rangle$ of H and $|\psi_n^{s=0}\rangle$ of $H_{s=0}$.

- Then we have $U(s) = \sum_n |\psi_n^s\rangle\langle\psi_n^{s=0}|$
  - in a basis like momentum space, this would give us the matrix element $\langle k|U(s)|k'\rangle = \sum_n \langle k|\psi_n^s\rangle\langle\psi_n^{s=0}|k'\rangle$
  - just an outer product.

- This works fine in practice but there are two other ways
  - i) Evolve O_s with its own flow equation
  - ii) evolve $U(s)$ " " " " " and then use $O_s = U(s) O U(s)^\dagger$

What are the equations? $\Rightarrow$ you do that for exercises!

7/11/2013 Thur-10

How do we know that SRG evolution of operators (including the Hamiltonian) must generate many-body terms?

- In exercises: thinking about physics. Here: formal discussion.

- Consider 2nd quantization. This is defined with two ingredients:
  1. a single-particle basis (e.g., plane waves [in a box] or HO w.f.s)
  2. a reference state that serves as the "vacuum".

examples:
- could be the actual vacuum
- or a filled core (Fermi sea or a closed shell)

- Kinetic energy: $T = \sum_i \frac{p_i^2}{2m} a_i^\dagger a_i$

- Two body potential: $\frac{1}{4} \sum V^{(2)}_{ijkl} a_i^\dagger a_j^\dagger a_l a_k$

- 3-body-potential: $\frac{1}{36} \sum V^{(3)}_{ijklmn} a_i^\dagger a_j^\dagger a_k^\dagger a_n a_m a_l$

- These operators have anti-commutation relations:

$$\{a_i, a_j^\dagger\} = a_i a_j^\dagger + a_j^\dagger a_i = \delta_{ij} \quad , \quad \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0$$

Claim, $$\frac{dV_s}{ds} = \Big[\Big[\underbrace{\sum a^\dagger a}_{G_s = T}, \underbrace{\sum a^\dagger a^\dagger a a}_{2\text{-body}}\Big], \underbrace{\sum a^\dagger a^\dagger a a}_{2\text{-body}}\Big] = \cdots + \underbrace{\sum a^\dagger a^\dagger a^\dagger a a a}_{3\text{ body!}}$$

- And this is just one time step!
- $\Rightarrow$ A-body operators generated

show this in the exercises

- Is this a problem?
  - depends: we need to be able to truncate $\Rightarrow$ need hierarchy
  - also need to be able to calculate with minimal (usually 3-body)

In-medium SRG $\Rightarrow$ - Alternative: Pick a different reference state $\Rightarrow$ reshuffles what is many-body!

Th2a-11

7/11/2013

SRG technology is to evolve 3-body forces.

Three methods now exist:

slides by Angelo Calci → i) evolve in a discrete harmonic oscillator basis — Eric Jurgenson

⇒ applied to No-Core Shell Model (tomorrow) — many developments in Darmstadt, P. Navratil

ii) evolve in a partial-wave momentum basis — Kai Hebeler

no disconnected pieces — ⇒ separate evolution of 2 and 3 body parts — (instructor

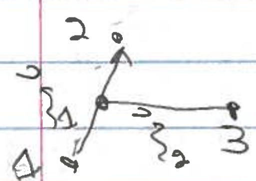
⇒ applied to neutron matter (next week) — next week!)

iii) evolve in a hyperspherical basis — Kyle Wendt

⇒ good features, visualization

- more later on these comparisons
- Recent! 4-body evolution (see Angelo Calci talk from Trento

- Oscillator evolution is in a 3-body Jacobi basis



- generalization of center-of-mass and relative

$\vec{\xi}_0 = \sqrt{\frac{1}{3}}\left[\vec{r}_1 + \vec{r}_2 + \vec{r}_3\right]$ ← potential doesn't depend

$\vec{\xi}_1 = \sqrt{\frac{1}{2}}\left[\vec{r}_1 - \vec{r}_2\right]$ ← relative between 1 and 2.

$\vec{\xi}_2 = \sqrt{\frac{2}{3}}\left[\underbrace{\tfrac{1}{2}(\vec{r}_1 + \vec{r}_2)}_{\text{com of } 1,2} - \vec{r}_3\right]$ ← relative b/wn 3 and com of 1 and 2

hard part! must antisymmetrize

HO basis: $|\alpha\rangle = |[(N_1 L_1 S_1) J_1 (N_2 L_2 S_2) J_2] J M_J, (T_1 T_2) T M_T\rangle$

- momentum space evolution $|p\, q\, \alpha\rangle_i \equiv |p_i q_i; [(LS)J(\ell s_i) j_i] \mathcal{J} \mathcal{J}_z (T t_i) T T_z\rangle$

$|p\,q\,\alpha\rangle_1$ $|p\,q\,\alpha\rangle_2$ $|p\,q\,\alpha\rangle_3$