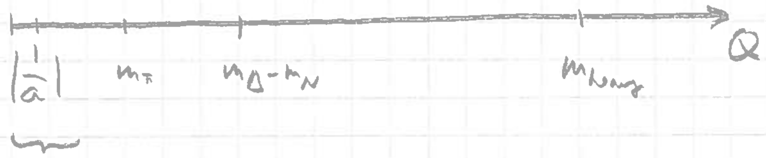
Lecture Renormalization and Universality

Scales in nuclear forces

see yesterday's lecture notes



pionless EFT $\quad Q \ll m_\pi \quad$ systematic expansion in $\dfrac{Q}{\Lambda_{\text{breakdown}}}$

breakdown scale of pionless EFT $\sim m_\pi$

Nonperturbative matching: leading order C_0

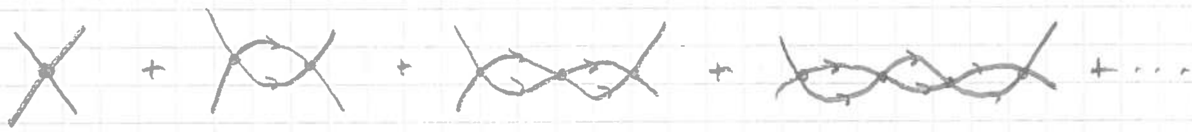
$\rightarrow$ unnatural case
expansion around $\left|\frac{1}{a}\right| = 0$

$$T = \frac{4\pi}{m_N} \frac{1}{\frac{1}{a} - \frac{1}{2} r_e k^2 + ik}$$

NN reduced mass

$\mu = \frac{m_N}{2}$

solve Lippmann-Schwinger equation

$T =$ 

$$= C_0 + C_0 I_0(k,\Lambda) C_0 + C_0 \left(I_0(k,\Lambda) C_0\right)^2 + C_0 \left(I_0(k,\Lambda) C_0\right)^3 + \cdots$$

$$= \frac{C_0}{1 - C_0 I_0(k,\Lambda)} = \frac{1}{\frac{1}{C_0} - I_0(k,\Lambda)}$$

exact solution of LS eqn for $V = C_0$

with $I_0(k,\Lambda) = -\frac{m}{4\pi}\left(ik + \frac{2}{\pi}\Lambda + \underbrace{\mathcal{O}\left(\frac{k^2}{\Lambda}\right)}_{}\right)$

$\hookrightarrow$ small for large Λ

Matching to scattering length only

$$\frac{4\pi}{m}\,\frac{1}{\frac{1}{a}+ik} \overset{!}{=} \frac{1}{\frac{1}{C_0}+\frac{m}{4\pi}\left(ik+\frac{2}{\pi}\Lambda+\mathcal{O}\left(\frac{k^2}{\Lambda}\right)\right)}$$

ik part from intermediate states matches

→ bound states = poles of T matrix are nonperturbative

to get ik part in denominator

matching C_0 to a:

$$\frac{1}{a} = \frac{4\pi}{m}\frac{1}{C_0} + \frac{2}{\pi}\Lambda \quad \Rightarrow \quad \boxed{C_0(\Lambda) = \frac{4\pi}{m}\,\frac{1}{\frac{1}{a}-\frac{2}{\pi}\Lambda}}$$

& running coupling $C_0(\Lambda)$

→ gives cutoff independent results at low k

power counting beyond leading order: resum C_0 interactions

+ treat higher-order 2-body interactions perturbatively

careful about 3-body interactions → Friday

Discuss $C_0(\Lambda)$ in strong and weak interaction limits

(i) strong interactions $\frac{1}{a}=0 \Rightarrow C_0(\Lambda) = -\frac{2\pi^2}{m\Lambda} < 0$

always attractive to give weakly bound or nearly bound state

fine-tuned $\sim \frac{1}{\Lambda}$ to give $\frac{1}{a}=0$

*)

(ii) weak interactions natural a, can choose $|\Lambda a| \ll 1$ for large cutoff range

↳ $T \approx V$

$$C_0(\Lambda) = \frac{4\pi a}{m}\,\frac{1}{1-\frac{2}{\pi}\Lambda a}$$

$$\approx \frac{4\pi a}{m}\left(1+\frac{2}{\pi}\Lambda a+\cdots\right)$$

→ see perturbative matching

*) nonperturbative renormalization for $\frac{1}{a} = 0$

$T =$  $+ \cdots$

$$= C_0(\Lambda) + C_0(\Lambda)\, I_0(k,\Lambda)\, C_0(\Lambda) + C_0(\Lambda)\left(I_0(k,\Lambda)\, C_0(\Lambda)\right)^2 + \cdots$$

$$\begin{array}{ccccc} \wr & & \wr & \wr & \wr \\ \frac{1}{\Lambda} & & \frac{1}{\Lambda} & (ik+\Lambda) & \frac{1}{\Lambda} \end{array}$$

$\underbrace{(ik+\Lambda)\frac{1}{\Lambda}}_{\sim 1}$ all orders equally important

$$= \frac{4\pi}{m} \frac{1}{ik + \mathcal{O}\left(\frac{k^2}{\Lambda}\right)}$$

$\Rightarrow$ Potential $V = V(\Lambda) = C_0(\Lambda)$ is not unique, not an observable

depends on resolution scale Λ $\longrightarrow$ scale dependence

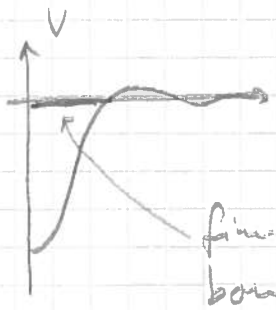
and scheme dependent Q: where did we choose a scheme?
$\longrightarrow$ sharp cutoff

$\longrightarrow$ need to use consistent scheme for currents and many-body forces

Q: Take $\frac{1}{a} = 0$ case. How does V look in coordinate space?

Infinite $\Lambda \rightarrow \delta(\vec{r})$ function

finite $\Lambda \rightarrow$ smeared out δ function



+ nonlocal part

Q: How large are the errors in LO pionless EFT?

Two sources: i) from omitted terms C_2, C_2'

scale as $\left(\frac{Q}{\Lambda_{breakdown}}\right)^2 \sim \left(\frac{Q}{m_\pi}\right)^2$

ii) from regularization: cutoff induces an effective range $\sim \left(\frac{Q}{\Lambda}\right)^2$

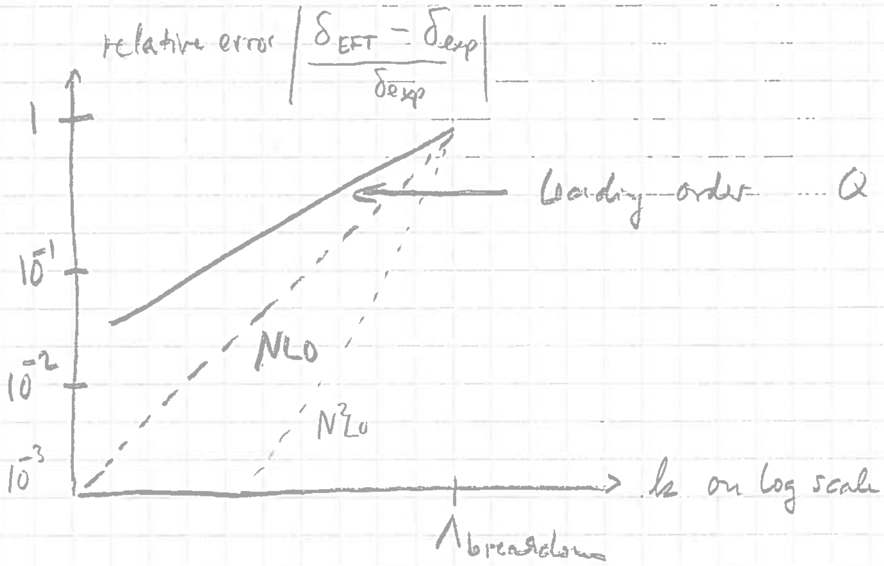
comparison to effective range expansion shows

$O\left(\frac{k^2}{\Lambda}\right)$ term $\rightarrow r_e \sim \frac{1}{\Lambda}$

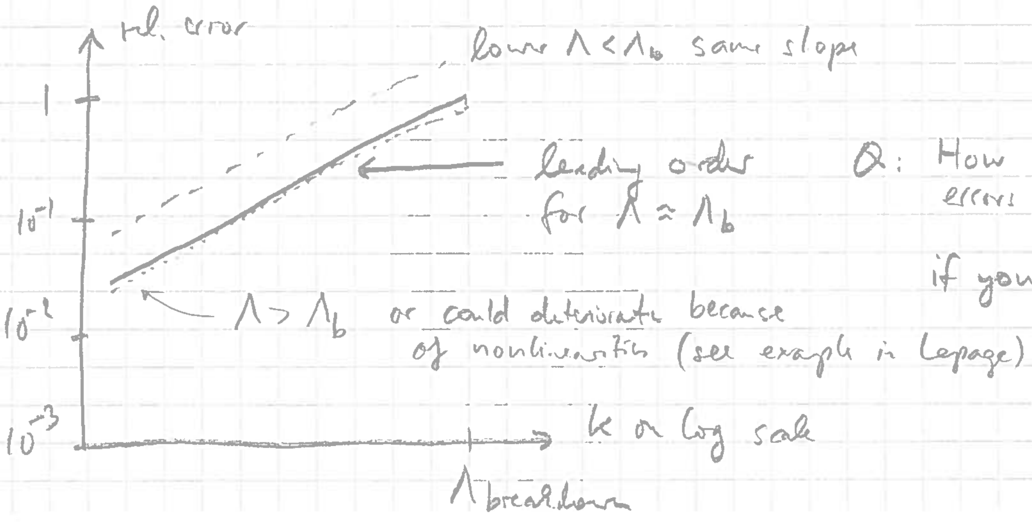
$\longrightarrow$ error $\max\left(\left(\frac{Q}{\Lambda_b}\right)^2, \left(\frac{Q}{\Lambda}\right)^2\right)$

$\Rightarrow$ As long as $\Lambda \gtrsim \Lambda_{breakdown}$ the regularization does not lead to errors larger than from the EFT truncation

Lepage plots $\longrightarrow$ see "How to renormalize the Schrödinger equation"



Q: How do you expect the errors of an NLO calculation to go?



Q: How do you expect the errors to change if you lower Λ?

if you increase Λ?

Can also derive a differential equation for how $C_0(\Lambda)$ runs with Λ:
Renormalization group equation (RG eqn.) by requiring $\frac{dT}{d\Lambda} = 0$

$$\Longleftrightarrow \quad \frac{d}{d\Lambda} \frac{1}{\frac{1}{C_0(\Lambda)} - I_0(k,\Lambda)} = 0 \quad \Longleftrightarrow \quad \frac{d}{d\Lambda}\frac{1}{C_0(\Lambda)} = \frac{d}{d\Lambda} I_0(k,\Lambda)$$

$$\Longleftrightarrow \quad -\frac{1}{C_0(\Lambda)^2}\frac{dC_0(\Lambda)}{d\Lambda} = -\frac{m}{2\pi^2}\Big(1 + \underbrace{\mathcal{O}\Big(\frac{k^2}{\Lambda^2}\Big)}_{\text{small at low } k}\Big)$$

$$\Longrightarrow \quad \frac{d}{d\Lambda} C_0(\Lambda) = \frac{m}{2\pi^2}\big(C_0(\Lambda)\big)^2$$ compare with QCD running coupling α_s

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generalize leading-order pionless EFT to spin

$$V = C_S + C_T\, \sigma_1 \cdot \sigma_2$$

antisymmetrized interaction to include exchange term 

$$V_{antisym.} = (1 - P_{12})\, V$$ with exchange operator $P_{12} = P_{k \leftrightarrow k'}\, P_{spin}$, $P_{spin} = \frac{1 + \sigma_1\cdot\sigma_2}{2}$

$$= (1 - P_{spin})\,(C_S + C_T\, \sigma_1\cdot\sigma_2)$$

use $(\vec{\sigma}_1\cdot\vec{\sigma}_2)^2 = 3 - 2\,\vec{\sigma}_1\cdot\vec{\sigma}_2$

$$= \frac{1}{2}\big(C_S - 3C_T + (3C_T - C_S)\,\vec{\sigma}_1\cdot\vec{\sigma}_2\big) = \begin{cases} 0\,, & S=1 \quad \text{Pauli principle} \\ 2(C_S - 3C_T)\,, & S=0 \end{cases}$$

$\Rightarrow$ so only one linearly independent combination

C_S, C_T are redundant, can pick any one, e.g. $C_T = 0$, or combination

$\Rightarrow$ LO pionless EFT with spin + isospin, $\overset{4}{\vee}$ possible operators

but only 2 S-waves $\qquad \mathbb{1}$, $\vec{\sigma}_1\cdot\vec{\sigma}_2$, $\vec{\tau}_1\cdot\vec{\tau}_2$, $\vec{\sigma}_1\cdot\vec{\sigma}_2\,\vec{\tau}_1\cdot\vec{\tau}_2$

$\Rightarrow$ can pick any 2 of the 4 operators (Fierz ambiguity)

conventional choice $V_{NN}^{LO} = C_S + C_T \sigma_1 \cdot \sigma_2$

NLO: 14 possible operators, but only 7 linearly independent

usual choice (but see Alex's lecture)

$$
\begin{aligned}
V_{NN}^{NLO} = {} & C_2 \tfrac{1}{2}(k^2 + k'^2) + C_2' \, k \cdot k' \\
& + C_2^S \tfrac{1}{2}(k^2 + k'^2)\, \vec{\sigma}_1 \cdot \vec{\sigma}_2 + C_2^{\prime S}\, \vec{k} \cdot \vec{k}'\, \vec{\sigma}_1 \cdot \vec{\sigma}_2 \\
& + i\, C_2^{LS} (\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{k} \times \vec{k}') \quad \rightarrow \text{spin-orbit interaction} \\
& \left.\begin{aligned}
& + C_2^T \, \sigma_1 \cdot (\vec{k}' - \vec{k})\, \sigma_2 \cdot (\vec{k}' - \vec{k}) \\
& + C_2'^T \, \sigma_1 \cdot (\vec{k}' + \vec{k})\, \sigma_2 \cdot (\vec{k}' + \vec{k})
\end{aligned}\right\} \text{lead to tensor interactions}
\end{aligned}
$$

Nonperturbative case $\frac{1}{a} = 0$ corresponds to maximally strong interactions

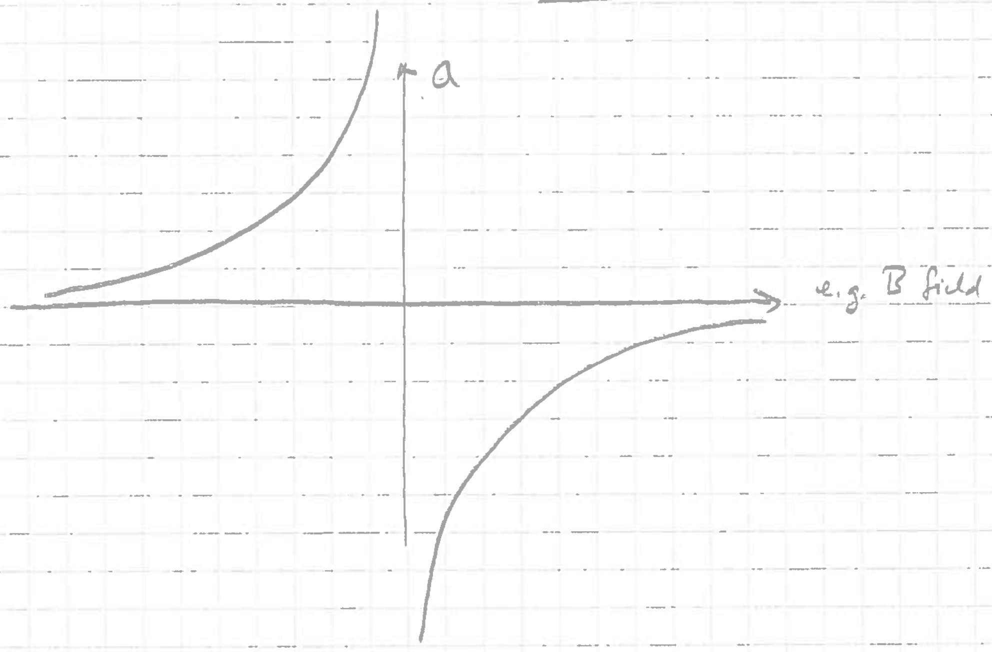
because for $\underbrace{\frac{1}{a} = 0 \text{ and } k r_e \ll 1}_{\delta = \frac{\pi}{2} \text{ for relevant energies (until } k r_e \text{ no longer} \ll 1)}$ $\quad \frac{d\sigma}{d\Omega} = \frac{1}{k^2}$ unitary limit of cross section

in this limit physics is independent of details of the interaction

$\rightarrow$ universal

Discussion problem to prepare for Alex's lecture tomorrow

in atomic gases it is possible to change scattering length a by varying a magnetic (or electric) field $\longrightarrow$ Feshbach resonance



How does the $V = C_0$ potential change across a Feshbach resonance?

keep Λ fixed.

weak $a > 0$ — on resonance $\frac{1}{a} = 0$ — weak $a < 0$