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Lecture T3a: MBPT continued + Evolution of Operators

① Take-away points from natural pionless EFT at finite density

② Traditional G-matrix and hole-line expansion MBPT
- brief discussion of Goldstone vs. Feynman diagrams
- power counting differences: hard vs. soft interaction
- G-matrix vs. low-momentum V
  - ⇒ why MBPT can work with softened V and DFT feasible

③ Operators at different resolution
- operator expectation values
- constructing consistent operators
- RG evolution and interpretation

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Take-away points about natural pionless EFT at finite density

- Low resolution view — coarse-grained impressions

① $\underbrace{\text{Renormalization of UV}}_{\text{high-momentum physics}}$ in free space carries over to $\underbrace{\text{finite density}}_{\text{low-momentum physics}}$

$\Rightarrow$ no new divergences $\Rightarrow$ no new sensitivities to Λ_c

② Energy (density) can be calculated from Feynman diagrams without external lines

reduce a diagram to an integral $\Rightarrow$ numerical in general

- Feynman rules $\rightarrow$ analogous to those from relativistic field theory
  - integrate over both frequency k_0 and 3-momentum $\vec{k}$
  - propagator from $\psi^\dagger\left(i\frac{\partial}{\partial t}+\nabla^2/2m\right)\psi$

$\leftarrow G^0 \propto$ inverse of ( )

find inverse by going to eigenbasis $\Rightarrow$ · take $\psi \propto e^{-ik_0 t}\, e^{i\vec{k}\cdot\vec{x}} \Rightarrow \left(i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}\right)\psi \rightarrow k_0-\frac{k^2}{2m} \equiv k_0-\omega_k$

$$\text{inverse} \xrightarrow[\text{conditions}]{\text{boundary}} \left(\frac{\theta(|\vec{k}|-k_F)}{k_0-\omega_k+i\varepsilon}+\frac{\theta(k_F-|\vec{k}|)}{k_0-\omega_k-i\varepsilon}\right)$$

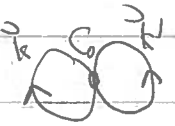
"Feynman propagator"

particle: propagate forward in time

hole: propagate backward in time

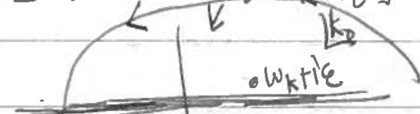
(cf. positive vs. negative energy Dirac propagator)

- integration over frequency $\Rightarrow$ pick up poles

LO: 

$$\frac{1}{2}C_0\left(1-\frac{1}{\nu}\right)\left(i\nu\int\frac{d^3k}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\left[\frac{\theta(|\vec{k}|-k_F)}{k_0-\omega_k+i\varepsilon}+\frac{\theta(k_F-|\vec{k}|)}{k_0-\omega_k-i\varepsilon}\right]e^{ik_0 0^+}\right)^2$$

$\nwarrow$ close in upper half plane



$= \frac{1}{2}C_0\left(1-\frac{1}{\nu}\right)C_0\rho^2$

$\Rightarrow$ residue $2\pi i\,\theta(k_F-|\vec{k}|)$

$\left[\text{since } \rho = \nu\int\frac{d^3k}{(2\pi)^3}\theta(k_F-|\vec{k}|)\right]$

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③ Power counting example $\Rightarrow$ systematic finite ρ example

- energy density diagrams scale as $\left(\frac{k_F}{\Lambda_b}\right)^\beta$ with $\beta = 5 + \sum_{n=0}^{\infty}\sum_{i=0}^{\infty}(3n+2i-5)V_{2i}^{n}$

n-body vertex; $2i$ derivatives; # of $n, 2i$ vertices

- $(3n+2i-5)V_{2i}^{n} \geq 1 \Rightarrow \beta \geq 6$

$n \geq 2$ $\quad 2i \geq 0$ $\quad \geq 1$

- Switch vertex for one with more derivatives

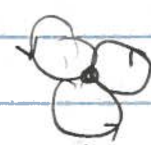
$C_0 \Rightarrow \beta = 6$ $\quad C_2(k^2+k'^2) \rightarrow 2i=2 \Rightarrow \beta = 8$ $\quad \Rightarrow 2i\uparrow, \beta\uparrow$ always

- Add a similar vertex

$V_0^2 = 3 \Rightarrow \beta = 8$ $\quad V_0^2 = 4 \Rightarrow \beta = 9 \Rightarrow V_{2i}^{n}\uparrow, \beta\uparrow$ always

- 3-body?

$\propto k_F^6 \propto \rho^2$ vs.



$\propto \rho^3 = k_F^9$

D_0

$n=3, i=0, V_0^3 = 1 \Rightarrow \beta = 5 + 3\cdot3 + 2\cdot0 - 5 = 9$ ✓

$\Rightarrow$ a finite # of diagram contribute at each order.

- power series? No, because term with $(\nu-2)(\nu-1)(k_F a_0)^4 \ln(k_F a_0)$ ← non-analytic
  - 3-body needed!

- An academic exercise?
  - Is this like low-density neutron matter?
    - no: if $a_0 \gg r_0$, then must sum all diagrams with C_0
      $\Rightarrow$ only numerically (at present)

*slides

- Anything like higher density? Don't we resolve pions? See DFT teaser slides!

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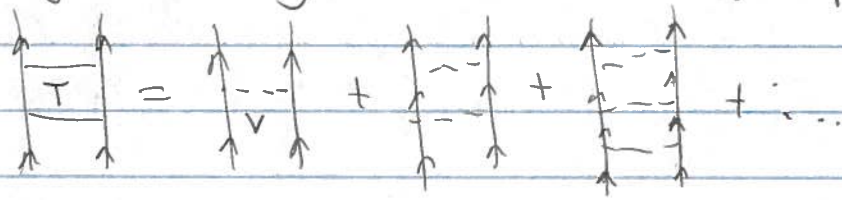
- Reminder: softened potentials at finite density
  - Weinberg eigenvalue analysis applied to the T-matrix Lippmann-Schwinger equation
    $$\hat{T}(\epsilon) = V + V\frac{1}{\epsilon - H_0}V + V\frac{1}{\epsilon - H_0}V\frac{1}{\epsilon - H_0}V + \ldots$$
    showed increased convergence (smaller eigenvalue) when SRG λ or $V_{\text{low-}k}$ Λ is reduced.
  - At finite density, the intermediate states are Pauli blocked $\Rightarrow$ changes the convergence even more $\Rightarrow$ perturbation theory in particle-particle ladders works.

- Diagrammatically, the Lippmann-Schwinger equation is



"particle-particle ladder diagrams"

- That's all there is in free space. At finite density many more diagrams are possible. (Why?)
- The question of MBPT is how to power count these diagrams:
  - What is an organizational principle that allows for systematic calculations?
  - How does the power counting depend on the resolution of the interaction?

- go to finite density by closing lines

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Traditional BBG power counting: G-matrix and hole-line expansion

Plan: Highlight the important features that depend on the interaction and how it changes with low-resolution interactions.

- For (much) more details on the original BBG theory, see the review by B. Day (RMP, 1978) and references therein and the many-body book by Negele and Orland.

- BBG → Brueckner - Bethe - Goldstone
  - Developed to deal with potentials with strongly repulsive cores
- Basics:
  - Write the Hamiltonian as $\hat{H} = \hat{H}_0 + \hat{H}_1$

    where

    $$\hat{H}_0 = \hat{T} + \hat{U} \quad \text{and} \quad H_1 = \hat{V} - \hat{U}$$

    with $\hat{U}$ a single-particle potential to be specified.
  - Great freedom to choose $\hat{U}$ (e.g. could be $\hat{U}_{HF}$ → Hartree-Fock)
    - $\hat{H}_0 |\Phi_0\rangle = E_0 |\Phi_0\rangle$ is the reference state (in a finite system where $\rho(\vec{x})$)
    - In DFT, need freedom to make the density of $|\Phi_0\rangle$ the same as the full, exact density, order-by-order in an expansion

      Kohn-Sham reference system has same density as exact system
    - In conventional BBG, freedom is needed to enhance convergence, so not available for DFT.

- Restatement of time-independent perturbation theory for ground-state energy E:

$$E = E_0 + \langle \Phi_0 | \hat{H}_1 \sum_{n=0}^{\infty} \left(\frac{1}{E_0 - \hat{H}_0} \hat{H}_1 \right)^n | \Phi_0 \rangle_{\text{connected}}$$

$\langle \Phi_0 | \hat{H}_0 | \Phi_0 \rangle$

looks like perturbation theory!

cf. Feynman diagrams
$\Rightarrow$ Feynman perturbation theory has time (or frequency) integrals. Do these and we get
$\downarrow$ time-ordered Goldstone diagrams

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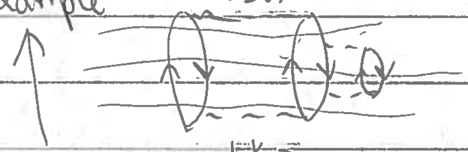
- Diagrammatic expansion called Goldstone diagrams
  - Example

$|\Phi_0\rangle$



$|\Phi_0\rangle$

with ------ $\Rightarrow V^{(2)}$ $\nwarrow$ 2-body

~~Emax~~
convergence
single-particle

- start with $|\Phi_0\rangle$ and apply $|\hat{H}_1\rangle$, which creates all possible particles and holes
  - For $V^{(2)}$, this is two particles and 2 holes
  - $\frac{1}{E_0 - H_0}$ propagates the state $\nwarrow$ particle lines up $\uparrow$ $\varepsilon > \varepsilon_F$
    $\Rightarrow$ involves single-particle energies ~~(sum of particle - sum of hole energies)~~ . hole lines down $\downarrow$ $\varepsilon < \varepsilon_F$

schematic! see refs for details $\Downarrow$

- "Connected" means $|\Phi_0\rangle$ is not an intermediate state
  - of put in $\hat{P} = 1 - |\Phi_0\rangle\langle\Phi_0|$ projector

$$\Rightarrow E_0 - E_0 = \sum_{\substack{\text{connected}\\ \text{diagrams}}} \frac{(-1)^{n_\ell + n_h}}{2^{n_e}} \prod_{(\text{product})} \frac{1}{-\left(\sum_a \varepsilon_a - \sum_A \varepsilon_A\right)} \prod \{ij|\hat{V}_{NN}|k\ell\}$$

anti-symmetrized

n_e = # energy denom.
n_ℓ = # of closed loops
n_h = # of hole lines

single particle energies according to H_0 $\Rightarrow$ hole energies $\uparrow$ particle energies $\uparrow$ $\Leftarrow$ sum of particle ~~- hole energies~~

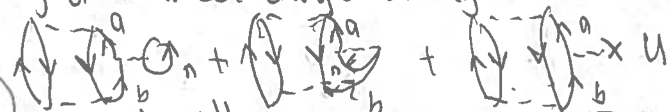
- The details are not so important to us as the basic organization and the consequences for a diagrammatic expansion.

For potentials ~~with~~ repulsive cores $\}$. What happens if you try to apply this in an expansion in the number of times $\hat{H}_1$ acts? Two infinite-order resummations needed:

① Successive particle-particle ladders within a series of diagrams are all the same size $\Rightarrow$ sum into G matrices

② Expansion in # of G matrices is still not perturbative: only adding an independent hole line to a diagram makes it smaller
$\Rightarrow$ sum all diagrams with a given number of hole lines (infinite!)

Choosing $\hat{U}$ to cancel diagrams. E.g. if $\hat{U} = \hat{U}_{HF}$, the Hartree Fock potential, then



same except middle: $\sum_{n \le \epsilon_F} \langle an|V|bn\rangle - \langle an|V|nb\rangle - \langle a|U_{HF}|b\rangle = 0$

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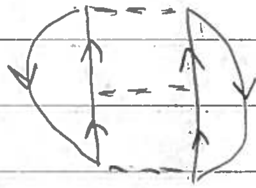
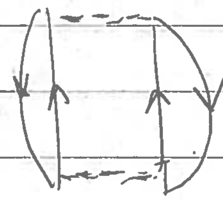
MBb-13

Let's start with numerical study of the ratio of

3rd order in $V^{(2)}$



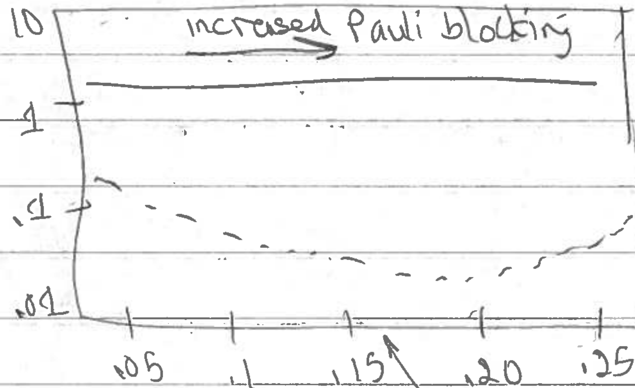
to 2nd order in $V^{(2)}$



Is it bigger or smaller? (And 4th to 3rd, 5th to 4th, etc.)

(see slides later)

[log plot]



$\frac{\text{3rd order}}{\text{2nd order}}$

← AV18 (ratio 1.7 independent of density)

← evolved by $V_{low\,k}$ RG to $\Lambda = 2.1\,\mathrm{fm}$

$\rho\,[\mathrm{fm}^{-3}]$

saturation density

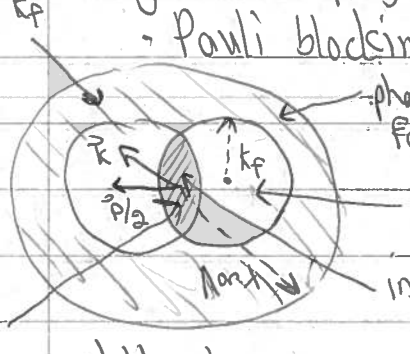
- Uses Hartree-Fock (HF) continuous single-particle spectrum
- contributions from higher-orders for AV18 have similar ratio

- N^3LO picture is much more favorable (less than unity) but still a dramatic reduction when evolved to $V_{low\,k}$ $\Lambda = 2\,\mathrm{fm}^{-1}$ or SRG $\lambda = 2\,\mathrm{fm}^{-1}$

- Why does decoupling high and low momentum lead to smaller contributions?
  - Pauli blocking and weaker interaction in relevant phase space for S-wave

com and relative momenta of pair

$|\frac{\vec{P}}{2} \pm \vec{k}| > k_F$ and $|\vec{k}| < \Lambda$



phase space for scattering

Pauli blocked volume

in Fermi sea

$|\frac{\vec{P}}{2} \pm \vec{k}| < k_F$

available phase space for in-medium NN scattering

- For large Λ and strong repulsive $V^{(2)}$, contribution is from large region and part excluded by Pauli blocking is small
  $\Rightarrow$ density independent.
- For low Λ, there is a smaller and smaller region as density increases, and matrix elements are weaker there

(MBb-14)

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- So SRG or $V_{low k}$ at $\lambda, \Lambda \lesssim 2\,\mathrm{fm}^{-1}$ is perturbative, each ladder rung added makes the diagram smaller
  - exactly consistent with Weinberg eigenvalues.

- For hard potentials, must add up the rungs, as we did in free space to form the T-matrix

$$\frac{1}{E-H_0} \text{ (free space)} \longrightarrow \frac{Q_F}{E'-H_0} \text{ (in-medium)}$$

Q_F ← Pauli-blocking operator

- It has been said in the literature that $V_{low k}$ interaction is just like a G-matrix,
  - But only true at low momentum (under certain assumptions for the treatment of the single-particle energies)
  - ✱ but there is still a lot of off-diagonal strength in the G-matrix and this makes the energy still non-perturbative in the G-matrix while perturbative in the $V_{low k}$ or V_{SRG} low-momentum potential

(see pictures)

- Hole-line expansion: power counting for the G-matrix
  - an analysis shows that the size of a diagram with conventional potentials doesn't relate to how many particle lines there are, but how many (independent hole lines)
  - see slides for examples of 4th order to third-order diagram, where one 4th order has another particle line while the other adds a hole line. ← extra hole line

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- The estimate for the ratio of 4^{th} to 3^{rd} order when a particle line is added goes like the defect wave function at the origin. (in coordinate space) [see B. Day, RMP (1967)]
  - This is the two-particle relative wf compared to non-interacting relative plane waves.
  - This defect is almost complete for highly repulsive cores but largely gone for low-momentum $^1S_0$ and greatly reduced for $^3S_1$

see graphs ⇒

  ⇒ adding an interaction doesn't reduce the diagram in the hard case (so resum all) but does in the soft case.

- When a hole line is added, the relevant expansion parameter is the excluded volume at short-range to the average volume occupied by a particle (proportional to $1/\rho$).
  - This is the so-called "wound-integral" κ
  - It is less than one ⇒ expansion for even hard potentials.
  - Still better for soft potentials (and no resummation)

- Bottom line: for soft potentials, adding another potential line reduces the size of the diagram (caveat: in all cases considered so far)

  ⇒ real perturbation theory may work. ⇐ tested for neutron matter so far with QMC comparison ✓

* • Regular hole-line expansion also needed to choose $\hat{U}$ to cancel diagrams to enhance convergence.
  - With low-momentum potentials we are free to choose it to make Kohn-Sham DFT work.

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MBPT in finite nuclei (teaser)

- Here we'll just briefly mention two applications of MBPT in finite nuclei with some slides.
  - Achim will have more to say later this week on the shell model applications.

- High-order Rayleigh-Schrödinger MBPT in nuclei (Roth et al.)
  - SRG-evolved two-body interactions based on an initial N^3LO interaction
  - Perturbation series diverges even for very soft potentials
  - A simple resummation with Padé approximants results in stable energies in agreement with exact NCSM calculations in the same harmonic oscillator model space.

X ⟹ see slide

- Direct use of perturbative methods in microscopic valence-shell calculations
  - a small number of nucleons outside a closed-shell core interact via an appropriate effective interaction treated in MBPT
    - includes nonperturbative transformation to remove the energy dependence of the MBPT effective Hamiltonian
  - one application: impact of 3NF on location of neutron dripline
  - mass prediction for calcium isotopes contradicted existing masses but validated by new, high precision measurements

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Operator expectation values (Exercise)

- Suppose we would like to know the contribution to the energy of a nucleus from the three-body force only?
  - e.g., If $\hat{H} = \hat{T} + \hat{H}^{(2)} + g_3\hat{H}^{(3)}$ and we want to know the expectation value of $g_3\hat{H}^{(3)}$ in the state $|\Psi\rangle$.
  - We could compare energies with and without $g_3\hat{H}^{(3)}$ included, but that would only be quantitatively correct if we knew $g_3\hat{H}^{(3)}$ was a small perturbation.

- Better: use the Hellmann-Feynman (or Feynman-Hellmann) theorem:

$$\frac{dE(\lambda)}{d\lambda} = \langle\Psi(\lambda)|\frac{\partial\hat{H}_\lambda}{\partial\lambda}|\Psi(\lambda)\rangle \quad \text{where } \hat{H}_\lambda|\Psi(\lambda)\rangle = E(\lambda)|\Psi(\lambda)\rangle$$

  - What would you choose for λ to find $\langle\Psi|g_3\hat{H}^{(3)}|\Psi\rangle$?

- How could you find $\langle\Psi|\hat{O}|\Psi\rangle$ for any operator $\hat{O}$ if you add it to the Hamiltonian $\hat{H} \to \hat{H} + \lambda\hat{O}$?

$\Rightarrow$ prove the theorem and work out the details as an exercise.

7/16/13 (T3a-5)

Constructing consistent operators

- How do we consistently match a Hamiltonian and operators for an observable? (e.g. for electromagnetic form factors)
  $\Rightarrow$ use EFT perspective and tools

- We built chirally symmetric Hamiltonians (Lagrangians) by identifying chiral building blocks and then constructing all operators.

- Example: electromagnetic current in deuteron (D.R. Phillips nucl-th/0503044)
  - for electromagnetism, use gauged derivative
    $\partial_\mu \rightarrow D_\mu = \partial_\mu + (\text{pion part from before}) - \frac{ie}{2}(1+\tau_3)A_\mu$
    heavy baryon formalism

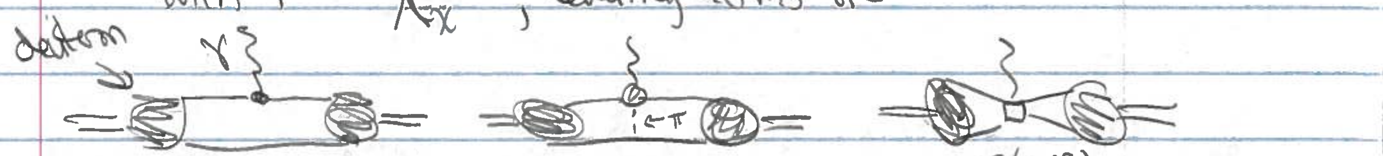
    and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
    $\Rightarrow$ include all terms in Lagrangian
  - organize (power count) by

$$J_\mu = e\sum_{i=1}^{\infty} c_i \frac{1}{\Lambda_\chi^{i-1}} O_\mu^{(i)}$$

order(1) coefficient

operator with $i-1$ powers of p (momentum of nucleons in deuteron), m_π, or Q (four-momentum transfer)

  - with $P \equiv \frac{p, Q, m_\pi}{\Lambda_\chi}$, leading terms are



$O(e)$ $\quad$ $O(eP^3)$ $\quad$ $O(eP^5)$

use same regulators, other vertices

  - model independent because complete (up to some order)
  - tells you when new info is needed (i.e., a new LEC required)
- Use $\overline{RG}$ as a tool to consistently evolve operators

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RG evolution of operators

- Recall SRG evolution of Hamiltonian:

$$\hat{H}_s = U(s)\hat{H}_{s=0}U^\dagger(s) \Rightarrow \frac{d\hat{H}_s}{ds} = [\eta_s, H_s] = [[G_s, H_s], H_s]$$

- From an exercise, operators evolve by

expect many-body operators. exercise: does one-body part ($a^\dagger a$) change? No!

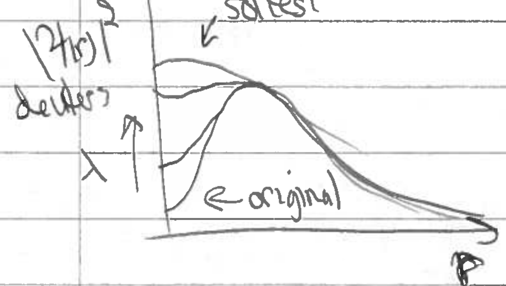
$$\hat{O}_s = U(s)\hat{O}U^\dagger(s) \Rightarrow \frac{d\hat{O}_s}{ds} = [[G_s, H_s], O_s]$$

or construct $U(s) = \sum_i |\psi_i(s)\rangle\langle\psi_i(0)|$ or $\eta_s = \frac{dU(s)}{ds}U^\dagger(s) \Rightarrow \frac{dU(s)}{ds} = \eta_s U(s)$

- If we evolve the operator, matrix elements are trivially unchanged $\langle\psi(s)|\hat{O}_s|\psi(s)\rangle = \langle\psi(0)|U^\dagger(s)U(s)\hat{O}U^\dagger(s)U(s)|\psi(0)\rangle$

$$= \langle\psi(0)|\hat{O}|\psi(0)\rangle$$

- What if we don't evolve the operator?
  - Recall that the wave function gets modified at short distances



$(\text{rms radius})^2 = \int |\psi(r)|^2 d^3r \cdot r^2$ changes!

or $\hat{Q} = \langle 3z^2 - r^2\rangle$ quadrupole moment

- changes because $\psi(\vec{r})$ does
- less change
- which answer is "correct"?

$\Rightarrow$ look at slides