TALENT/INT Course on Nuclear Forces
Exercises and Discussion Questions W2

[Last revised on July 10, 2013 at 01:41:51.]

Wednesday 2: RG 1; Nuclear forces from lattice QCD

We have again grouped all of the two-minute and discussion questions toward the beginning. But remember to spend only about an hour working on questions and then try some of the other problems as well. When you need a break, go back and try another question!

1. Two minute and discussion questions:

   (a) Give some reasons "why the renormalization group is a good thing" for nuclear physics calculatinons.

   (b) If we only need to use a potential at low energies, why not just set to zero its matrix elements that couple low to high energies?

   (c) We can repeat any nuclear structure calculation for different values of the SRG λ. Why might this be a useful (i.e., informative) thing to do? [Hint: Observables are supposed to be unchanged under a unitary transformation. What does it tell us if they *do* change with different λ?]

   (d) How big does a lattice have to be to do nuclear physics with $A > 1$? Are the large scattering lengths a fatal problem?

   (e) Dimensionality of quark propagator matrix. A typical lattice volume used in LQCD calculations of nuclear systems is $48 \times 48 \times 48$ where 48 indicates that the lattice spatial extent in each direction is 48 times the lattice spacing used in the calculation. Given that the quark propagator is a matrix in the space of spin, color and spatial coordinates, evaluate the dimensionality of this matrix for a u-quark propagator.

   (f) Signal to noise issue. What would be the signal to noise ratio for a LQCD calculation of the correlation function of a nucleus with A nuclei? Estimate this ratio for ^{4}He and ^{208}Pb for a lattice time $t = \frac{T}{2} = \frac{64\times 0.1\,\mathrm{fm}}{2}$. (Be careful about dimensions. Use the conversion factor $\hbar c$ if necessary.)

   (g) Exponential corrections. How large do the lattice volumes need to be to be able to obtain the scattering phase shifts up to 1% percent error at physical pion mass from the Luscher formalism? How about for $m_\pi = 300$ MeV and $m_\pi = 800$ MeV? (Consider the systematic uncertainty due to the range of interaction only and assume other systematics are lower than percent level.)

   (h) Deuteron. How large do the lattice volumes need to be to be able to obtain the deuteron binding energy up to 2% percent error at physical pion mass? (Consider the systematic uncertainty due to the finite size of the deuteron and assume other systematics are lower than percent level.)

2. Fill in all the steps of the derivation of the SRG flow equation

$$\frac{dH_s}{ds} = [\eta(s), H_s]$$

where $H_s = U_s H_{s=0} U_s^\dagger$ and

$$\eta(s) \equiv \frac{dU(s)}{ds} U^\dagger(s) \ .$$

Note that T is the kinetic energy and V_s is *defined* as $H_s - T$.

3. Start with the SRG equation

$$\frac{dV_s}{ds} = [[T_{\rm rel}, H_s], H_s] \ ,$$

and derive the two-body partial wave flow equation:

$$\frac{dV_s(k,k')}{ds} = -(k^2 - k'^2)^2 V_s(k,k') + \frac{2}{\pi}\int_0^\infty q^2\, dq\, (k^2 + k'^2 - 2q^2) V_s(k,q) V_s(q,k') \ ,$$

where $V_s(k,k') \equiv \langle klm|V_s|k'lm\rangle$ and we use units where $\hbar^2/m = 1$.

(a) What is $T_{\rm rel}|k\rangle$ in these units?

(b) Show that if the first term on the right side dominates for $k \neq k'$, than the solution to the equation is

$$V_s(k,k') = V_{s=0}(k,k') e^{-s[(\epsilon_k - \epsilon_{k'})]^2} \ ,$$

where $\epsilon_k \equiv k^2/m$.

(c) What does this last result imply about the width in k^2 of the non-zero part of V_s in terms of $\lambda = 1/s^{1/4}$, which has dimension of momentum? [Hint: how does it compare to λ^2?]

4. Large volume expansion of Luscher formula for S-wave scattering. This problem guides you through a simple derivation of the Luscher formula for S-wave scattering in the large volume limit. The derivation is entirely based on a quantum mechanical perturbation theory. Work out the problem in parts (d) and (e) to first order in perturbation theory. The second order approximation is recommended for those who are interested to fully understand the essence of Luscher formalism. The problem is based on Luscher 1986 paper, and a University of Washington qualification exam problem in 2007!

The Hamiltonian of a system is $\hat{H} = \hat{H}_0 + \hat{V}$, with $\hat{V}$ a perturbation. Eigenstates $\left|p^{(0)}\right\rangle$ of the unperturbed Hamiltonian $\hat{H}_0$ are labeled by some index p and satisfy $\hat{H}_0 \left|p^{(0)}\right\rangle = E_p^{(0)} \left|p^{(0)}\right\rangle$. Consider a three-dimensional cubic volume with sides of length L. Particles are placed inside this volume subject to periodic boundary conditions.

(a) What are the eigenstates of a single particle confined to this volume?

(b) What are the eigenstates of two non-interacting particles confined to this volume with zero total momentum?

(c) What is the ground state energy of two particles confined to this volume that interact with each other via the interaction

$$\hat{V} = \eta \delta^3(\hat{\mathbf{r}_1} - \hat{\mathbf{r}_2}) \tag{1}$$

evaluated out to first (second) order in perturbation theory, assuming that $\eta/L \ll 1$. Do not evaluate any sums or integrals that may occur.

Now consider very low-energy scattering of two particles interacting via this potential.

(d) Using the Lippman-Schwinger equation, determine the scattering amplitude to first (second) order in the potential, and thereby show that

$$\eta = -\frac{4\pi a}{M}\left[1 - 4\pi a \int \frac{d^3p}{(2\pi)^3}\frac{1}{|\mathbf{p}|^2 + i\epsilon}\right] \tag{2}$$

where a is the scattering length.

(e) Use this result to show that the energy shift of two particles in the cubic volume is

$$\Delta E_0 = -\frac{4\pi a}{ML^3}\left[1 + \frac{a}{L}\left(\sum_{\mathbf{n}\neq 0}^{\Lambda}\frac{1}{|\mathbf{n}|^2} - 4\pi\Lambda\right) + ...\right] \tag{3}$$

where the sum extends over all integer triplets, $\mathbf{n} = (n_1, n_2, n_3)$ (which are not identically zero) with $|\mathbf{n}| = \sqrt{\mathbf{n}.\mathbf{n}} < \Lambda$, and it is understood that the limit $\Lambda \to \infty$ is to be taken.

(f) Show that to leading (next to leading) order in a/L expansion, the Luscher formula presented in the lecture reduces to Eq. (3).

5. In slide 34 from the first lecture, titled "Run to Lower λ via SRG $\Longrightarrow \approx$ Universality", the curves in the final figure on the left disagree for some values of k while they are almost all the same in the figure on the right. How do you explain this?

6. Weinberg eigenvalue analysis of convergence. We would like to know if the SRG is making the two-body problem more perturbative. A measure of perturbativeness is provided by Weinberg eigenvalues.

(a) Consider the operator Born series for the T-matrix $T(E)$ at energy E:

$$T(E) = V_s + V_s\frac{1}{E - T_{\text{rel}}}V_s + V_s\frac{1}{E - T_{\text{rel}}}V_s\frac{1}{E - T_{\text{rel}}}V_s + \cdots$$

(warning: don't confuse $T(E)$ with the kinetic energy T_{rel}!). As a review, show that this is equivalent to the operator Schrödinger equation. [Hint: First you need to show that the series is the expansion of the Lippmann-Schwinger equation $T = V + VG_0T$, where $G_0(E) = 1/E - T_{\text{rel}}$.]

(b) At fixed $E \le 0$, suppose we have found all of the (Weinberg) eigenvalues $\eta_\nu(E)$ and eigenvectors $|\Lambda_\nu\rangle$ of

$$\frac{1}{E - T_{\rm rel}} V_s \; .$$

What is the result of acting with $T(E)$ on one of these eigenvectors?

(c) Use the result of the last part to explain how the values of $\eta_\nu(E)$ tell us whether (and how rapidly) the Born series converges. [Hint: if all eigenvalues have magnitude less than 1, the Born series converges.]

(d) Can you devise a way to numerically find these Weinberg eigenvalues and eigenvectors? [Hint: think about solving a matrix equation.]