TALENT/INT Course on Nuclear Forces
Exercises and Discussion Questions T1

[Last revised on July 9, 2013 at 19:45:37.]

Tuesday 1: Nuclear forces 1; Scattering theory 2

1. Two minute and discussion questions.
   (a) Why is a term like $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\tau_2}$ not allowed in the nuclear potential?
   (b) Keeping in mind that $\boldsymbol{\tau}$ matrices do not commute, is $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = \boldsymbol{\tau}_2 \cdot \boldsymbol{\tau}_1$? [Hint: be clear about what spaces these act in!]
   (c) Why do we not consider terms in the potential of the form $(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)^n$ for $n > 1$ or $(\boldsymbol{L} \cdot \boldsymbol{S})^n$ for $n > 2$?
   (d) What are the electromagnetic interactions for np and nn scattering?
   (e) Add to the list of physical situations where a parameter is taken either to 0 or infinity to simplify a problem (and then one can expand around this limit).
   (f) You will often hear that effective (field) theories exploit a "separation of scales". What does this mean?
   (g) How high in energy do we need to know NN phase shifts to do nuclear structure?
   (h) Why should a strong repulsive short-range potential make (some) many-body calculations of nuclei more difficult? Would a mean-field approximation be accurate?
   (i) You often hear it said that the problem must be nonperturbative because there are bound states. Why can't you find bound states in perturbation theory?
   (j) Why is the deuteron bound but two neutrons are not bound? Why are two protons not bound?

2. Two-minute questions on the effective range expansion and phase shifts.
   (a) If you had two candidate potentials that had the same effective range expansion coefficients (out to the order you could measure them), how could you tell them apart experimentally? [Hint: trick question!]
   (b) Find the scattering length and effective range and shape parameter (i.e., the first three terms of the effective range expansion) for a hard sphere of radius R (that is, the potential is infinite for $r < R$ and zero for $r > R$). Are the parameters natural or unnatural? (E.g., can they all be reasonably estimated by naive dimensional analysis.) What is the radius of convergence for the expansion in this case? (That is, for what range of k does the expansion converge?)

(c) Why is the deuteron binding energy different from $\frac{\hbar^2}{m a_S^2}$, where a_S is the scattering length?

(d) Which S-waves contribute to np scattering?

(e) The central part of the one-pion-exchange NN potential is given by $V_{\mathrm{OPE}}(r) \sim m_\pi^2\, e^{-m_\pi r}/r$. Discuss how the pion-mass dependence can explain why the scattering length in the ${}^1S_0$ channel is more attractive for np, $a_S = -23.7\,\mathrm{fm}$, than for nn, $a_S = -18.5\,\mathrm{fm}$.

(f) List all coupled channels for $J \leqslant 4$.

(g) Why do only central interactions contribute to the average P-wave phase shift (defined in the scattering theory 2 lecture)?

3. Basic skills.

(a) Verify:

$$e^{i\widehat{S}}\widehat{O}e^{-i\widehat{S}} = \widehat{O} + i[\widehat{S},\widehat{O}] + \frac{i^2}{2!}[\widehat{S},[\widehat{S},\widehat{O}]] + \frac{i^3}{3!}[\widehat{S},[\widehat{S},[\widehat{S},\widehat{O}]]] + \cdots$$

explicitly to this order (or until you run out of patience :).

(b) If $U = e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{G}}$ is a symmetry operation, show that it is necessary and sufficient that $[\boldsymbol{G},\widehat{O}] = 0$ for an operator $\widehat{O}$ to be invariant under the symmetry.

4. Devise a way to estimate the range of the parts of the NN interaction from pion exchange, "σ" exchange (which is generated by correlated two-pion exchange at a mass of about 500 MeV), and ω exchange (look up the mass online if necessary). Are your results consistent with the pictures of the ${}^1\mathrm{S}_0$ potentials in the lecture notes?

5. What do we mean when we say that probes or particles at low energy do not "resolve" fine detail? What size structure would a 5 MeV neutron resolve? [Hint: recall diffraction.]

6. What is the first (i.e., lowest energy) source of inelasticity in NN scattering? Find an expression in terms of the pion and nucleon masses for the laboratory energy of the inelastic threshold. Why are phenomenological potentials typically fit to elastic phase shifts for 50 MeV above this threshold?

7. The usual plot you see of the central nucleon-nucleon (NN) potential is in the ${}^1\mathrm{S}_0$ channel (e.g., in the figures shown in the lectures). The notation is ${}^{2S+1}L_J$, with $S = 0$ (singlet) or $S = 1$ (triplet), $L = 0, 1, 2, \ldots$ (with the corresponding letter), and J takes on values consistent with L and S.

(a) What are the possible channels for $L = 0$, 1, and 2 for neutron-proton scattering?

(b) Same question, but for neutron-neutron scattering. (Hint: nucleons are fermions, so the total two-particle wave function must be antisymmetric.)

(c) Experimental: What are the difficulties and advantages of scattering neutrons from proton targets versus protons from neutron targets?

8. Questions about nucleon-nucleon phase shift plots (e.g., from NN-Online).

   (a) Is the partial wave analysis leading to these graphs model independent? That is, are there assumptions made? If so, what are they?

   (b) Go to the NN-OnLine website linked under Miscellany→Links→Nuclear Resources and generate graphs of neutron-proton scattering phase shifts from 0 to 350 MeV lab energy for some different channels.

   (c) For the D-wave phase shifts, which ones have attractive interactions and which ones have repulsive interactions?

   (d) Compare ${}^1\mathrm{S}_0$ to ${}^1\mathrm{D}_2$ and use the results to estimate the radial extent of the repulsive core for a local potential. (Hint: What does the centrifugal barrier in the D state do? You can use a classical argument.) [It would be great to test out such an argument with the numerical calculations!]

   (e) Extract the scattering lengths (and effective ranges, if possible) from the phase shift data for np scattering and compare to quoted answers.

   (f) Neutrons form Cooper pairs in neutron stars. At low densities/momenta, neutrons pair in the ${}^1\mathrm{S}_0$ channel where the NN interaction is most attractive. As the S-wave interaction becomes repulsive with increasing density/momentum, in which channel are neutrons expected to pair?

9. Repulsive-attractive square well as a test laboratory using the VPA. By this we mean a combined repulsive square well of radius R_c and height V_c (the "core") and an attractive square well of radius R_0 and depth $-V_0$. This is already implemented for the VPA in the Mathematica notebook `square_well_scattering.nb` and in an iPython notebook.

   (a) Play with the value of V_c with R_c set to a reasonable value (given that we are in units where $R_0 = 1$); what do you observe?

   (b) Convince yourself that a local potential must have a strongly repulsive core to be consistent the observed sign change of the phase shifts in the NN ${}^1\mathrm{S}_0$ channel.

   (c) Suppose we have a momentum dependent potential without a hard core as an alternative. Can we still get a sign change?

10. How can you generate an infinite positive scattering length with a square well potential? Is it possible for a potential to lead to an infinite positive scattering length without having a bound state?

11. Estimate the radius and energy of hydrogen-like atoms using dimensional analysis. This is on p. 7 of the lecture notes on QCD1 (but we didn't cover it in the lecture).

12. We claim the following hierarchy of (three) scales for a hydrogen atom:

   - electron mass $m_e \approx 0.511\,\mathrm{MeV}$
   - characteristic momentum $p \sim \alpha m_e \approx 3.6\,\mathrm{keV}$
   - characteristic energy $B \sim \frac{1}{2}\alpha^2 m_e \approx 13.6\,\mathrm{eV}$

   Derive the scaling with α and m_e by simple scaling arguments (that could be applied to other systems). In particular,

   (a) Apply the uncertainty relation to relate the momentum p to the characteristic radius R (in this case it is the Bohr radius, but we pretend we don't know it yet).

   (b) Use this to eliminate p from the total energy (sum of kinetic and potential) to find $E(R)$.

   (c) Minimize $E(R)$ to find R and therefore p and then the value at the minimum, verifying the results quoted above.

   (d) For this example, why is there a hierarchy? Would there be a hierarchy of the same type in QCD with the strong coupling α_s instead of the fine structure constant?

   (e) What is the analogous hierarchy exploited in chiral (i.e., pionful) effective field theory?

13. T-matrix for a separable potential [adapted from Taylor, Scattering Theory]. A separable potential has the form

$$\widehat{V} = g|\eta\rangle\langle\eta| \,,$$

   where we usually choose $|\eta\rangle$ to be a normalized vector given, for example, by its momentum space function $\eta(\mathbf{k}) \equiv \langle\mathbf{k}|\eta\rangle$ (note that we're not in partial waves here). Recall the Lippmann-Schwinger equation for the operator $T(z)$ described in the Scattering Theory 1 notes (long version):

$$\widehat{T}(z) = \widehat{V} + \widehat{V}\frac{1}{z-\widehat{H}_0}\widehat{T}(z) = \widehat{V} + \widehat{V}\frac{1}{z-\widehat{H}_0}\widehat{V} + \widehat{V}\frac{1}{z-\widehat{H}_0}\widehat{V}\frac{1}{z-\widehat{H}_0}\widehat{V} + \cdots \,.$$

   (a) Show that $T(z)$ is given explicitly by

$$T(z) = \frac{g\,|\eta\rangle\langle\eta|}{1-g\Delta(z)} \,,$$

   where

$$\Delta(z) = \langle\eta|\frac{1}{z-H_0}|\eta\rangle = \int d^3k\,\frac{|\eta(\mathbf{k})|^2}{z-E_k}$$

   with $E_k = k^2/2\mu$. [Hint: substitute the separable form for $\widehat{V}$ into the Born series for $T(z)$ and note the form of each term.]

   (b) Show that the Born series for $T(z)$ is convergent for g small but divergent for g large.

   (c) The poles of $T(z)$ as a function of complex energy z tells of about the bound states of the potential (see the end of the Scattering Theory 1 notes). Show that the separable potential has either one or no bound states.

14. [Advanced] Deriving the Coulomb potential from QED by actually integrating out the photon field. (This is an alternative to matching QED and potential calculations of scattering, which we could also do.) Consider the QED Lagrangian including gauge-fixing:

$$\begin{aligned}\mathcal{L}_{\text{QED}} &= \frac{1}{2}A_\mu[g^{\mu\nu}\partial_\lambda\partial^\lambda - (\xi^{-1}-1)\partial^\mu\partial^\nu]A_\nu - j_e^\mu A_\mu + \overline{\psi}(i\partial\!\!\!/ - m)\psi \\ &= \frac{1}{2}A_\mu[D_{\text{F}}^{\mu\nu}]^{-1}A_\nu - j_e^\mu A_\mu + \overline{\psi}(i\partial\!\!\!/ - m)\psi\ ,\end{aligned}$$

with electromagnetic current (charge e) $j_e^\mu = e\overline{\psi}\gamma^\mu\psi$. (The second line defines D_{F} as the inverse of the operator in the first line, where F means to use Feynman boundary conditions.) The physics of electrons and photons can be derived (e.g., Feynman diagrams) from the functional (path) integral:

$$Z = \int \mathcal{D}\overline{\psi}\,\mathcal{D}\psi\,\mathcal{D}A\ \exp[iS(\overline{\psi},\psi,A)]$$

(after adding external sources, which we omit here). But suppose we "integrate out" the photon field A_μ (which we can do because it appears at most quadratically):

$$\exp[iS_{\text{eff}}(\overline{\psi},\psi)] = \int \mathcal{D}A\ \exp[iS(\overline{\psi},\psi,A)]\ .$$

(a) Complete the square to show (with $j_e^\mu = e\overline{\psi}\gamma^\mu\psi$)

$$S_{\text{eff}} = \int d^4x\,\overline{\psi}(x)(i\partial\!\!\!/ - m)\psi(x) + \frac{1}{2}\int d^4x\,d^4y\,j_e^\mu(x)D_{\text{F}\mu\nu}(x-y)j_e^\nu(y)\ ,$$

where

$$[g_{\mu\nu}\partial_\lambda\partial^\lambda - (\xi^{-1}-1)\partial_\mu\partial_\nu]D_{\text{F}}^{\nu\rho}(x-y) = i\delta_\mu{}^\rho\delta^{(4)}(x-y)$$

or (after a Fourier transform)

$$[-k^2 g_{\mu\nu} + (1-\frac{1}{\xi})k_\mu k_\nu]D_{\text{F}}^{\nu\rho}(k) = i\delta_\mu{}^\rho$$

which leads to (check that this works!)

$$D_{\text{F}}^{\mu\nu}(k) = \frac{-i}{k^2+i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)\ .$$

(b) Can we directly identify the last term in S_{eff} as (with particle density $\rho = \psi^\dagger\psi$)

$$-\frac{1}{2}\int dt\int d^3x\,d^3y\,\rho(\mathbf{x},t)V(\mathbf{x}-\mathbf{y})\rho(\mathbf{y},t)$$

and in doing so identify the potential V? (Think a bit about it but then come back to this part after doing the next two sections.)

(c) If we consider a classical *static* distribution $j_e^\mu \to e(\rho,\mathbf{0})$, show that

$$V(\mathbf{x}-\mathbf{y}) = -e^2\int dt'\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik^\mu(x-y)_\mu}}{k_0^2-\mathbf{k}^2+i\epsilon} = e^2\int\frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{\mathbf{k}^2} = \frac{e^2}{4\pi|\mathbf{x}-\mathbf{y}|}\ ,$$

which is the Coulomb interaction. Is making the distribution static a big assumption?

(d) Really the current density is a quantum-mechanical operator. What does this imply for defining $V(\mathbf{x}-\mathbf{y})$ quantum mechanically? Where are the ambiguities in defining V?

(e) Suppose we were exchanging a massive boson instead of a massless photon? How would the derivation change? Do we always get a local potential?

15. Discussion questions for those who have had a quantum field theory course. (To be explained to those who haven't!)

(a) In what ways is nonrelativistic quantum mechanical scattering simpler than what you learn in relativistic quantum field theory?

(b) In what ways is the nonrelativistic quantum mechanical bound state problem simpler than solving for bound state in relativistic QFT? For example, you might consider solving for positronium in QED versus using an effective potential. [Hint: Look up the Bethe-Salpeter equation.]