TALENT/INT Course on Nuclear Forces
Exercises and Discussion Questions Th1

[Last revised on July 4, 2013 at 09:22:17.]

Thursday 1: Cold atoms, QMC; Tensor, SO, deuteron

We have grouped all of the two-minute and discussion questions at the beginning, as usual. However, for today you should only spend about an hour working on questions and then try some of the other problems as well. When you need a break, go back and try another question!

1. Two-minute and discussion questions.

   (a) What is the average density of a neutron star? Do you expect the density to be uniform?

   (b) Would you consider a neutron star to be hot or cold? What is an appropriate comparison temperature or energy?

   (c) Is a neutron star made entirely of neutrons?

   (d) Consider ^{6}Li and ^{7}Li atoms. Which is a fermion and which is a boson?

   (e) What are the natural units for a and r_e for cold atoms?

   (f) What is the value of the pairing gap for a normal system?

   (g) Why can't you use a perturbative expansion in $k_{\rm F}a$ to find the gap $\Delta^0_{\mathbf{BCS}}(k)$?

   (h) Why would you use a pionless EFT as opposed to an EFT with pions, which would appear to include more physics? Which EFT has the higher resolution?

   (i) What do you deduce from the S-wave scattering lengths being negative for $a_{np}(^1\mathrm{S}_0)$ and positive for $a_{np}(^3\mathrm{S}_1)$? [E.g., recall the diagrams on scattering length and zero-energy wave functions from the lecture on scattering.]

   (j) Why is the deuteron not spherically symmetric?

   (k) Why is the deuteron bound but two neutrons are not bound?

   (l) Explain how the uncertainty principle can justify replacing contributions from omitted high-energy states with changes in the coupling constants.

2. Basic skills.

   (a) For pure neutron matter, the density is related to the Fermi momentum $k_{\rm F}$ by

   $$\rho = \frac{k_{\rm F}^3}{3\pi^2} \ .$$

   Derive this for a non-interacting Fermi gas and find the analogous expression for the density for symmetric nuclear matter.

(b) In the lecture, the average energy of a Fermi gas $E_{\rm FG}$ was related to the Fermi energy E_F by

$$E_{\rm FG} = \frac{3}{5} N E_F \qquad \text{where} \qquad E_F = \frac{\hbar^2 k_{\rm F}^2}{2m} \,.$$

What would the expression be if there were four rather than three space dimensions?

(c) Find the relation between the critical temperature T_C and the gap. [That is, look it up online.]

(d) If you know the energies of three consecutive systems of cold atoms or atomic nuclei, you can find the gap from the formula

$$\Delta(N) = E(N+1) - \frac{1}{2}[E(N) + E(N+2)] \,,$$

where N is the number of atoms or nucleons. Find a typical pairing gap size for nuclei in the tin region. [You can use an online table of nuclides to find the energies you need.]

3. A tensor interaction proportional to S_{12} arises from one-pion exchange. This interaction has important consequences for nuclear properties. In what NN partial waves is it non-zero? (I.e., does it contribute in $^1\mathrm{S}_0$ or $^3\mathrm{S}_1$?). What NN partial waves does it *couple*? (I.e., it does not conserve orbital angular momentum, so there are non-zero matrix elements between different partial waves.)

4. Tensor matrix elements in different partial waves. Consider this table of matrix elements of $S_{12}(\hat{r})$ in different partial waves.

	$l = J-1$	$l = J$	$l = J+1$
$l' = J-1$	$-\frac{2}{3}\frac{J-1}{2J+1}$	0	$2\frac{\sqrt{J(J+1)}}{2J+1}$
$l' = J$	0	$\frac{2}{3}$	0
$l' = J+1$	$2\frac{\sqrt{J(J+1)}}{2J+1}$	0	$-\frac{2}{3}\frac{J+2}{2J+1}$

(a) What does the sign of the quadrupole moment of the deuteron tell you about the sign of the T = 0 tensor interaction at low energies?

(b) Using the tensor matrix elements from the table, what do you deduce about the sign of the $T = 1$ tensor interaction? Recall that in lecture T1b we could not explain the attractive $^3\mathrm{P}_0$ partial wave (at low energies) with only central and spin-orbit forces.

5. Derive the coordinate space one-pion exchange potential from the momentum-space version.

(a) Show that the Fourier transform of $\frac{1}{q^2+m_\pi^2}$ is given by a Yukawa function:

$$\int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2+m_\pi^2} e^{i\boldsymbol{q}\cdot\boldsymbol{r}} = \frac{1}{4\pi}\frac{e^{-m_\pi r}}{r}$$

Hint: Calculate the radial (dq) integral by appropriately closing the integration contour in the complex plane.

(b) Now do the same for the spin-dependent part of OPE (ignoring for a moment the isospin dependence).

$$\int \frac{d^3q}{(2\pi)^3} \frac{(\boldsymbol{\sigma}_1 \cdot \mathbf{q})(\boldsymbol{\sigma}_2 \cdot \mathbf{q})}{q^2 + m_\pi^2} e^{i\boldsymbol{q}\cdot\boldsymbol{r}} = -\frac{m_\pi^2}{12\pi} \frac{e^{-m_\pi r}}{r} \left(1 + \frac{3}{m_\pi r} + \frac{3}{(m_\pi r)^2}\right) S_{12} - \frac{m_\pi^2}{12\pi} \frac{e^{-m_\pi r}}{r} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$$

where S_{12} is the tensor operator, $S_{ij} = 3(\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}_{12})(\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}_{12}) - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$

Initial Hint: write out the cartesian components, pull out momentum derivatives, and use part (a).

6. One-boson-exchange interaction for σ exchange. For the exchange of a scalar-isoscalar "σ" meson ($J^P = 0^+$, $T = 0$), the interaction Hamiltonian is given by the same form as for the coupling of a pion to a nucleon (see lecture notes), except in the case of a "σ" meson the coupling to a nucleon is replaced by:

$$i\frac{g_A}{2f_\pi} \boldsymbol{\sigma} \cdot \mathbf{q} \boldsymbol{\tau} \quad \longrightarrow \quad g_S \left(1 - \frac{\boldsymbol{\sigma} \cdot \mathbf{k}' \, \boldsymbol{\sigma} \cdot \mathbf{k}}{4m^2}\right)$$

where g_S is a scalar coupling and m is the nucleon mass.

Following the derivation for one-pion exchange, show that the exchange of a scalar-isoscalar "σ" meson leads to a one-boson-exchange potential given by:

$$V_\sigma = -g_S^2 \frac{1}{q^2 + m_\sigma^2} \left(1 - \frac{\mathbf{k} \cdot \mathbf{k}'}{2m^2} - \frac{i}{4m^2} (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot (\mathbf{k}' \times \mathbf{k}) + \mathcal{O}\left((k \, \text{or} \, k')^4\right)\right)$$

What type of interactions do the terms in V_σ correspond to?

[Hint: A useful expression for the product of two Pauli matrices is $\sigma^i \sigma^j = \delta_{ij} + i\epsilon_{ijk}\sigma^k$.]

7. Mechanics of DMC.

(a) Derive the expression for the $3N$-dimensional Gaussian free-particle Green's function:

$$G_0(\mathbf{R}, \mathbf{R}') = \langle \mathbf{R} | e^{-T\Delta\tau} | \mathbf{R}' \rangle = \left[\sqrt{\frac{m}{2\pi\hbar^2 \triangle\tau}}\right]^{3N} exp\left[\frac{-m(\mathbf{R} - \mathbf{R}')^2}{2\hbar^2 \triangle\tau}\right] . \qquad (1)$$

[Extended Hint: Start with the one-dimensional one-particle analogue, $G_0(x, x') = \langle x | exp(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Delta\tau) | x' \rangle$, insert two resolutions of the identity, complete the square, and do the gaussian integral.]

(b) One possible short-time (Trotter) approximation for the many-body interacting propagator amounts to using $e^{-V\Delta\tau/2} e^{-T\Delta\tau} e^{-V\Delta\tau/2}$ instead of $e^{-H\Delta\tau}$. Taylor expand $e^{-H\Delta\tau}$ and then separately do the same thing for $e^{-V\Delta\tau/2} e^{-T\Delta\tau} e^{-V\Delta\tau/2}$ to see up to which order the two expressions match. What is the structure of the first terms that begin to disagree in the two approaches?

(c) Prove that the mixed estimate leads to the ground-state expectation value in the limit of large imaginary times. In others words show that

$$\langle H \rangle_M \equiv \frac{\langle \Psi_V | H | \Psi(\tau) \rangle}{\langle \Psi_V | \Psi(\tau) \rangle} \to \frac{\langle \Psi(\tau) | H | \Psi(\tau) \rangle}{\langle \Psi(\tau) | \Psi(\tau) \rangle} \tag{2}$$

8. Spin-orbit force in momentum space.

(a) Show that the Fourier transform of a spin-orbit force,

$$V = V(r)\, \mathbf{S} \cdot (\mathbf{r} \times \boldsymbol{\nabla}/i)$$

is of the following form in momentum space

$$\widetilde{V} = \langle \mathbf{k} | V | \mathbf{k}' \rangle \sim i\mathbf{S} \cdot (\mathbf{k} \times \mathbf{k}')$$

where $\mathbf{k}$ and $\mathbf{k}'$ are the relative momenta and plane waves $\langle \mathbf{r} | \mathbf{k} \rangle = (2\pi)^{-3/2} e^{i\mathbf{k}\cdot\mathbf{r}}$.

(b) Can you re-write the result for $\widetilde{V}$ in terms of the two new momenta $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ and $\mathbf{p} = (\mathbf{k} + \mathbf{k}')/2$?

9. The quadrupole moment of the deuteron Q_d is given by the matrix element in the deuteron

$$Q_d = \langle \psi_d \, M = J = 1 | \sqrt{\frac{16\pi}{5}} e \frac{r^2}{4} Y_{20}(\theta, \phi) | \psi_d \, M = J = 1 \rangle \,,$$

where the deuteron wave function $|\psi_d\rangle$ has an S-wave radial part $u(r)$ and a D-wave radial part $w(r)$ with normalization

$$\int_0^\infty dr \, [u(r)^2 + w(r)^2] = 1 \,.$$

The goal here is to derive Q_d in terms of u and w:

$$Q_d = \frac{e}{20} \int_0^\infty dr \, r^2 w(r) [\sqrt{8} u(r) - w(r)] \,.$$

(a) Decouple the total angular momentum $(l, S)J, M$ to the l, m_l, S, m_S basis. This involves a Clebsch-Gordan coefficient for each deuteron wave function.

(b) Use the following formula to carry out the integral over the angles,

$$\int_0^{2\pi} d\phi \int_0^\pi d\theta \, \sin\theta \, Y_{l_1,m_1}(\theta,\phi) Y_{l_2,m_2}(\theta,\phi) Y^*_{l_3,m_3}(\theta,\phi) = \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l_3+1)}} \mathcal{C}^{l_3 0}_{l_1 0 l_2 0} \mathcal{C}^{l_3 m_3}_{l_1 m_1 l_2 m_2} \,,$$

where $\mathcal{C}^{LM}_{l_1 m_1 l_2 m_2}$ are Clebsch-Gordan coefficients.

(c) Use Mathematica, MATLAB, or anything else to sum over the m_l and m_S quantum numbers in the Clebsch-Gordan coefficients.