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# MODIFIED DEBYE-HÜCKEL ELECTRON SHIELDING AND PENETRATION FACTOR

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## ABSTRACT

A screened potential modified by nonstandard electron cloud distributions, which are responsible for the shielding effect on fusion of reacting nuclei in astrophysical plasmas, is derived. The case of clouds with depleted tails in space coordinates is discussed. The modified screened potential is obtained both from statistical mechanics arguments based on fluctuations of the inverse of the Debye-Hückel radius and from the solution of a Bernoulli equation used in generalized statistical mechanics. Plots and tables useful for evaluating the penetration probability at any energy are provided.

*Subject headings:* atomic processes — nuclear reactions, nucleosynthesis, abundances — plasmas

## 1. INTRODUCTION

Nuclear fusion reaction cross sections and rates are sensitive to the screening effect of the electron cloud around the reacting nuclei, an effect that has been widely investigated both theoretically and experimentally since the early work of Salpeter (Salpeter 1954; Salpeter & Van Horn 1969). Different situations arise when fusion reactions take place: (1) in laboratory experiments, where a metal or gaseous target of a given element is bombarded by an ionic or charged-particle beam and electrons are for the most part bound in atomic orbits, such that few of them can be considered free; (2) in stellar cores and other space and astrophysical plasmas, where ions and nuclei are embedded in an electronic environment made by mainly free electrons; (3) in a deuterated metal or other solid-state matrix, where an impinging deuteron beam reacts with implanted deuterons.

1. In laboratory experiments, penetration through a screened Coulomb potential at center-of-mass energy $E$ is known to be equivalent to that of bare nuclei at energy $E + U_e$, where $U_e = Z_1 Z_2 e^2/R_a$ and $R_a$ is the atomic radius, or the radius of the innermost electrons. $Z_1 e$ and $Z_2 e$ are the charges of the two reacting nuclei, and $U_e$ is usually taken as constant in the evaluation of cross sections and rates at any energy. Very often, at very low energy, fusion cross sections measured in laboratory experiments are higher than the value calculated by means of the usual Debye-Hückel (D-H) screening factor. The stopping power of the incoming beam and temperature reached after energy deposition are important quantities for the correct measurement of the cross sections. Modifications of the electron distribution can be induced. The screening effect must be taken into account if one is to obtain the correct astrophysical factor at very low energies (Assenbaum et al. 1987; Carraro et al. 1988; Bracci et al. 1990; Shoppa et al. 1993; Strieder et al. 2001).

2. In astrophysical plasmas, which can be considered ideal, free electrons move around the reacting nuclei and occupy a sphere of D-H radius $R_{\rm DH} = [kT/(4\pi e^2 n Z_\rho)]^{1/2}$, which is on the order of $R_a$, with $n$ the particle density and $Z_\rho = \sum_i (Z_i^2 + Z_i) X_i/A_i$, where the sum is over all positive ions and $X_i$ is the mass fraction of nuclei of type $i$. Only with decreasing radius does the effect of screening become important. A screening factor for the reaction rate can be derived when the energy of the Gamow peak $E_{\rm G} > U_e$. It is given by the Debye factor $f = \exp\,(U_e/kT)$ with, this time, $U_e = Z_1 Z_2 e^2/R_{\rm DH}$ (Rolfs & Rodney 2005; Ichimaru 1993; Castellani et al. 1997; Opher & Opher 2000). Recently, it has been clarified through numerous experimental observations that the velocity distribution function of electrons (and possibly also of ions) in stellar atmospheres, and in space and astrophysical plasmas, may deviate from a Maxwell-Boltzmann distribution in the high-energy tail if nonlocal thermodynamic effects are nonnegligible (Oxenius 1986; Collins 1989; Peyraud-Cuenca 1992; Chevallier 2002). In stellar atmospheres, atomic processes such as radiative and dielectronic recombination exhibit rates that indicate deviations from a Maxwellian distribution of electrons (Maero et al. 2006). In stellar cores, signals of possible deviations in the ion distributions are evident and, although small, should be considered because they are capable of meaningfully influencing nuclear fusion reaction rates (Ferro & Quarati 2005; Lissia & Quarati 2005).

3. In deuterated metals or solid-state matter, the effect of strong screening has yet to be clearly understood and discussed, although a few interesting descriptions have recently been brought forward to reproduce experimental results (Raiola et al. 2004; Coraddu et al. 2004a, 2004b, 2006; Kim & Zubarev 2006). The approach we are referring to here is very useful in understanding the fusion rates in this matter, which could simulate some high-density astrophysical plasmas. However, this application deserves a separate, detailed paper, and we will not pursue this case here.

An important issue in the shielding of electrostatic potential in plasmas concerns the investigation of nonlinear charge-screening effects that can induce modifications in the D-H potential, usually derived by linearizing the Poisson equation (Cravens 1997; Gruzinov & Bahcall 1998). One of the first studies that, on a microscopic basis, demonstrated deviations from the D-H factor was carried out by Johnson et al. (1992; see also Shaviv 2004; Shaviv & Shaviv 2000, 2001, 2002; Chitanvis 2007). The different approaches mentioned so far are based on the assumption that electrons are distributed in space according to a Boltzmann factor. A few authors (Bryant 1996; Treumann et al. 2004; Leubner 2004; Kim & Jung 2004; Rubab & Murtaza 2006) have assumed a stationary $k$-Lorentz distribution to describe significant deviations from the standard distribution, constructed from experimental distributions, due to the presence of enhanced high-energy tails. In a recent paper (Rubab & Murtaza 2006), for instance, one can find citations to many works in which such distributions are reported. By analyzing energy profiles, Rubab & Murtaza derive an effective length smaller than the standard D-H radius, depending on the $k$-parameter. This result is obtained by solving a Poisson equation in the weak linear approximation. The consequence must be an increased barrier penetration factor.

Our approach differs. We assume that the electron cloud is spatially distributed following a generalized steady state distribution of the $q$-type, which reduces to an exponential distribution when the $q$-parameter (also known as the Tsallis entropic parameter) approaches the limit $q \to 1$ (Tsallis 1988; Tsallis & Borges 2003). We refer the reader to Leubner & Vörös (2005), Leubner (2005), and Burlaga et al. (2006) for a detailed description of Tsallis generalized statistics and some of its applications to astrophysical problems. We then calculate the modified screening potential by considering two different approaches. One concerns the use of a generalized Poisson equation or Bernoulli equation, as used by Tsallis & Borges (2003); the other is based on superstatistics (Beck 2001, 2004; Wilk & Włodarczyk 2001, 2007), considering fluctuations of an *intensive* parameter (the inverse of the D-H radius). This implies fluctuations in the temperature and density of the components of the plasma. Once we obtain the modified potential, we calculate the penetration probability through that potential. In this paper we limit ourselves to values $0 < q < 1$ (a distribution in spatial coordinates with a depleted tail cut off at about $\langle 1/R_{\rm DH}\rangle$), leaving aside the case $q > 1$, which describes an enhanced distribution in space coordinates at long distances.

In the standard D-H shielding approach, after linearization of the Poisson equation the electrostatic screened potential behaves as $V_{\rm DH} \sim r^{-1} \exp(-r/R_{\rm DH})$, where $r$ is the coordinate with which to evaluate the D-H potential. The assumptions made to derive the above relation are, among other things, that (Cravens 1997; Bellan 2006) plasmas are collisionless; the induced perturbation is slow and depends slowly on time (slowness); only electrostatic fields are present, while induced fields are negligible; temperature is spatially uniform and the plasma remains in equilibrium during a perturbation; a temperature can always be defined; the number of particles inside the D-H sphere is large and therefore fluctuations are small; and, although ions and electrons have random thermal motion, perturbations induced around the equilibrium, which are responsible for small spatial variations in the electrostatic potential, can be neglected.

Of course, if the real situation differs from that required by one or more of these assumptions, the use of the D-H potential may produce errors in the evaluation of the penetration factor and nuclear reaction rates. Deviations from the conditions imposed by the assumptions are taken into account in this work by choosing the inverse of the D-H length, $1/R_{\rm DH}$, as a fluctuating parameter. The electrostatic quantity $rV(r)$ is asymptotically given, in this case, by a power law instead of an exponential because it must satisfy a differential equation, the Bernoulli equation (or a special case of it), that has power-law functions as solutions. We can derive the modified D-H potential $V_q(r)$ and, a posteriori, the charge distribution $\rho_q$ as asymptotically power-law functions. Also, in the standard D-H approach two equations are needed, one coming from electromagnetism (the Poisson equation), the other from statistical mechanics (the Boltzmann factor). The penetration factor $\Gamma(E)$ can be calculated by means of the WKB approach using the modified D-H potential. We obtain results that differ from those calculated with the standard D-H potential and which will be useful in the interpretation of experimental results on atomic and nuclear rates in several astrophysical systems and processes. We also derive the equivalent energy $U_q$, which we give in the form of an interpolating analytical expression, a plot, and a tabulation. The energy $U_q$ turns out to be a function of the variable $D/E$, where $D$ is defined by $D = Z_1 Z_2 e^2 \langle 1/R_{\rm DH}\rangle$. It is easy to observe that $U_q$ depends, for a wide range of values of $D/E$, on $1/(kT)^{1/2}$ and $n^{1/2}$.

In § 2, we explain how to derive the modified potential on the basis of superstatistics arguments with the inverse D-H length as a fluctuating parameter; we then introduce a nonlinear differential equation, to be associated with the Poisson equation, whose solution coincides with the potential derived directly from the superstatistics approach. In § 3, we derive the penetration factor, which can be used for the evaluation of nuclear fusion rates in astrophysical plasmas, and we indicate the range of validity of the approximations adopted. In § 4 we discuss some representative examples, while in § 5 we report our conclusions.

## 2. MODIFIED DEBYE-HÜCKEL POTENTIAL

Combining Gauss's law and the relation that links the electrostatic field to the electric potential $V(r)$ of a point test unitary charge at the origin in a vacuum, from the Poisson equation one obtains the pure Coulomb potential. After a sufficiently long time, electrons and ions rearrange themselves in response to the forces on them. The ion density eventually becomes uniform, while the electron density near the test charge increases. At the new thermal equilibrium, the distribution of electrons in the electrostatic field is assumed to be given by the well-known Boltzmann factor. Assuming the Boltzmann factor for all the particles, after linearization and using the condition of neutrality, the Poisson equation can be written

$$\frac{1}{r}\frac{d^2}{dr^2}[rV(r)] = \frac{1}{R_{\rm DH}^2}V(r), \tag{1}$$

with a solution given by the D-H potential, and the charge density $\rho_{\rm DH}$ is expressed as

$$\rho_{\rm DH} \sim -\frac{1}{rR_{\rm DH}} \exp\left(-\frac{r}{R_{\rm DH}}\right). \tag{2}$$

When one or more of the linearity constraints are violated or relaxed, a different description of the screening is required.

If we assume that nonlinear effects produce fluctuations in the inverse D-H radius, by following the development usually adopted in superstatistics for the inverse temperature $\beta = 1/kT$ (Beck 2001, 2004), we can describe the plasma around the test charge as being made of cells in which $R_{\rm DH}$ is approximately constant, and the system can be described by ordinary statistical mechanics, in this case by the exponential (Boltzmann) factor $\exp(-r/R_{\rm DH})$. In the long term, the system is described by a spatial average over the mean of the fluctuating quantity $1/R_{\rm DH}$. A fluctuation of the inverse D-H radius also implies a fluctuation of the plasma parameter, given by

$$\gamma = \frac{1}{nR_{\rm DH}^3} = \left(\frac{4\pi e^2 Z_\rho}{kT}\right)^{3/2}\sqrt{n}. \tag{3}$$

With few changes, we follow the approach of Wilk & Włodarczyk for the case of distributions with depleted tails ($q < 1$; Wilk &

Włodarczyk 2001, 2007). Here we focus our attention on the $q < 1$ distribution because this exhibits a depleted tail with a cutoff, that is, the spatial distribution we assume for the electrons. We assume that a certain variable $r$ of the system is confined between 0 and $[(1-q)\lambda_0]^{-1}$, where $\lambda_0$ is a constant parameter. We define the function

$$\mathcal{F}_{q<1}(r, \lambda_0) = C_q \int_0^\infty f_{q<1}(r, \lambda; \lambda_0) \exp(-\lambda r) \, d\lambda, \tag{4}$$

where $C_q$ is a normalization factor and $f_{q<1}(r, \lambda; \lambda_0)$ is the probability density for observing a certain value $\lambda$ of the system, which is spread around the value $\lambda_0$. The expression we choose for $f_{q<1}(r, \lambda; \lambda_0)$ is a gamma distribution:

$$f_{q<1}(r, \lambda; \lambda_0) = \frac{A_q(r; \lambda_0)^{1/(1-q)}}{\Gamma(1/(1-q))} \times \lambda^{[1/(1-q)]-1} \exp[-\lambda A_q(r; \lambda_0)], \tag{5}$$

where

$$A_q(r; \lambda_0) = \frac{2-q}{1-q} \lambda_0^{-1} - r \tag{6}$$

and $q$ is the entropic Tsallis parameter.

Inserting the function $f_{q<1}(r, \lambda; \lambda_0)$ into $\mathcal{F}_{q<1}(r; \lambda_0)$, we obtain the normalized power-law distribution

$$\mathcal{F}_{q<1}(r; \lambda_0) = \lambda_0 \left(1 - \frac{1-q}{2-q} \lambda_0 r\right)^{1/(1-q)}. \tag{7}$$

The average value and variance of $\lambda$ depend on the variable $r$ according to

$$\bar{\lambda} = \frac{1}{(1-q) A_q(r; \lambda_0)}, \quad \overline{\lambda^2} = \frac{2-q}{[(1-q) A_q(r; \lambda_0)]^2}, \tag{8}$$

where $\overline{x(r)} = \int x(r, \lambda) f_{q<1}(r, \lambda; \lambda_0) d\lambda$ is evaluated by means of the distribution $f_{q<1}(r, \lambda; \lambda_0)$. However, the relative variance depends only on $q$:

$$\omega = \frac{\overline{\lambda^2} - \bar{\lambda}^2}{\bar{\lambda}^2} = 1 - q. \tag{9}$$

We note that the quantity $\lambda_0$ coincides with the spatial average of $\bar{\lambda}$, that is,

$$\langle \bar{\lambda} \rangle = \lambda_0, \tag{10}$$

where $\langle x \rangle = \int \overline{x(r)} \, \mathcal{F}_{q<1}(r; \lambda_0) dr$ is evaluated by means of the distribution $\mathcal{F}_{q<1}(r; \lambda_0)$, which is the weighted average of the exponential (or Boltzmann-like) factor $\exp(-\lambda r)$ with weight equal to $f_{q<1}(r, \lambda; \lambda_0)$ and coincides with the Laplace transform of $f_{q<1}(r, \lambda; \lambda_0)$.

Some special limiting cases are

$$f_{q\to1}(r, \lambda; \lambda_0) = \delta(\lambda - \lambda_0), \tag{11}$$

$$\mathcal{F}_{q\to1}(r; \lambda_0) = \lambda_0 \exp(-\lambda_0 r), \tag{12}$$

the ordinary Boltzmann-like factor, with $\overline{\lambda^2} = \bar{\lambda}^2$, and where $A_0(r; \lambda_0) = 2/\lambda_0 - r$ and

$$\mathcal{F}_{q\to0}(r; \lambda_0) = \lambda_0(1 - \lambda_0 r), \tag{14}$$

a linear function, with $\overline{\lambda^2} = 2\bar{\lambda}^2$.

Let us now identify the functional $\mathcal{F}_{q<1}(r; \lambda_0)$ with the quantity $rV_q(r)$ and replace $\lambda$ with $1/R_{\rm DH}$, so that $\lambda_0 = \langle 1/R_{\rm DH} \rangle$ coincides with the spatial average of the fluctuation in D-H radius. By setting

$$\zeta_q = (2-q)\langle 1/R_{\rm DH} \rangle^{-1}, \tag{15}$$

a characteristic length of the system under inspection, which reduces to $R_{\rm DH}$ in the $q \to 1$ limit, we obtain the following expression:

$$V_q(r) = \frac{1}{r} \left[1 - (1-q)\frac{r}{\zeta_q}\right]^{1/(1-q)}, \tag{16}$$

which for $q \to 1$ reduces to the standard D-H potential. We also have the charge distribution,

$$\rho_q(r) \sim -\frac{1}{(2-q) r \zeta_q^2} \left[1 - (1-q)\frac{r}{\zeta_q}\right]^{1/(1-q)}, \tag{17}$$

which for $q \to 1$ reduces to $\rho_{\rm DH}$, the charge distribution in the D-H approximation.

Therefore, by considering $1/R_{\rm DH}$ to be subject to fluctuations described by a gamma distribution, the quantity $rV(r)$ related to the potential energy barrier is modified from the exponential D-H expression to a power-like law, typical of the generalized $q < 1$ distribution. If we can establish the functional relation between density $n$ and temperature $kT$, as, for instance, in a solar-like star, where the quantity $n/(kT)^3$ is constant along the star's profile (Ricci et al. 1995), by means of the relative variance we can establish a link between the inverse D-H radius and temperature fluctuations and the parameter $q$ through

$$\frac{\Delta(1/R_{\rm DH})}{1/R_{\rm DH}} = \frac{\Delta\sqrt{n/kT}}{\sqrt{n/kT}} = \frac{\Delta(kT)}{kT} = (1-q)^{1/2}. \tag{18}$$

Diffusion of matter between layers with different temperatures induces local temperature fluctuations and density perturbations; fluctuations around an equilibrium or a steady state matter profile induce fluctuations of the quantity $1/R_{\rm DH}$ in the D-H sphere, particularly in regions where the number of particles inside the D-H sphere is small. Density and temperature fluctuations do not alter the macroscopic plasma parameters and must agree with the requirements of the constraints imposed by the macroscopic observations. For the solar interior, Gruzinov & Bahcall (1998) have calculated the D-H radius to be about $2 \times 10^{-9}$ cm, therefore containing a small number of particles with a correspondingly nonnegligible particle fluctuation.

Let us now justify equations (16) and (17) by means of another approach, which considers a generalized version of the Poisson equation. In fact, in the standard case the D-H potential can be obtained from the solution of a second-order differential equation of the type

$$\frac{dy}{dr} = ay \quad \text{or} \quad \frac{d^2y}{dr^2} = a^2 y \tag{19}$$

with $y \equiv rV(r) = \exp ar$. We replace the above linear equation with the following one:

$$\frac{dy}{dr} = a_q y^q \quad \text{or} \quad \frac{d^2y}{dr^2} = q a_q^2 y^{2q-1} \tag{20}$$

with

$$y = \exp_q a_q r \equiv [1 - (1-q)a_q r]^{1/(1-q)}, \tag{21}$$

where $q$ is a real-number parameter and coincides with the Tsallis parameter. For $q \to 1$, we obtain $\exp_1 ar \equiv \exp ar$.

To be explicit, for our case we generalize equation (1) to

$$\frac{d^2}{dr^2}[rV_q(r)] = \frac{q}{\zeta_q^2}[rV_q(r)]^{2q-1}, \tag{22}$$

which is a generalized Poisson equation whose solution coincides with equation (16). We report, for completeness, that by considering the Bernoulli equation introduced by Tsallis & Borges (2003),

$$\frac{dy}{dr} = a_1 y + a_q y^q \quad \text{or} \quad \frac{d^2y}{dr^2} = a_1^2 y + (1+q)a_1 a_q y^q + q a_q^2 y^{2q-1}, \tag{23}$$

the solution is

$$y = \left[e^{(1-q)a_1 r} + \frac{a_q}{a_1}\left(e^{(1-q)a_1 r} - 1\right)\right]^{1/(1-q)}. \tag{24}$$

By imposing $y = rV(r)$, we obtain $rV(r) = [1 + (1-q)a_q r]^{1/(1-q)}$ when $a_1 = 0$ and $rV(r) = \exp a_1 r$ when $a_q = 0$.

## 3. PENETRATION PROBABILITY

We calculate the probability of penetrating through the repulsive barrier—one of the terms we need in order to evaluate the fusion reaction rates—using the WKB approach and following Bahcall et al. (1998). The fusion cross section of two isolated reacting nuclei is written as

$$\sigma(E) = \frac{S(E)}{E} e^{-2\pi\eta(E)}, \tag{25}$$

where $S(E)$ is the astrophysical factor, $E$ is the center-of-mass energy of the fusing nuclei, of charge $Z_1 e$ and $Z_2 e$, colliding with relative velocity $v/c = [2E/(\mu c^2)]^{1/2}$ and reduced mass $\mu$, and

$$\eta(E) = \frac{1}{\hbar c}\frac{Z_1 Z_2 e^2}{\sqrt{E}}\sqrt{\frac{\mu c^2}{2}}. \tag{26}$$

First of all we define the penetration factor for a pure Coulomb electrostatic potential energy barrier $\hat{V}_C(r) = Z_1 Z_2 e^2 \neq r$, that is, when the reacting nuclei are isolated:

$$\Gamma_C(E) = e^{-2\pi\eta(E)} = \exp\left(-\frac{2}{\hbar c}\int_0^{r_C}\{2\mu c^2[\hat{V}_C(r) - E]\}^{1/2} dr\right), \tag{27}$$

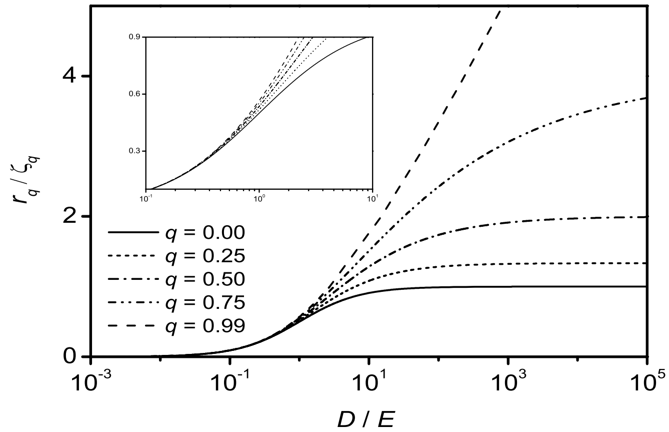


FIG. 1.—Log-linear plot of the quantity $r_q/\zeta_q$ as a function of $D/E$ for several values of $q$ (in the inset, the region $10^{-1} < D/E < 10$ is expanded).

where $r_C$ is the classical turning point, whose value is fixed by the relation $\hat{V}_C(r_C) = E$.

Secondly, by using the standard D-H potential and still taking the turning point $r_{DH} = r_C$, a relation that is valid only for $E > D$, for small values of $r_C/R_{DH}$ we have

$$\Gamma_{DH}(E) = \exp\left[-\pi\left(2 + \frac{r_C}{R_{DH}}\right)\eta(E)\right]. \tag{28}$$

If we consider the rates instead of the cross sections, the factor $\exp(-\pi\eta r_C/R_{DH})$ can be evaluated at the most probable energy $E_0$ in such a way that the rate can be factorized as the product of $\Gamma_C(E)$ and a factor fixed at $E = E_0$.

Finally, we consider the deformed D-H potential energy barrier

$$\hat{V}_q(r) = \frac{D}{r\langle 1/R_{DH}\rangle}\left[1 - (1-q)\frac{r}{\zeta_q}\right]^{1/(1-q)} \tag{29}$$

and the penetration factor

$$\Gamma_q(E) = \exp\left(-\frac{2}{\hbar c}\int_0^{r_q}\{2\mu c^2[\hat{V}_q(r) - E]\}^{1/2} dr\right), \tag{30}$$

where $r_q$ must be derived from the relation $\hat{V}_q(r_q) = E$. Because we consider $q < 1$, the potential energy $\hat{V}_q(r)$ has a cutoff, and as a consequence, $0 < r < \zeta_q/(1-q)$. We write equation (30) as

$$\Gamma_q(E) = \exp[-2\pi\eta(E)\tau_q], \tag{31}$$

where the function $\tau_q$, which depends on the quantity $D/E$, goes to 1 for $D/E \to 0$.

The evaluation of $r_q$ and $\Gamma_q(E)$ can be worked out only numerically, although for small deformations ($q \approx 1$), the penetration factor can be worked out analytically and is given as the product of $\Gamma_C(E)$ and a correction factor. However, we do not report this expression here, for simplicity's sake. In the case $q = 0$, which represents the greatest deformation with respect to the exponential function, $\tau_q$ has the simplest analytical solution,

$$\tau_{q=0} = \frac{E}{E + D}. \tag{32}$$

In Figure 1, we plot the quantity $r_q/\zeta_q$ as a function of $D/E$ for a few values of $q$ between zero and one. In Figure 2, the quantity $1 - \tau_q$ is plotted as a function of $D/E$.

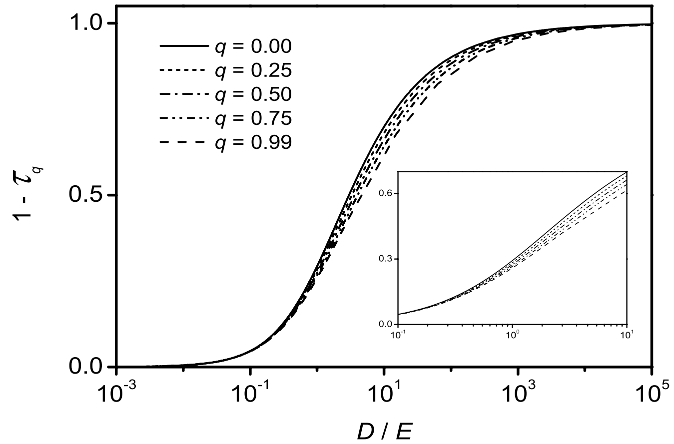


FIG. 2.—Same as Fig. 1, but for the quantity $1 - \tau_q$.

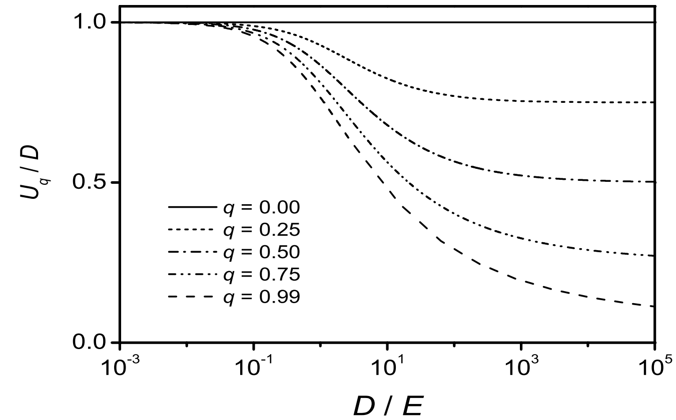


FIG. 3.—Log-linear plot, in arbitrary units, of $U_q/D$ as a function of $D/E$ for several values of $q$.

Penetration of the potential energy barrier $\hat{V}_q(r)$ at energy $E$ is equivalent to penetration of the pure Coulomb barrier at an effective energy $E + U_q$, where

$$U_q = \frac{D}{r_q\langle 1/R_{\rm DH}\rangle} - E \tag{33}$$

is a function of $D/E$. We have also calculated numerically the equivalent energy $U_q$. The behavior of $U_q/D$, as a function of $D/E$, for a few values of $q$ is plotted in Figure 3. Finally, a quantitative comparison of the quantities $r_q/\zeta_q$, $U_q/D$, and $1 - \tau_q$, corresponding to the values of $q$ depicted in the figures, can be obtained from Table 1.

For $q = 0$, the potential $V_{q=0}$ is cut off at $r = 2\langle 1/R_{\rm DH}\rangle^{-1}$, and we have $U_q = D$ for any value of energy the $E$. In the other cases of $0 < q < 1$, we have the following: At high energy, $E > D$, $U_q$ approaches $D$. At low energy, $E < D$, $U_q$ approaches the value $(1 - q)D$, meaning that $U_1$ (the D-H exponential potential) approaches zero. Therefore, when the electron distribution is a deformed or generalized distribution of the $q$-type, with a cutoff at $r = (2 - q)/[(1 - q)\langle 1/R_{\rm DH}\rangle]$, the penetration factor is enhanced even at very low energies, except for the case $q = 1$. The enhancement depends on $U_q$, which is an energy proportional to $1/(kT)^{1/2}$ and $n^{1/2}$.

We have also interpolated the function $U_q/D$ (for values $0 < q < 0.8$), which is well described by the following analytical function:

$$U_q/D = 1 - q + q[1 - (1 - q)a(D/E)^b]^{-c/(1-q)}, \tag{34}$$

where the parameters $a(q)$, $b(q)$, and $c(q)$ are given by

$$\begin{aligned} a(q) &= 0.484012 + 1.984961q + 42.933001q^8, \\ b(q) &= 0.958079 - 0.129717q - 0.038993q^5 + 0.804107q^6, \\ c(q) &= 0.760938 - 1.272798q + 0.493549q^2. \end{aligned} \tag{35}$$

The interpolating function allows us to find the value of $U_q(E)$ once we know the temperature and electron density, fixed at $Z_1$, $Z_2$, and energy $E$ for a certain value of $q$. In the range $0 < D/E \leq 10^2$, for $0.8 < q < 1$ this function (eq. [34]) still gives a good approximation of $U_q/D$, but with more complicated relationships for $a(q)$, $b(q)$, and $c(q)$. We omit the details.

Calculation of nuclear fusion rates (quantities weighted over the distribution of reacting nuclei, which in many cases is a generalized distribution with a proper ionic parameter $q_i$ close to 1) requires the insertion into the average integral of the penetration factor, which is a function of $E$. In the case of a pure Coulomb barrier, the screening factor can be factorized. In our general case this factorization is not possible, and the behavior of $U_q$ as a function of $D/E$ must be considered with care, both for nonresonant and resonant reactions. The same consideration is valid if instead of $U_q$ we calculate the rates by use of the plotted and tabulated function $\tau_q(E)$.

## 4. REPRESENTATIVE EXAMPLES OF PENETRATION PROBABILITY

We report some representative examples of the evaluation of the electronic screening factor for nonresonant and resonant fusion reactions of interest in the solar core and in other dense astrophysical plasmas. Resonant fusions are influenced, aside from electron screening, by a resonance screening factor (Cussons et al. 2002). Furthermore, Maxwellian rates can be corrected by nonstandard ionic distributions. In this work we are interested in the modified D-H potential, and here we limit the discussion to the electron screening factor.

The enhancement of the penetration factor $\Gamma_q(E)$ over the pure Coulomb penetration $\Gamma_{\rm C}(E)$ can be expressed, using equations (27) and (30), by the ratio

$$\begin{aligned} f_{q,{\rm C}}(E) &= \frac{\Gamma_q(E)}{\Gamma_{\rm C}(E)} \\ &= \exp\left[-2\pi\frac{Z_1Z_2e^2}{\hbar c}\sqrt{\frac{\mu c^2}{2}}\left(\frac{1}{\sqrt{E+U_q}} - \frac{1}{\sqrt{E}}\right)\right], \end{aligned} \tag{36}$$

and the enhancement of $\Gamma_q(E)$ over $\Gamma_{\rm DH}(E) \equiv \Gamma_{q\to 1}(E)$ is given by

$$\begin{aligned} f_{q,{\rm DH}}(E) &= \frac{\Gamma_q(E)}{\Gamma_{\rm DH}(E)} \\ &= \exp\{-2\pi\eta(E)[\tau_q(E) - \tau_{q\to 1}(E)]\}. \end{aligned} \tag{37}$$

It is evident from Figure 2 and from Table 1 that an important enhancement over the D-H potential comes from the energy range $E \simeq 2 \times 10^{-4}D$ to $D$, with a maximum at about $E \simeq D/15$. The most effective burning energy is at $E_0 = [E_{\rm G}(kT)^2/4]^{2/3}$, where $E_{\rm G}$ is the Gamow energy.

TABLE 1

A Few Numerical Values of the Quantities $r_q/\zeta_q$, $U_q/D$, and $1 - \tau_q$ in the Energy Range $0.1 \leq D/E \leq 30$ for Several Values of $q$

| | $q = 0.25$ | | | $q = 0.50$ | | | $q = 0.75$ | | | $q = 0.099$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $D/E$ | $r_q/\zeta_q$ | $U_q/D$ | $1 - \tau_q$ | $r_q/\zeta_q$ | $U_q/D$ | $1 - \tau_q$ | $r_q/\zeta_q$ | $U_q/D$ | $1 - \tau_q$ | $r_q/\zeta_q$ | $U_q/D$ | $1 - \tau_q$ |
| 0.10...................... | 0.091 | 0.988 | 0.046 | 0.091 | 0.977 | 0.045 | 0.091 | 0.966 | 0.045 | 0.091 | 0.956 | 0.045 |
| 0.20...................... | 0.167 | 0.978 | 0.085 | 0.167 | 0.958 | 0.084 | 0.168 | 0.938 | 0.083 | 0.168 | 0.920 | 0.082 |
| 0.30...................... | 0.232 | 0.969 | 0.120 | 0.233 | 0.941 | 0.118 | 0.235 | 0.915 | 0.116 | 0.236 | 0.891 | 0.114 |
| 0.40...................... | 0.288 | 0.961 | 0.151 | 0.291 | 0.927 | 0.148 | 0.294 | 0.894 | 0.145 | 0.297 | 0.866 | 0.142 |
| 0.50...................... | 0.338 | 0.955 | 0.178 | 0.343 | 0.914 | 0.174 | 0.347 | 0.877 | 0.170 | 0.351 | 0.844 | 0.166 |
| 0.60...................... | 0.382 | 0.948 | 0.203 | 0.389 | 0.902 | 0.198 | 0.395 | 0.861 | 0.193 | 0.401 | 0.825 | 0.188 |
| 0.70...................... | 0.421 | 0.942 | 0.226 | 0.430 | 0.892 | 0.220 | 0.439 | 0.846 | 0.214 | 0.447 | 0.807 | 0.208 |
| 0.80...................... | 0.457 | 0.937 | 0.246 | 0.468 | 0.882 | 0.239 | 0.479 | 0.834 | 0.232 | 0.489 | 0.792 | 0.226 |
| 0.90...................... | 0.489 | 0.932 | 0.265 | 0.503 | 0.874 | 0.257 | 0.517 | 0.822 | 0.250 | 0.529 | 0.778 | 0.243 |
| 1.00...................... | 0.518 | 0.928 | 0.283 | 0.535 | 0.866 | 0.274 | 0.552 | 0.811 | 0.266 | 0.566 | 0.765 | 0.258 |
| 2.00...................... | 0.716 | 0.896 | 0.407 | 0.763 | 0.809 | 0.393 | 0.809 | 0.735 | 0.380 | 0.850 | 0.675 | 0.367 |
| 3.00...................... | 0.826 | 0.876 | 0.482 | 0.902 | 0.774 | 0.465 | 0.977 | 0.689 | 0.449 | 1.047 | 0.621 | 0.434 |
| 4.00...................... | 0.898 | 0.863 | 0.534 | 1.000 | 0.750 | 0.516 | 1.102 | 0.657 | 0.498 | 1.198 | 0.584 | 0.482 |
| 5.00...................... | 0.949 | 0.852 | 0.572 | 1.073 | 0.731 | 0.554 | 1.200 | 0.633 | 0.535 | 1.321 | 0.556 | 0.518 |
| 6.00...................... | 0.988 | 0.844 | 0.603 | 1.131 | 0.717 | 0.583 | 1.281 | 0.613 | 0.564 | 1.426 | 0.534 | 0.546 |
| 7.00...................... | 1.019 | 0.838 | 0.627 | 1.179 | 0.705 | 0.608 | 1.349 | 0.598 | 0.588 | 1.517 | 0.516 | 0.570 |
| 8.00...................... | 1.043 | 0.832 | 0.648 | 1.219 | 0.695 | 0.628 | 1.408 | 0.584 | 0.608 | 1.597 | 0.500 | 0.590 |
| 9.00...................... | 1.064 | 0.828 | 0.665 | 1.253 | 0.686 | 0.646 | 1.461 | 0.573 | 0.626 | 1.670 | 0.487 | 0.607 |
| 10.00.................... | 1.081 | 0.824 | 0.680 | 1.283 | 0.679 | 0.661 | 1.507 | 0.563 | 0.641 | 1.735 | 0.476 | 0.622 |
| 15.00.................... | 1.140 | 0.810 | 0.733 | 1.390 | 0.652 | 0.715 | 1.684 | 0.526 | 0.695 | 1.996 | 0.434 | 0.676 |
| 20.00.................... | 1.174 | 0.801 | 0.766 | 1.459 | 0.635 | 0.749 | 1.806 | 0.503 | 0.730 | 2.188 | 0.406 | 0.711 |

The first example concerns $p$-$p$ fusion in the solar core. At center-of-mass energies above $E_0 = 5.9 \times 10^{-3}$ MeV, $U_q$ can be taken to be constant and equal to $D = 0.7 \times 10^{-4}$ MeV. The factor $f_{q,\mathrm{C}}(E)$ is of order a few percent above unity at high energies, for ions with energy above $E_0$ and belonging to the distribution tail and having fusion probability greater than those belonging to the head of distribution. However, at these high energies $f_{q,\mathrm{DH}}(E)$ is practically equal to 1 also with $q \ll 1$. Therefore, the rate is not modified significantly with respect to the standard value. At energies below $E_0$, the enhancement is not negligible and $U_q$, which depends on $E$, goes to $(1 - q)D$ as $E \to 0$. We have, for instance, $f_{q,\mathrm{DH}}(E = D) = 1.12$ and $f_{q,\mathrm{DH}}(E = 0.1D) = 1.65$, when $q = 0.5$. However, protons with these energies have a very small probability of fusing, and the rate, in conclusion, cannot change with respect to the standard evaluation by more than a few percent, as also required by luminosity constraints. The above discussion is valid for all the other reactions of the hydrogen burning cycle, the effective burning energies being in the range $E_0 \simeq 15$–$30$ keV with $D < E_0$.

Among the reactions of the CNO cycle, we consider $^{14}\mathrm{N}(p, \gamma)^{15}\mathrm{O}$. The astrophysical factor recently measured (Runkle et al. 2005; Imbriani et al. 2005) has important consequences for the evolution of stars, estimates of the age of globular clusters, and, of course, the evaluation of CNO neutrino flux (Liolios 2000; Degl'Innocenti et al. 2004). At solar-core conditions, we have that in the high-energy region $U_q = D \ll E_0 = 27.0$ keV, and therefore $\Gamma_q(E) \approx \Gamma_{\mathrm{DH}}(E)$. At low energy $U_q \to (1 - q)D$, and for $E = D/15 = 0.04$ MeV we obtain $\Gamma_q(E) = 2.63 \times 10^{-4}$ with $f_{q,\mathrm{DH}}(E) = 3.33$ at $E = 0.047$ MeV and $q = 0.5$. Of course, $f_{q,\mathrm{DH}}(E) \to 1$ in the limit $q \to 1$, as we can expect from laboratory experiments.

Fusion reactions, such as $^{7}\mathrm{Li}(p, \alpha)\alpha$ and $^{6}\mathrm{Li}(d, \alpha)\alpha$, are of particular interest for their implications in astrophysics. Experimental measurements in the laboratory of the screening potential (Engstler et al. 1992; Pizzone et al. 2003) have indicated a value of 350–400 eV, greater than the adiabatic theoretical one of 186 eV. We should evaluate $D$ at the experimental conditions of target temperature and density. The quantity $U_q$ is a fraction of $D$, decreasing to $(1 - q)D$ and not to zero as in the standard D-H approach. Assuming $q = 0.5$, we need $D = 800$ eV to obtain $U_q \simeq 400$ eV. Unfortunately, we do not know the correct target temperature and density to calculate $D$. This difficulty is a motivation to consider questionable the application of this approach to laboratory experiments.

We consider now the important nonresonant reaction $^{12}\mathrm{C}(\alpha, \gamma)^{16}\mathrm{O}$ at a temperature of $kT = 17.2$ keV and density $\rho = 10^{3.5}$ g cm$^{-3}$ (as in the core plasma of helium-burning red giant stars) with a mass fraction $X(\mathrm{He}) = \frac{1}{2}$ and $X(\mathrm{C}) = \frac{1}{2}$. We have $D = 0.573$ keV. With $q = 0.5$, $U_q$ goes from about $U_q = 0.286$ keV at very low energy to $U_q = 0.496$ keV at $E = D$ and to $U_q \simeq D$ at high energy. The greatest value of $f_{q,\mathrm{DH}}(E)$ is at $E \approx 4 \times 10^{-2}$ keV. At such low energy, the screening factor $f_{q,\mathrm{DH}}(E) \approx 10^{55}$, the Coulomb penetration factor being practically zero and $\Gamma_{\mathrm{DH}}(E) \approx 10^{-458}$.

Next we consider the example of $^{12}\mathrm{C}$-$^{12}\mathrm{C}$ fusion in laboratory experiments and in massive stars (in the classical thermonuclear regime, without considering, for simplicity's sake, effects due to the presence of resonances and degeneration; Cussons et al. 2002; Itoh et al. 2003; Ferro et al. 2004). In the experimental study of $^{12}\mathrm{C}$-$^{12}\mathrm{C}$ fusion near the Gamow energy, the target temperature is $kT \simeq 6 \times 10^{-8}$ MeV and the graphite density is $\rho \sim 1.7$ g cm$^{-3}$ (Spillane et al. 2007), and therefore we have

TABLE 2

Penetration Factors for $^{12}\mathrm{C}$-$^{12}\mathrm{C}$ Fusion in Laboratory Experiments with a Graphite Target

| $E$ (MeV) | $\Gamma_{\mathrm{C}}(E)$ | $\Gamma_{\mathrm{DH}}(E)$ | $\Gamma_{q=0.50}(E)$ | $\Gamma_{q=0.99}(E)$ |
|---|---|---|---|---|
| 0.0059...... | $8.16 \times 10^{-494}$ | $2.16 \times 10^{-349}$ | $5.20 \times 10^{-285}$ | $1.75 \times 10^{-303}$ |
| 0.0590...... | $1.18 \times 10^{-156}$ | $2.13 \times 10^{-149}$ | $9.70 \times 10^{-141}$ | $2.16 \times 10^{-141}$ |
| 0.5900...... | $4.91 \times 10^{-50}$ | $8.63 \times 10^{-50}$ | $2.00 \times 10^{-49}$ | $1.97 \times 10^{-49}$ |
| 2.2000...... | $2.92 \times 10^{-26}$ | $3.16 \times 10^{-26}$ | $3.56 \times 10^{-26}$ | $3.16 \times 10^{-26}$ |

TABLE 3

Penetration Factors for $^{12}$C-$^{12}$C Fusion in Massive Stars

| $E$ (MeV) | $\Gamma_{q=0.25}(E)$ | $\Gamma_{q=0.99}(E)$ |
|---|---|---|
| 0.0059 | $1.89 \times 10^{-43}$ | $3.07 \times 10^{-82}$ |
| 0.0590 | $7.50 \times 10^{-41}$ | $2.42 \times 10^{-54}$ |
| 0.5900 | $2.19 \times 10^{-31}$ | $1.08 \times 10^{-33}$ |
| 2.2000 | $6.64 \times 10^{-22}$ | $2.62 \times 10^{-22}$ |

approximately $R_{\rm DH} \simeq 0.31 \times 10^4$ fm and $D \simeq 0.0168$ MeV. From Assenbaum et al. (1987), we have $U_e$=5900 eV, $R_a$=0.88×10$^4$ fm, $\Gamma_{\rm C}(E) = \exp(-87.21/\sqrt{E})$, and $\Gamma_{\rm DH}(E) = \exp[-87.21/(E+U_e)^{1/2}]$. We assume $q$ = 0.5, representing an average deformation of the electron distribution. While $U_e$ is fixed, $U_q$ depends on the energy $E$. The ratios $\Gamma_{\rm DH}(E)/\Gamma_{\rm C}(E)$ coincide with those of Spillane et al. (2007). In Table 2, we report the values of $\Gamma_{\rm C}(E)$, $\Gamma_{\rm DH}(E)$, $\Gamma_{q=0.50}(E)$, and $\Gamma_{q=0.99}(E)$. Our value $\Gamma_{q\to 1}(E)$ differs from $\Gamma_{\rm DH}(E)$ because the first factor is calculated using $U_{q\to 1}(E)$ and the second is calculated with $U_e$ held fixed.

In massive stars (Gasques et al. 2005), the strong screening effect in a dense plasma can be simulated by assuming $q$ = 0.25, a value that represents a large electron deformation. At $kT = 30 \times 10^{-3}$ MeV and $\rho \simeq 10^9$ g cm$^{-3}$, we have $R_{\rm DH} \simeq 0.5 \times 10^2$ fm and $D \simeq 1.04$ MeV. Enhancement of $\Gamma_{q=0.25}(E)$ over $\Gamma_{q=0.99}(E)$ is very evident; at $E$ = 2.2 MeV, the enhancement is by a factor of about 2.5. In Table 3 we report, for several energies, $\Gamma_{q=0.25}(E)$ and $\Gamma_{q=0.99}(E)$.

Finally, Type Ia supernova environmental conditions give $kT = 5 \times 10^{-3}$ MeV, a central density $\rho = 3 \times 10^9$ g cm$^{-3}$, and $X$(C) = $\frac{1}{3}$. Therefore, $D$ = 1.5 MeV, the greatest deviation is at $E$ = 0.1 MeV, and consequently we have at this energy $\Gamma_{\rm C} = 7.9 \times 10^{-125}$, $\Gamma_{\rm DH} = 6.5 \times 10^{-78}$, and $\Gamma_{q=0.5} = 4.5 \times 10^{-73}$ with $f_{q,{\rm DH}} = 6.9 \times 10^4$.

Of course, in all the above examples $q$ is an arbitrary parameter whose value should be determined a priori. Rates of fusion reactions will be evaluated using the parametric expression of $U_q(E)$ given in equation (34).

## 5. CONCLUSIONS

We have shown that in those systems where, in space coordinates, stationary electron distributions deviate from the standard exponential one, the shielded electrostatic Coulomb potential is modified with respect to the standard D-H potential derived using linear conditions and constraints. We have used two different approaches, which produce the same results. One consists of associating a Poisson equation having an unknown electron density distribution with a Bernoulli equation for the quantity $rV_q(r)$, whose solution is asymptotically a power law. This result can also be obtained, and justified, by means of the superstatistics approach recently developed within generalized statistical mechanics by considering $1/R_{\rm DH}$, the inverse D-H radius, as a fluctuating intensive parameter with relative variance $\omega$ and parameter $q$, given by the relation $q = 1 + \omega$, characterizing the fluctuation.

The $q$-modified D-H potential has asymptotic power-law behavior, and we have discussed its meaning for the case $q < 1$, which holds when the electrostatic potential has a depleted and cut-off tail. Fluctuations of the inverse D-H radius may be due to temperature and density fluctuations of the electrons surrounding the reacting nuclei in astrophysical plasmas.

Exact evaluation of the penetration factor has to be carried out numerically. However, in the case of small deformations one can calculate $\Gamma(E)$ as the pure Coulomb penetration factor $\Gamma_{\rm C}(E)$ times a correction that may meaningfully differ from the standard D-H correction. We have presented detailed plots and numerical tables of several useful quantities that allow the evaluation of the penetration factor and the reaction rates, which, being weighted integrals, depend meaningfully on the behavior of the penetration factor as a function of $E$.

The penetration factor $\Gamma_q(E)$ can be given as the pure Coulomb penetration factor at an equivalent energy $E + U_q$. The energy $U_q$ is not a constant and can be evaluated only numerically. We have given a useful fit for it. $U_q$ is a function of energy $D$, characteristic of reacting nuclei, their density, and fusion temperature, the constant of proportionality depending on $q$ and varying with $D/E$. $U_q$ is proportional to $1/(kT)^{1/2}$ and to $n^{1/2}$, as observed experimentally in fusion reactions in metal matrices. The complete and correct expression for the penetration factor or for $U_q$ or $1 - \tau_q$ reported in this paper is necessary to evaluate the fusion rates at any energy. In fact, a possibly deformed ion distribution, together with a deformed electron distribution, may meaningfully affect the values of the rates. In the evaluations of the rates of fusion reactions, such as in the examples of $p$-$p$, $^{14}$N($p, \gamma$)$^{15}$O, $^7$Li($p, \alpha$)$\alpha$ and $^6$Li($d, \alpha$)$\alpha$, $^{12}$C($\alpha, \gamma$)$^{16}$O, and $^{12}$C-$^{12}$C fusion, mentioned in the previous section, important enhancements may occur, in addition to possible enhancements or decrements due to non-Maxwellian energy-momentum ion distributions, if the electron clouds surrounding the reacting nuclei are spatially modified by an assumed $q < 1$ distribution. Specific applications of our approach are in progress; they are comprehensive of discussions of the recent experimental results of the LUNA collaboration (Spillane et al. 2007; Imbriani et al. 2005; Gyurky 2007) to derive astrophysical factors in the stellar energy range and will be the topic of a further work. We are confident that our approach and the numerical tables presented here will be used to study astrophysical processes where temperature and density fluctuations cannot be neglected to evaluate the D-H shielding effect on reaction rates.

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