Nuclear astrophysics II

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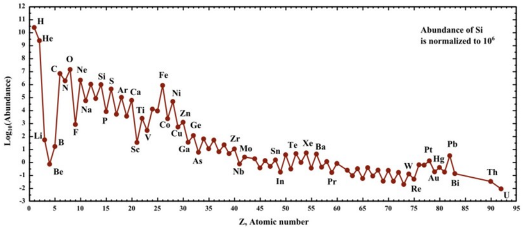
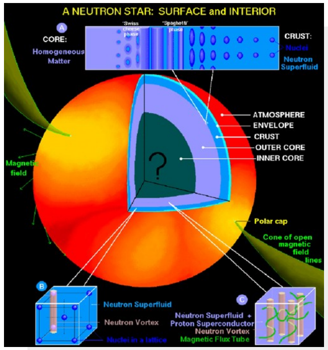
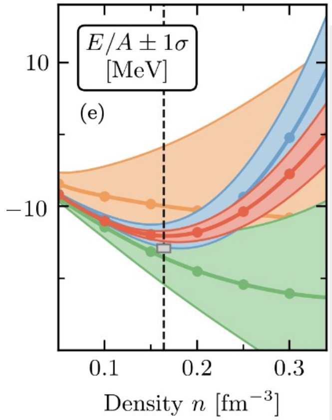
Nuclear astrophysics



Element formation



Neutron stars



Exotic matter

Stars

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- **Normal stars**: thermal pressure (via nuclear energy generation) to counteract gravity

- **Compact stars**: mostly degeneracy pressure (since nuclear fuel basically spent) counteracts gravity (if at all)
  - *White dwarfs*: electron degeneracy pressure
  - *Neutron stars*: neutron degeneracy pressure (and repulsion)
  - *Black holes*: the struggle is real

MASS

Neutron astrophysics

How does a neutron star stay there?

Reminder: thermal pressure not an option. Instead, we need **degeneracy pressure**.

Degeneracy pressure

One-dimensional box

Periodic boundary conditions:

$$\psi(x) = \psi(x + L)$$

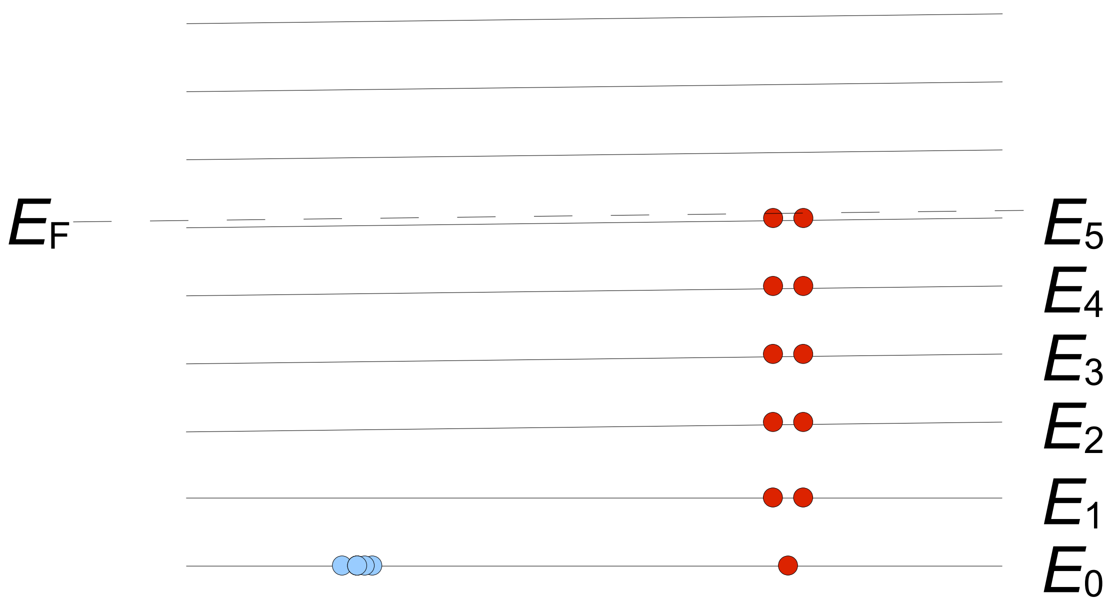
Wave function is:

$$\psi(x) = \frac{1}{\sqrt{L}} e^{ik_x x}$$

Characterized by:

$$k_x = \frac{2\pi}{L} n_x$$

$$E = \sum_{k_x}^{k_x \le k_\mathrm{F}} \frac{\hbar^2 k_x^2}{2m}$$



bosons

you *can* place as many as you want in the same state

fermions

you *cannot* place as many as you want in the same state

Degeneracy pressure

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Three-dimensional box

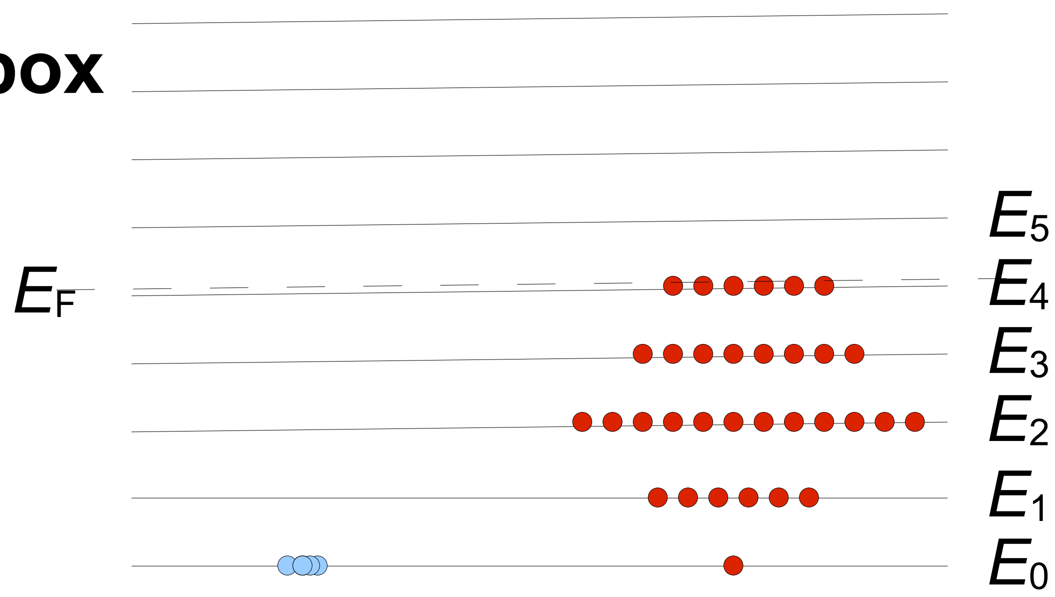
In periodic boundary conditions, the wave function is:

$$\psi(\mathbf{r}) = \frac{1}{\sqrt{L^3}} e^{i\mathbf{k}\cdot\mathbf{r}}$$

characterized by:

$$\mathbf{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

$$E = \sum_{k_x,k_y,k_z}^{|\mathbf{k}|\leq k_{\mathrm{F}}} \frac{\hbar^2\mathbf{k}^2}{2m}$$



bosons

you *can* place as many as you want in the same state

fermions

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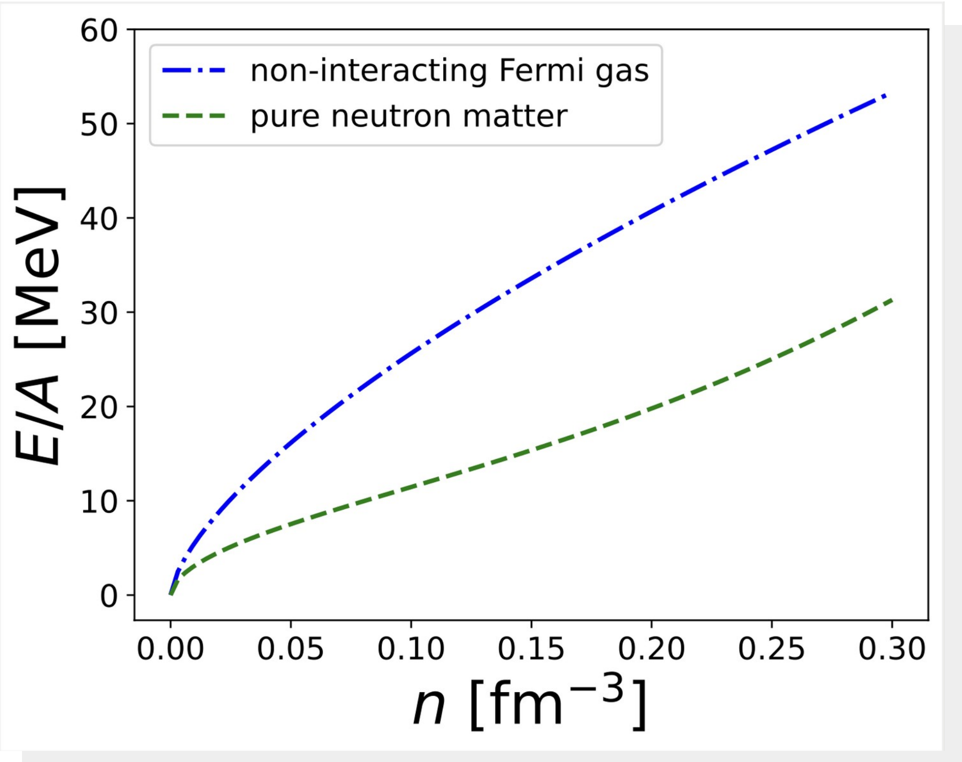
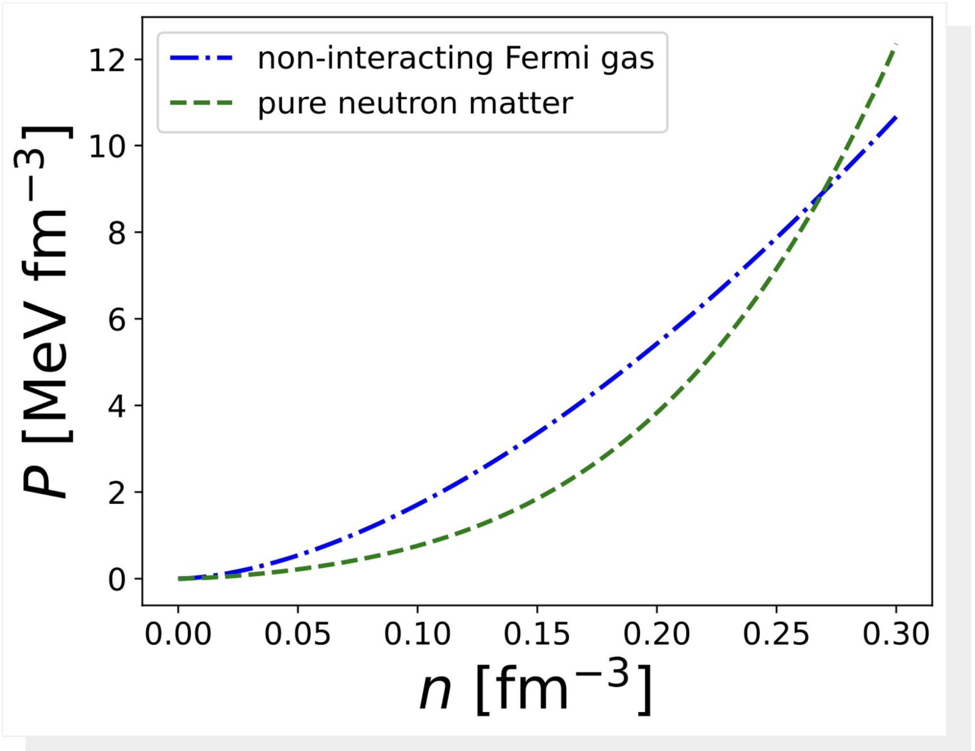
From free fermions to neutron matter

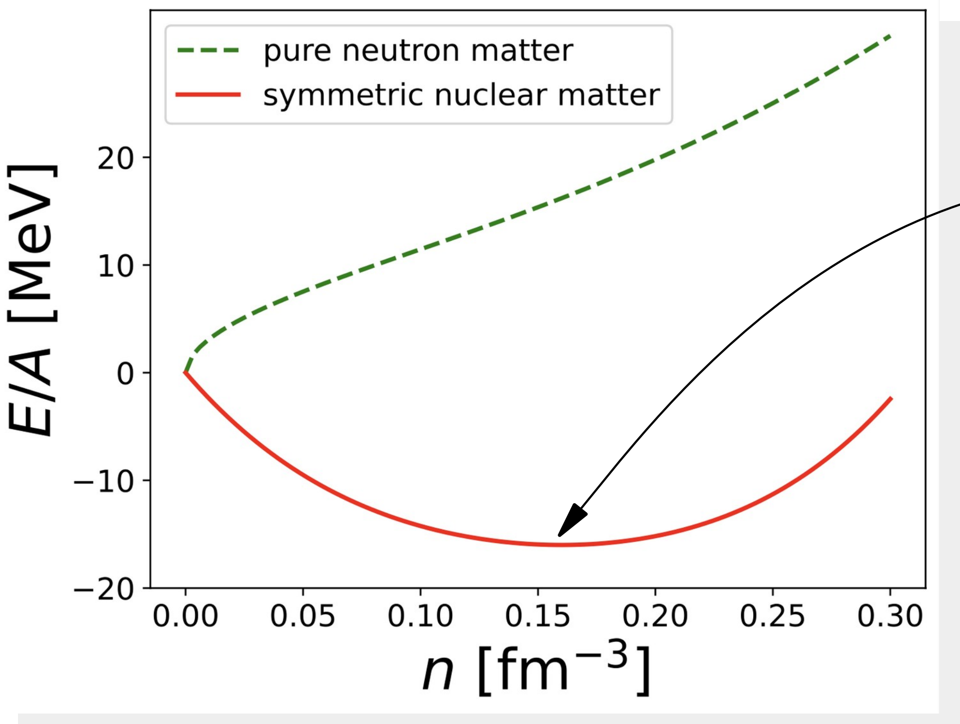
$$P = n^2 \frac{\partial(E/A)}{\partial n}$$





Pressure always positive

From the liquid drop to nuclear matter

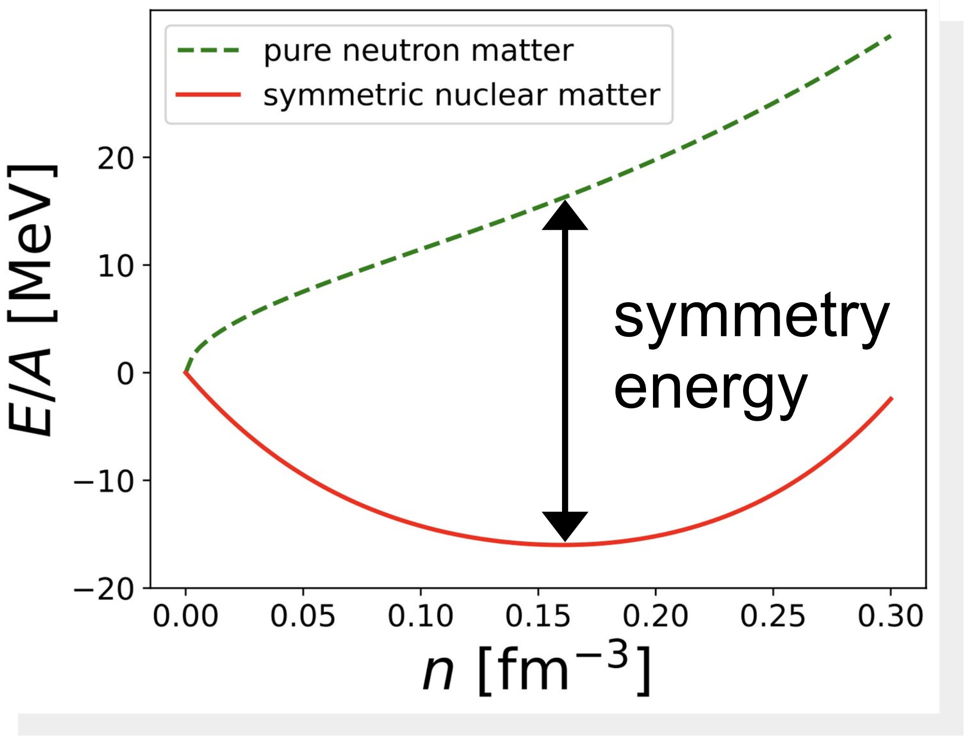


Energy always negative, exhibits a minimum

Saturation point (at 0.16 fm^{-3}) has an energy per nucleon of -16 MeV, consistent with taking *N* and *Z* to infinity in the liquid-drop formula:

$$\frac{E_B}{N+Z} = a_V - \frac{a_S}{(N+Z)^{1/3}}$$

From the liquid drop to nuclear matter



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From the liquid drop to nuclear matter

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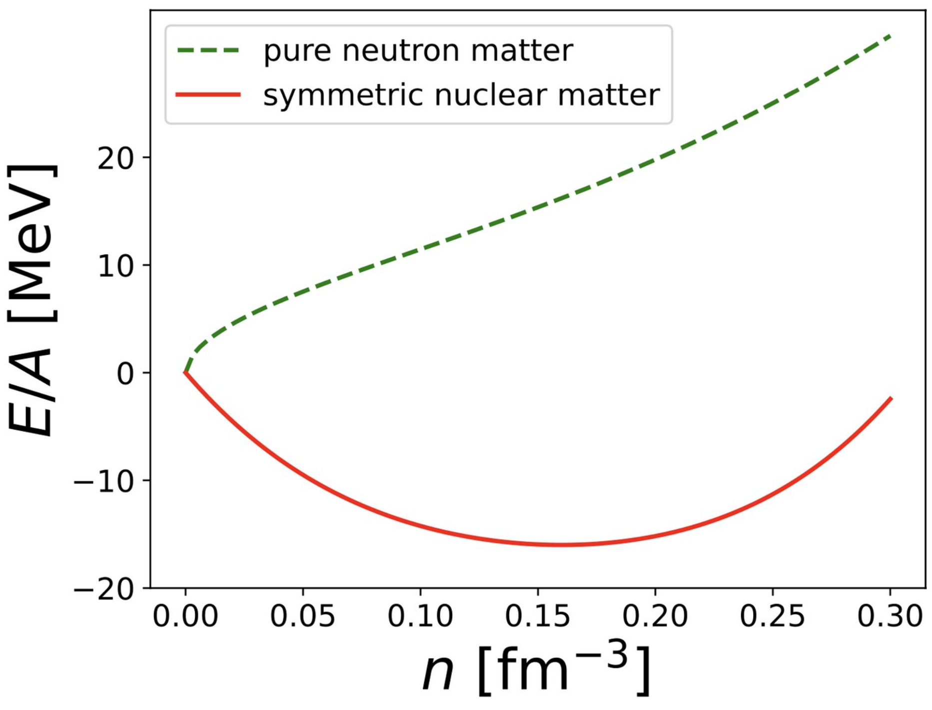

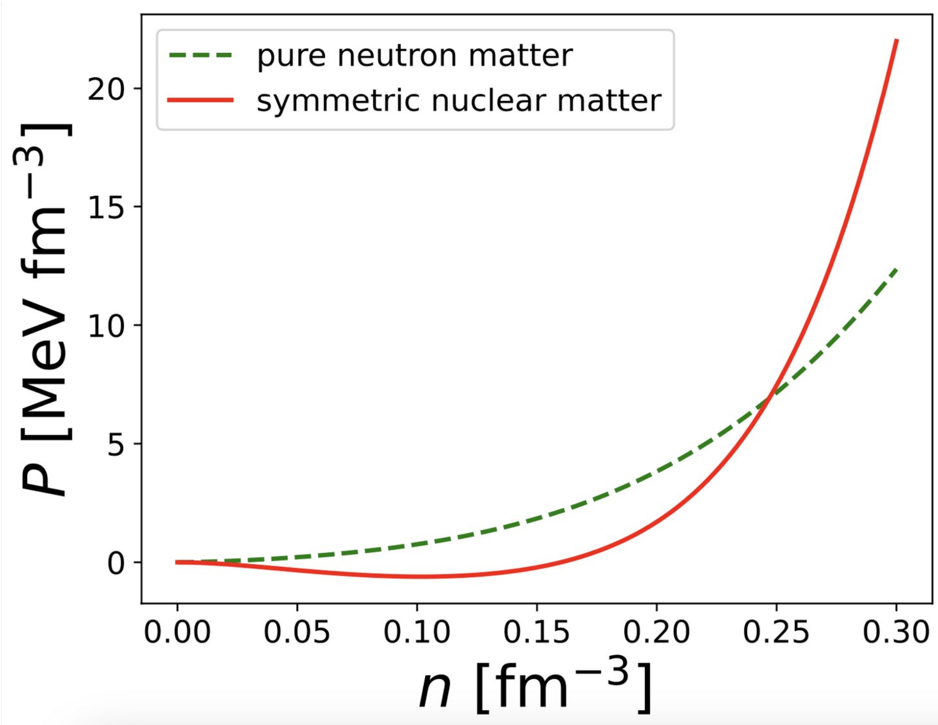

Pressure zero at saturation point
(discuss shoulder)

Nucleonic matter

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In general, the energy per particle will depend on the total density *n* and on the asymmetry parameter $\delta = \frac{n_n - n_p}{n_n + n_p}$

One can do a bivariate Taylor expansion, to find:

$$\frac{E}{A}(n,\delta) = E_0 + E_{\rm sym}\delta^2 + L_{\rm sym}\delta^2\left(\frac{n-n_0}{3n_0}\right) + \frac{1}{2}\left[K_0 + K_{\rm sym}\delta^2\right]\left(\frac{n-n_0}{3n_0}\right)^2$$

Current estimates:

$$E_0 = -15.8 \pm 0.3 \text{ MeV}$$
$$E_{\rm sym} = 32 \pm 2 \text{ MeV}$$
$$L_{\rm sym} = 60 \pm 15 \text{ MeV}$$
$$n_0 = 0.155 \pm 0.005 \text{ fm}^{-3}$$
$$K_0 = 230 \pm 20 \text{ MeV}$$
$$K_{\rm sym} = -100 \pm 100 \text{ MeV}$$

J. Piekarewicz & M. Centelles, *Phys. Rev. C* **79**, 054311 (2009)
J. Margueron et al, *Phys. Rev. C* **97**, 025805 (2018)

Beta equilibrium

$$\frac{E}{A}(n,\delta) = E_0 + E_{\rm sym}\delta^2 + L_{\rm sym}\delta^2\left(\frac{n-n_0}{3n_0}\right) + \frac{1}{2}\left[K_0 + K_{\rm sym}\delta^2\right]\left(\frac{n-n_0}{3n_0}\right)^2$$

This allows us to obtain the EOS for any density and any population asymmetry. But which value of *δ* is actually relevant to neutron-star matter?

In matter made up of *n*, *p*, and *e*, we impose:

1) charge neutrality $n_e = n_p$

2) β-equilibrium $\mu_n = \mu_p + \mu_e$

$$n \to p^+ + e^- + \bar{\nu}_e$$

$$p^+ + e^- \to n + \nu_e$$

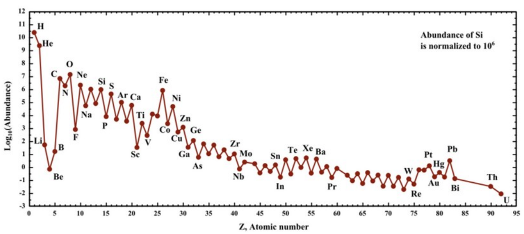
Protons are typically less than 10% of the total, hence the appellation *neutron* stars

Neutron astrophysics

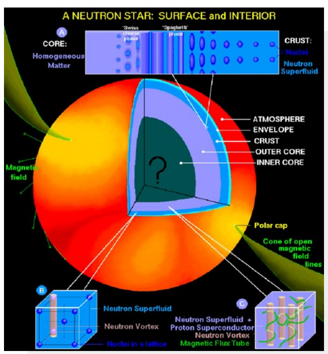
Up to this point, the EOS for pure neutron matter and symmetric nuclear matter dropped from the sky (no pun intended).

To see where they actually come from, we'll need a crash course in modern nuclear theory.

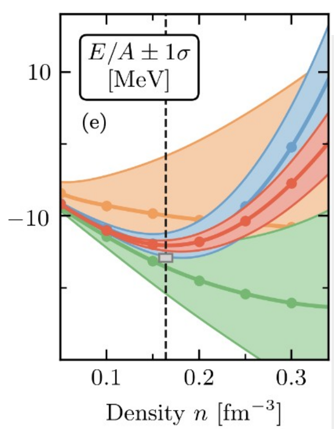
Nuclear astrophysics



Element formation



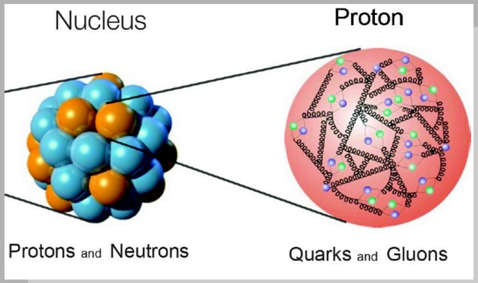
Neutron stars



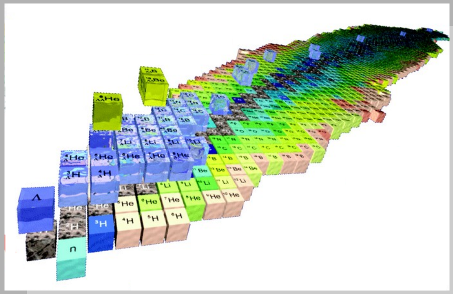
Exotic matter

Physical systems studied

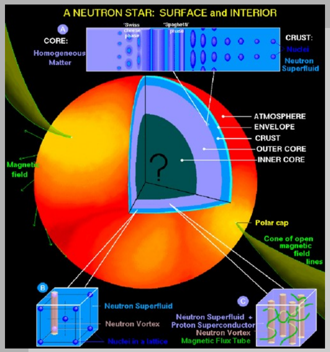
Nuclear forces

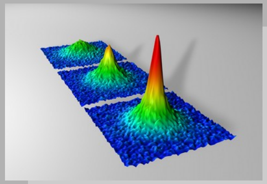


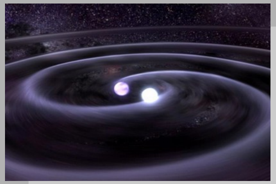
Nuclear structure



Nuclear astrophysics

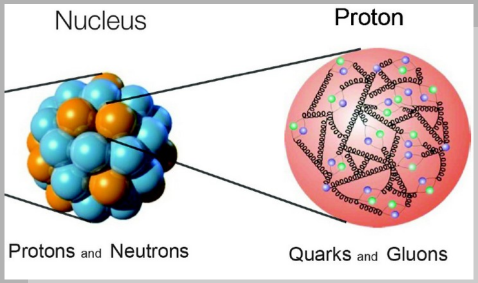




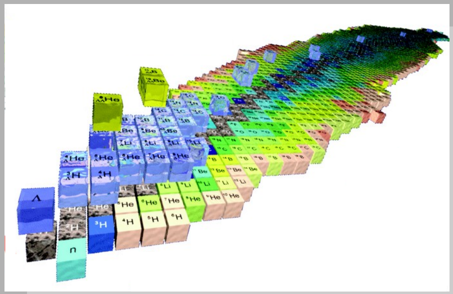


Physical systems studied

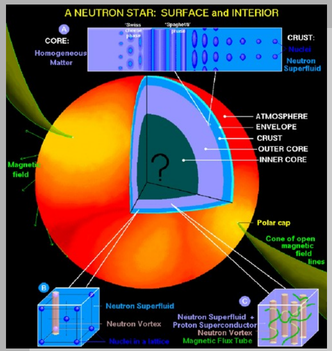
Few nucleons

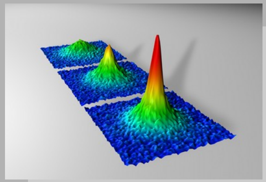


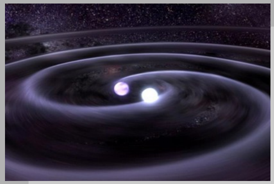
Many nucleons



Very many nucleons







Quotes on degrees of freedom

"The underlying physical laws necessary for the mathematical theory of a large part of physics and the whole of chemistry are thus completely known**, and the difficulty is only that the exact application of these laws leads to equations much too complicated to be soluble.**"

– Paul Dirac

"To understand macroscopic properties of matter based on understanding these microscopic laws is just unrealistic. Even though the microscopic laws are, in a strict sense, controlling what happens at the larger scale, they are not the right way to understand that."

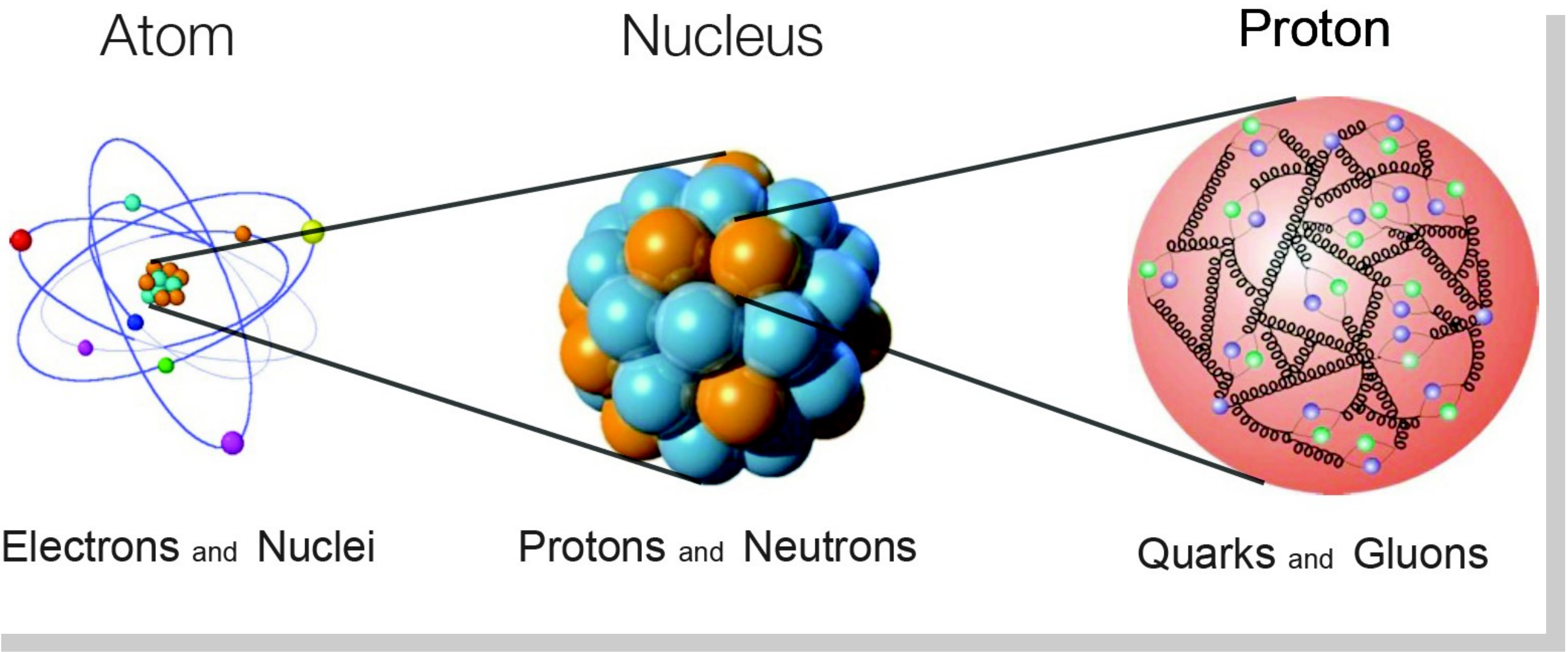
– John Schwarz

"only a fool would imagine that one should try to understand the properties of waves in the ocean in terms of Feynman-diagram calculations in the standard model, even if the latter understanding is possible 'in principle'."

– Tom Banks

Degrees of freedom

What does *from first principles* mean?



Steven Weinberg's Third Law of Progress in Theoretical Physics:

You may use any degrees of freedom you like to describe a physical system, but if you use the wrong ones, you'll be sorry!

Nuclear interactions 1

Historically

"Effective Interactions" were employed in the context of mean-field theory.

Phenomenological

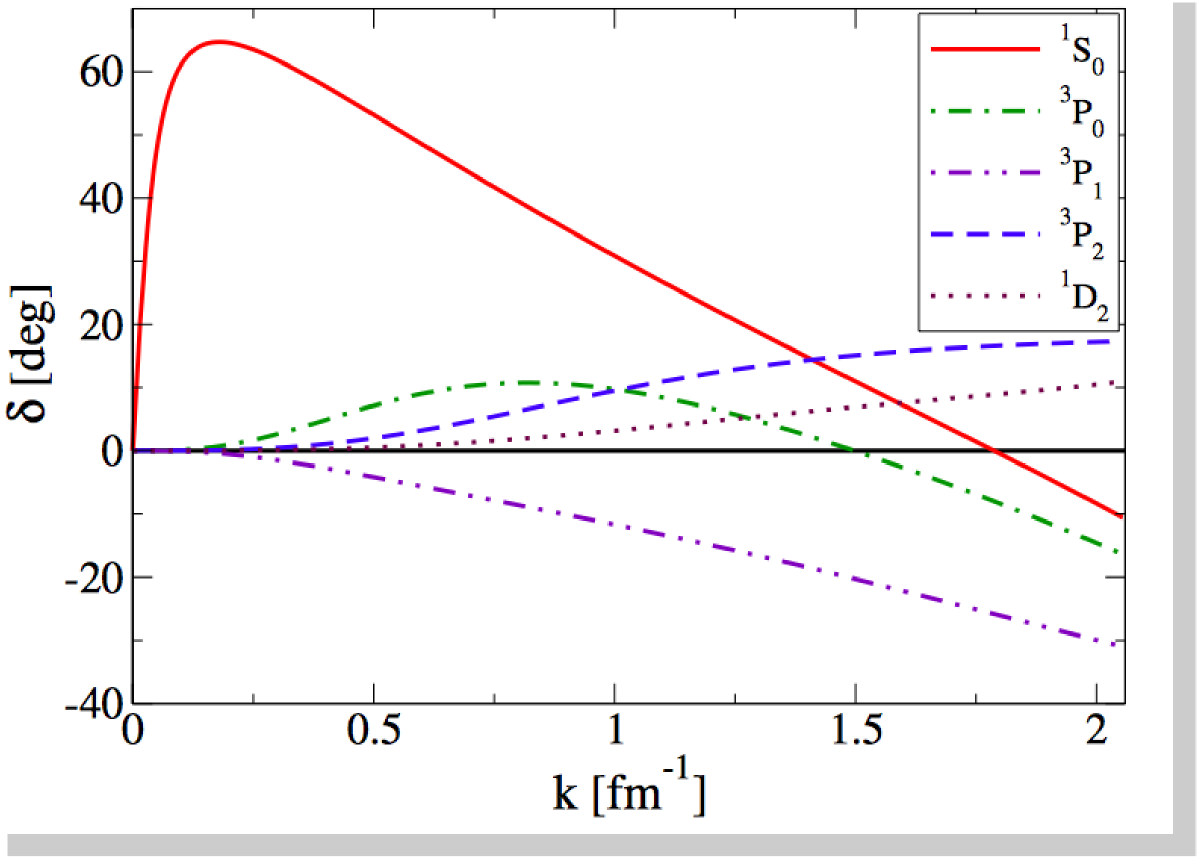
NN interaction fit to N-body experiment

Non-microscopic

NN interaction does not claim to (and will not) describe np scattering

Nuclear physics is difficult

Scattering phase shifts: different “channels” have different behavior.



Any potential that reproduces them must be spin (and isospin) dependent

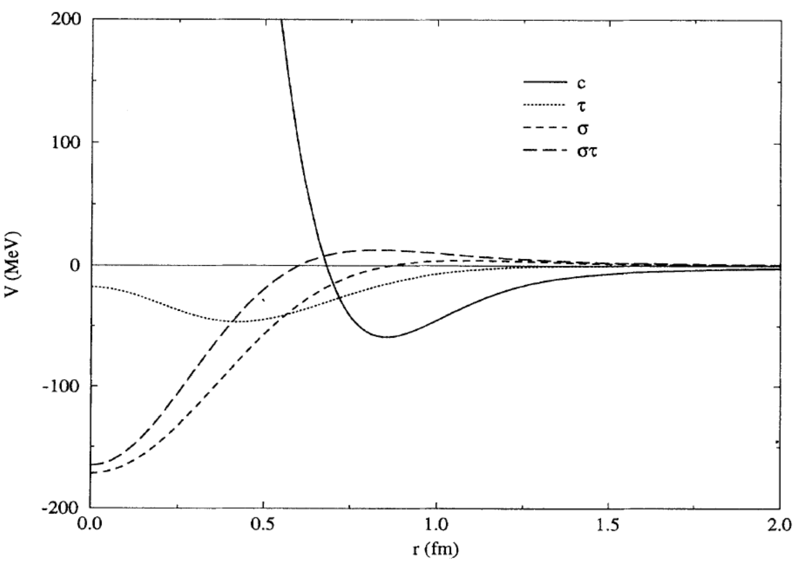
Nuclear interactions 2

Different approach: phenomenology treats NN scattering without connecting with the underlying level

$$V_2 = \sum_{j<k} v_{jk} = \sum_{j<k} \sum_{p=1}^{8} v_p(r_{jk}) O^{(p)}(j,k)$$

$$O^{p=1,8}(j,k) = (1, \sigma_j \cdot \sigma_k, S_{jk}, \mathbf{L}_{jk} \cdot \mathbf{S}_{jk}) \otimes (1, \tau_j \cdot \tau_k)$$

Such potentials are hard, making them non-perturbative at the many-body level (which is a problem for most methods on the market).



Softer, momentum-space formulations like CD-Bonn very popular

How to go beyond?

Historically, fit NN interaction to N-body experiment

Parallel approach, fit NN interaction to 2-body experiment, ignoring underlying level of quarks and gluons

Natural goal: fit NN interaction to 2-body experiment, without ignoring underlying level

Chiral effective field theory

Nuclear Hamiltonian: chiral EFT

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How to build on QCD in a systematic manner?

Exploit separation of scales: $a_{^1S_0} = (11\ \mathrm{MeV})^{-1}$

$$m_\pi = 140\ \mathrm{MeV}$$

$$\Lambda_\chi \approx m_\rho \approx 800\ \mathrm{MeV}$$

Chiral Effective Field Theory approach:

Use nucleons and pions as degrees of freedom

Systematically expand in $\dfrac{Q}{\Lambda_\chi}$

Program introduced by S. Weinberg, now taken over by the nuclear community

Nuclear Hamiltonian: chiral EFT

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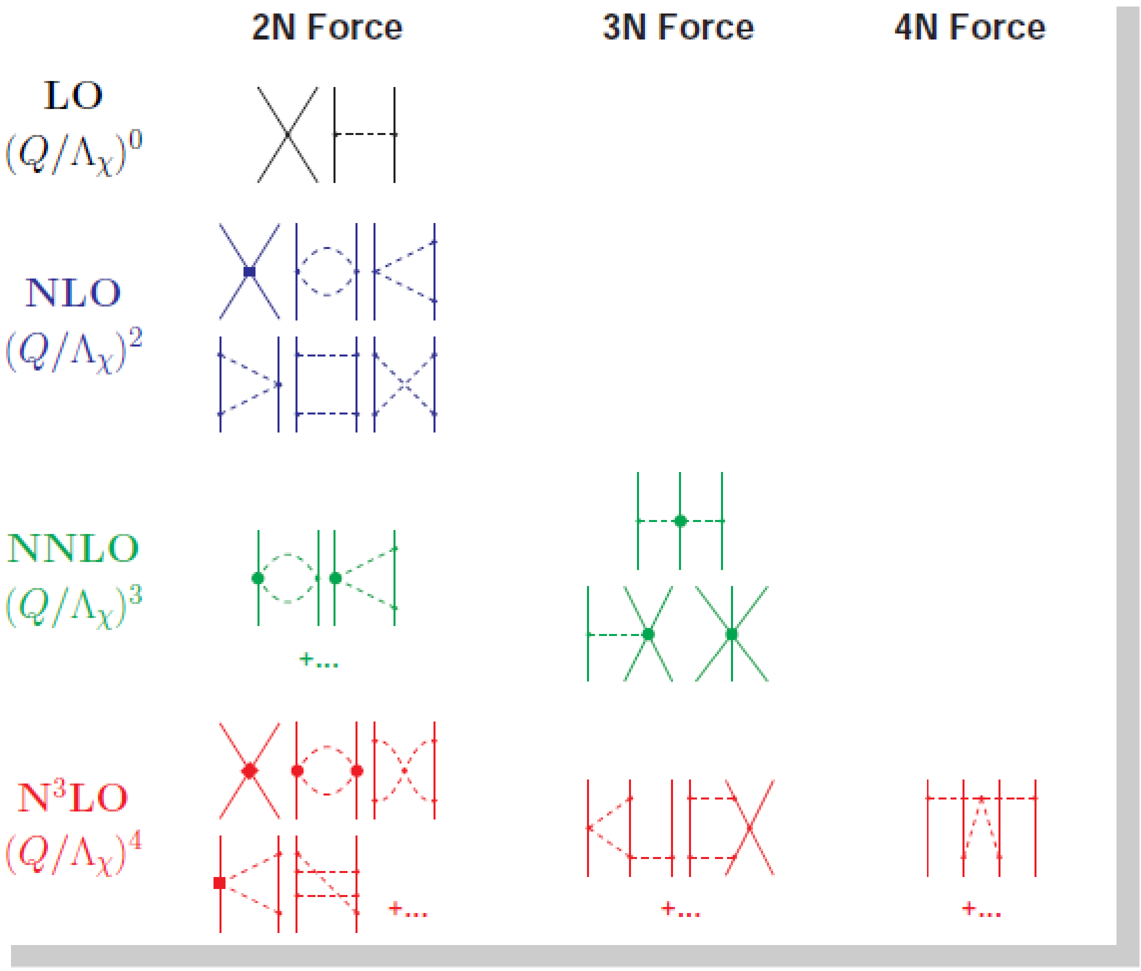


- Attempts to connect with underlying theory (QCD)
- Systematic low-momentum expansion
- Consistent many-body forces
- Low-energy constants from experiment or lattice QCD
- Until now non-local in coordinate space, so unused in continuum QMC
- Power counting's relation to renormalization tackled only very recently

Digression

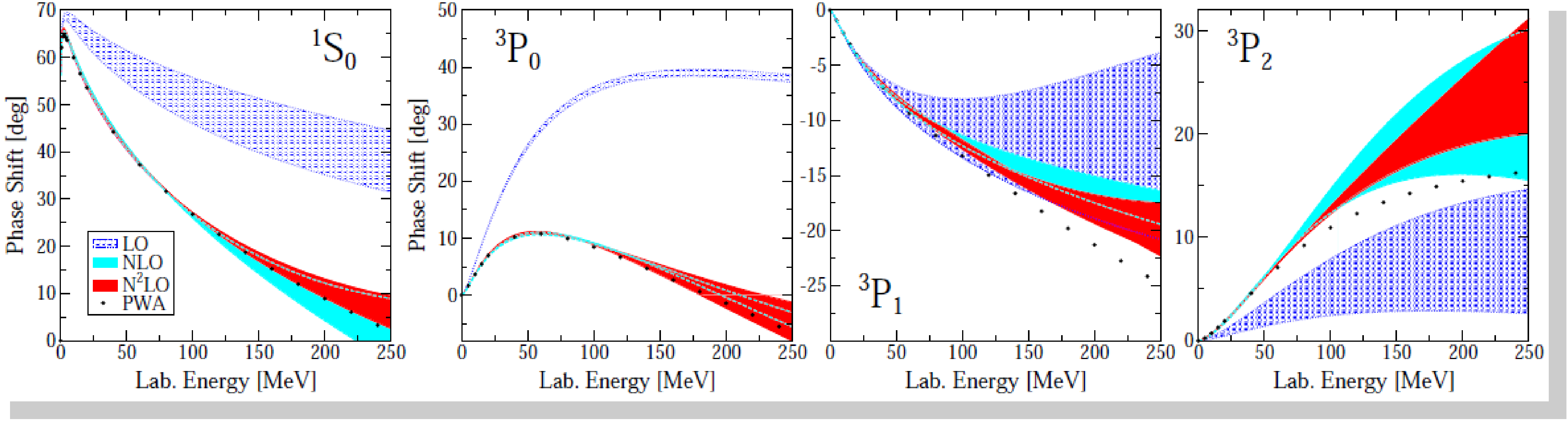
Probably the most famous line in the Unix kernel code:

```
/*
 * You are not expected to understand this.
 */
```

How to go beyond?

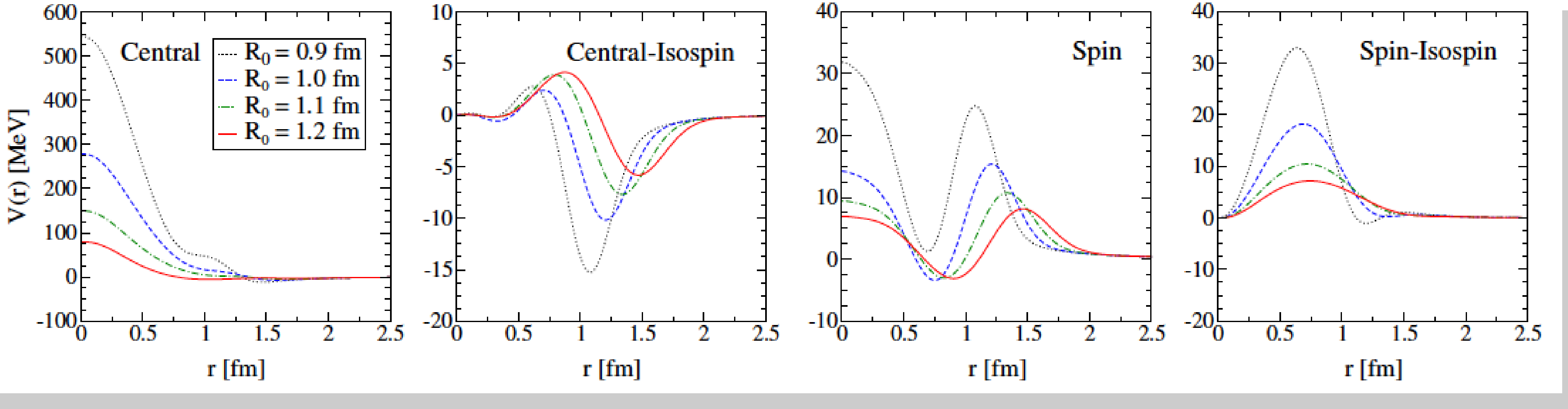
Combine power of Quantum Monte Carlo with consistency of chiral Effective Field Theory

- Write down a local energy-independent NN potential
- Before doing many-body calculations, fit to NN phase shifts

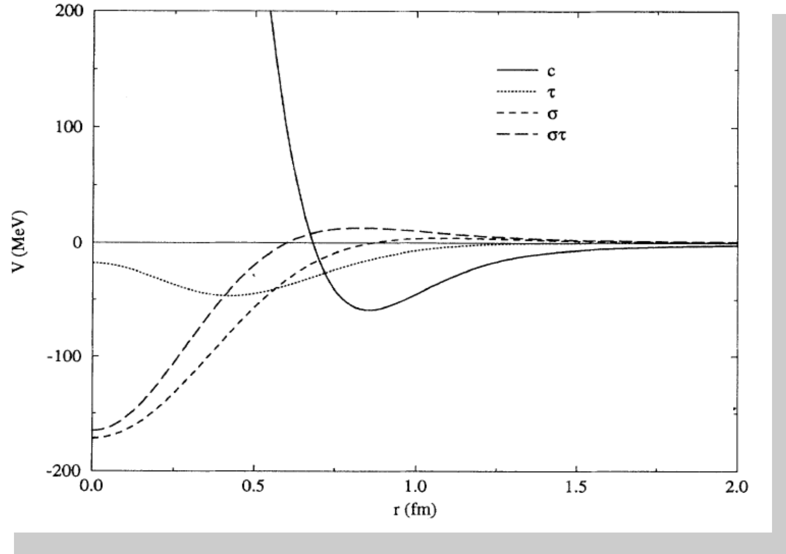


A. Gezerlis, I. Tews, E. Epelbaum, S. Gandolfi, K. Hebeler, A. Nogga, A. Schwenk, Phys. Rev. Lett. **111**, 032501 (2013).

Since it's local, let's plot it (N^2LO)



Compare with AV18



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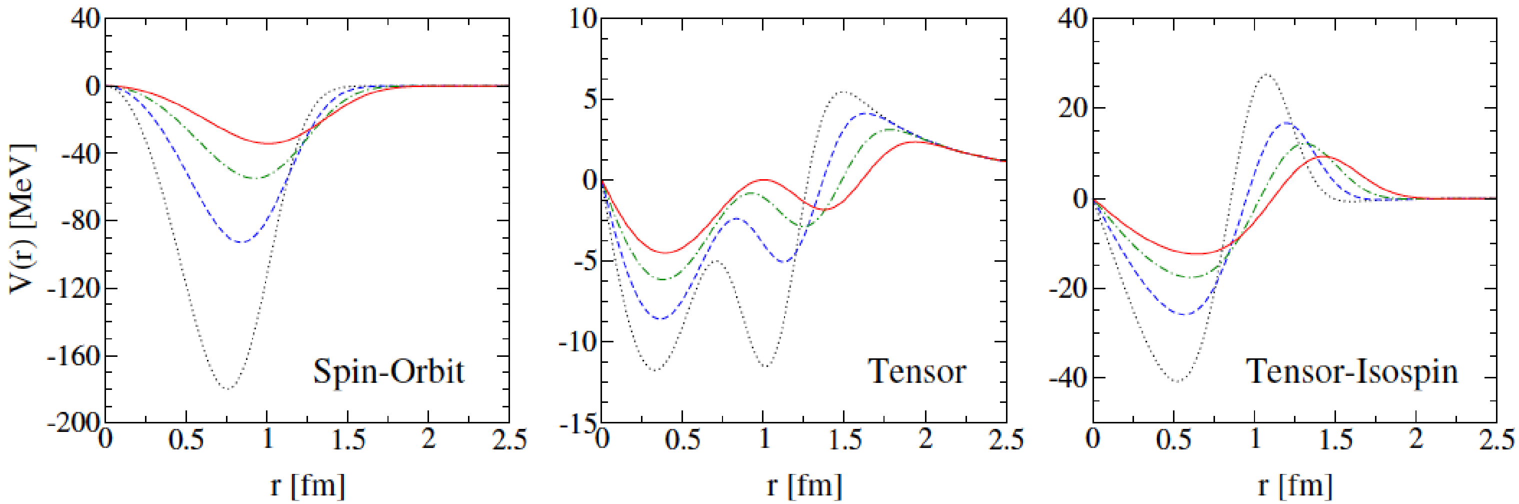

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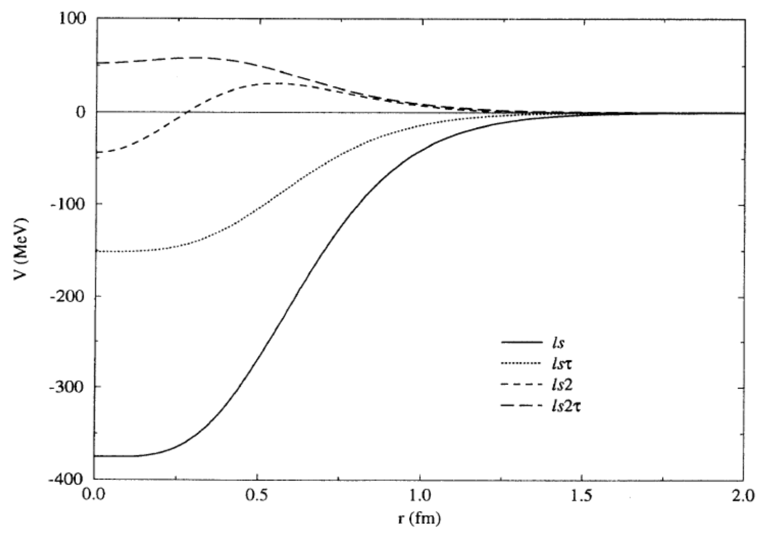

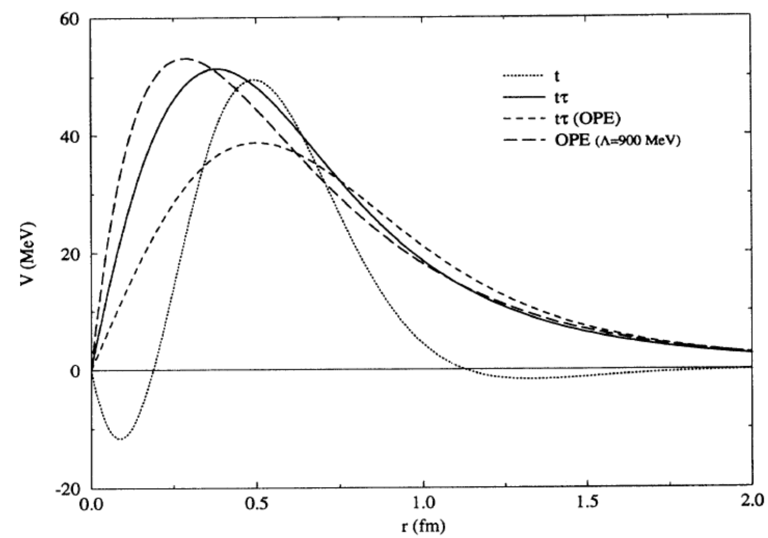

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But even with the interaction in place, how do you solve the many-body problem?

Nuclear many-body problem

$$H\Psi = E\Psi$$

where $$H = \sum_i K_i + \sum_{i<j} v_{ij} + \sum_{i<j<k} V_{ijk} + \cdots$$

so

$$H\Psi(\mathbf{r}_1, \cdots, \mathbf{r}_A; s_1, \cdots, s_A; t_1, \cdots, t_A) = E\Psi(\mathbf{r}_1, \cdots, \mathbf{r}_A; s_1, \cdots, s_A; t_1, \cdots, t_A)$$

i.e. $2^A \binom{A}{Z}$ complex coupled second-order differential equations

Nuclear many-body methods

Phenomenological (fit to A-body experiment)

Ab initio (fit to few-body experiment)

Nuclear many-body methods

Phenomenological (fit to A-body experiment)

- **Shell model**
  mainstay of nuclear physics, still very important

- **Hartree-Fock/Hartree-Fock-Bogoliubov (HF/HFB)**
  mean-field theory, a priori inapplicable, unreasonably effective

- **Energy-density functionals (EDF)**
  like mean-field but with wider applicability

Nuclear many-body methods

Ab initio (fit to few-body experiment)

- **Quantum Monte Carlo (QMC)**
  stochastically solve the many-body problem “exactly”

- **Perturbative Theories (PT)**
  first few orders only

- **Resummation schemes (e.g. SCGF)**
  selected class of diagrams up to infinite order

- **Coupled cluster (CC)**
  generate np-nh excitations of a reference state

- **No-core shell model (NCSM)**
  fully ab initio, in contradistinction to traditional SM

Modern phenomenology:
Skyrme energy-density functionals

Skyrme EDF

Energy-density functional

Slater determinant $\phi(x_1, \ldots, x_N) = \frac{1}{\sqrt{N!}} \det|\phi_i(x_j)|$

Total energy $E = \langle\phi| \hat{H} |\phi\rangle = \int \mathcal{H}(\mathbf{r}) d^3 r$

Energy density $\mathcal{H} = \mathcal{K} + \mathcal{H}_0 + \mathcal{H}_3 + \mathcal{H}_{\text{eff}} + \mathcal{H}_{\text{fin}}$

Skyrme EDF

Energy-density functional

Energy density $$\mathcal{H} = \mathcal{K} + \mathcal{H}_0 + \mathcal{H}_3 + \mathcal{H}_{\text{eff}} + \mathcal{H}_{\text{fin}}$$

$$\mathcal{K} = \frac{\hbar^2}{2m}\tau(\mathbf{r}),$$

$$\mathcal{H}_0 = \left(C_0^{\rho,0} + C_1^{\rho,0}\right)\rho^2(\mathbf{r}) = \frac{1}{4}t_0(1-x_0)\rho^2(\mathbf{r}),$$

$$\mathcal{H}_3 = \left(C_0^{\rho,\alpha} + C_1^{\rho,\alpha}\right)\rho^{2+\alpha}(\mathbf{r}) = \frac{1}{24}t_3(1-x_3)\rho^{2+\alpha}(\mathbf{r})$$

$$\mathcal{H}_{\text{eff}} = \left(C_0^{\tau} + C_1^{\tau}\right)\rho(\mathbf{r})\tau(\mathbf{r})$$

$$= \frac{1}{8}\left[t_1(1-x_1) + 3t_2(1+x_2)\right]\rho(\mathbf{r})\tau(\mathbf{r}), \text{ and}$$

$$\mathcal{H}_{\text{fin}} = -\left(C_0^{\Delta\rho} + C_1^{\Delta\rho}\right)\left(\nabla\rho(\mathbf{r})\right)^2$$

$$= \frac{3}{32}\left[t_1(1-x_1) - t_2(1+x_2)\right]\left(\nabla\rho(\mathbf{r})\right)^2$$

Skyrme EDF

Energy-density functional Expressed in terms of:

Number density
$$\rho(\mathbf{r}) = \sum_{i,\sigma} |\phi_i(\mathbf{r},\sigma)|^2$$

Kinetic density
$$\tau(\mathbf{r}) = \sum_{i,\sigma} |\nabla\phi_i(\mathbf{r},\sigma)|^2$$

Single-particle orbitals determined self-consistently
Minimize total energy:

$$\frac{\delta}{\delta\phi_i^*(\mathbf{r},\sigma)}\left(E - \sum_i e_j \int d\mathbf{r} \sum_\sigma |\phi_j(\mathbf{r},\sigma)|^2\right) = \frac{\partial\mathcal{H}_v}{\partial\phi_i^*(\mathbf{r},\sigma)} - \nabla\cdot\frac{\partial\mathcal{H}_v}{\partial\nabla\phi_i^*(\mathbf{r},\sigma)} - e_i\phi_i(\mathbf{r},\sigma) = 0$$

Hartree-Fock equation(s):

$$-\nabla\cdot\left(\frac{\hbar^2}{2m^*(\mathbf{r})}\nabla\phi_i(\mathbf{r},\sigma)\right) + U(\mathbf{r})\phi_i(\mathbf{r},\sigma) + v(\mathbf{r})\phi_i(\mathbf{r},\sigma) = e_i\phi_i(\mathbf{r},\sigma)$$

Modern *ab initio*: quantum Monte Carlo

Quantum Monte Carlo

Big picture

stochastically solve the many-body Schrödinger equation in a fully non-perturbative manner

Rudiments of Diffusion Monte Carlo

$$\Psi(\tau \to \infty) = \lim_{\tau \to \infty} e^{-(\mathcal{H}-E_T)\tau} \Psi_V$$

$$= \lim_{\tau \to \infty} \sum_i \alpha_i e^{-(E_i - E_T)\tau} \Psi_i$$

$$\to \alpha_0 e^{-(E_0 - E_T)\tau} \Psi_0$$

Probabilistic foundations

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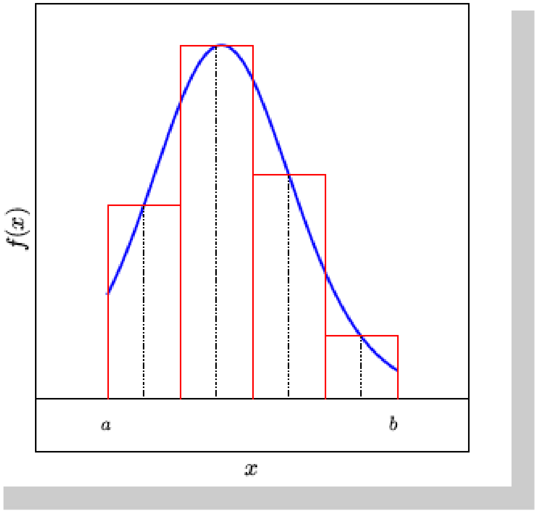
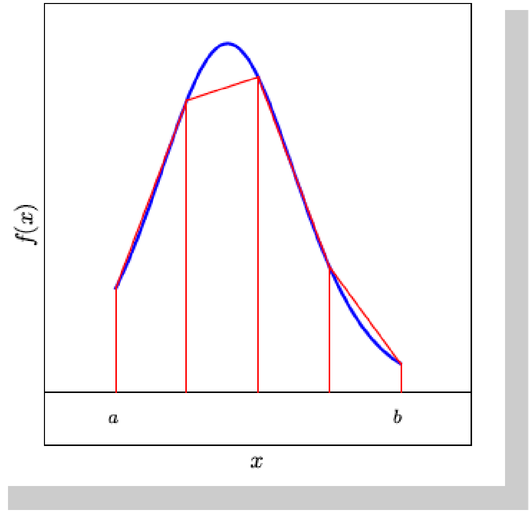
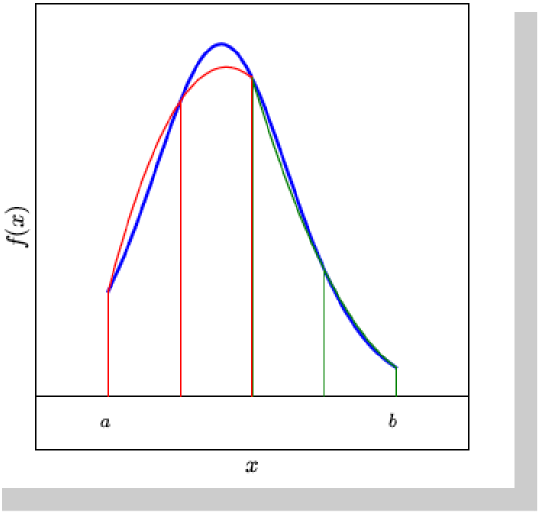
Midpoint rule



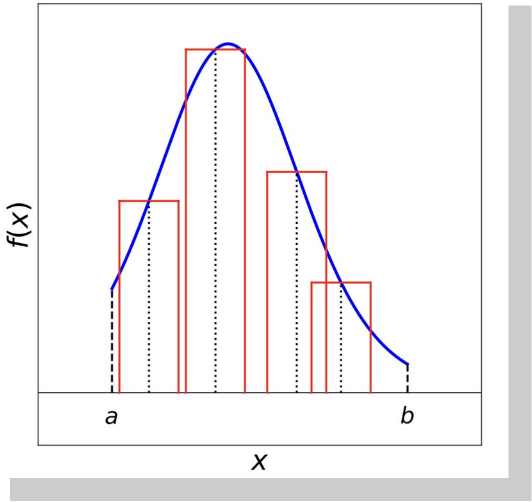
Trapezoid rule



Simpson's rule



Monte Carlo quadrature



Probabilistic foundations

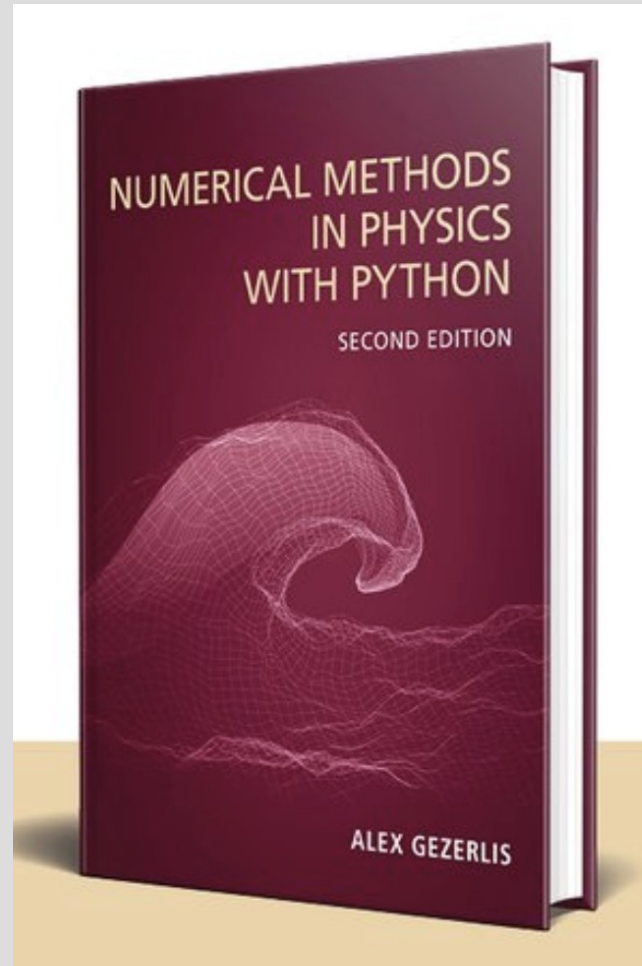
Midpoint rule **Trapezoid rule** **Simpson's rule**

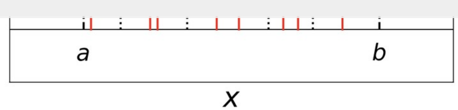
For more on foundational techniques, see:

Alex Gezerlis

Numerical Methods in Physics with Python

(2nd ed, Cambridge University Press, 2023)





Conclusions

- Rich connections between physics of nuclei and that of compact stars
- Exciting time in terms of interplay between nuclear interactions, QCD, and many-body approaches
- Ab initio and phenomenology are mutually beneficial

Takeaway

[F]aut pousser à une porte, pour sçavoir qu'elle nous est close.

– Michel Eyquem de Montaigne
Essais, Livre III, Chapitre 13 (De l’experience)

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