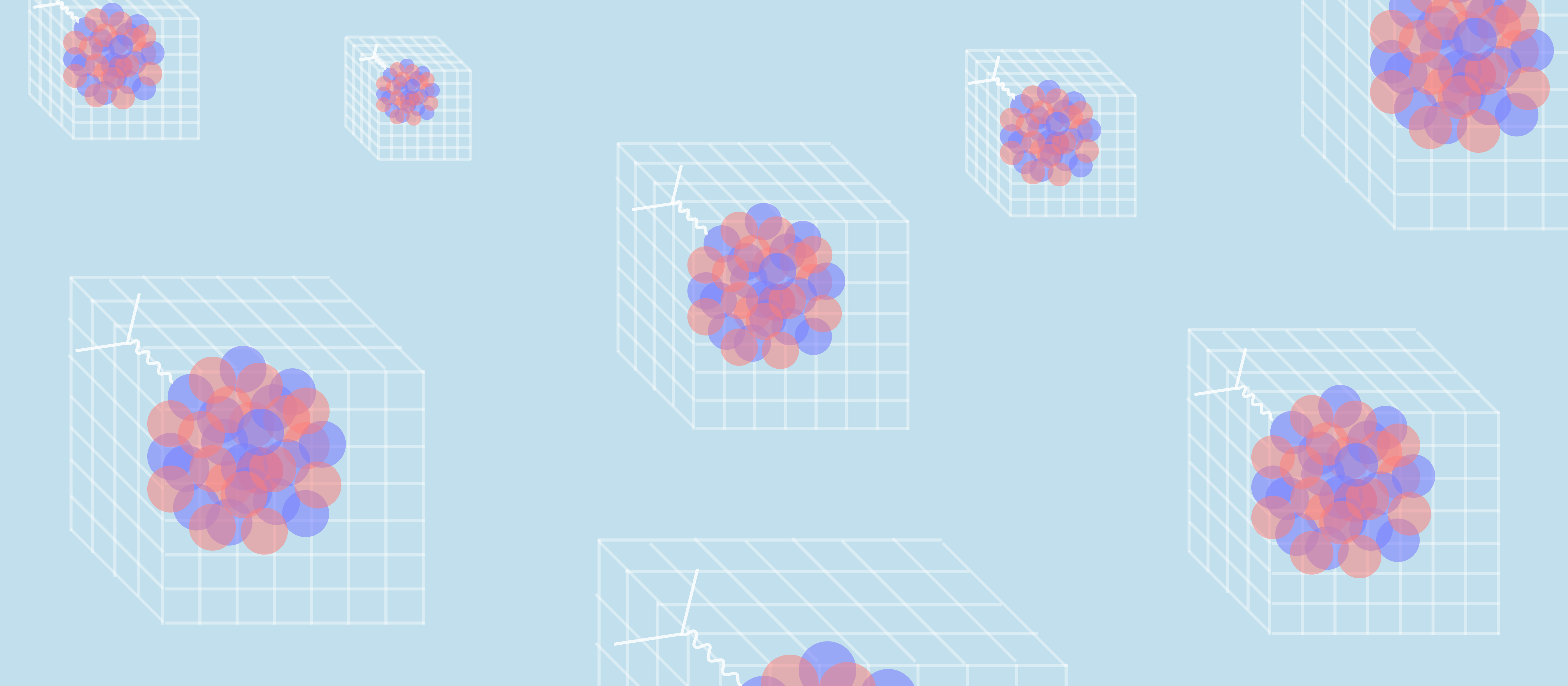

THE 2021 NATIONAL NUCLEAR PHYSICS SUMMER SCHOOL

LATTICE QCD
AND
NUCLEON(US) STRUCTURE

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UNIVERSITY OF MARYLAND AND RIKEN FELLOW

LECTUTE I:
LATTICE QCD FORMALISM AND METHODOLOGY

LECTUTE II:
NUCLEON STRUCTURE FROM LATTICE QCD

LECTUTE III:
TOWARDS NUCLEAR STRUCTURE FROM LATTICE QCD

LECTUTE I:
LATTICE QCD FORMALISM AND METHODOLOGY

Quantum chromodynamics (QCD) in continuum:

QCD is a SU(3) Yang-Mills theory augmented with several flavors of massive quarks:

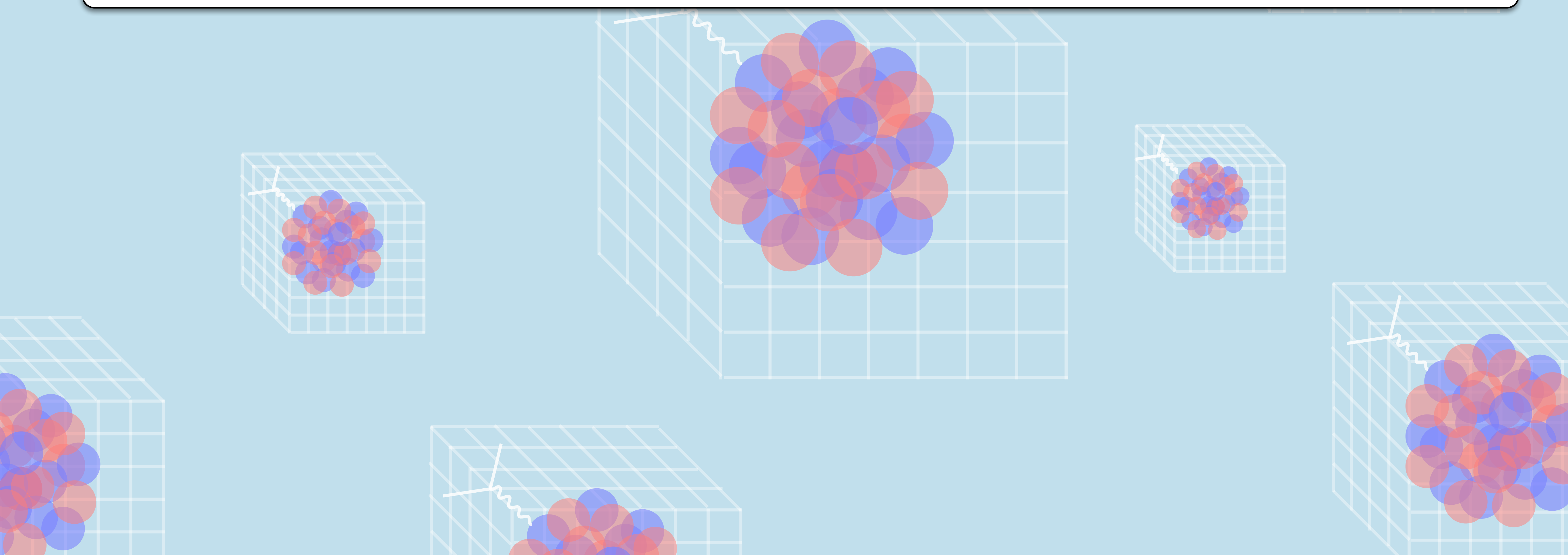
Quark kinetic and mass term

Quark/gluon interactions

$$\mathcal{L}_{QCD} = \sum_{f=1}^{N_f} \left[\bar{q}_f (i\gamma^\mu \partial_\mu - m_f) q_f - g A^i_\mu \bar{q}_f \gamma^\mu T^i q_f \right]$$

$$-\frac{1}{4} F^i_{\mu\nu} F^{i\mu\nu} + \frac{g}{2} f_{ijk} F^i_{\mu\nu} A^{i\mu} A^{j\nu} - \frac{g^2}{4} f_{ijk} f_{klm} A^j_\mu A^k_\nu A^{l\mu} A^{m\nu}$$

Gluons kinetic and interaction terms



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Gluons kinetic and interaction terms

Observe that:

i) There are only $1 + N_f$ input parameters plus QCD coupling. Fix them by a few quantities and all strongly-interacting aspects of nuclear physics is predicted (in principle)!

ii) QCD is asymptotically free such that: $\alpha_s(\mu') = \frac{1}{2b_0 \log \frac{\mu'}{\Lambda_{QCD}}}$

Positive constant for $N_f \leq 16$

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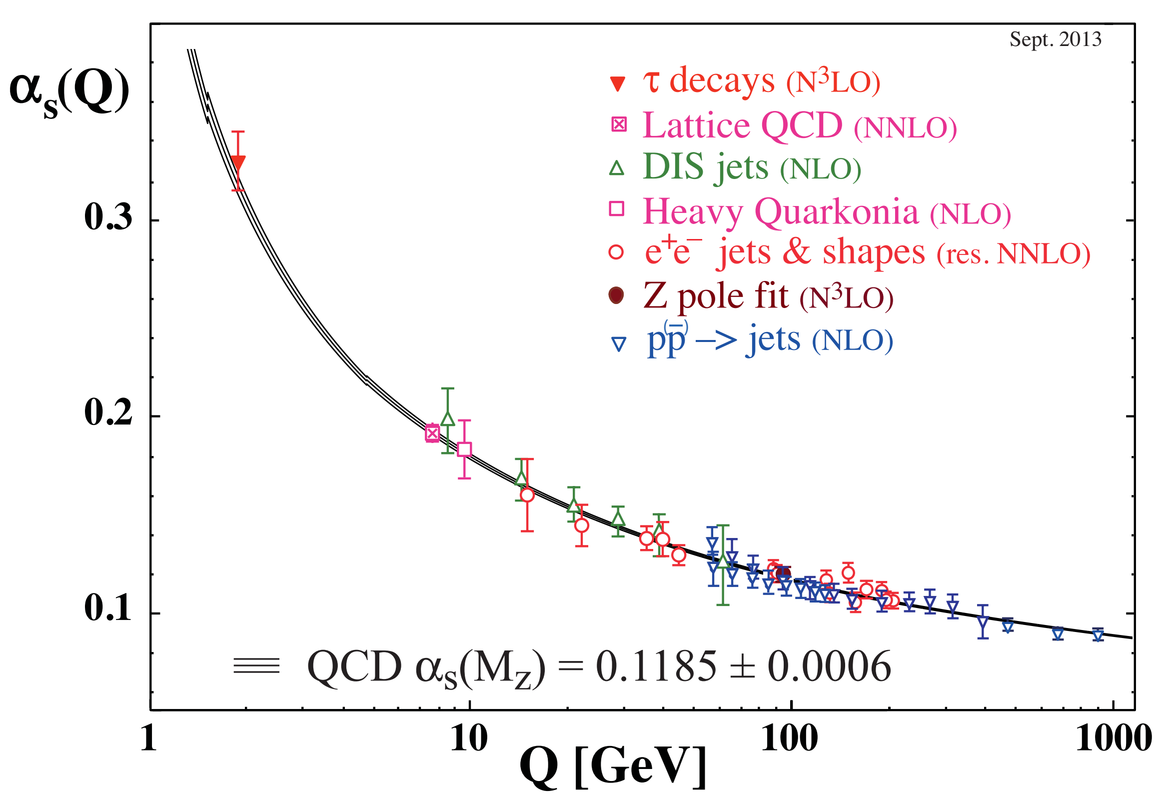

Let's enumerate the steps toward numerically simulating this theory nonperturbatively…

Step I: Discretize the QCD action in both space and time. Consider a finite hypercubic lattice. Wick rotate to imaginary times.

Step II: Generate a large sample of thermalized decorrelated vacuum configurations.

Step III: Form the correlation functions by contracting the quark fields. Need to specify the interpolating operators for the state under study.

Step IV: Extract energies and matrix elements from correlation functions.

Step V: Make the connection to physical observables, such as scattering amplitudes, decay rates, etc.

See e.g., ZD, arXiv:1409.1966 [hep-lat]

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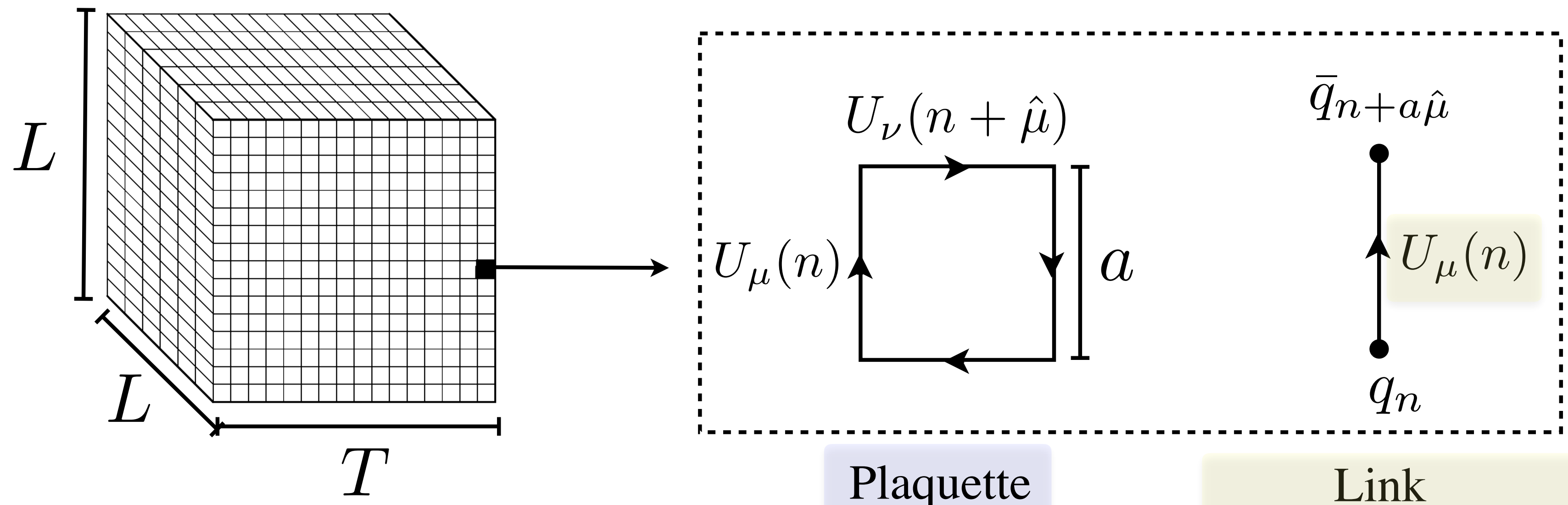

Plaquette

Link
$U_\mu(n) \equiv e^{igA_\mu(n)}$

Two conditions:

$T, L \gg m_\pi^{-1}$ $\qquad a \ll \Lambda_{QCD}^{-1}$

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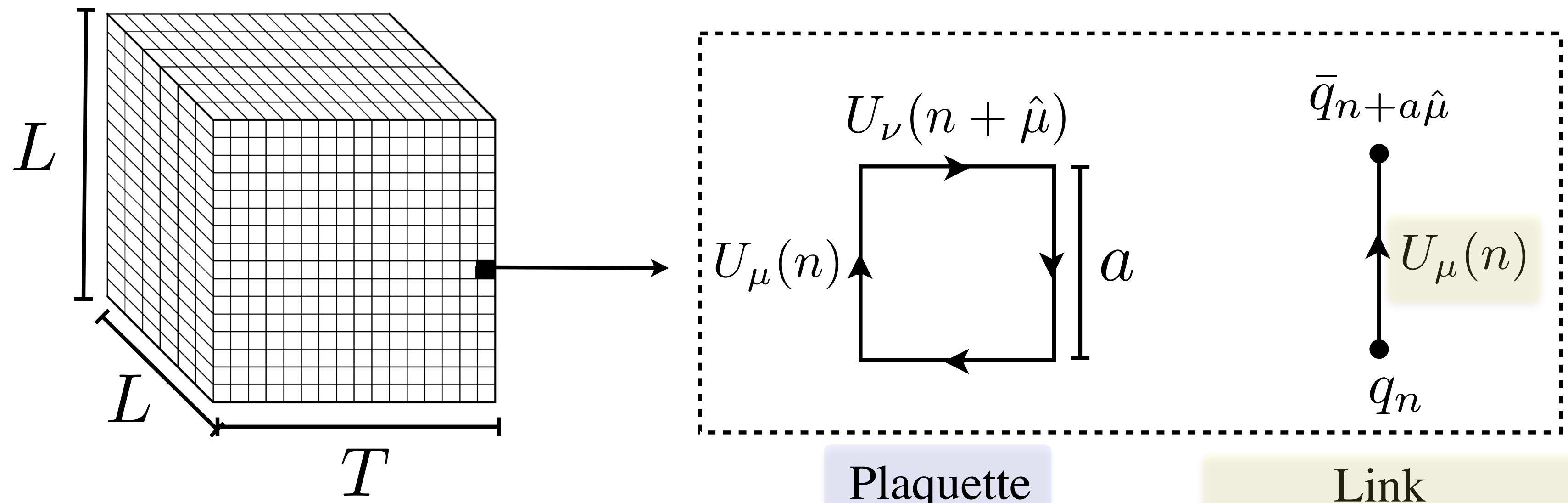

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An example of a discretized action by K. Wilson:

$\beta = 2/g^2$

$$S^{(E)}_{\text{Wilson}} = \frac{\beta}{N_c}\sum_n \sum_{\mu<\nu} \Re\text{Tr}[\mathbb{1} - P_{\mu\nu;n}]$$
$$- \sum_n \bar{q}_n[\overline{m}^{(0)} + 4]q_n + \sum_n \sum_\mu \left[\bar{q}_n \frac{r-\gamma_\mu}{2} U_\mu(n) q_{n+\hat{\mu}} + \bar{q}_n \frac{r+\gamma_\mu}{2} U^\dagger_\mu(n-\hat{\mu}) q_{n-\hat{\mu}}\right]$$

r: Wilson parameter. Gives the naive action if set to zero and has doublers problem.

`For discussions of actions consistent with chiral symmetry of continuum see: Kaplan, arXiv:0912.2560 [hep-lat].`

Step II: Generate a large sample of thermalized decorrelated vacuum configurations.

$$\langle\hat{\mathcal{O}}\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}U_\mu \mathcal{D}q\mathcal{D}\bar{q}\; e^{-S^{(G)}_{\mathrm{lattice}}[U]-S^{(F)}_{\mathrm{lattice}}[U,q,\bar{q}]}\;\hat{\mathcal{O}}[U,q,\bar{q}]$$

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Quark part of expectation values

Define: $\langle \hat{\mathcal{O}} \rangle_F = \frac{1}{\mathcal{Z}_F} \int \mathcal{D}q\mathcal{D}\bar{q}\; e^{-S^{(F)}_{\text{lattice}}[U,q,\bar{q}]} \mathcal{O}[q,\bar{q},U]$

$\mathcal{Z}_F = \int \mathcal{D}q\mathcal{D}\bar{q}\; e^{-S^{(F)}_{\text{lattice}}[U,q,\bar{q}]} = \prod_f \det D_f$ Dirac matrix

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⇩

$$\langle \hat{\mathcal{O}} \rangle = \frac{1}{N} \sum_i^N \langle \hat{\mathcal{O}} \rangle_F [U^{(i)}]$$

N number of $U^{(i)}$ sampled from the distribution: $\frac{1}{\mathcal{Z}} e^{-S^{(G)}_{\text{lattice}}[U]} \prod_f \det D_f$

Steps II is computationally costly…

Example: Consider a lattice with: $L/a = 48,\ T/a = 256$

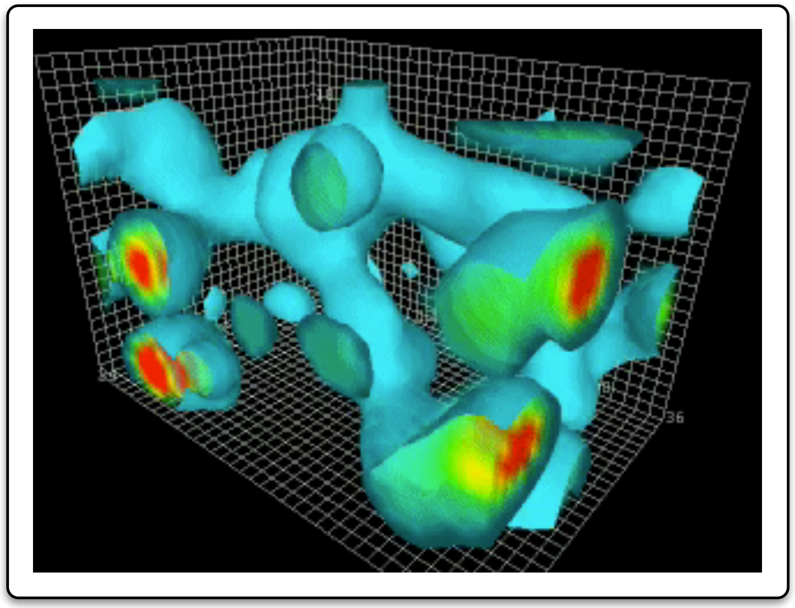
Sampling SU(3) matrices. Already for one sample requires storing

$$8 \times 48^3 \times 256 = 226,492,416$$

c-numbers in the computer!

Requires calculating determinant of a large matrix.

Requires tens of thousands of uncorrelated samples. Molecular-dynamics-inspired hybrid Monte Carlo sampling algorithms often used.



Step III: Form the correlation functions by contracting the quarks. Need to specify the interpolating operators for the state under study.

$$\langle \hat{\mathcal{O}} \rangle_F = \frac{1}{\mathcal{Z}_F} \int \mathcal{D}q \mathcal{D}\bar{q}\; e^{-S^{(F)}_{\text{lattice}}[U,q,\bar{q}]} \mathcal{O}[q,\bar{q},U]$$

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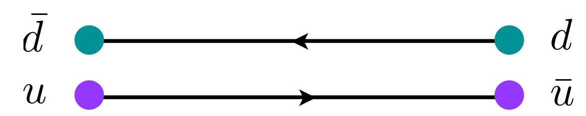
$$\Downarrow$$

e.g., $\boxed{\hat{O} = \overline{u}\gamma_5 d}$



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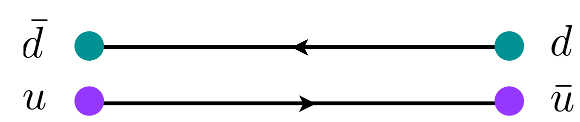
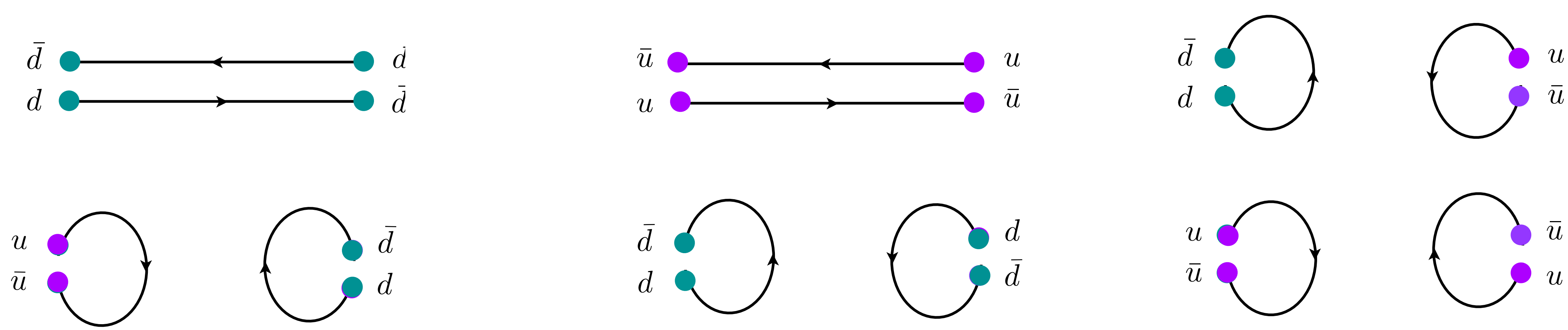


e.g., $\boxed{\hat{O} = \frac{1}{\sqrt{2}}(\overline{u}\gamma^5 u - \overline{d}\gamma^5 d)}$



Quark disconnected diagrams. Require expensive all-to-all propagators.

Steps III is computationally costly…

Example: Consider a lattice with: $L/a = 48, \ T/a = 256$

Solving

$$[D(U)]_{X,Y}[S(U)]_{Y,X_0} = G_{X,X_0}$$

Dirac matrix

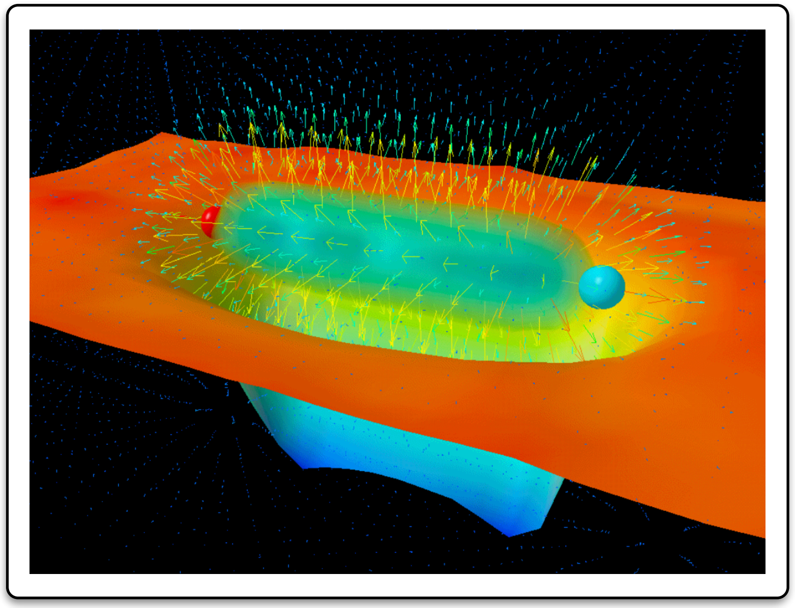
Quark propagator

Source

Requires taking determinant and inverting a matrix with dimensions:

$$(4 \times 3 \times 48^3 \times 256)^2 =$$

$$339,738,624 \times 339,738,624$$



EXERCISE 1

Show that for the correlation function of the charged pion:

$$\langle \hat{O}^{\pi^+}(n)\hat{O}^{\pi^+\dagger}(0)\rangle_F = -\mathrm{Tr}\left[D_u^{-1}(n,0)D_d^{-1}(n,0)\right]$$

where D_u^{-1} and D_d^{-1} denote the the inverse Dirac matrix (the quark propagator) for the u and d quarks, respectively. Trace is over spin and color degrees of freedom.

BONUS EXERCISE 1

Show that for the correlation function of the neutral pion:

$$\begin{aligned}\langle \hat{O}^{\pi^0}(n)\hat{O}^{\pi^0\dagger}(0)\rangle_F = & -\frac{1}{2}\mathrm{Tr}\left[\gamma^5 D_u^{-1}(n,0)\gamma^5 D_u^{-1}(0,n)\right] \\ & +\frac{1}{2}\mathrm{Tr}\left[\gamma^5 D_u^{-1}(n,n)\right]\mathrm{Tr}\left[\gamma^5 D_u^{-1}(0,0)\right] \\ & -\frac{1}{2}\mathrm{Tr}\left[\gamma^5 D_u^{-1}(n,n)\right]\mathrm{Tr}\left[\gamma^5 D_d^{-1}(0,0)\right] + \{u \leftrightarrow d\}\end{aligned}$$

Step IV: Extract energies and matrix elements from correlation functions

$$C_{\hat{\mathcal{O}},\hat{\mathcal{O}}'}(\tau;\mathbf{d}) \;=\; \sum_{\mathbf{x}} e^{2\pi i\mathbf{d}\cdot\mathbf{x}/L}\langle 0|\hat{\mathcal{O}}'(\mathbf{x},\tau)\hat{\mathcal{O}}^{\dagger}(\mathbf{0},0)|0\rangle = \mathcal{Z}_0'\mathcal{Z}_0^{\dagger}e^{-E^{(0)}\tau} + \mathcal{Z}_1'\mathcal{Z}_1^{\dagger}e^{-E^{(1)}\tau} + \ldots$$

Ground state and a tower of excited states are, in principle, accessible!

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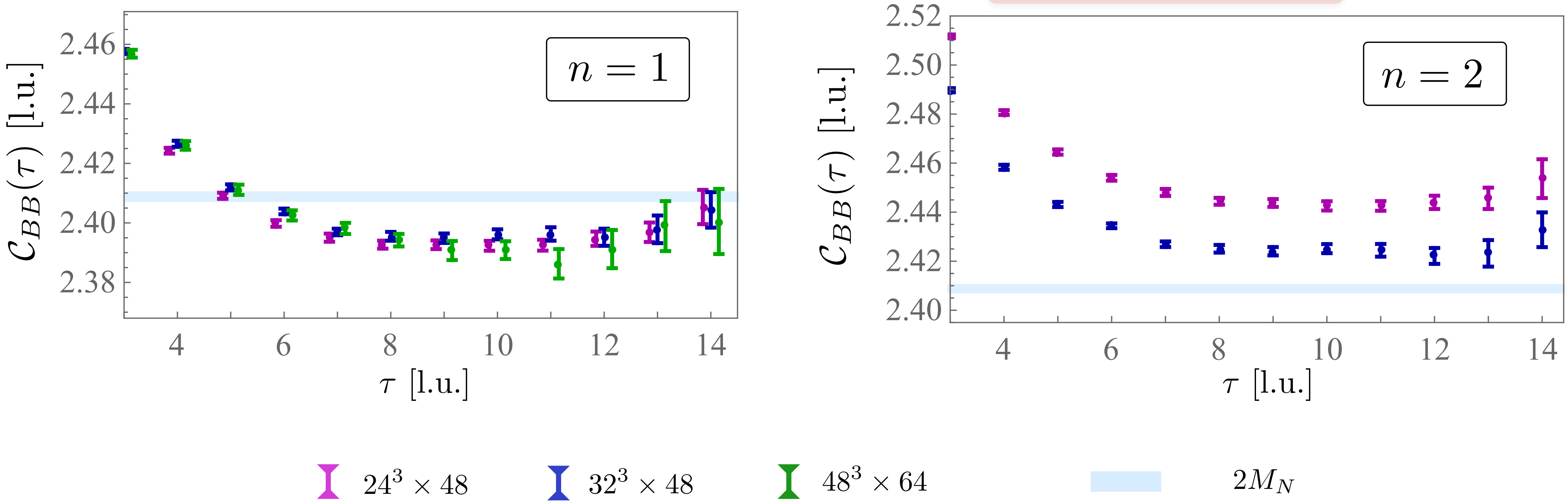
Example: $NN\ (^1S_0)$

What should we make of the volume dependence?



Beane et al (NPLQCD), arXiv:1705.09239, Wagman et al (NPLQCD), arXiv:1706.06550.