Bayesian Model/Data Analysis
Big Experiment vs Big Theory
Lessons from Relativistic Heavy Ion Collisions

Models and Data Analysis Initiative
http://madai.us

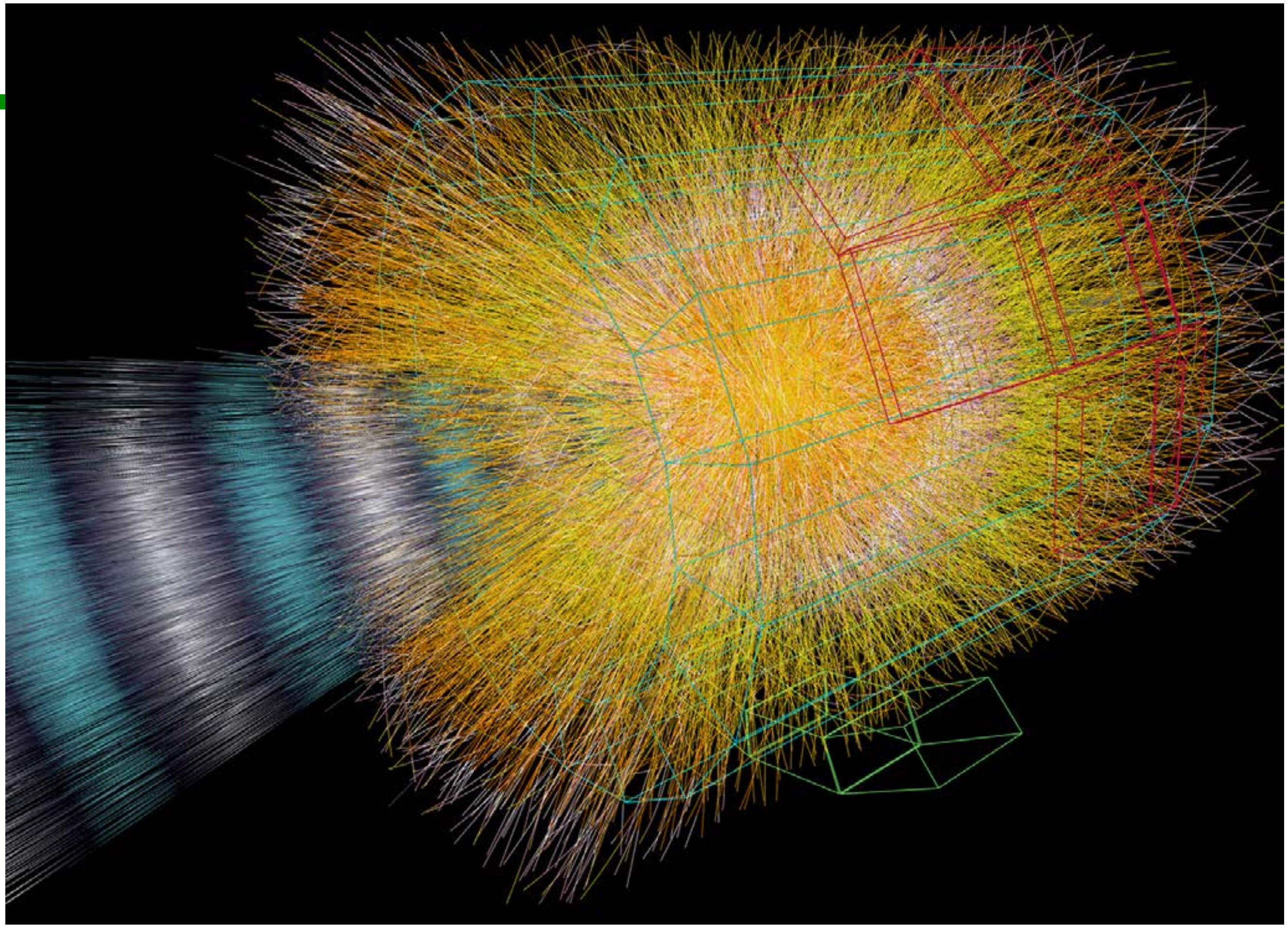
Cyber-enabled Discovery and Innovation

MICHIGAN STATE UNIVERSITY | Duke UNIVERSITY | THE UNIVERSITY of NORTH CAROLINA at CHAPEL HILL | renci

1st MADAI Collaboration Meeting, SANDIA 2010

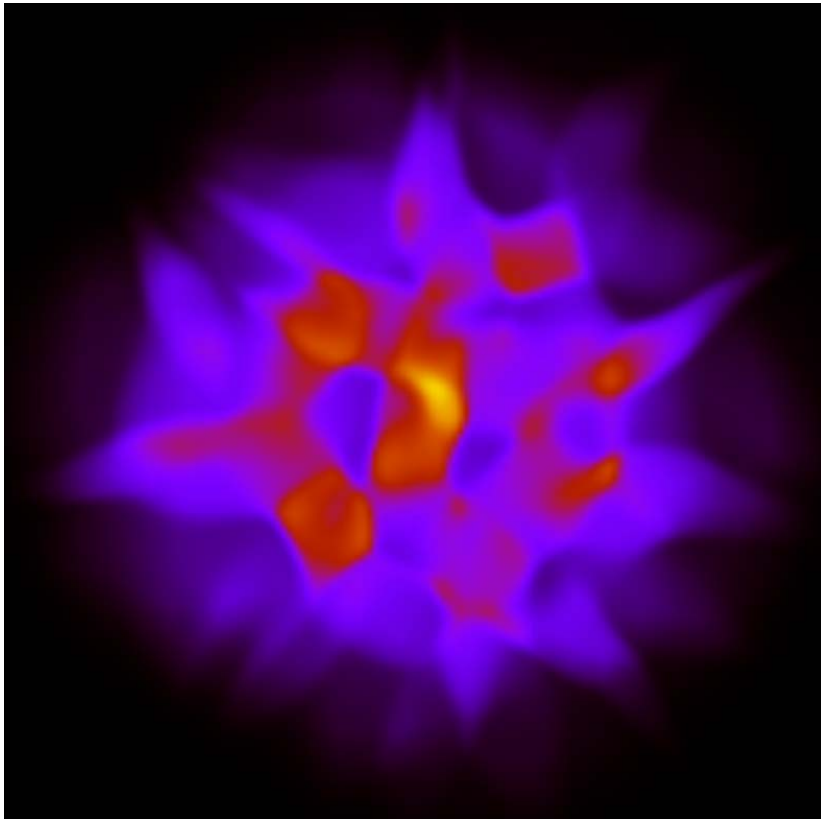
BIG EXPERIMENT

Large, Heterogenous Data Sets

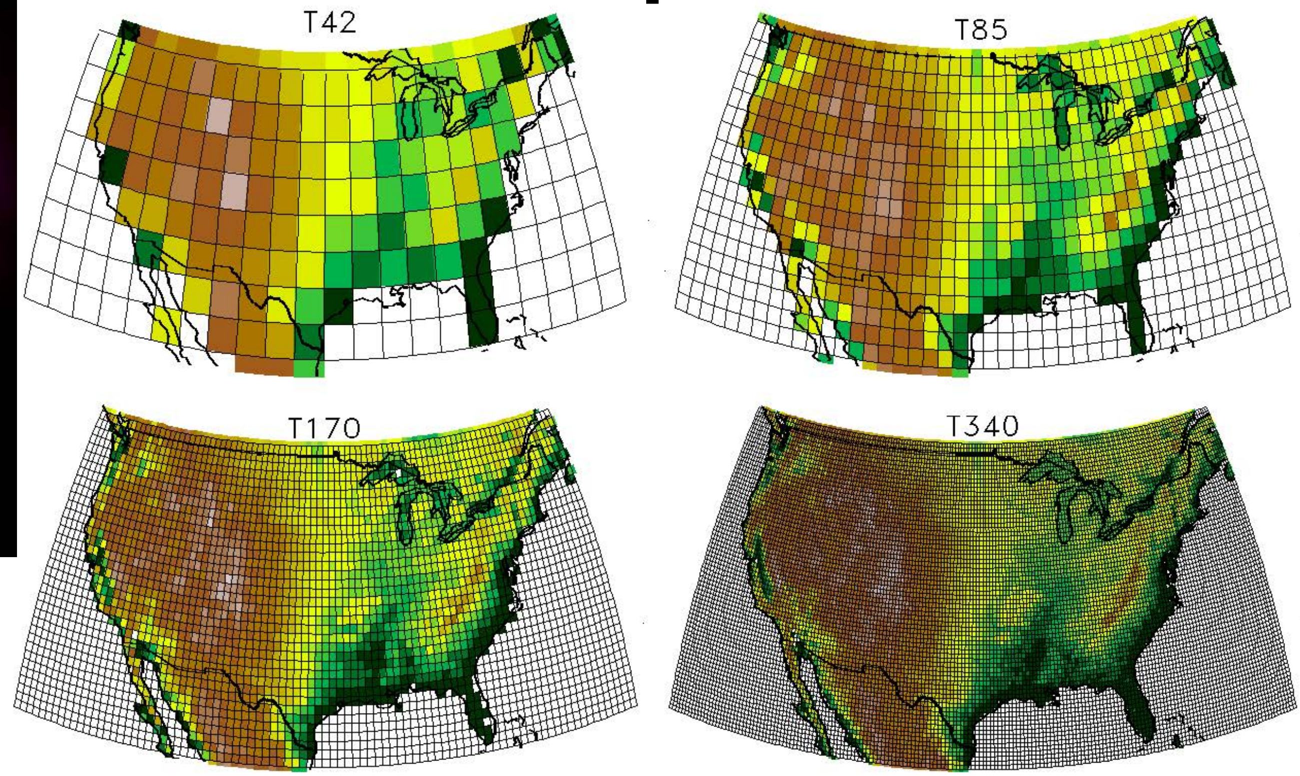


BIG MODELS

Numerous parameters
Assumptions



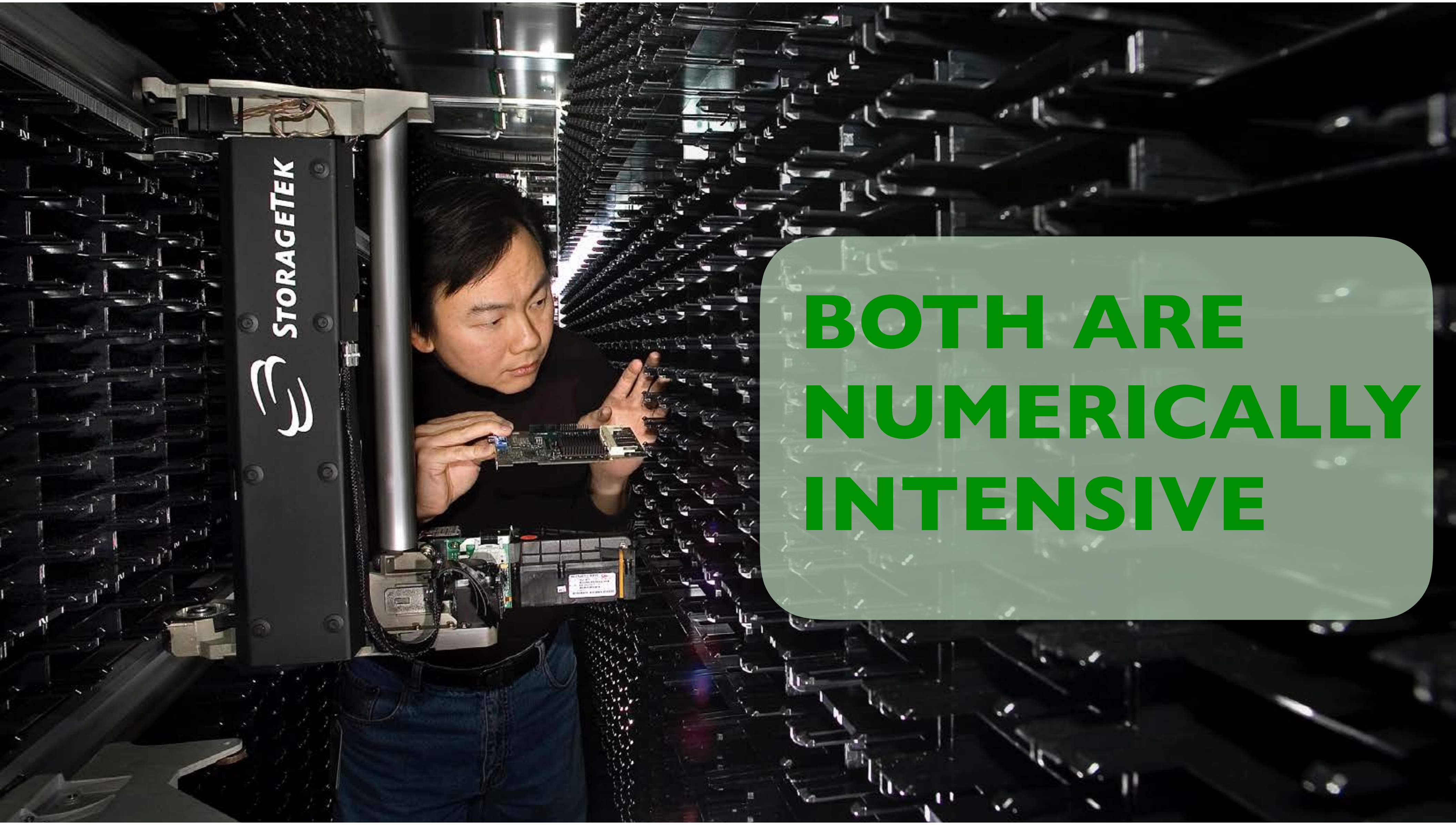
heavy ion collisions



Atmospheric modeling

GOALS

- **Determine parameters**
- **Test assumptions**
- **Identify connections between observables and parameters**



BOTH ARE NUMERICALLY INTENSIVE

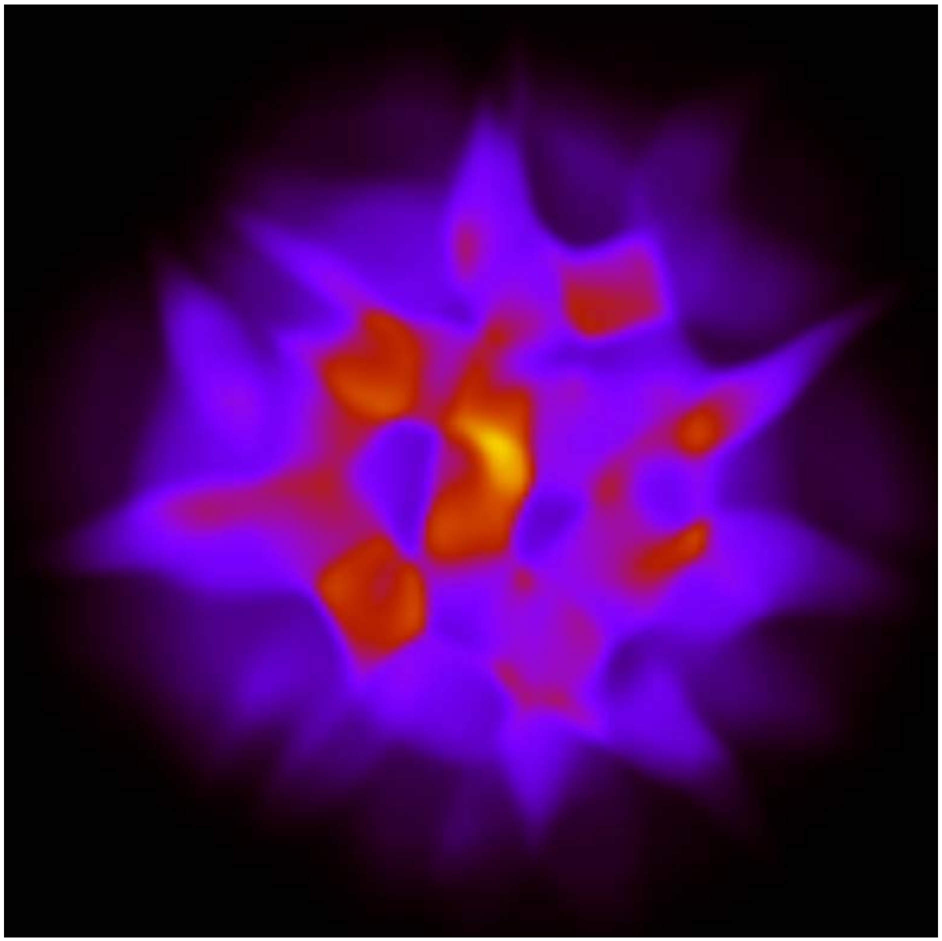
BAYES theorem

$$P(A\&B) = P(A|B)P(B) = P(B|A)P(A)$$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

P(A|B) is probability of data given theory(parameters)
P(B|A) is probability of theory(parameters) given data

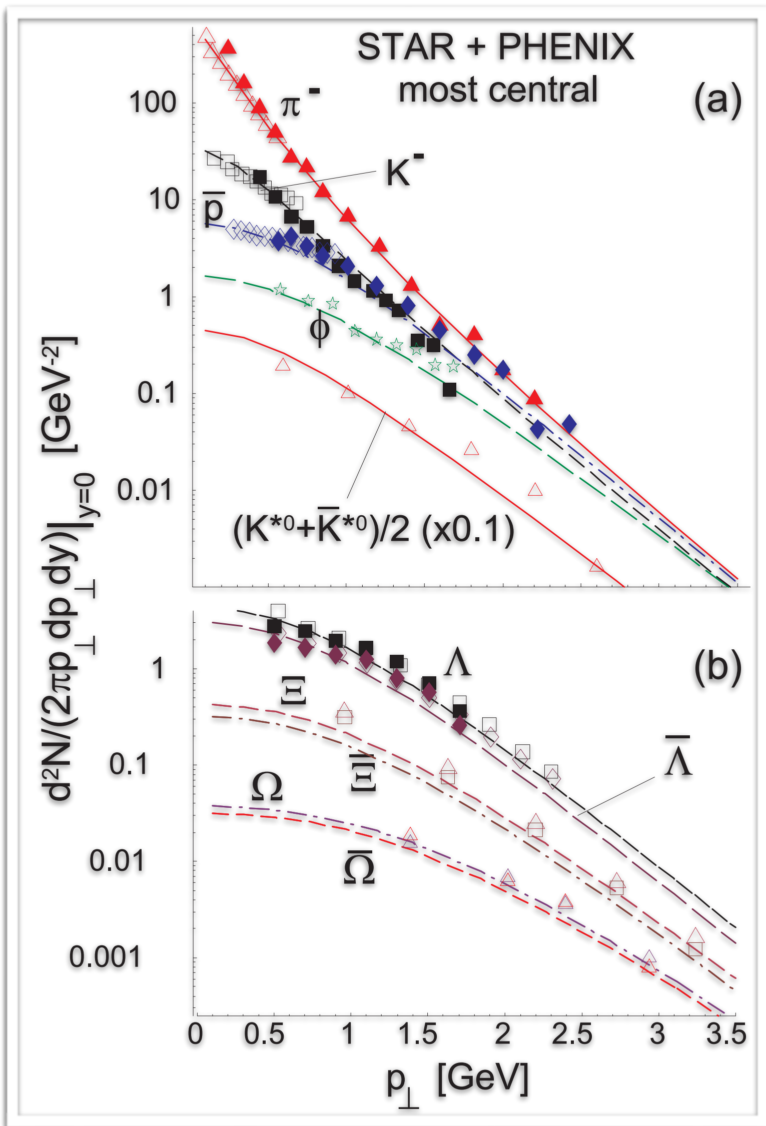
MODEL COMPONENTS



MODEL COMPONENTS

- **Thermalization**
  (First fm/c)
- **Viscous Hydrodynamics**
  (First ~5-10 fm/c)
- **Hadron Simulation**
  (Dissolution & Breakup)
- Numerous parameters
  (up to few dozen)
- ~Days of CPU to study one point in parameter space

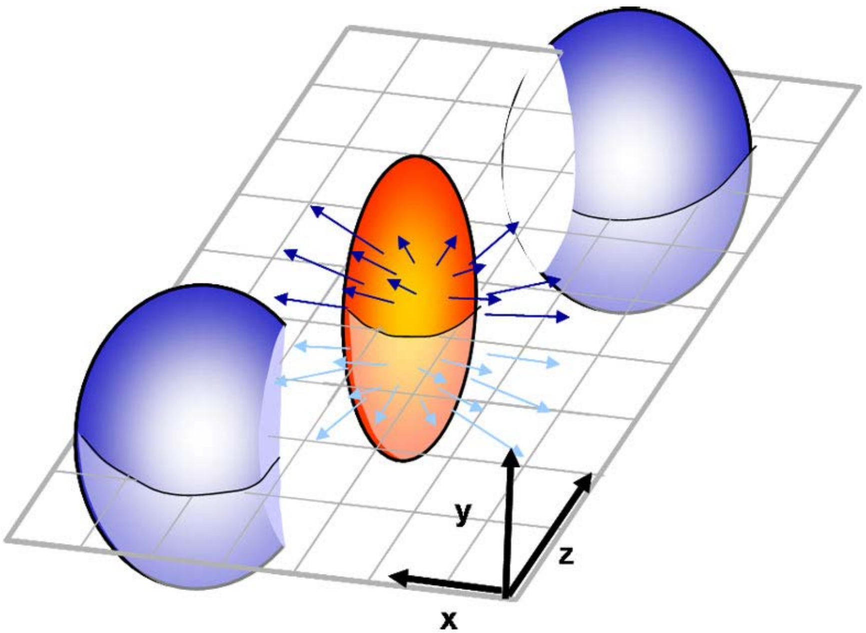
OBSERVABLES: Spectra



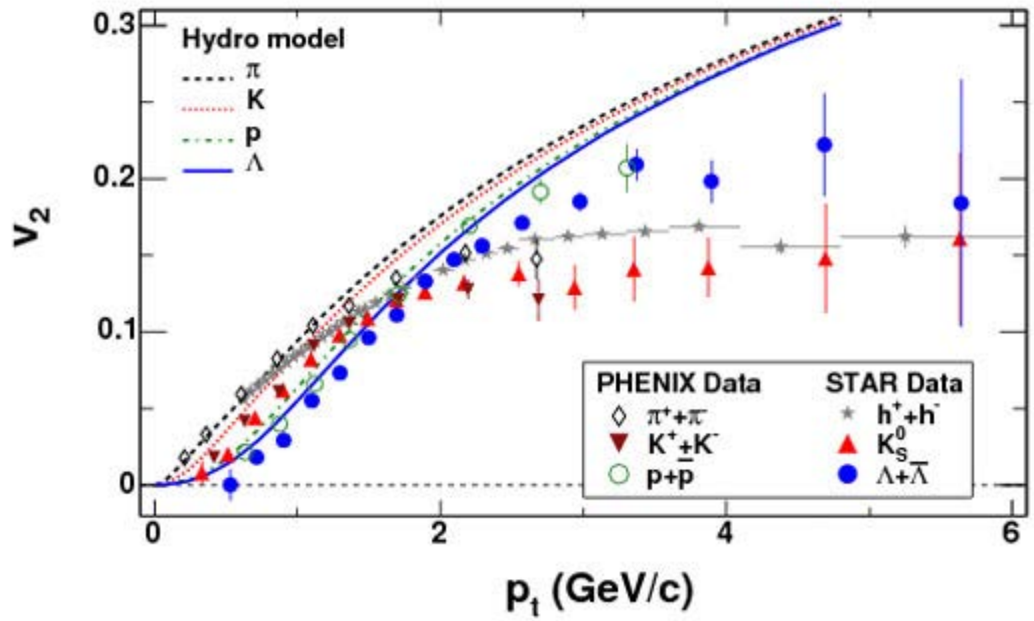
W.Florkowski & W. Bronkowski, NPA 2003

feature	infer
Yields	entropy
$\langle p_t \rangle$	T and flow
$\langle p_t \rangle$ of heavy particles	more flow than T

OBSERVABLES: v2 (elliptic flow)



$$v_2 = \langle \cos 2\phi \rangle$$

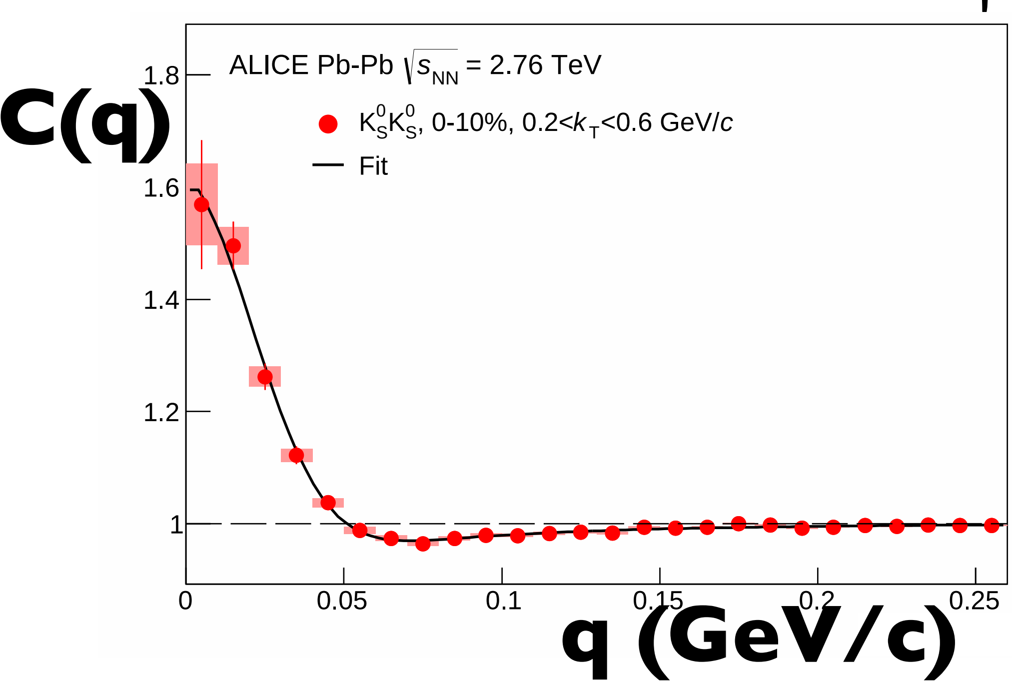


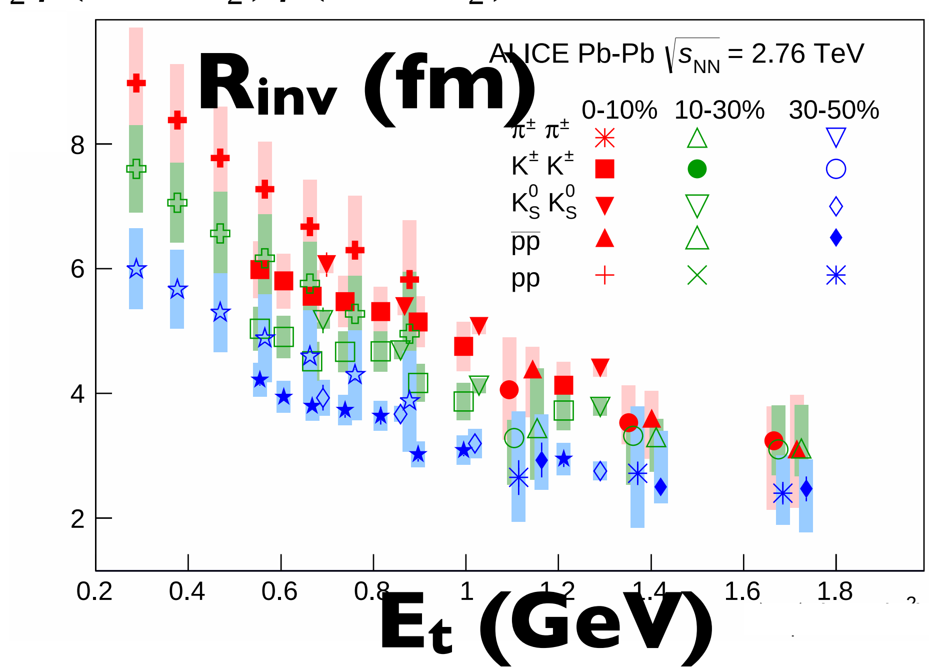
Sensitive to viscosity

OBSERVABLES: Femtoscopic Correlations

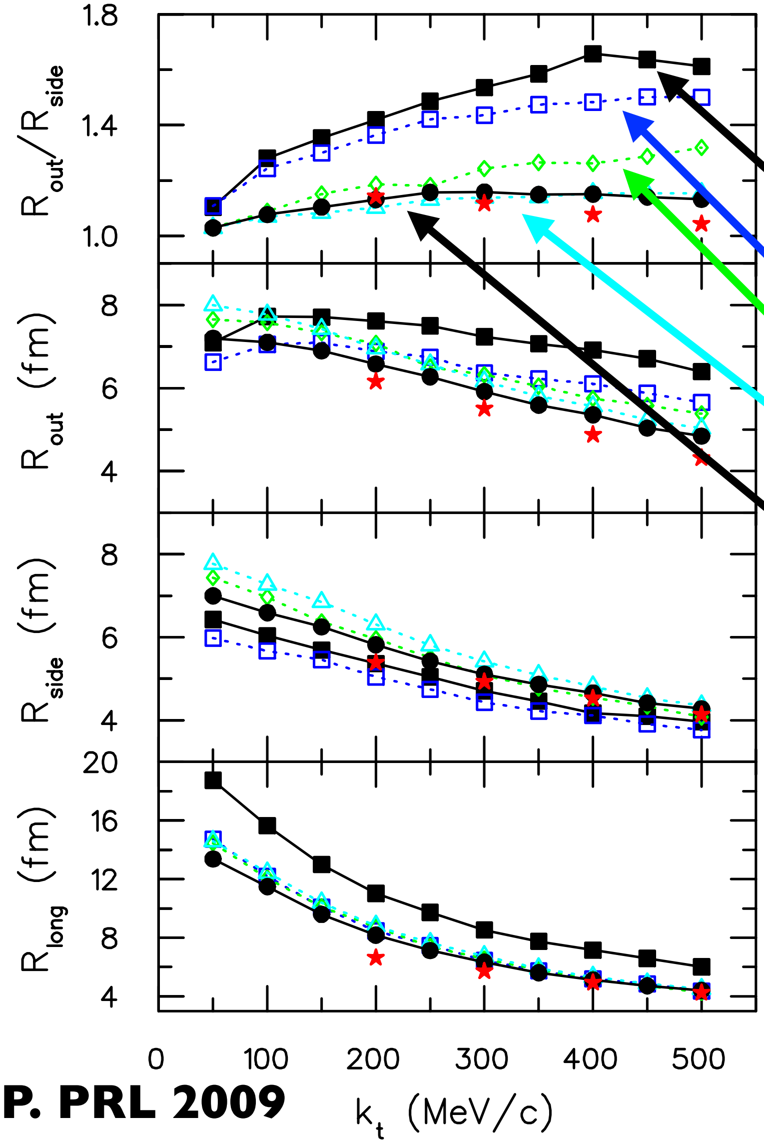
$$C(P = p_1 + p_2, q = p_1 - p_2) = \frac{d^2N / dp_1dp_2}{(dN / dp_1)(dN / dp_2)}$$

$$C(P,q) \to S(P,r) = \frac{\int dr_1dr_2\, f(P/2,r_2)\, f(P/2,r_2)\, \delta(r - [r_1 - r_2])}{\int dr_1dr_2\, f(P/2,r_2)\, f(P/2,r_2)}$$





OBSERVABLES: Femtoscopic



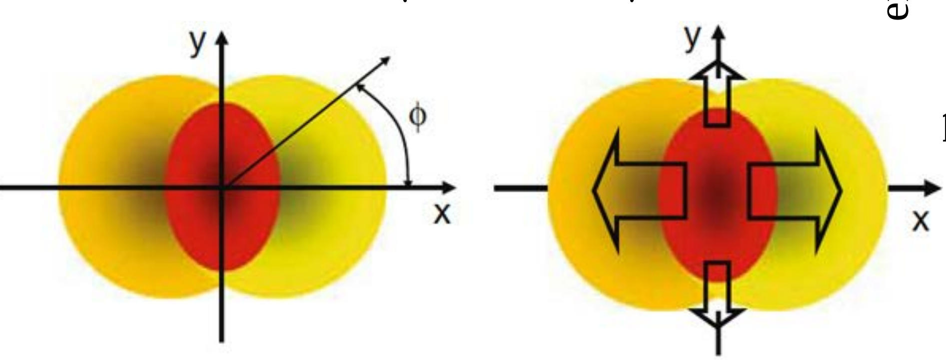
S.P. PRL 2009

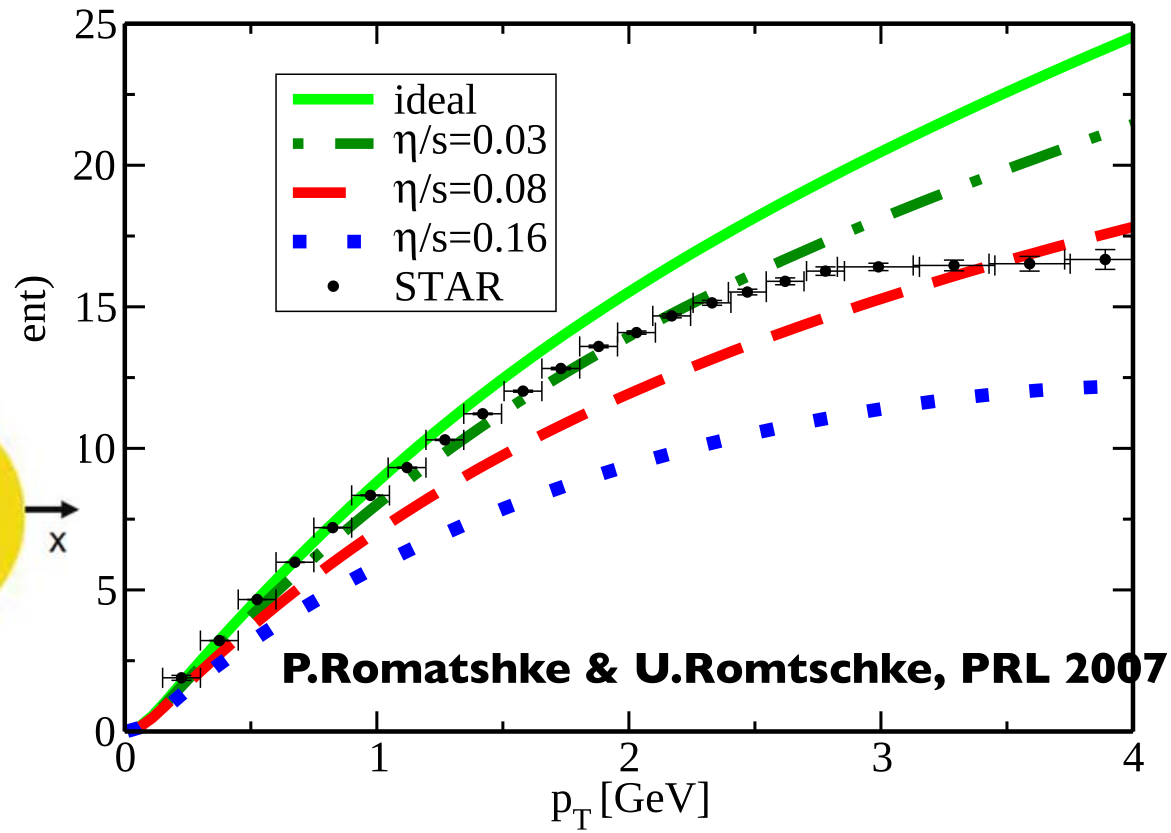
I[st] generation hydro model
+ initial acceleration
+ realistic Eq. of State
+ viscosity
+ improved wave function

★STAR

How this was done before (*v*2 and η/s)

$$v_2 \equiv \langle \cos 2\phi \rangle$$





PROBLEM

v2 depends on ....

- **viscosity**
- **saturation model**
- **pre-thermal flow**
- **Eq. of State**
- **T-dependence of η/s**
- **initial T_{xx}/T_{zz}**
- . . . .

e.g. Drescher, Dumitru, Gombeaud and Ollitrault
PRC 2007

Correct Way (MCMC)

- Simultaneously vary *N* model parameters $\boldsymbol{x_i}$
- Perform random walk weight by likelihood

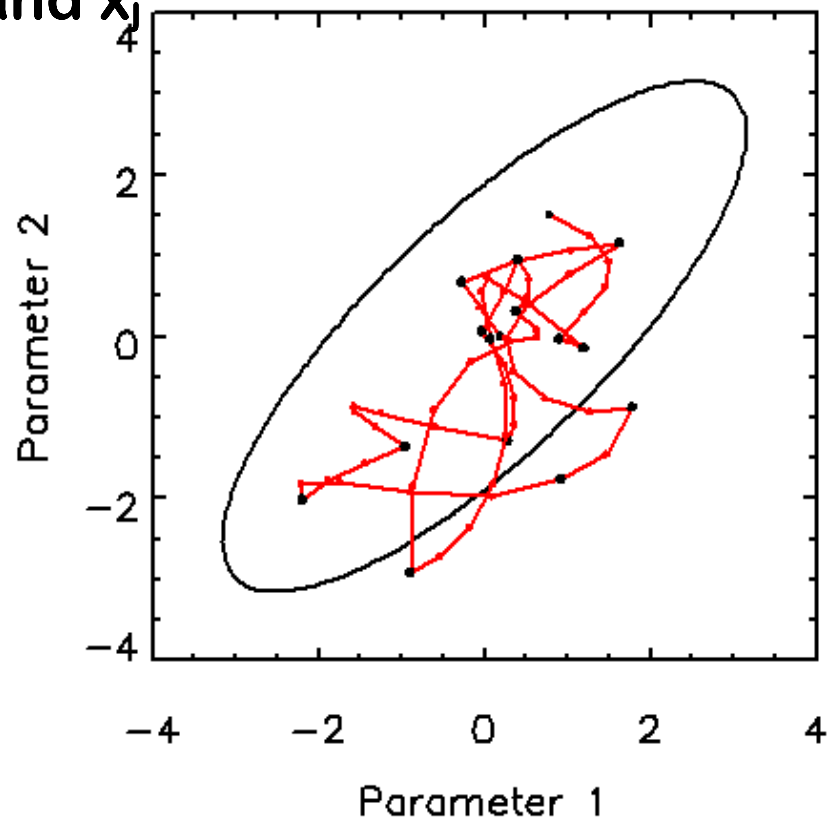
$$\mathcal{L}(\mathbf{x}|\mathbf{y}) \sim \exp\left\{-\sum_a \frac{(y_a^{(\text{model})}(\mathbf{x}) - y_a^{(\text{exp})})^2}{2\sigma_a^2}\right\}$$

- Use all observables $\boldsymbol{y_a}$
- Obtain representative sample of posterior

MCMC Metropolis algorithm

Imagine N➜∞ instances (samplings) of parameters x with probability P(x)
Consider two point in parameter space x_i and x_j
Rates of i➜j and j➜i are

$$R(i \to j) = P(i) R(i \to j|i)$$
$$R(j \to i) = P(j) R(j \to i|j)$$
$$R(j \to i|j) = R(i \to j|i) \frac{P(i)}{P(j)}$$



If in state "i" do random step
if P(j)>P(i), keep 100%
if(P(i)>P(j), keep with prob. P(j)/P(i)

Difficult Because...

I. Too Many Model Runs
Requires running model ~10^6 times

II. Many Observables
Could be hundreds of plots,
each with dozens of points
Complicated Error Matrices

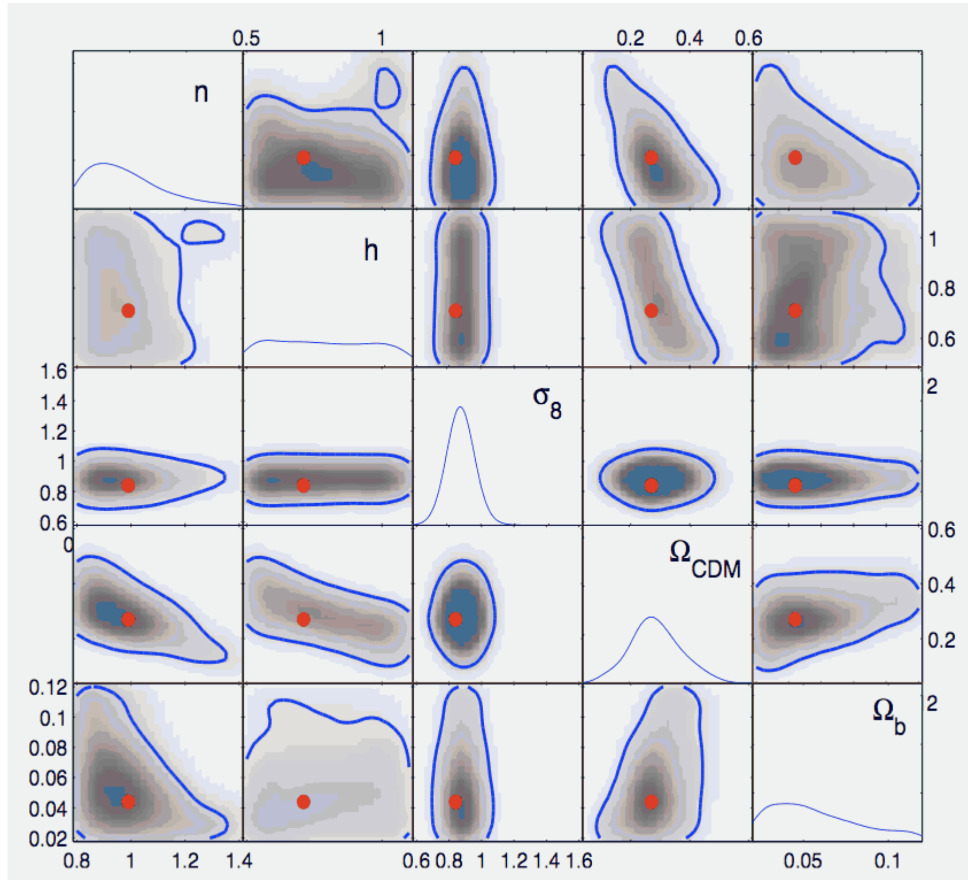
Model Emulators

1. **Run the model ~1000 times**
   Semi-random points (LHS sampling)

2. **Determine Principal Components**
   $(y_a - \langle y_a \rangle)/\sigma_a \rightarrow z_a$

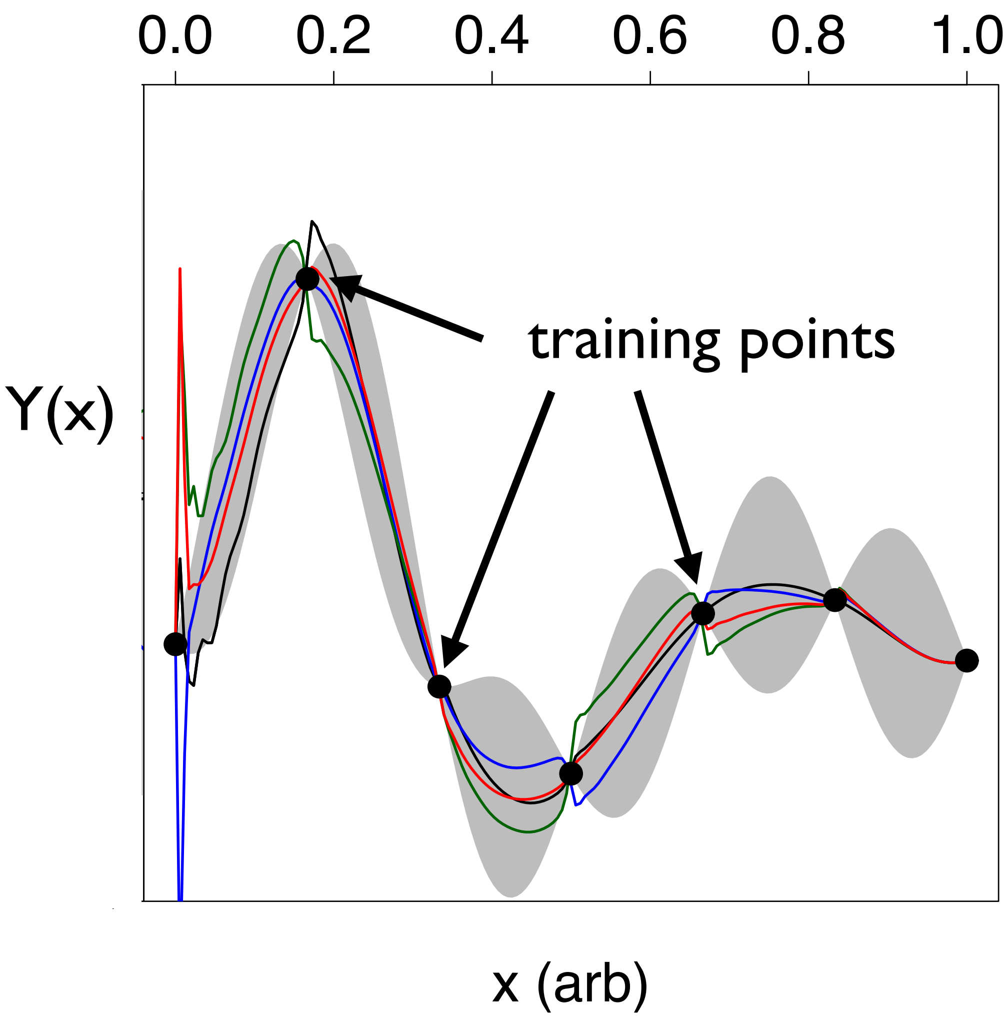
3. **Emulate z_a (Interpolate) for MCMC**
   Gaussian Process…

$$\mathcal{L}(\mathbf{x}|\mathbf{y}) \sim \exp\left\{-\frac{1}{2}\sum_a (z_a^{(\text{emulator})}(\mathbf{x}) - z_a^{(\exp)})^2\right\}$$



S. Habib, K. Heitman, D. Higdon, C. Nakhleh & B. Williams, PRD(2007)

Emulator



- **Gaussian Process**
  - **Reproduces training points**
  - **Assumes localized Gaussian covariance**
  - **Must be trained, i.e. find "hyper parameters"**
- **Other methods also work**

14 Parameters

- 5 for Initial Conditions at RHIC
- 5 for Initial Conditions at LHC
- 2 for Viscosity
- 2 for Eq. of State

30 Observables

- π,K,p Spectra
  $\langle p_t \rangle$, Yields
- Interferometric Source Sizes
- v_2 Weighted by p_t

Initial State Parameters

$$\epsilon(\tau = 0.8\text{fm}/c) = f_{\text{wn}}\epsilon_{\text{wn}} + (1 - f_{\text{wn}})\epsilon_{\text{cgc}},$$

$$\epsilon_{\text{wn}} = \epsilon_0 T_A \frac{\sigma_{\text{nn}}}{2\sigma_{\text{sat}}} \{1 - \exp\left(-\sigma_{\text{sat}} T_B\right)\} + (A \leftrightarrow B)$$

$$\epsilon_{\text{cgc}} = \epsilon_0 T_{\text{min}} \frac{\sigma_{\text{nn}}}{\sigma_{\text{sat}}} \{1 - \exp\left(-\sigma_{\text{sat}} T_{\text{max}}\right)\}$$

$$T_{\text{min}} \equiv \frac{T_A T_B}{T_A + T_B},$$

$$T_{\text{max}} \equiv T_A + T_B,$$

$$u_\perp = \alpha\tau \frac{\partial T_{00}}{2T_{00}}$$

$$T_{zz} = \gamma P$$

5 parameters for RHIC, 5 for LHC

Equation of State and Viscosity

$$c_s^2(\epsilon) = c_s^2(\epsilon_h) + \left(\frac{1}{3} - c_s^2(\epsilon_h)\right) \frac{X_0 x + x^2}{X_0 x + x^2 + X'^2},$$

$$X_0 = X' R c_s(\epsilon)\sqrt{12},$$

$$x \equiv \ln \epsilon/\epsilon_h$$

$$\frac{\eta}{s} = \left.\frac{\eta}{s}\right|_{T=165} + \kappa \ln(T/165)$$

2 parameters for EoS, 2 for η/s

DATA Distillation

1. **Experiments reduce PBs to 100s of plots**
2. **Choose which data to analyze**
   Does physics *factorize*?
3. **Reduce plots to a few representative numbers, y_a**
4. **Transform to principal components, z_a**

$$\mathcal{L} \sim \exp\left\{ \frac{-1}{2} \sum_a (z_a - z_a^{(\exp)})^2 \right\}$$

5. **Resolving power of RHIC/LHC data reduced to $\lesssim$10 numbers!**

Principal Component Analysis (PCA)

Many observables y_a
All change as function of parameters x_i
BUT, some linear combinations change — some don't

$$\delta\tilde{y}_a = \frac{(y_a - \langle y_a \rangle)}{\sigma_a}$$ **average over x**

$$M_{ab} = \langle \delta\tilde{y}_a \delta\tilde{y}_b \rangle$$

Diagaonalize M

$$M \rightarrow \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$$

Principal Component Analysis (PCA)

$$\delta\tilde{y}_a = \frac{(y_a - \langle y_a \rangle)}{\sigma_a}$$

$$M_{ab} = \langle \delta\tilde{y}_a \delta\tilde{y}_b \rangle$$

Diagaonalize M

$$M \rightarrow \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$$

In new basis, $y_a \leftrightarrow z_a$

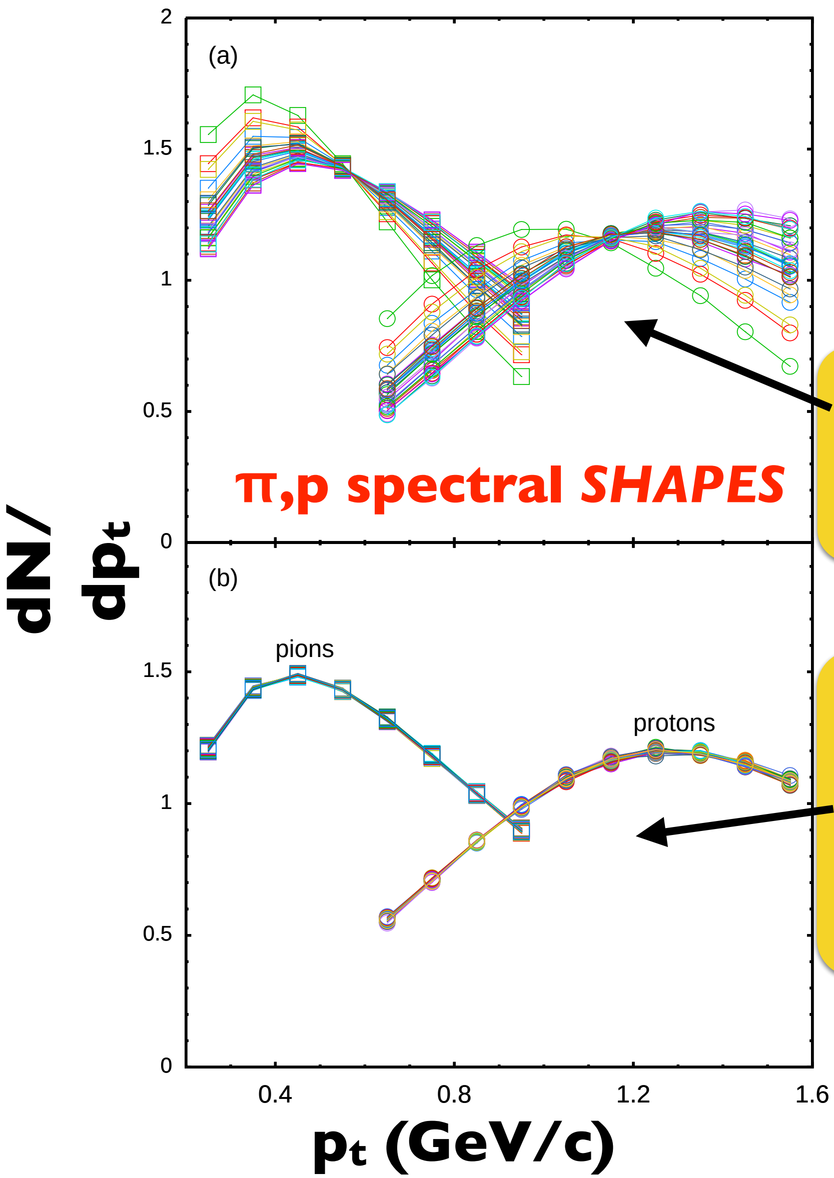
z_a are known as principal components

If $\lambda_a >> 1$, good resolving power

If $\lambda_a << 1$, little resolving power, no need to analyze

Checking the Distillation

Spectral information
encapsulated by two numbers,
dN/dy & ⟨p_t⟩



model spectra from 30 random points in parameter prior

**74 pion spectra:
with 573<⟨p_t⟩$_\pi$< 575 MeV**

**44 proton spectra:
with 1150<⟨p_t⟩$_p$< 1152 MeV**

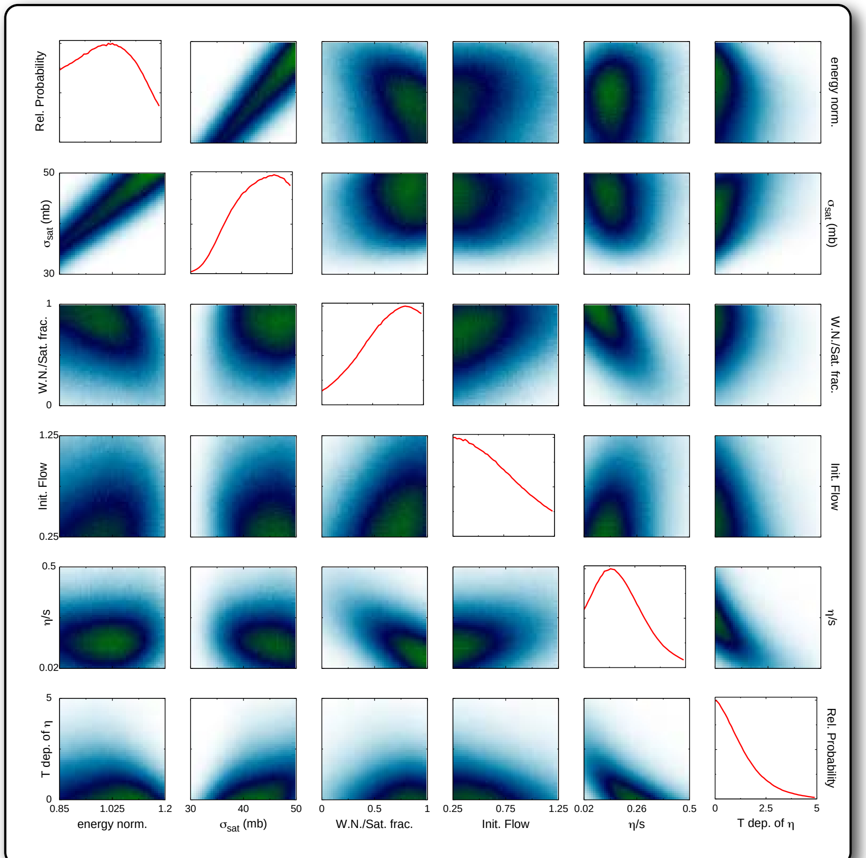
Review the Grand Plan

1. Choose observables
2. Distill Data
3. Parameterize model
4. Run full model hundreds of times
   (Latin hyper-cube sampling)
5. Build & Tune emulator
6. Perform MCMC with emulator
7. Analyze sensitivities

Two Calculations

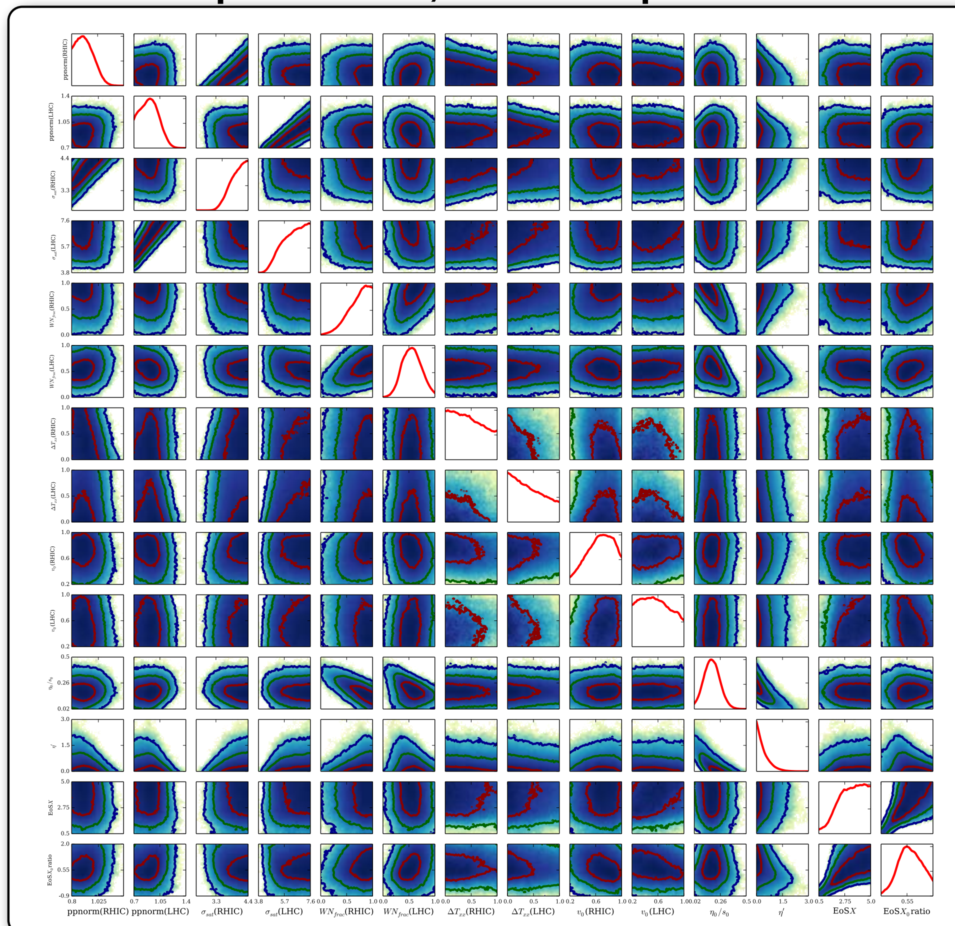
J.Novak, K. Novak, S.P., C.Coleman-Smith & R.Wolpert, PRC 2014

RHIC Au+Au Data
6 parameters

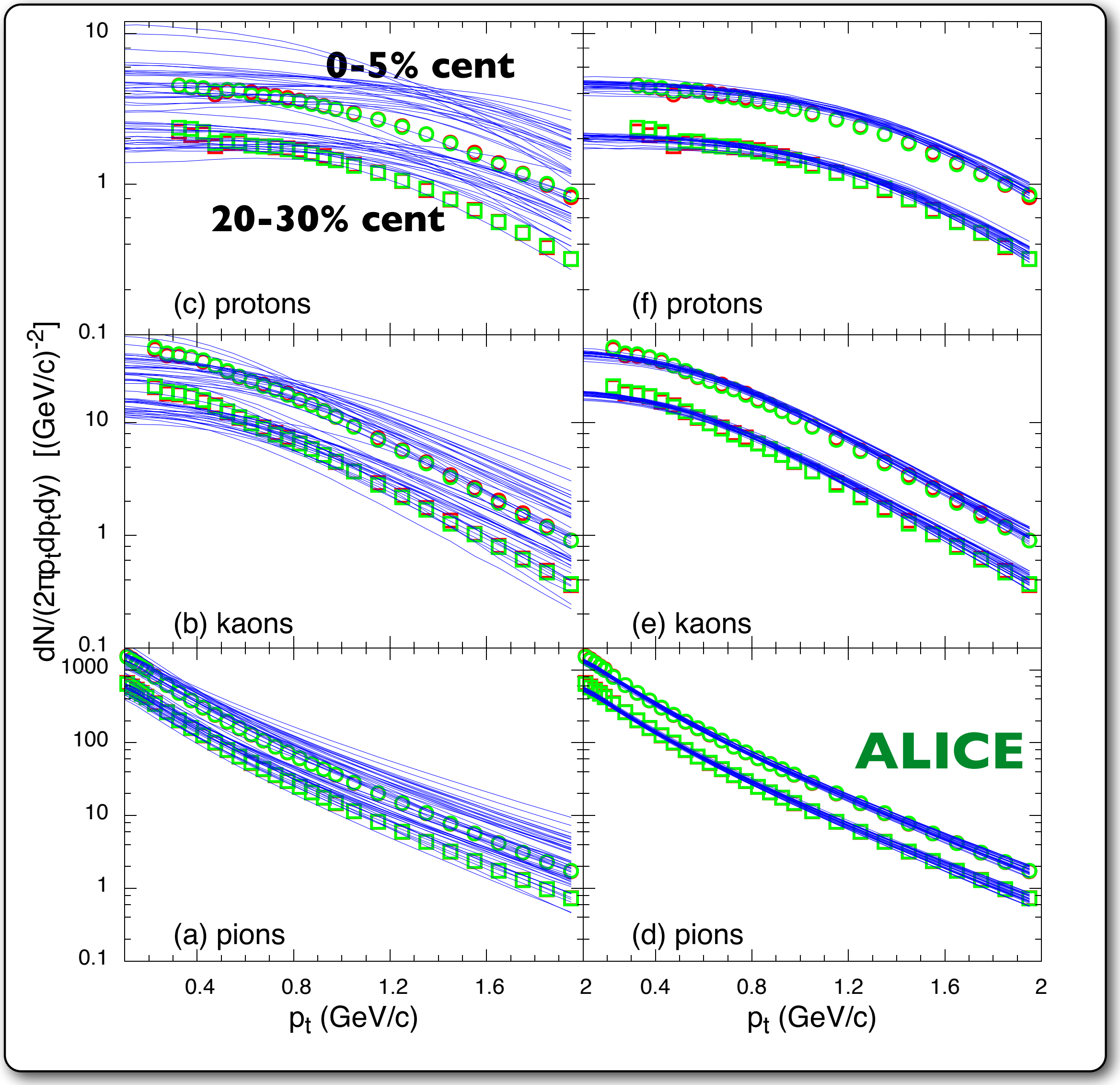


S.P., E.Sangaline, P.Sorensen & H.Wang, PRL 2015

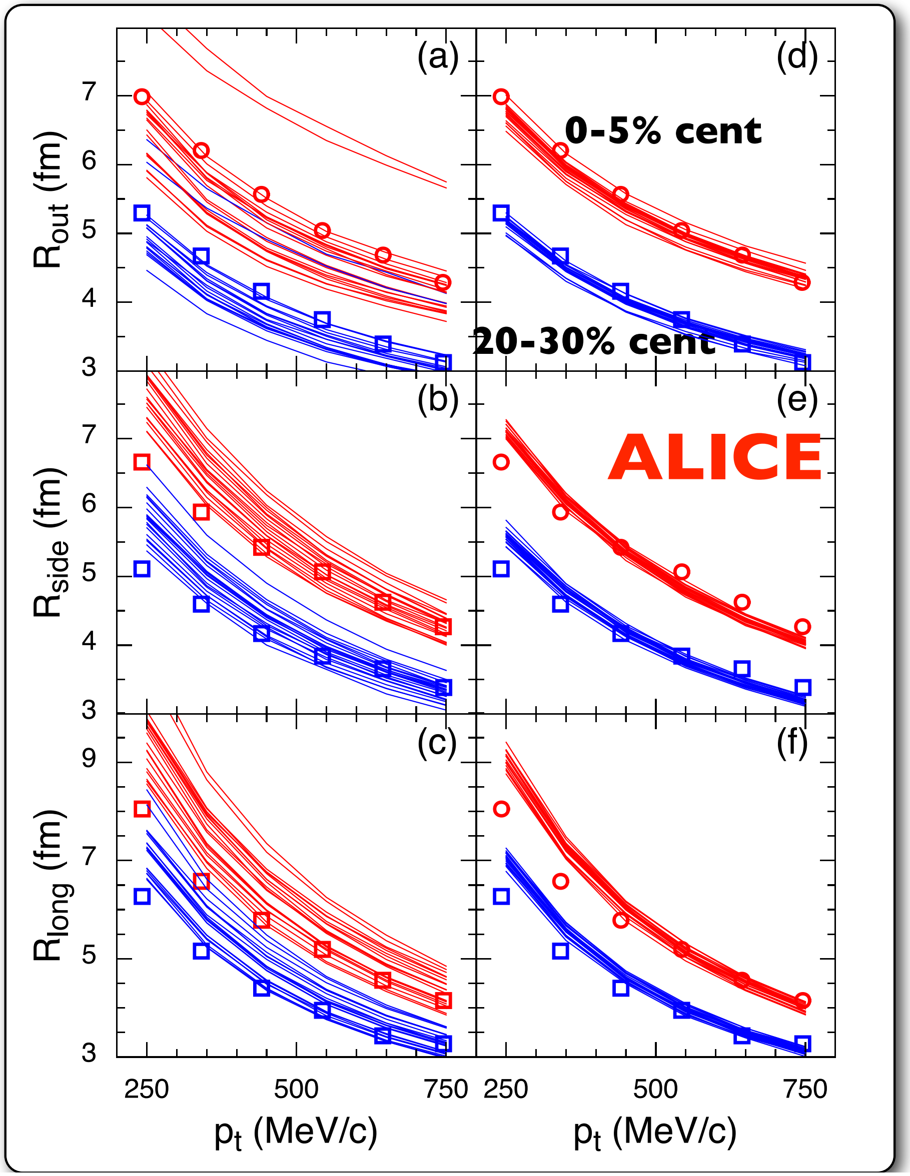
RHIC Au+Au and LHC Pb+Pb Data
14 parameters, include Eq. of State



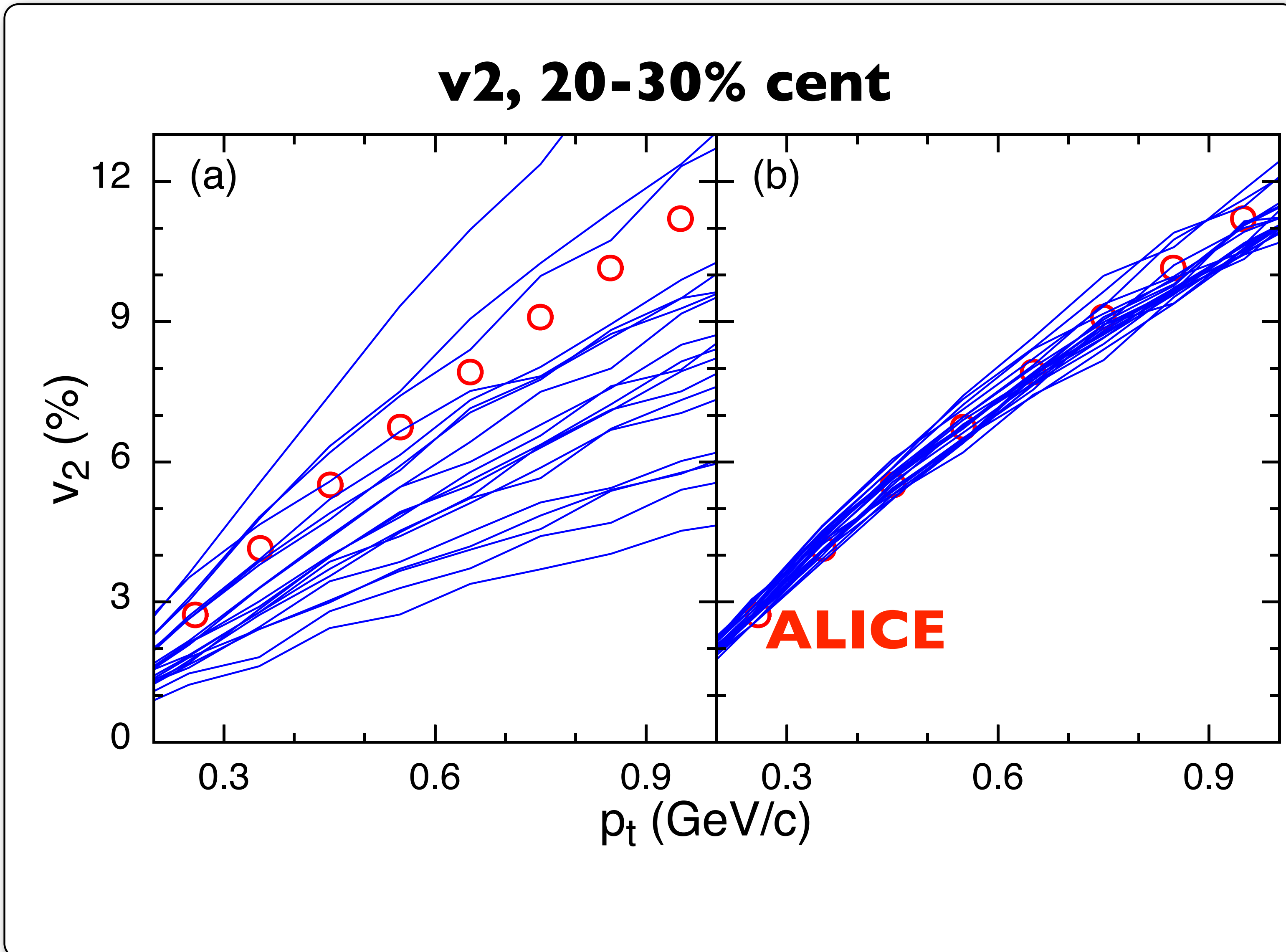
Sample Spectra from Prior and Posterior



Sample HBT from Prior and Posterior



Sample V2 from Prior and Posterior

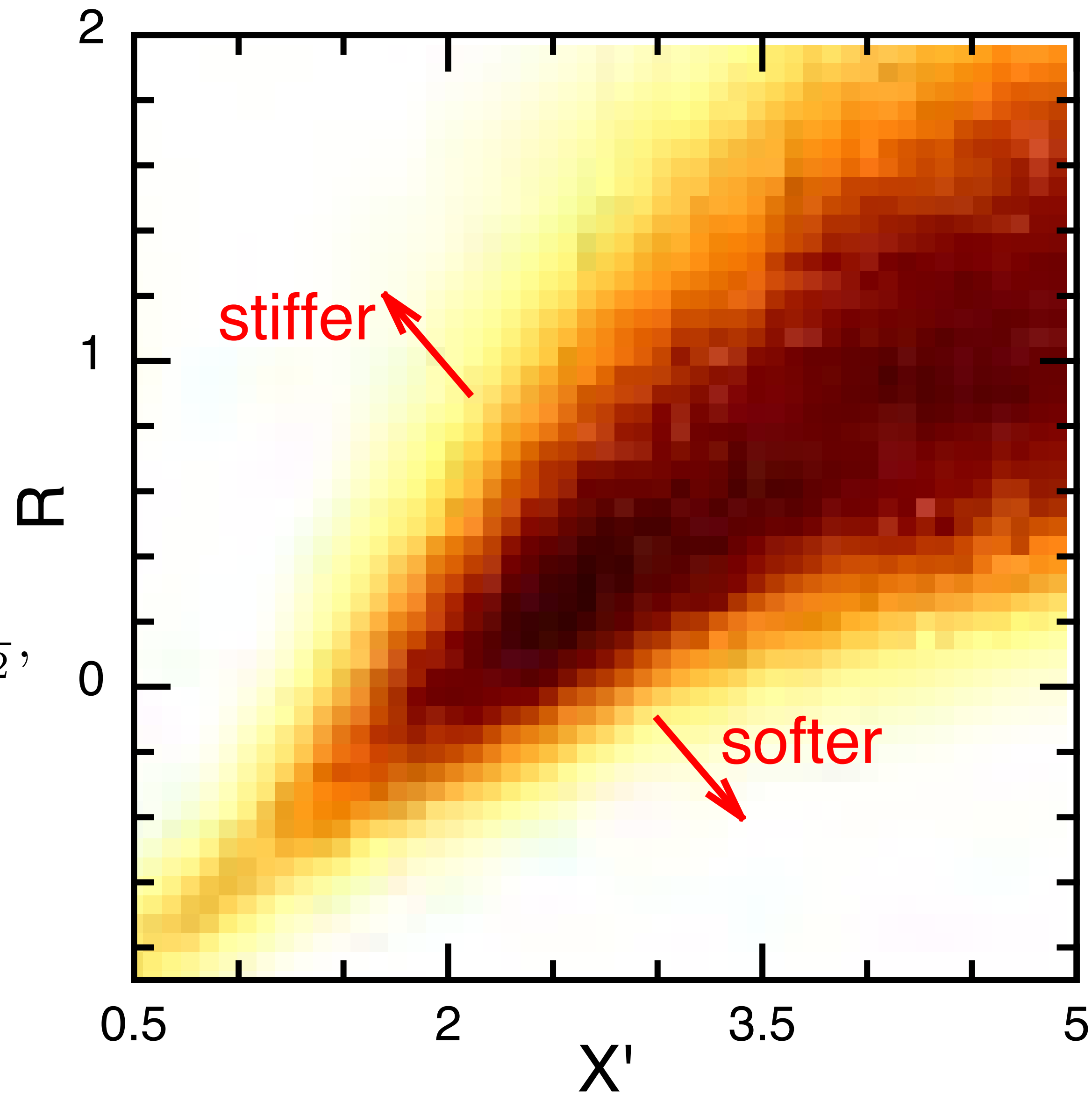


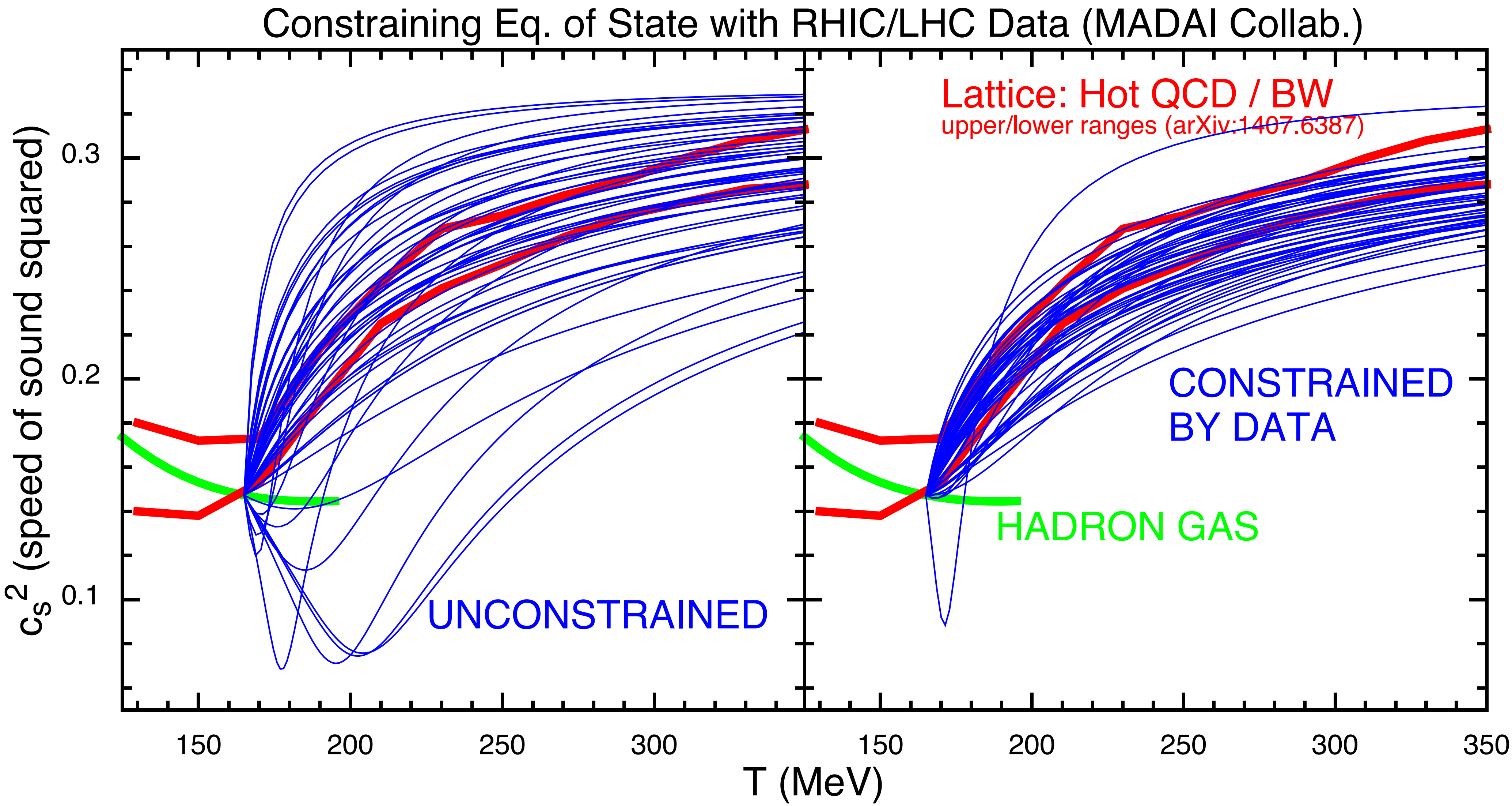
Eq. of State

$$c_s^2(\epsilon) = c_s^2(\epsilon_h) + \left(\frac{1}{3} - c_s^2(\epsilon_h)\right) \frac{X_0 x + x^2}{X_0 x + x^2 + X'^2},$$

$$X_0 = X' R c_s(\epsilon)\sqrt{12},$$

$$x \equiv \ln \epsilon/\epsilon_h$$

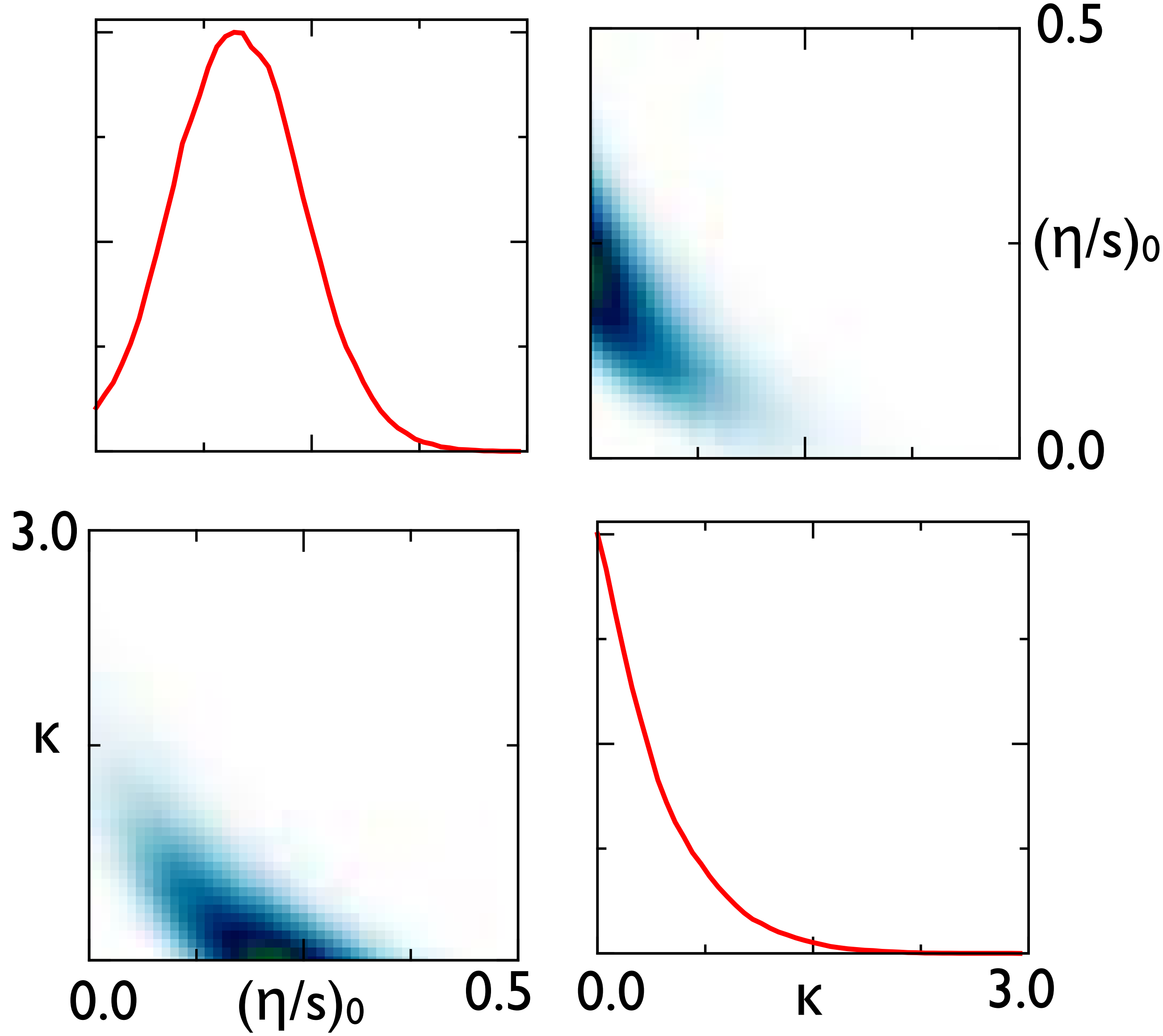




Constraining Eq. of State with RHIC/LHC Data (MADAI Collab.)
Lattice: Hot QCD / BW
upper/lower ranges (arXiv:1407.6387)
UNCONSTRAINED
CONSTRAINED
BY DATA
HADRON GAS
c_s^2 (speed of sound squared)
0.1
0.2
0.3
150
200
250
300
350
T (MeV)

η/s(T)

$$\eta/s = (\eta/s)_0 + \kappa \ln(T/165)$$

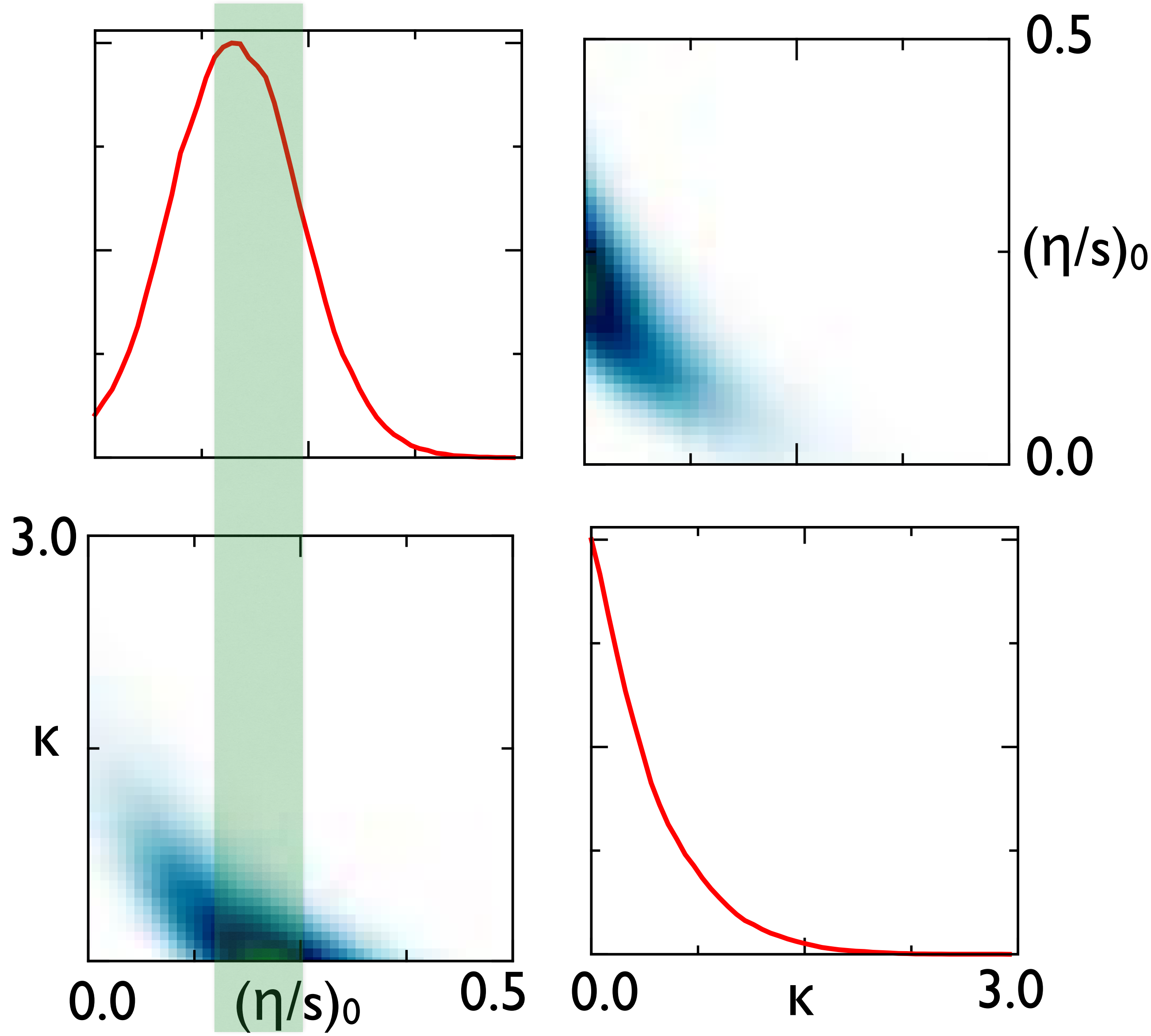


What should you expect for η/s at T=165 MeV?

- ADS/CFT: 0.08
- Perturbative QCD: > 0.5 ($\sigma \approx$ 3 mb)
- Hadron Gas: $\approx$ 0.25 ($\sigma \approx$ 30 mb)

Extracted η/s at T=165 MeV consistent with expectations for hadron gas!

Does not rise strongly in QGP



RESOLVING POWER OF OBSERVABLES

How does changing $y_{a,exp}$ or σ_a alter $\langle\langle x_i \rangle\rangle$ or $\langle\langle \delta x_i \delta x_j \rangle\rangle$?

We need

$$\frac{\partial}{\partial y_a^{(\mathrm{exp})}} \langle\langle x_i \rangle\rangle$$

NOT

$$\frac{\partial}{\partial x_i} y_a^{(\mathrm{mod})}$$

E.Sangaline and S.P., arXiv 2015

RESOLVING POWER OF OBSERVABLES

$$\langle\langle x_i \rangle\rangle = \frac{\langle x_i \mathcal{L} \rangle}{\langle \mathcal{L} \rangle}$$

$$\frac{\partial}{\partial y_a^{(\mathrm{exp})}} \langle\langle x_i \rangle\rangle = \langle\langle x_i (\partial_a \mathcal{L})/\mathcal{L} \rangle\rangle - \langle\langle x_i \rangle\rangle \langle\langle (\partial_a \mathcal{L})/\mathcal{L} \rangle\rangle$$

$$= \langle\langle \delta x_i (\partial_a \mathcal{L})/\mathcal{L} \rangle\rangle$$

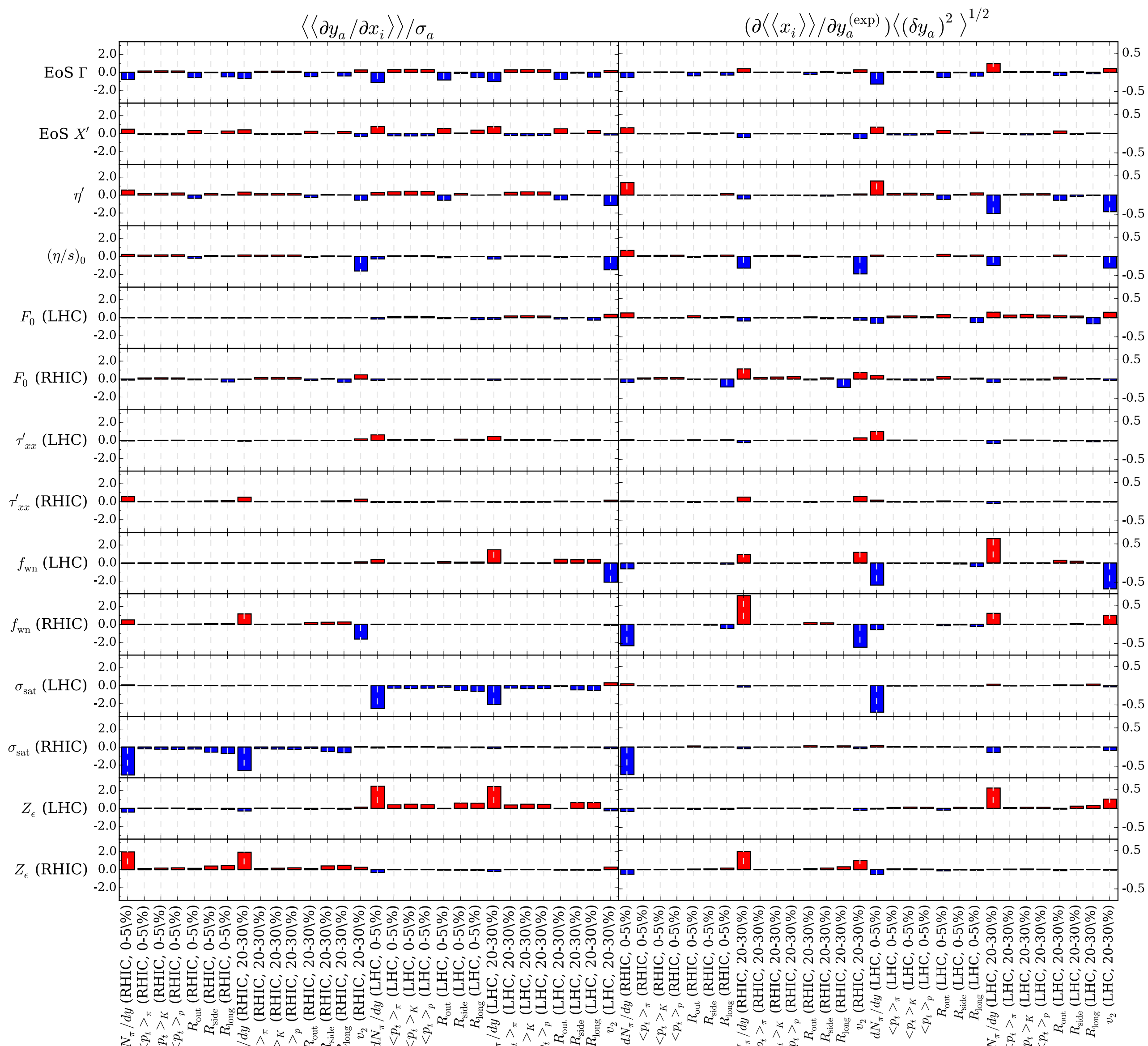
$$= -\Sigma_{ab}^{-1} \langle\langle \delta x_i \delta y_b \rangle\rangle \qquad (\text{for Gaussian})$$

$$\delta x_i = x_i - \langle\langle x_i \rangle\rangle, \quad \delta y_a = y_a - y_a^{(\mathrm{exp})}$$

can find similar relation for $\frac{\partial}{\partial \sigma_a} \langle\langle \delta x_i \delta x_j \rangle\rangle$

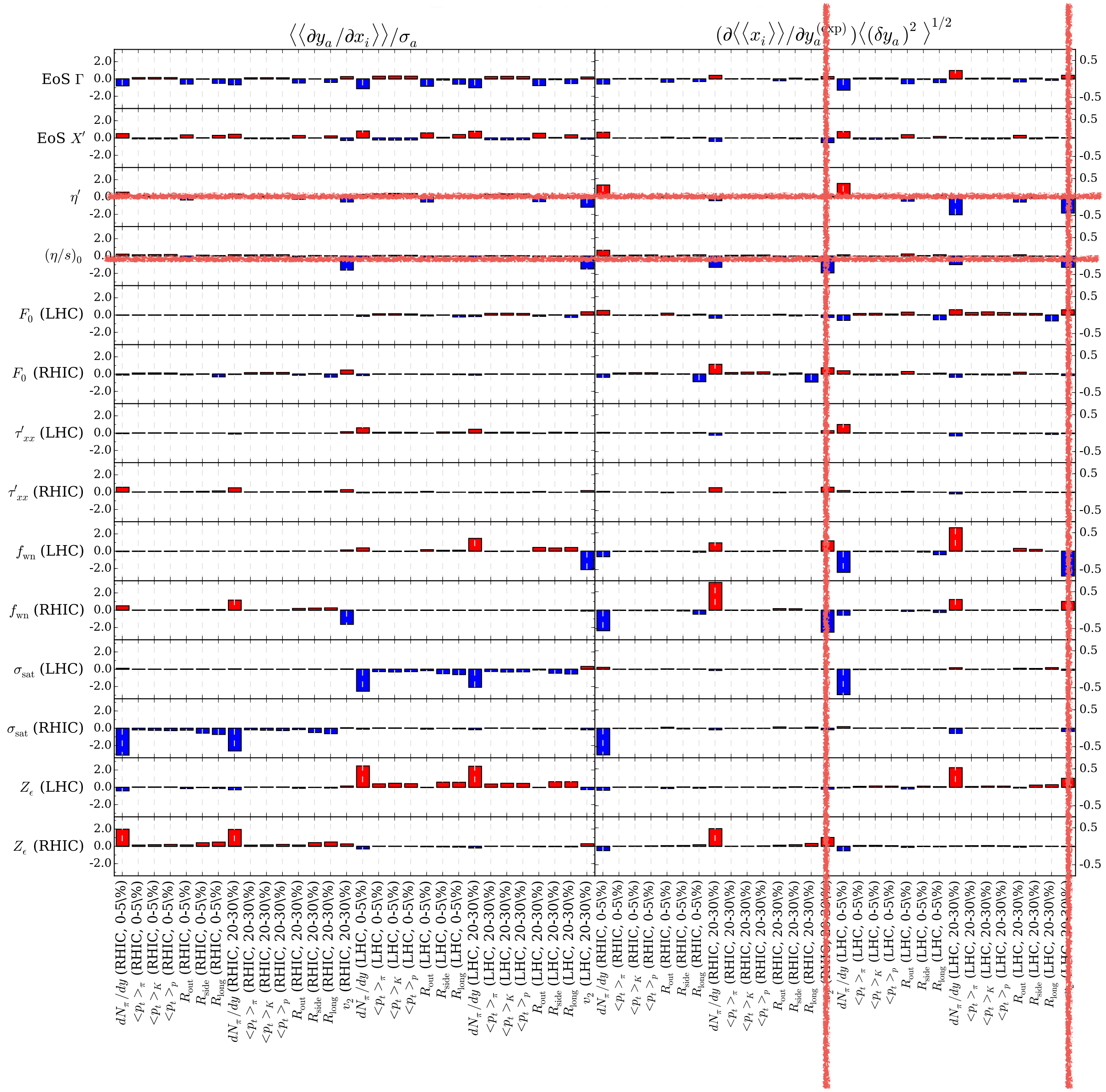
E.Sangaline and S.P., PRC 2016

$$\frac{1}{\sigma_a}\frac{\partial y_a}{\partial x_i}\bigg|_{y_{b\neq a}}$$

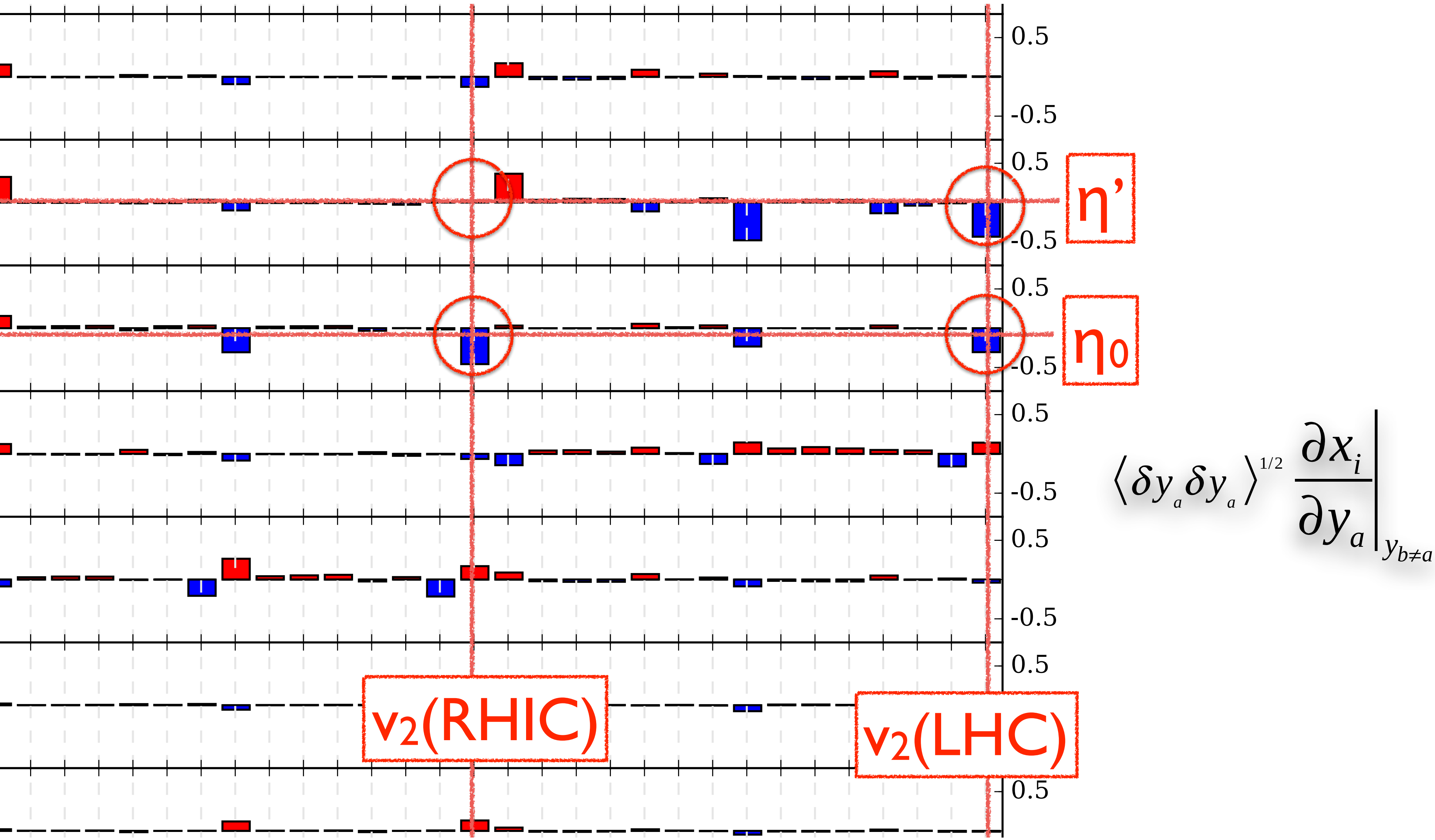


$$\langle \delta y_a \delta y_a\rangle^{1/2}\frac{\partial x_i}{\partial y_a}\bigg|_{y_{b\neq a}}$$

$$\frac{1}{\sigma_a}\frac{\partial y_a}{\partial x_i}\bigg|_{y_{b\neq a}}$$



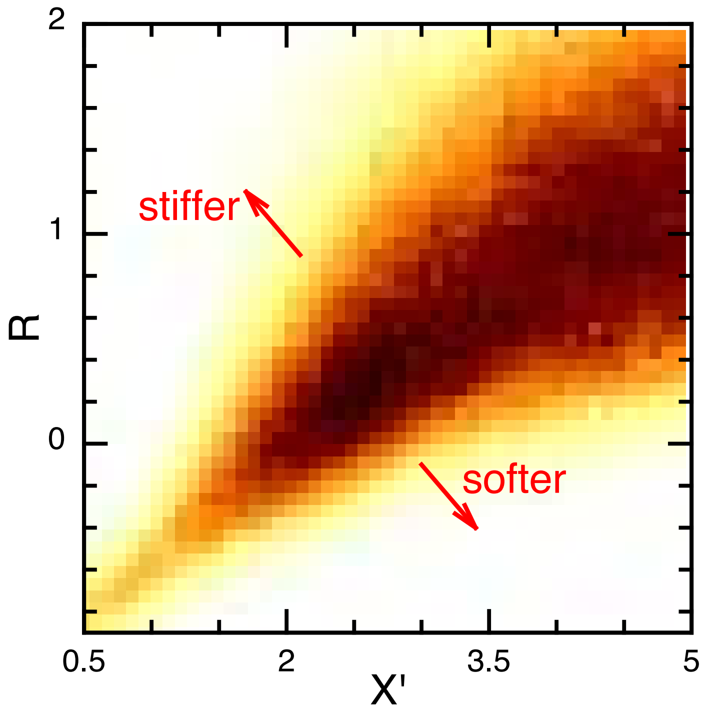
$$\langle \delta y_a \delta y_a \rangle^{1/2} \frac{\partial x_i}{\partial y_a}\bigg|_{y_{b\neq a}}$$

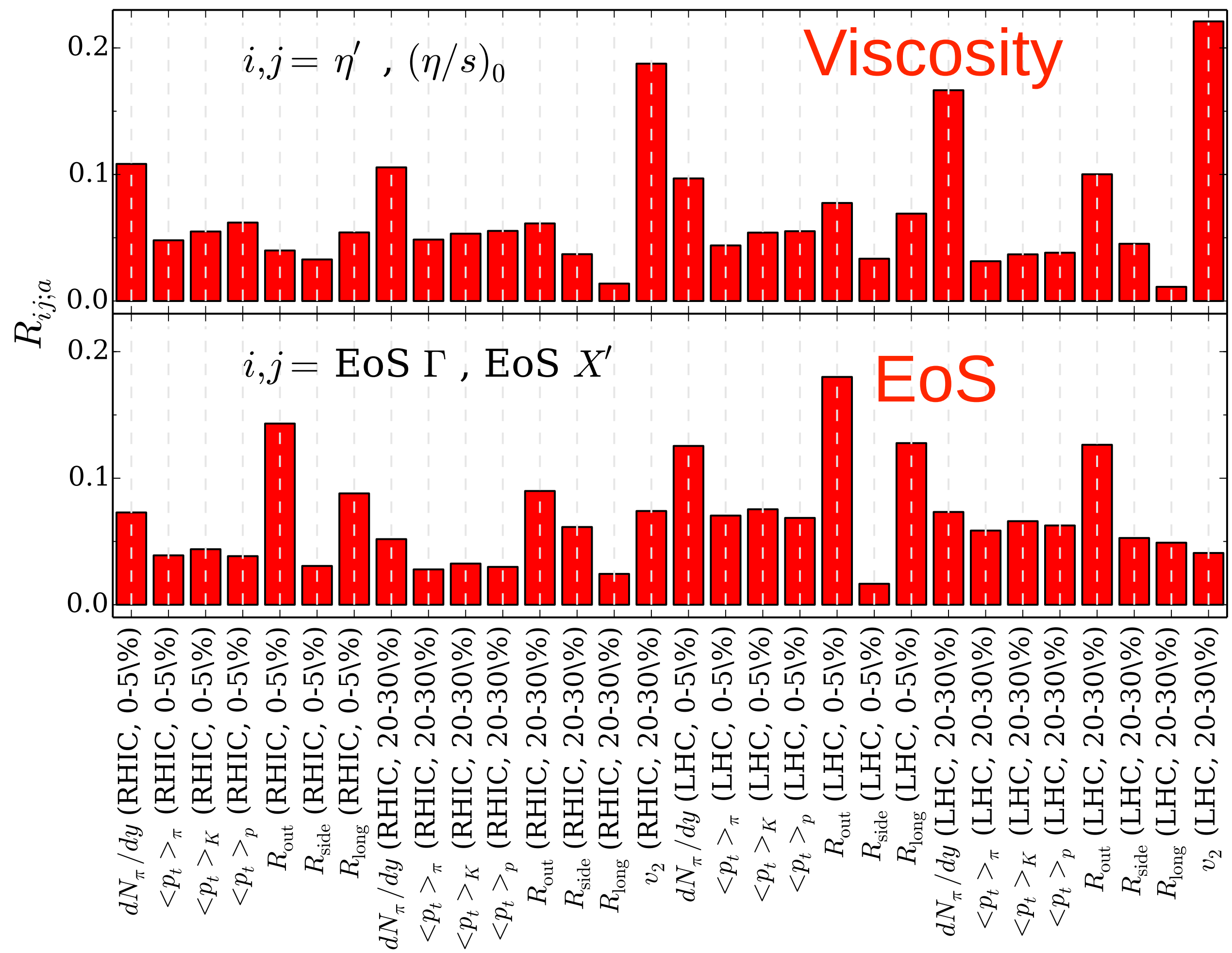


0.5
-0.5
η'
η0
v2(RHIC)
v2(LHC)
$\left\langle \delta y_a \delta y_a \right\rangle^{1/2} \left.\frac{\partial x_i}{\partial y_a}\right|_{y_{b\neq a}}$

$$\frac{d}{d\sigma_y}\sqrt{\begin{vmatrix} \langle\langle \delta x_1 \delta x_1 \rangle\rangle & \langle\langle \delta x_1 \delta x_2 \rangle\rangle \\ \langle\langle \delta x_1 \delta x_2 \rangle\rangle & \langle\langle \delta x_2 \delta x_2 \rangle\rangle \end{vmatrix}}\,\langle \delta y \delta y \rangle^{1/2}$$

2-Parameter Resolving Power





What determines viscosity?

- Both v_2 and multiplicities
- T-dependence comes from LHC v_2

What determines EoS?

- **Lots of observables**
- **Femtoscopic radii are important**

CONCLUSIONS

- **Robust, emulation works splendidly**
- **Scales well to more parameters & more data**
- **Eq. of State and Viscosity can be extracted from data**
- **Eq. of State consistent with lattice gauge theory**
- **Other parameters not as well constrained**
- **Heavy-Ion Physics can be a Quantitative Science!!!!**

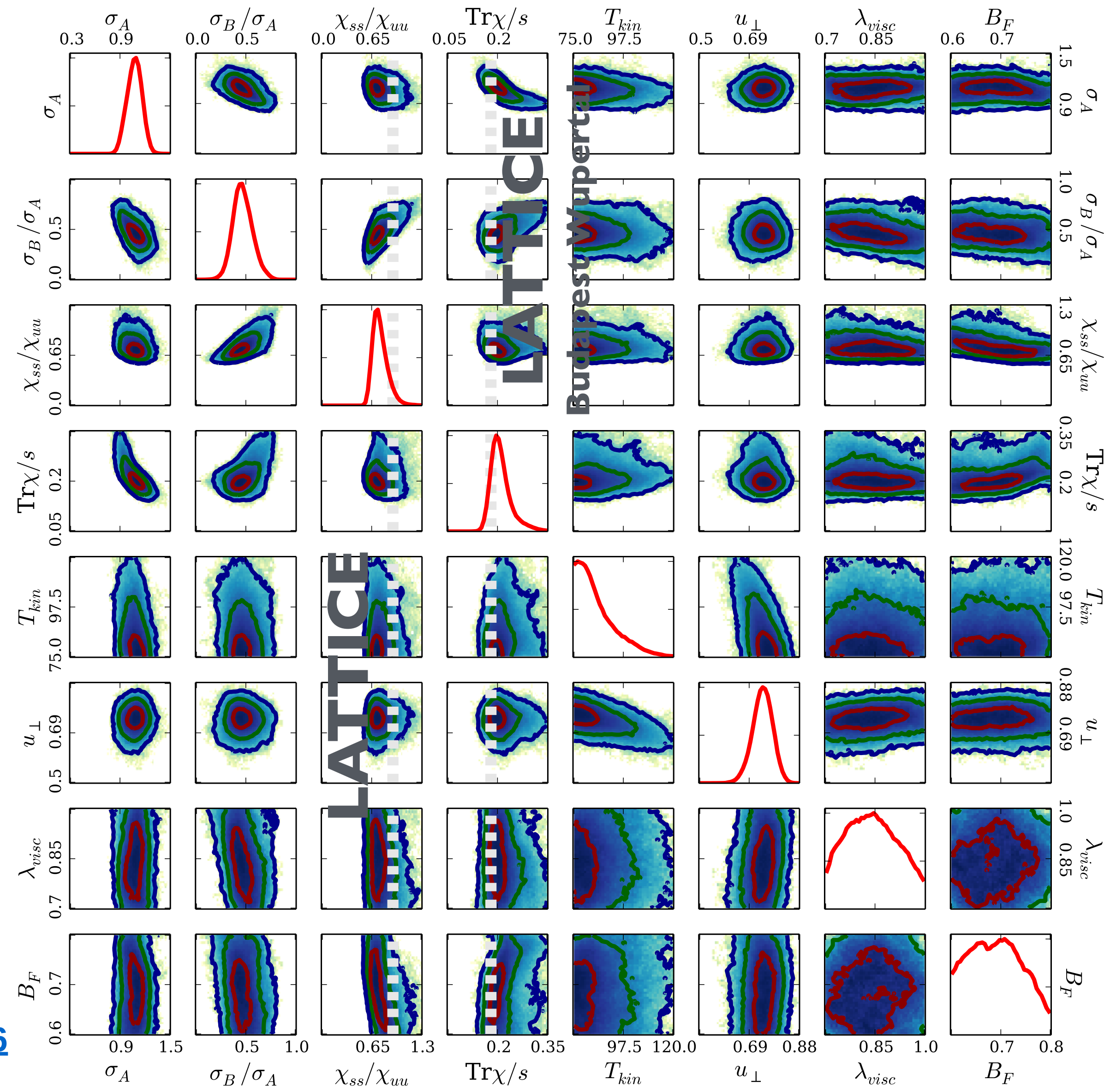
FUTURE

- Improve statement of uncertainties
- Add parameters
  many related to hadronization region
- Consider more data
  - more observables
  - Beam energy scan (Yikes!!!!)
- Improve models
  - Lumpy initial conditions
  - 3D calculations for lower energies
  - Fill in missing physics (e.g. bulk viscosity)

THINGS TO KEEP IN MIND...

- Remember what you're trying to do
  - — parameter constraint?
  - — identify weakness of models?
  - — predictivity?
  - — not draw lines through data!
  - — more parameters(physics) are better
- Data can be redundant
  - — correlated uncertainties
  - — underestimates of uncertainty
- Models have systematic uncertainties
  - — requires objective self doubt

Additional slide: Charge BFs and charge susceptibilities



S.P., C. Ratti and W.McCormack, PRC 2016