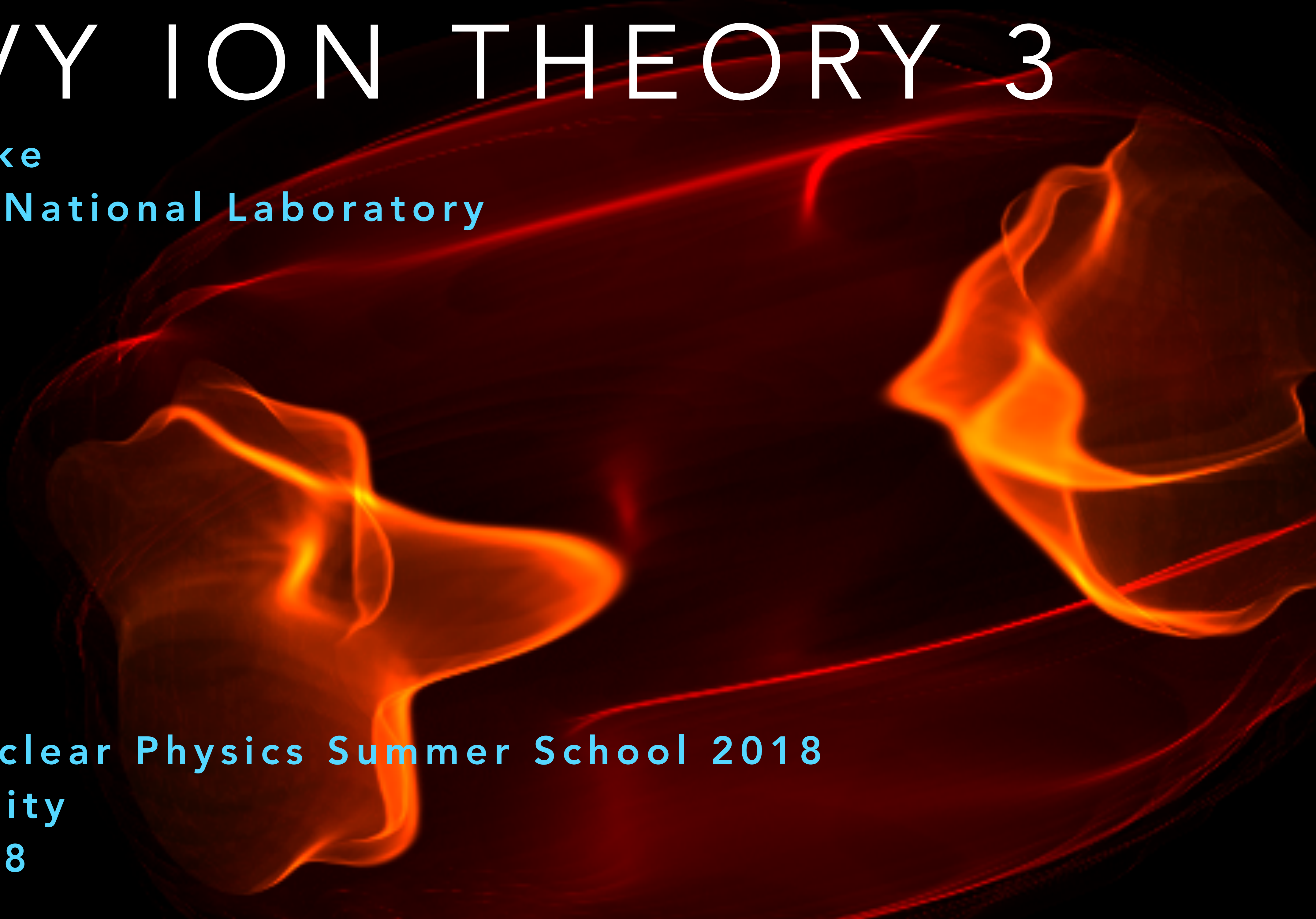
HEAVY ION THEORY 3

Björn Schenke
Brookhaven National Laboratory



National Nuclear Physics Summer School 2018
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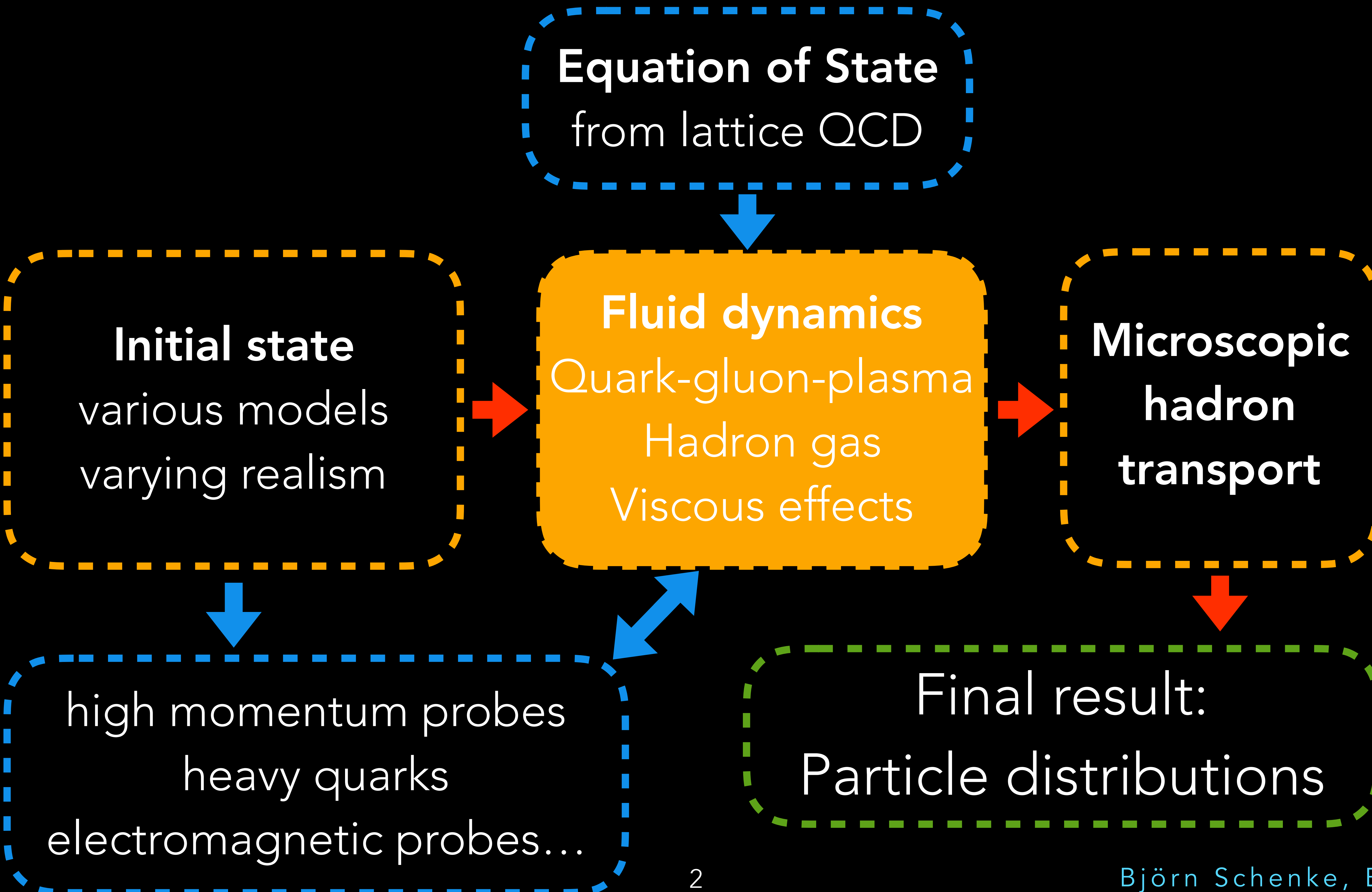
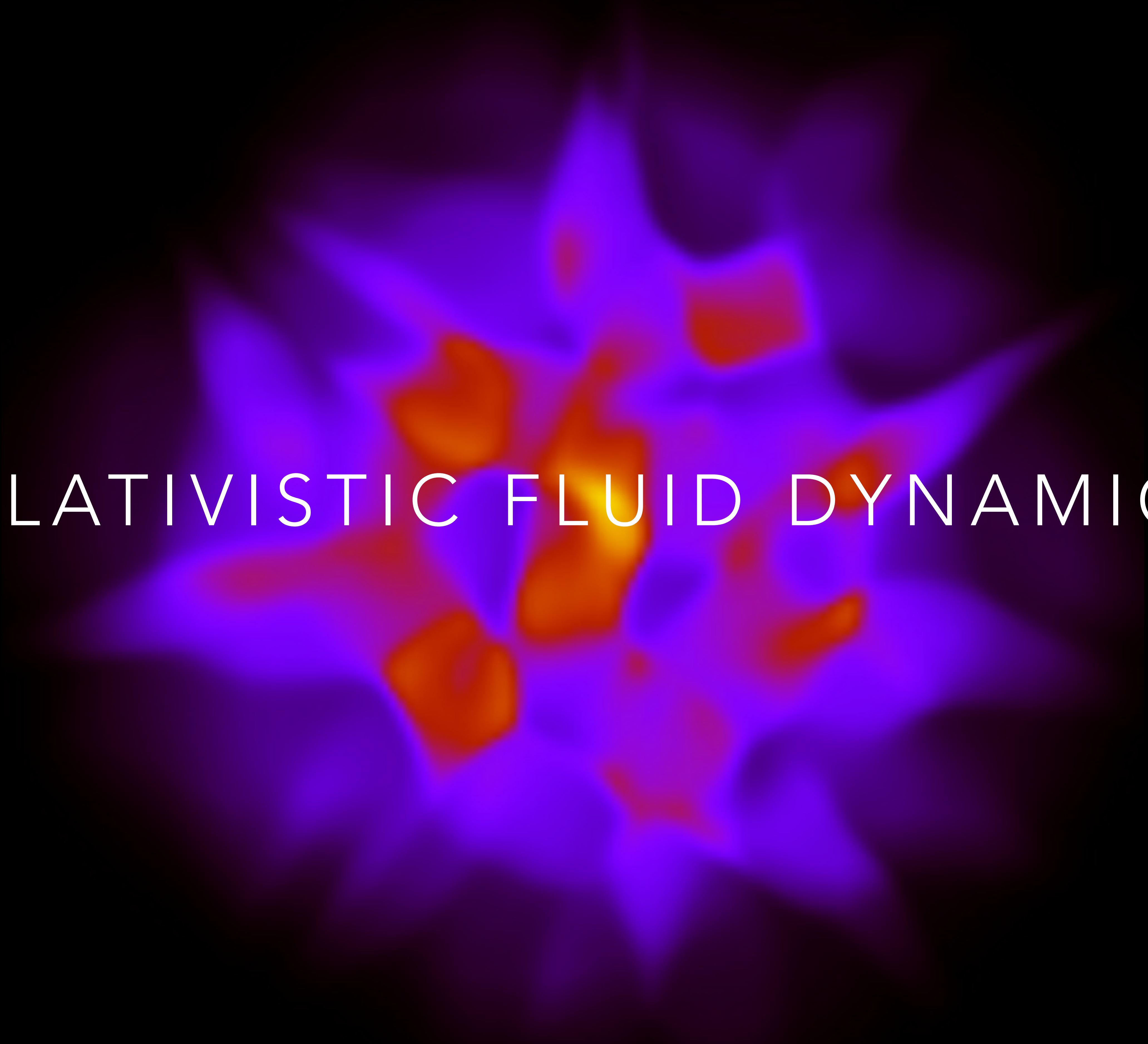
INGREDIENTS TO DESCRIBE HEAVY ION COLLISIONS



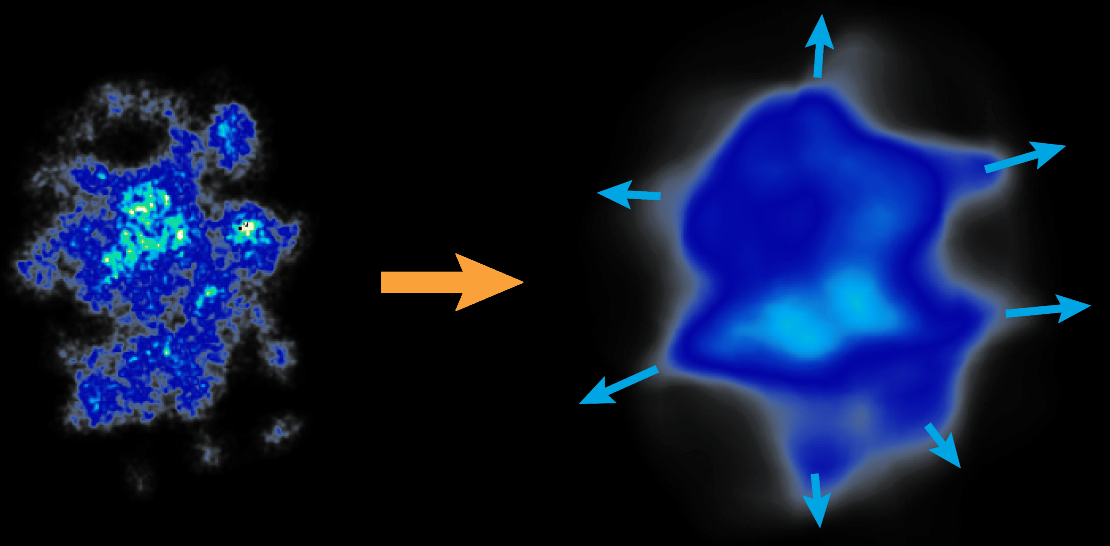
RELATIVISTIC FLUID DYNAMICS



RELATIVISTIC FLUID DYNAMICS

As discussed, experimental data on anisotropic azimuthal momentum distributions can only be explained with the existence of a strongly interacting final state (at least in large A+A systems).

Quantitatively, it is well described by fluid dynamics with a very small viscosity.



FLUID APPROXIMATION

- Treat an ensemble of particles as a single fluid
- Compute ensemble's mean velocity at each point $u(r, t)$
- Lost information about the spread of velocities around that mean: But if we have local thermodynamic equilibrium (LTE), that spread is described by the temperature $T(r, t)$
- LTE requires that particles are in equilibrium locally:
  Mean free path needs to be smaller than any length scale of interest

EQUATIONS OF FLUID DYNAMICS

Non-relativistic case

Conservation of mass:

Variation of mass in the volume V is due to in- and out- flow through the surface ∂V

$$\frac{\partial}{\partial t}\int \rho dV = -\int_{\partial V} \rho\vec{u}\cdot\vec{n}dA$$

use Gauss' theorem:

$$\frac{\partial}{\partial t}\int \rho dV = -\int_{V} \vec{\nabla}\cdot(\rho\vec{u})dV$$

true for all V, so

$$\partial_t\rho + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

continuity equation

EQUATIONS OF FLUID DYNAMICS

Non-relativistic case

Conservation of momentum:

Momentum density: $\rho\vec{u}$ Same procedure as for mass, but also include the force from the outside fluid on the surface ∂V

$$\frac{\partial}{\partial t}\int(\rho\vec{u})dV = -\int_{\partial V}(\rho\vec{u})\vec{u}\cdot\vec{n}dA - \int_{\partial V}p\vec{n}dA$$

use Gauss' theorem:

$$\frac{\partial}{\partial t}\int(\rho\vec{u})dV = -\int_{V}\vec{\nabla}\cdot(\rho\vec{u})\vec{u} + \vec{\nabla}pdV$$

true for all V: $\partial_t(\rho\vec{u}) + \vec{\nabla}\cdot(\rho\vec{u})\vec{u} + \vec{\nabla}p = 0$. Use continuity equation:

$$\boxed{\partial_t\vec{u} + \vec{u}(\vec{\nabla}\cdot\vec{u}) = -\frac{1}{\rho}\vec{\nabla}p}$$

Euler equation

RELATIVISTIC FLUID DYNAMICS

In a relativistic system, mass density is not a good degree of freedom: It does not account for kinetic energy which can be large for motions close to the speed of light *c*.

So replace ρ by the total energy density ε.

Also, $\vec{u}$ does not transform correctly under Lorentz transformations and should be replaced by the Lorentz four-vector

$$u^\mu = \frac{dx^\mu}{d\mathcal{T}}$$

$d\mathcal{T}$ is the proper time increment:

$$(d\mathcal{T})^2 = g_{\mu\nu} dx^\mu dx^\nu = (dt)^2 - (d\vec{x})^2$$
$$= (dt)^2 \left[1 - \left(\frac{d\vec{x}}{dt}\right)^2\right] = (dt)^2(1 - \vec{u}^2)$$

$g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$

$c = k_B = \hbar = 1$

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FLOW VELOCITY

$$u^\mu = \frac{dx^\mu}{d\mathcal{T}} = \frac{dt}{d\mathcal{T}}\frac{dx^\mu}{dt} = \frac{1}{\sqrt{1-\vec{u}^2}}\begin{pmatrix}1\\ \vec{u}\end{pmatrix} = \gamma(\vec{u})\begin{pmatrix}1\\ \vec{u}\end{pmatrix}$$

In the local rest frame $u^\mu = (1,\vec{0})$

Because u^μ obeys the relation

$$u^2 = u^\mu g_{\mu\nu} u^\nu = \gamma^2(\vec{u})(1-\vec{u}^2) = 1$$

there is no need for an additional equation when replacing $\vec{u}$ by u^μ

ENERGY MOMENTUM TENSOR

The ideal energy momentum tensor (no viscosity) has to be built from the pressure p, the energy density ε, and u^μ, as well as $g^{\mu\nu}$.

Properties: symmetric, transforms like a Lorentz-tensor

So the *most general* form is

$$T^{\mu\nu} = \epsilon(c_0 g^{\mu\nu} + c_1 u^\mu u^\nu) + p(c_2 g^{\mu\nu} + c_3 u^\mu u^\nu)$$

Constraints:

$T^{00} = \varepsilon$ and $T^{0i} = 0$ and $T^{ij} = \delta^{ij} p$ in the local rest frame.

ENERGY MOMENTUM TENSOR

$$T^{\mu\nu} = \epsilon(c_0 g^{\mu\nu} + c_1 u^\mu u^\nu) + p(c_2 g^{\mu\nu} + c_3 u^\mu u^\nu)$$

$T^{00} = \varepsilon$ and $T^{0i} = 0$ and $T^{ij} = \delta^{ij} p$ in the local rest frame.

$$T^{00} = \varepsilon(c_0 + c_1) + p(c_2 + c_3) = \varepsilon$$
$$\Rightarrow c_0 = 1 - c_1 \text{ and } c_2 = -c_3$$

$$T^{ij} = -\varepsilon c_0 \delta^{ij} - p c_2 \delta^{ij} = \delta^{ij} p$$
$$\Rightarrow c_0 = 0 \text{ and } c_2 = -1$$
$$\Rightarrow c_1 = 1 \text{ and } c_3 = 1$$

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - p(g^{\mu\nu} - u^\mu u^\nu)$$

FLUID DYNAMIC EQUATIONS

Relativistic case

Introduce the projector on the space orthogonal to u^μ

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$

Then

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - p\Delta^{\mu\nu}$$

Without external sources $T^{\mu\nu}$ is conserved:

$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

Energy and momentum conservation

ideal relativistic hydrodynamic equations

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FLUID DYNAMIC EQUATIONS

Relativistic case

To identify equations of motion analogous to the non-relativistic ones, project on directions parallel and perpendicular to u^μ.

Parallel:

$$u_\nu \partial_\mu T^{\mu\nu} = u^\mu \partial_\mu \varepsilon + \varepsilon(\partial_\mu u^\mu) + \varepsilon u_\nu u^\mu \partial_\mu u^\nu - p u_\nu \partial_\mu \Delta^{\mu\nu}$$
$$= (\varepsilon + p)\partial_\mu u^\mu + u^\mu \partial_\mu \varepsilon = 0$$

Perpendicular:

$$\Delta^\alpha_\nu \partial_\mu T^{\mu\nu} = \varepsilon u^\mu \Delta^\alpha_\nu \partial_\mu u^\nu - \Delta^{\mu\alpha}(\partial_\mu p) + p u^\mu \Delta^\alpha_\nu \partial_\mu u^\nu$$
$$= (\varepsilon + p) u^\mu \partial_\mu u^\alpha - \Delta^{\mu\alpha}\partial_\mu p = 0$$

Introducing $D = u_\mu \partial^\mu$

and $\nabla^\alpha = \Delta^{\mu\alpha}\partial_\mu$

$$\Rightarrow \quad D\varepsilon + (\varepsilon + p)\partial_\mu u^\mu = 0$$
$$(\varepsilon + p) D u^\alpha - \nabla^\alpha p = 0$$

RELATION TO NON-RELATIVISTIC CASE

In the non-relativistic limit

$$D\varepsilon + (\varepsilon + p)\partial_\mu u^\mu = 0$$
$$(\varepsilon + p)Du^\alpha - \nabla^\alpha p = 0$$

become continuity and Euler equations. For $|\vec{u}| \ll 1$ we have

$$D = u^\mu \partial_\mu \simeq \partial_t + \vec{u} \cdot \vec{\nabla} + \mathcal{O}(\vec{u}^2)$$

$$\nabla^i = \Delta^{i\mu}\partial_\mu \simeq \partial^i + \mathcal{O}(|\vec{u}|)$$

Imposing also a non-relativistic equation of state, where p « **ε** and assuming that energy density is dominated by mass density **ε** ≃ **ρ**, we get

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{u}) = 0$$
$$\partial_t \vec{u} + \vec{u}(\vec{\nabla} \cdot \vec{u}) = -\frac{1}{\rho}\vec{\nabla} p$$

remember?

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RELATIVISTIC VISCOUS FLUID DYNAMICS

To include dissipative (viscous) effects, we write

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + \Pi^{\mu\nu}$$

where $T^{\mu\nu}_{(0)}$ is the ideal part we considered before
$\Pi^{\mu\nu}$ is the viscous stress tensor

In a system without conserved charges (or e.g. at $\mu_B=0$), all momentum density is due to flow of energy density:

$$u_\mu T^{\mu\nu} = \varepsilon u^\nu \rightarrow \boxed{u_\mu \Pi^{\mu\nu} = 0}$$

In a more general system this corresponds to choosing the Landau-Lifshitz frame (local frame where energy density is at rest)

Alternatively, in the Eckart frame
the charge density (if there is one) is at rest

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RELATIVISTIC VISCOUS FLUID DYNAMICS

Equations of motion are obtained by projections of $\partial_\mu T^{\mu\nu} = 0$ as in the ideal case:

$$u_\nu \partial_\mu T^{\mu\nu} = D\varepsilon + (\varepsilon + p)\partial_\mu u^\mu + u_\nu \partial_\mu \Pi^{\mu\nu} = 0$$

$$\Delta^\alpha_\nu \partial_\mu T^{\mu\nu} = (\varepsilon + p)Du^\alpha - \nabla^\alpha p + \Delta^\alpha_\nu \partial_\mu \Pi^{\mu\nu} = 0$$

Simplify the first equation using:

- $u_\nu \partial_\mu \Pi^{\mu\nu} = \partial_\mu (u_\nu \Pi^{\mu\nu}) - \Pi^{\mu\nu}\partial_\mu u_\nu = \partial_\mu (u_\nu \Pi^{\mu\nu}) - \Pi^{\mu\nu}\partial_{(\mu} u_{\nu)}$
- $u_\nu \Pi^{\mu\nu} = 0$ (frame)
- $\partial_\mu = u_\mu D + \nabla_\mu$

$$D\varepsilon + (\varepsilon + p)\partial_\mu u^\mu - \Pi^{\mu\nu}\nabla_{(\mu} u_{\nu)} = 0$$

$$(\varepsilon + p)Du^\alpha - \nabla^\alpha p + \Delta^\alpha_\nu \partial_\mu \Pi^{\mu\nu} = 0$$

Symmetrization $A_{(\mu} B_{\nu)} = \frac{1}{2}(A_\mu B_\nu + A_\nu B_\mu)$ not necessary but helpful later
OK, because $\Pi^{\mu\nu}$ is already symmetric

$\Pi^{\mu\nu}$ FROM THERMODYNAMICS

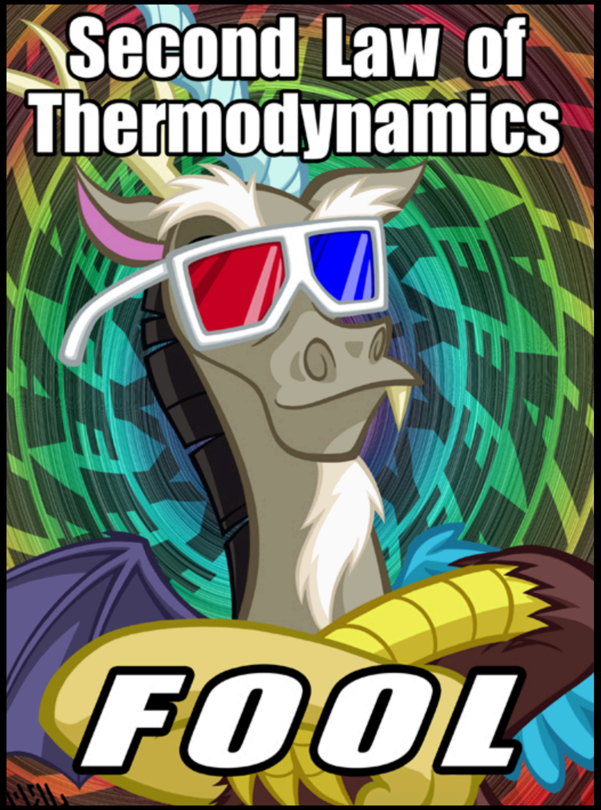
$\Pi^{\mu\nu}$ has not yet been specified

We can use the second law of thermodynamics to get it:

Entropy must always increase locally

For μ_B=0:

$$\varepsilon + p = Ts \text{ and } Tds = d\varepsilon$$



Second law in covariant form:

$$\partial_\mu s^\mu \geq 0$$

with $s^\mu = su^\mu$

$$\partial_\mu s^\mu = Ds + s\partial_\mu u^\mu = \frac{1}{T}D\varepsilon + \frac{\varepsilon + p}{T}\partial_\mu u^\mu = \frac{1}{T}\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} \geq 0$$

using $D\varepsilon + (\varepsilon + p)\partial_\mu u^\mu - \Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} = 0$

$\Pi^{\mu\nu}$ FROM THERMODYNAMICS

Now split $\Pi^{\mu\nu}$ into a traceless part and the rest

$$\Pi^{\mu\nu} = \pi^{\mu\nu} - \Delta^{\mu\nu}\Pi$$

Let's define

$$\nabla_{<\mu}u_{\nu>} = 2\nabla_{(\mu}u_{\nu)} - \frac{2}{3}\Delta_{\mu\nu}\nabla_\alpha u^\alpha$$

the second law becomes

see backup slides

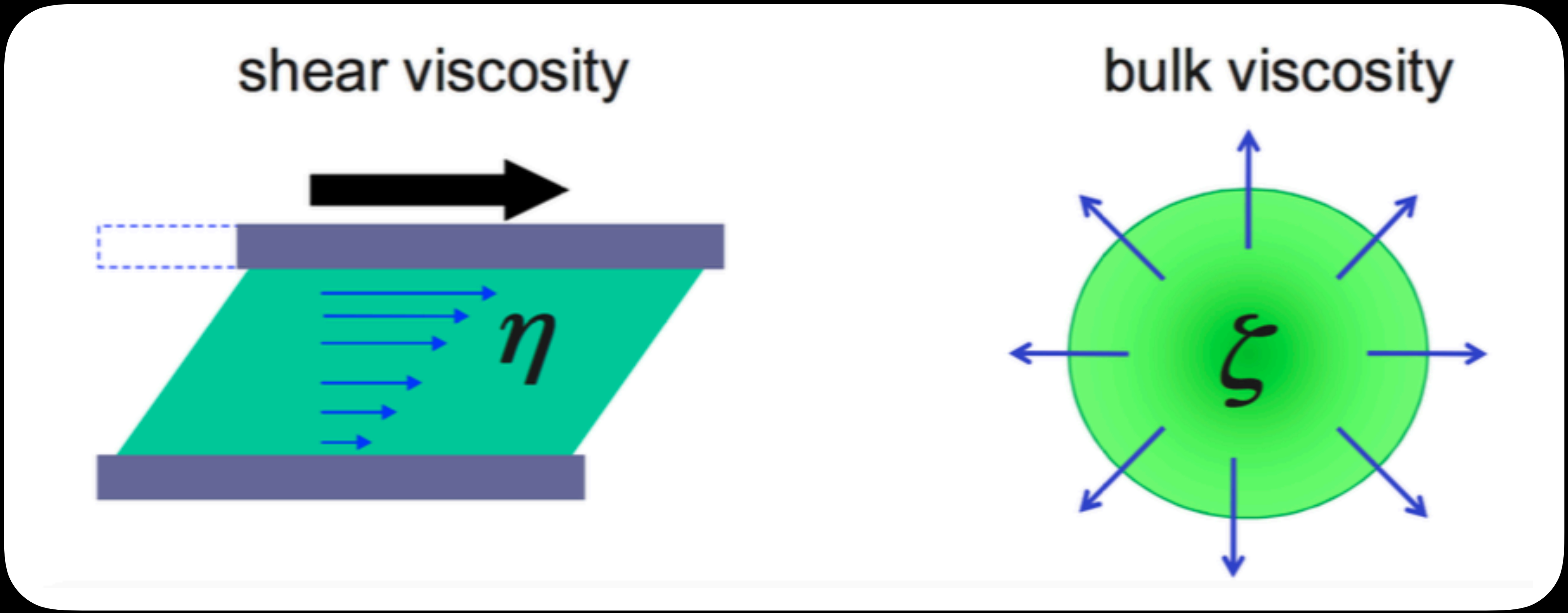
$$\partial_\mu s^\mu = \frac{1}{T}\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} = \frac{1}{2T}\pi^{\mu\nu}\nabla_{<\mu}u_{\nu>} - \frac{1}{T}\Pi\nabla_\alpha u^\alpha \geq 0$$

using $\Delta_{\mu\nu}\Delta^{\mu\nu} = 3$ and $\Delta_{\mu\nu}\pi^{\mu\nu} = \pi^\mu_\mu - u_\mu\pi^{\mu\nu}u_\nu = 0$

Fulfilled by

$\pi^{\mu\nu} = \eta\nabla^{<\mu}u^{\nu>}$ and $\Pi = -\zeta\nabla_\alpha u^\alpha$ with $\eta \geq 0, \zeta \geq 0$

SHEAR AND BULK VISCOSITY

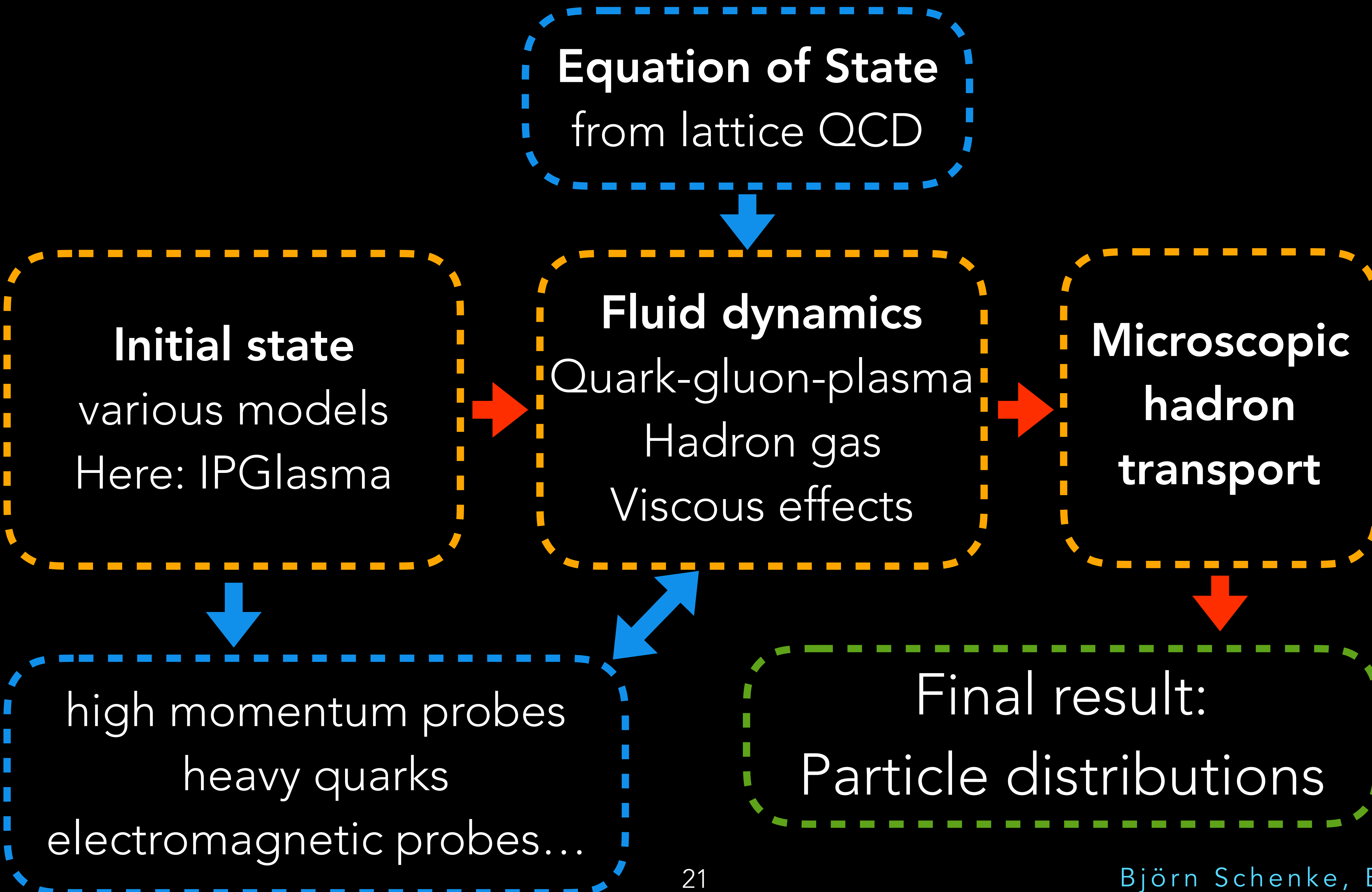


Shear viscosity describes the resistance to shear flow
Bulk viscosity describes the resistance to expansion

SECOND ORDER VISCOUS HYDRODYNAMICS

- Relativistic Navier Stokes equations exhibit acausal behavior (see backup slides) because large k modes travel faster than light
- Problem for numerical stability
- Must go to at least second order in gradients
- Can be done by including viscous corrections to the entropy current when employing the second law of thermodynamics
- 2nd order hydrodynamics can also be derived from kinetic theory (see backup slides)

INGREDIENTS TO DESCRIBE HEAVY ION COLLISIONS



MATCHING IP-GLASMA TO HYDRODYNAMICS

- Compute all components of the field energy momentum tensor

$$T^{\mu\nu}_{\rm CYM} = -g^{\mu\alpha} g^{\nu\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} + \frac{1}{4} g^{\mu\nu} g^{\alpha\gamma} g^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta}$$

- Determine ε and u^μ from

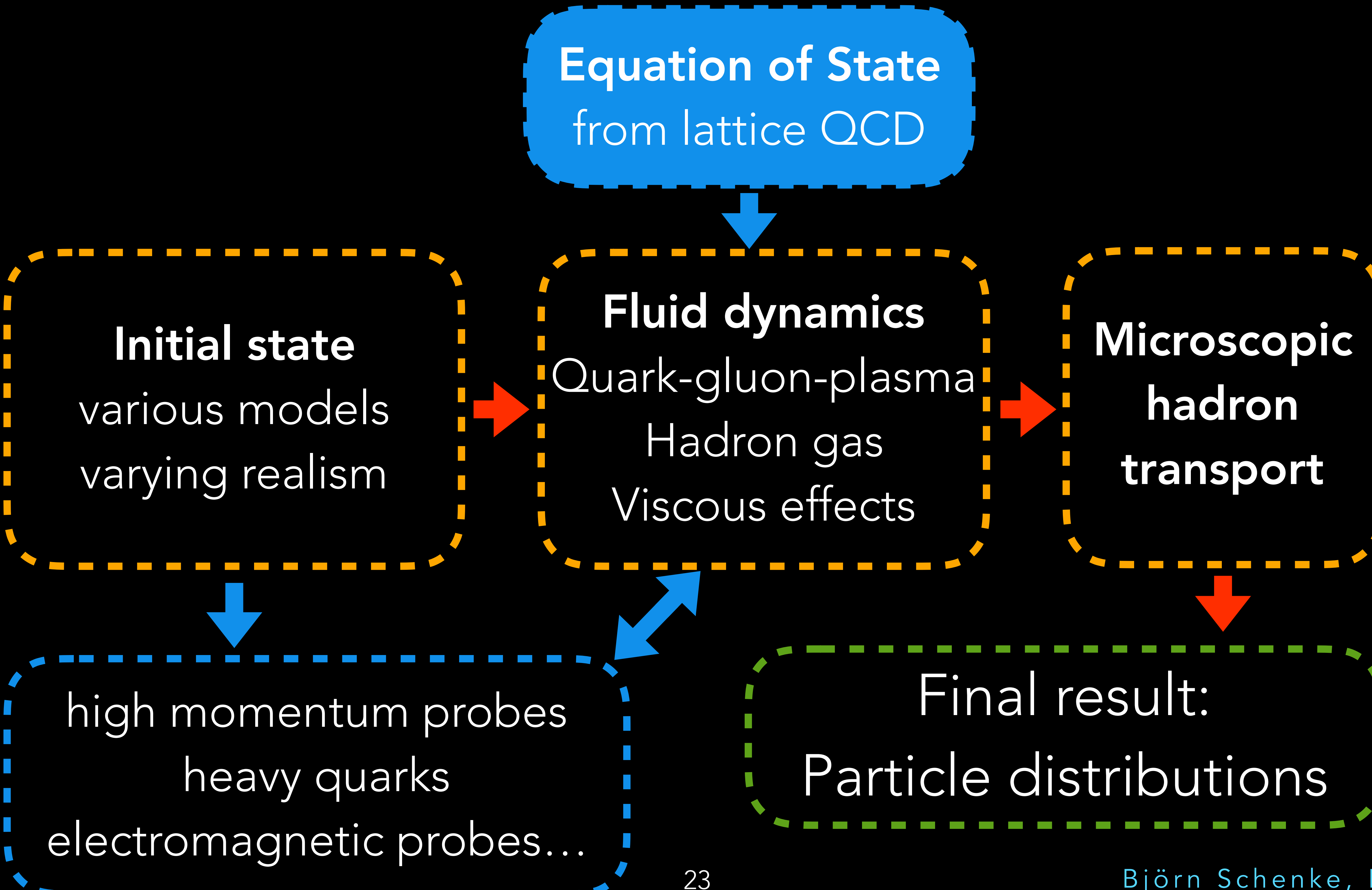
$$\varepsilon u^\nu = u_\mu T^{\mu\nu}_{\rm CYM}$$

- Then, using P=ε/3 (Yang-Mills system is conformal):

$$\pi^{\mu\nu} = T^{\mu\nu}_{\rm CYM} - \frac{4}{3} \varepsilon u^\mu u^\nu + \frac{\varepsilon}{3} g^{\mu\nu}$$

- Finally one can define bulk stress as $\Pi = \frac{\varepsilon}{3} - P$ using P from the EoS used in hydrodynamics to match to all components of the Classical Yang-Mills (CYM) $T^{\mu\nu}$

INGREDIENTS TO DESCRIBE HEAVY ION COLLISIONS



EQUATION OF STATE

To close the equations we need P(ε)

We use an EoS constructed from lattice QCD data

S. Borsanyi, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg, K. K. Szabo, Phys. Lett. B730, 99–104 (2014)
S. Borsanyi, Z. Fodor, S. D. Katz, S. Krieg, C. Ratti, and K. Szabo, JHEP 01, 138 (2012)

$$\frac{P(T)}{T^4} = \frac{P(T_{\rm low})}{T_{\rm low}^4} + \int_{T_{\rm low}}^{T} \frac{dT'}{T'} \frac{\varepsilon - 3P}{T'^4}$$

measured trace anomaly

smoothly matched to a hadron resonance gas around $T_{\rm match} \approx 150$ MeV

The EoS is also constructed at non-zero μ_B which we do not need here

EQUATION OF STATE

Different lattice results and constructions

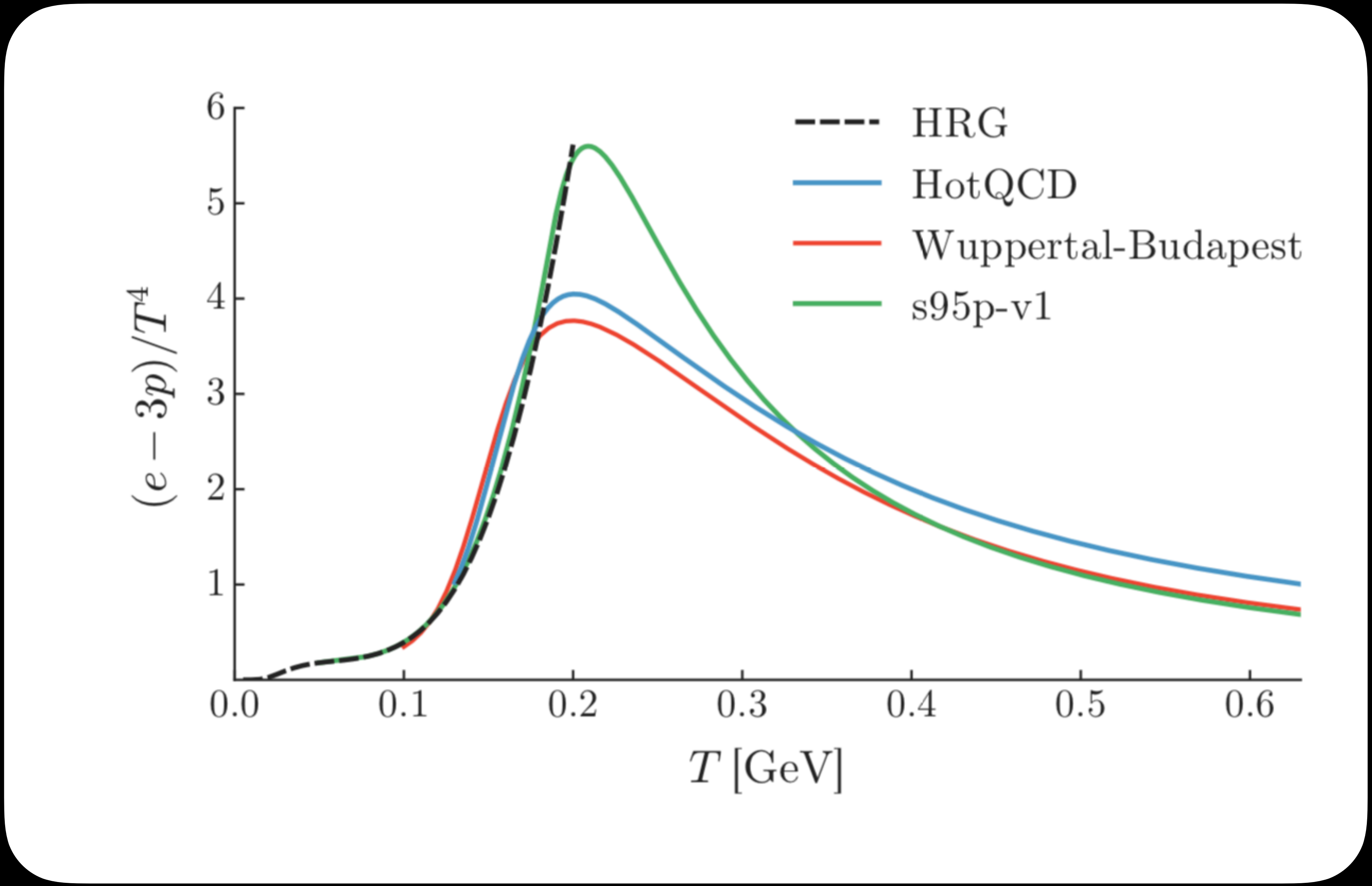
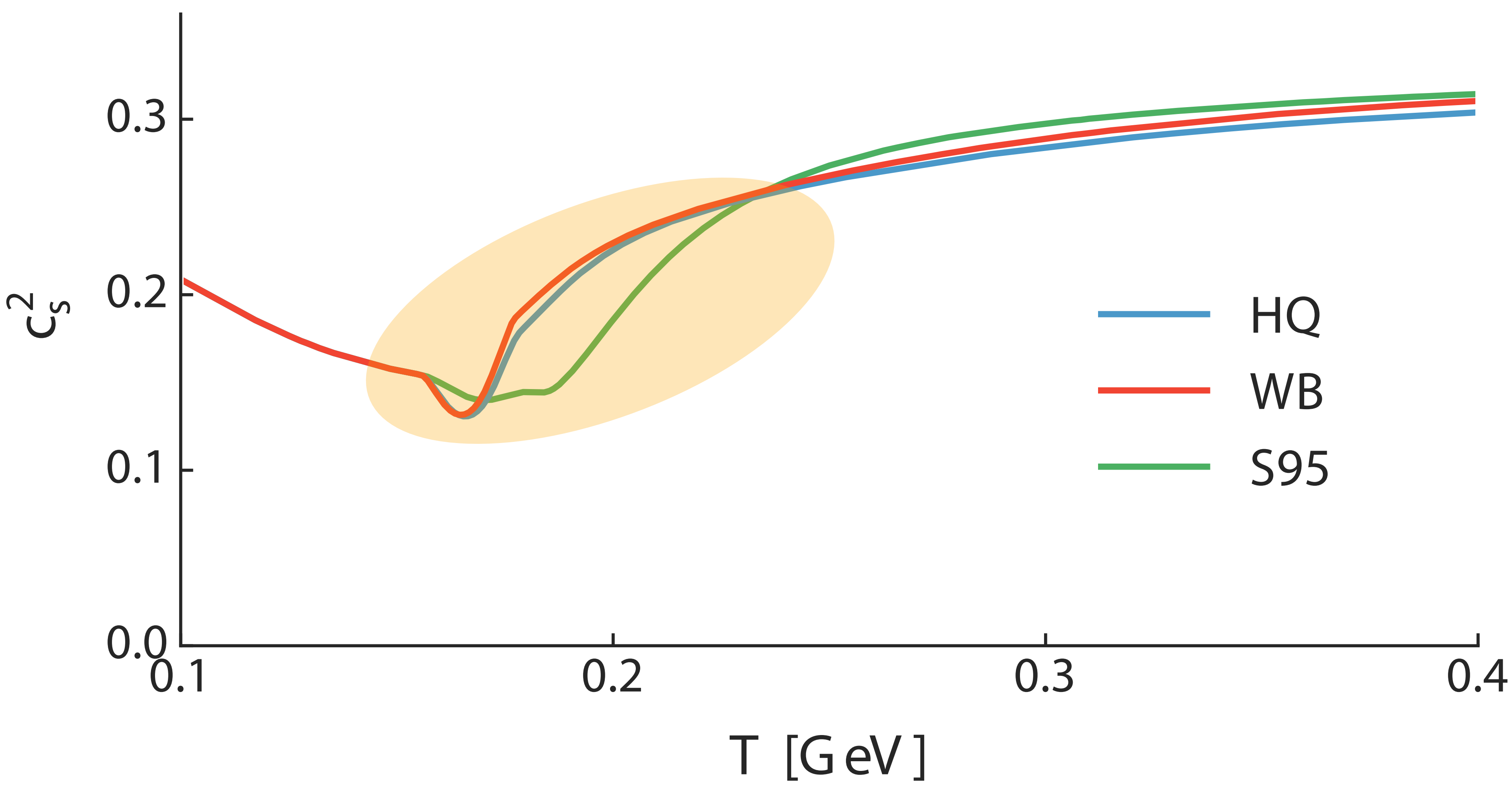


Figure from J.S. Moreland, R.A. Soltz, Phys.Rev. C93, 044913 (2016)
s95p-v1: P. Huovinen and P. Petreczky, Nucl. Phys. A837, 26–53 (2010)
HotQCD: A. Bazavov et al. (HotQCD), Phys. Rev. D90, 094503 (2014)
WB: S. Borsanyi, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg, K. K. Szabo, Phys. Lett. B730, 99–104 (2014)

EQUATION OF STATE

Smaller speed of sound in s95p leads to weaker flow



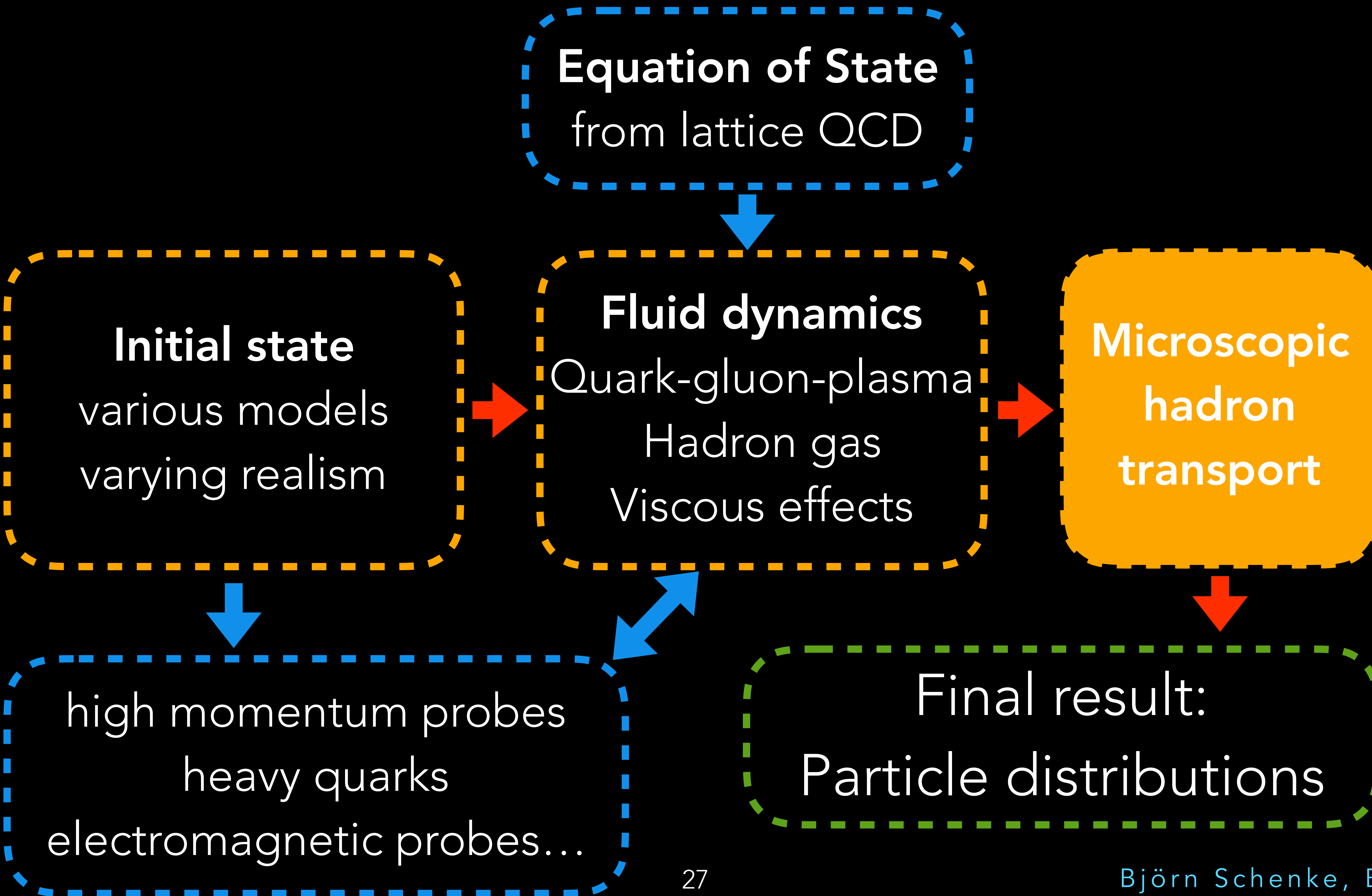
affects extraction of viscosities:

v_n change by 10%

η/s needs to change by ~50%

Figure from J.S. Moreland, R.A. Soltz, Phys.Rev. C93, 044913 (2016)
s95: P. Huovinen and P. Petreczky, Nucl. Phys. A837, 26–53 (2010)
HQ: A. Bazavov et al. (HotQCD), Phys. Rev. D90, 094503 (2014)
WB: S. Borsanyi, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg, K. K. Szabo, Phys. Lett. B730, 99–104 (2014)

INGREDIENTS TO DESCRIBE THE BULK OF HEAVY ION COLLISIONS



MICROSCOPIC HADRON CASCADE

Sample particles on the freeze-out surface (surface of constant (low) energy density) according to

$$E\frac{dN_i}{d^3p} = \frac{g_i}{(2\pi)^3}\Delta^3\Sigma_\mu p^\mu (f_i^{(0)} + \delta f_i)$$

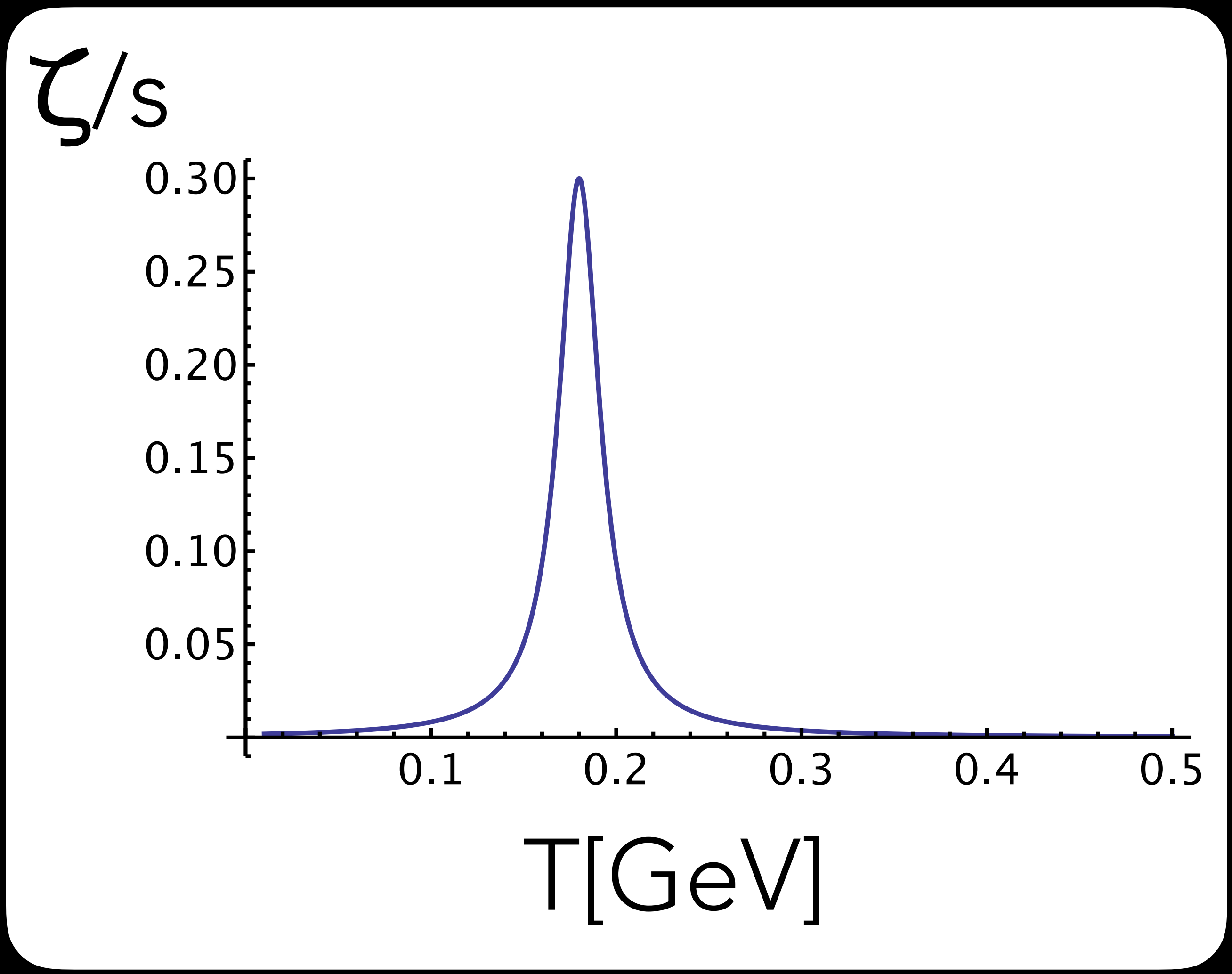
then feed particles into UrQMD

S. A. Bass et al., Prog. Part. Nucl. Phys. 41, 255–369 (1998)
M. Bleicher et al., J. Phys. G25, 1859–1896 (1999)

which performs resonance decays and scattering according to hadronic cross sections

VISCOSITIES

We use a constant **η**/s =0.13
(up from 0.095 used earlier because of change in EoS)

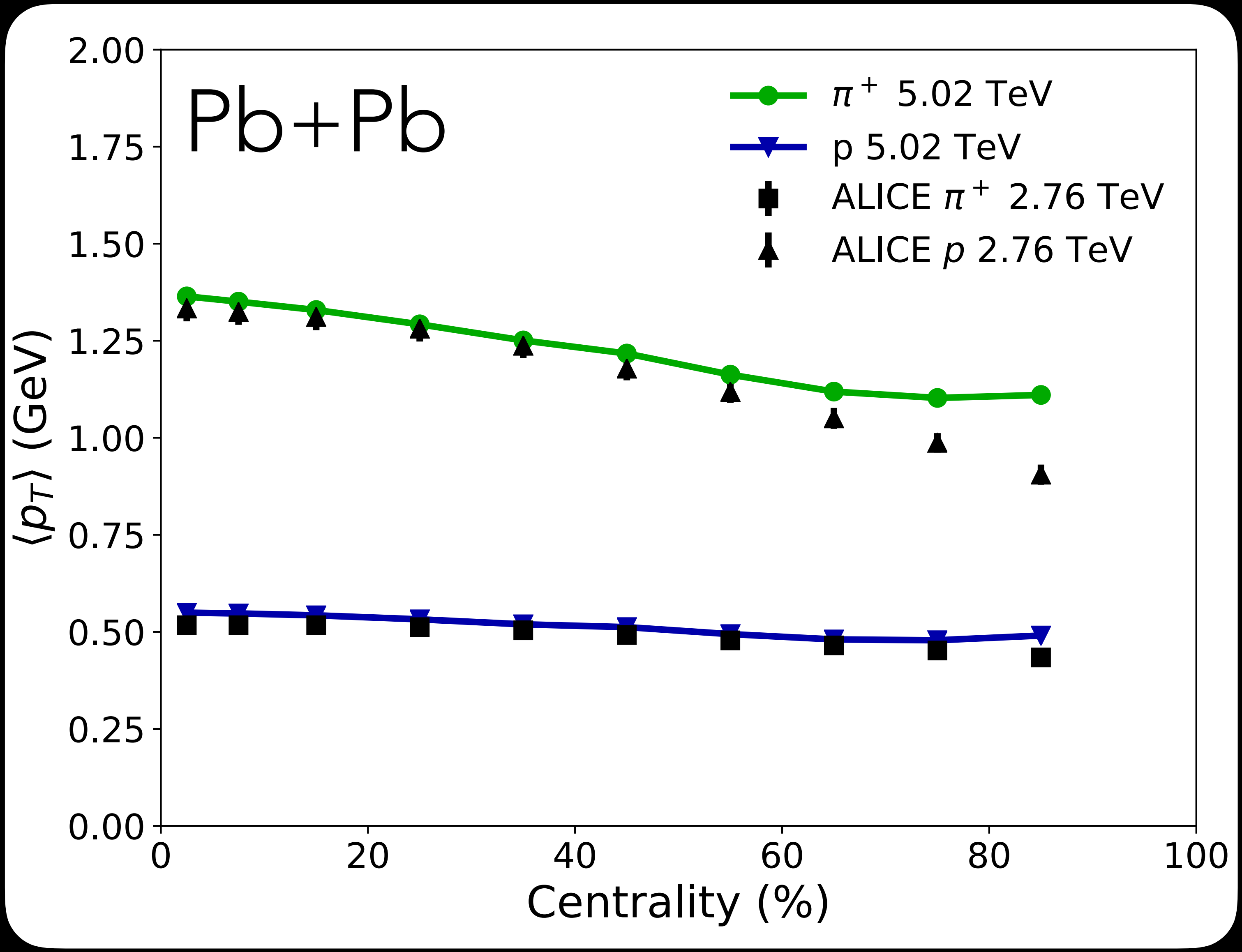


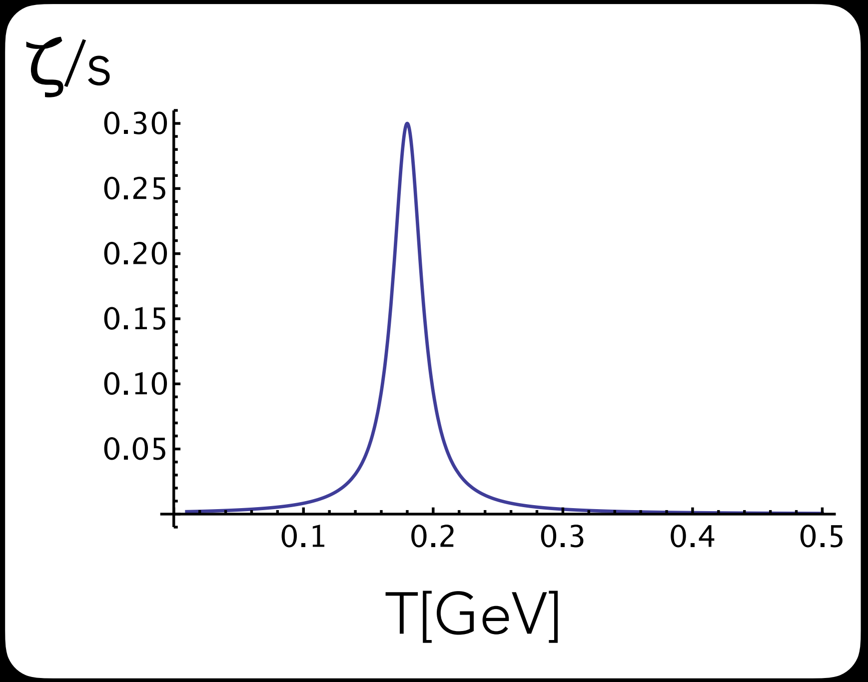
Bulk viscosity peaks at 180 MeV

Width and position are free parameters

Mean transverse momentum

B. Schenke, C. Shen, P. Tribedy, in preparation



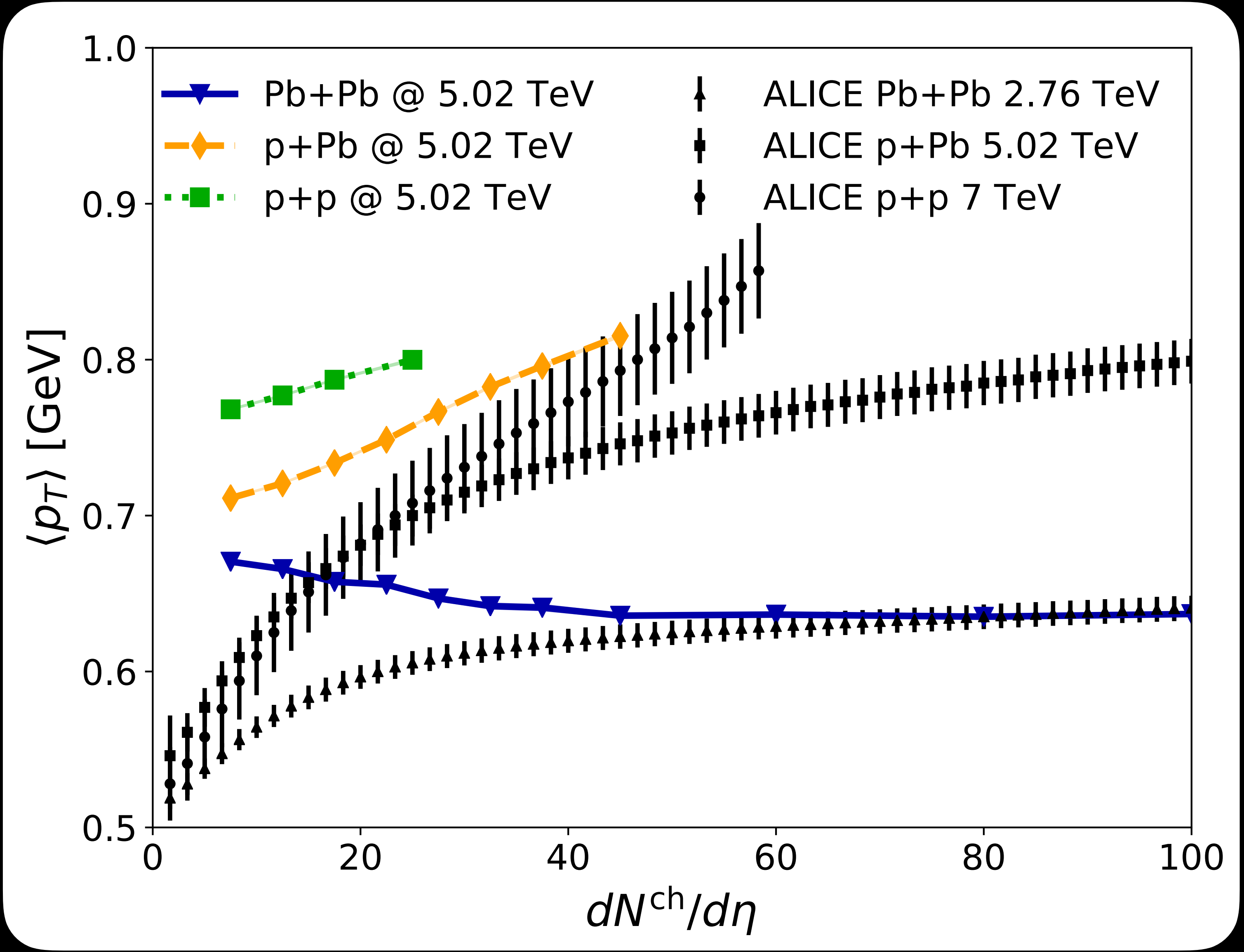


Bulk viscosity smaller at low T (low multiplicity)

UrQMD may have too little bulk viscosity

Mean transverse momentum

B. Schenke, C. Shen, P. Tribedy, in preparation



p+Pb $\langle p_T \rangle$ ~8% > than data

p+p $\langle p_T \rangle$ ~12% >

Pb+Pb $\langle p_T \rangle$ over-estimated at low N_{ch}

Bulk viscosity smaller at low T (low multiplicity)

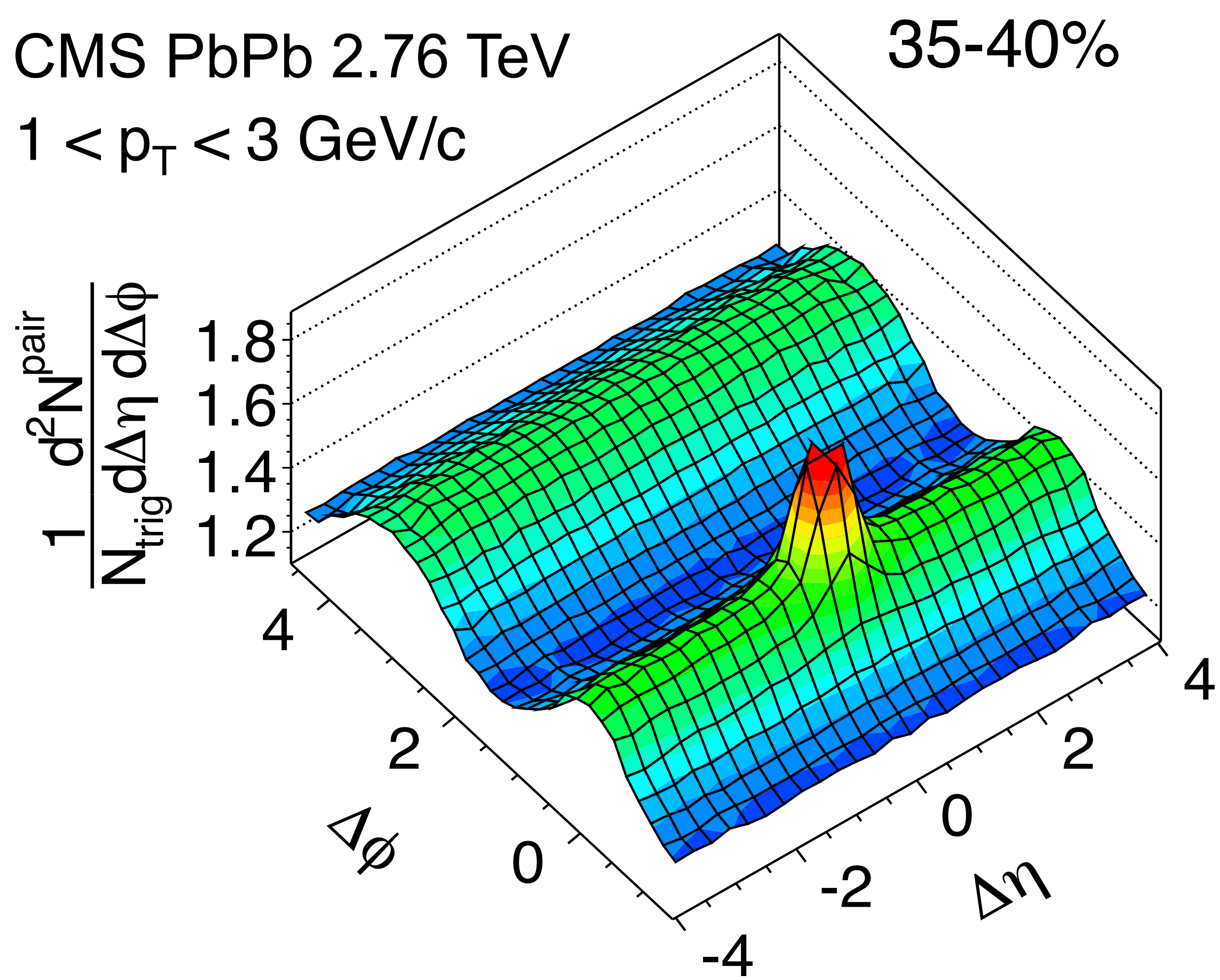
UrQMD may have too little bulk viscosity

MULTI-PARTICLE CORRELATIONS

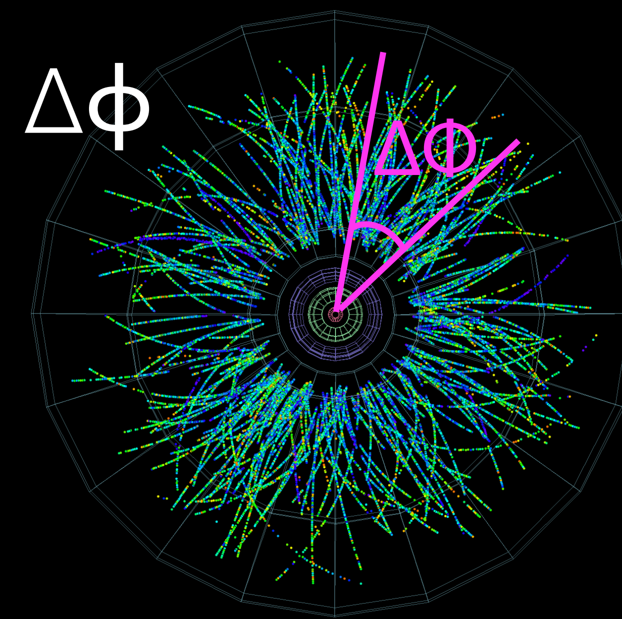
2-particle correlation as a function of Δη and Δϕ

Δη: DIFFERENCE IN PSEUDO-RAPIDITY

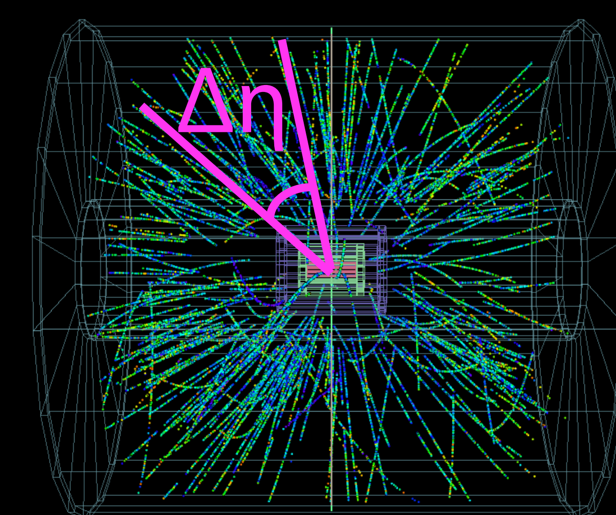
ΔΦ: DIFFERENCE IN AZIMUTHAL ANGLE



CMS COLL., EUR. PHYS. J. C72 (2012)



ΔΦ: DIFFERENCE IN AZIMUTHAL ANGLE



Δη: DIFFERENCE IN PSEUDO-RAPIDITY

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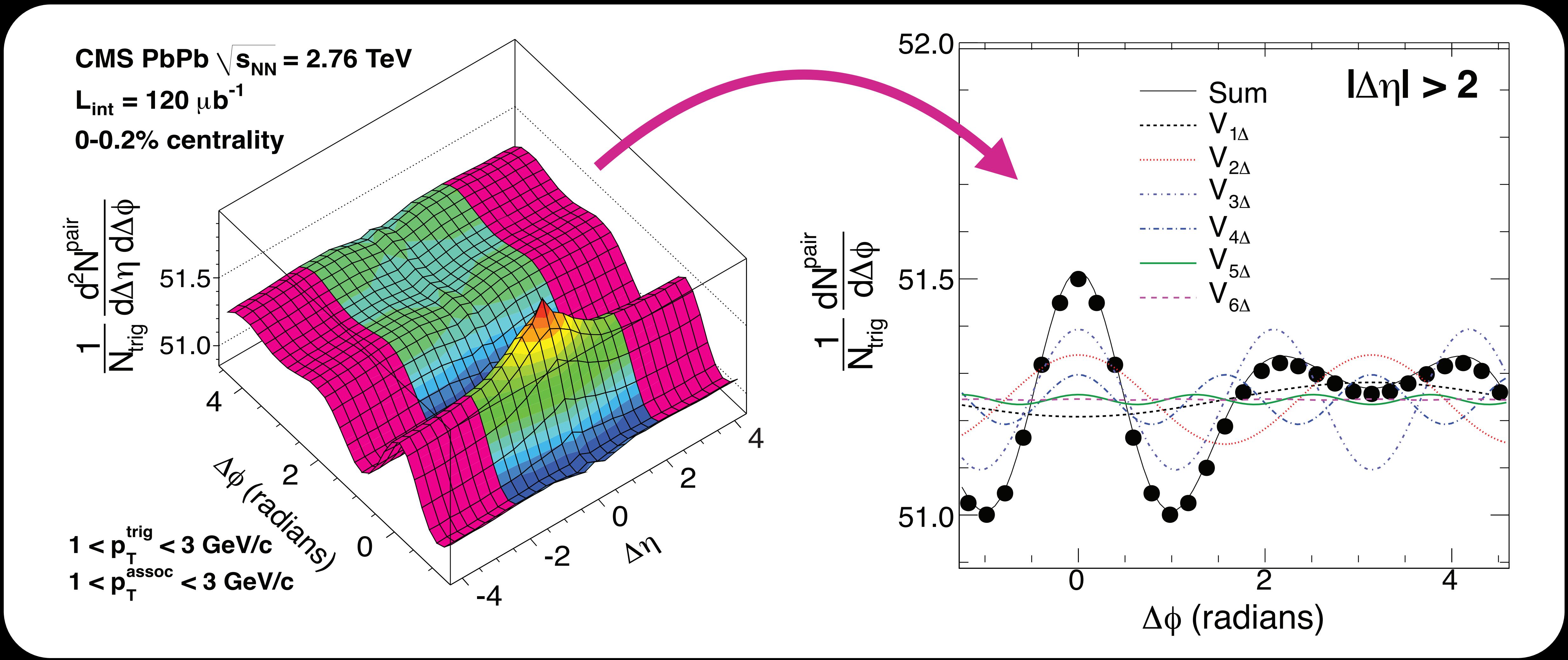
FOURIER EXPANSION

Azimuthal structure quantified using Fourier expansion

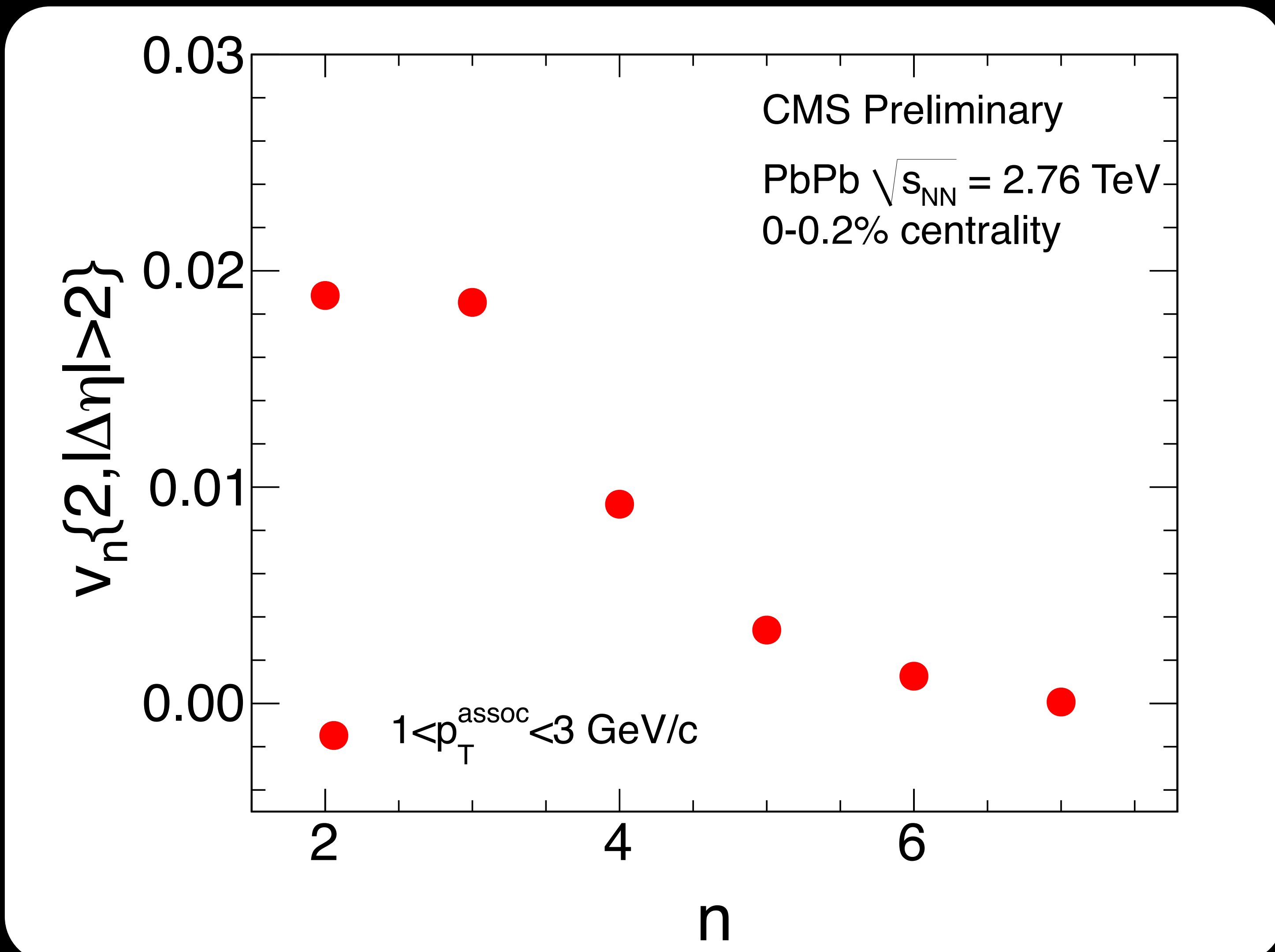


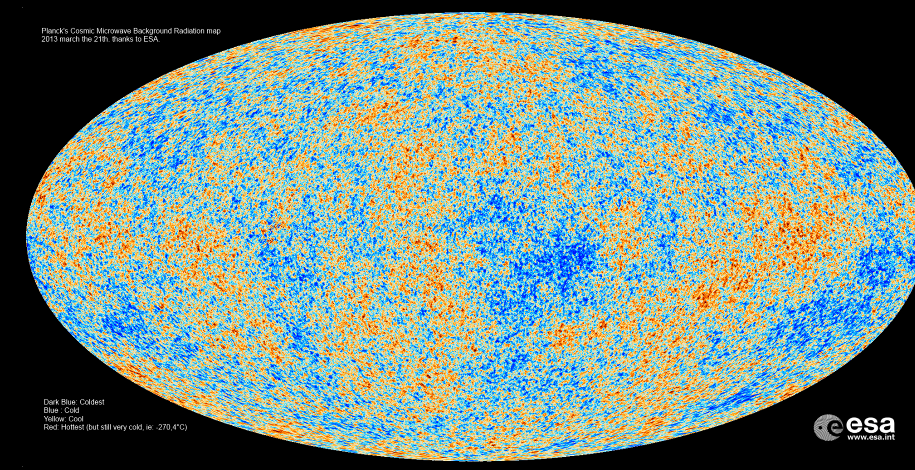
$$\frac{1}{N_{\text{trig}}}\frac{dN^{\text{pair}}}{d\Delta\phi} \sim 1 + 2\sum_{n=1}^{n=\infty} V_{n\Delta}(p_T^{\text{trig}}, p_T^{\text{assoc}})\cos(n\Delta\phi) \qquad v_n = \sqrt{V_{n\Delta}}$$

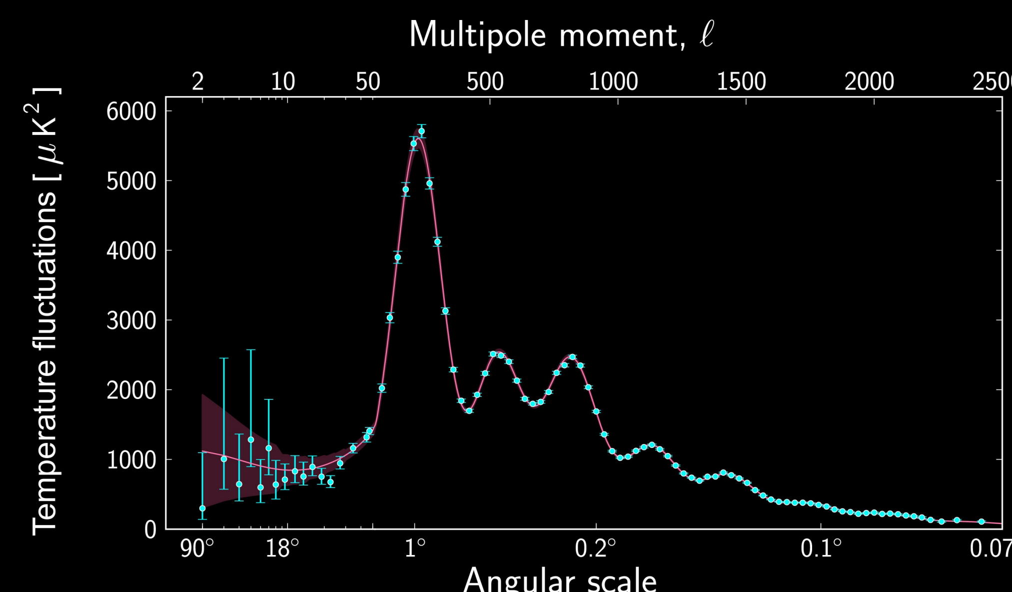
CMS COLLABORATION, JHEP 02 (2014) 088

Ridge structure and anisotropic flow

v_n as a function of n in central Pb+Pb collisions







PLANCK Coll. A&A 571, A16 (2014)

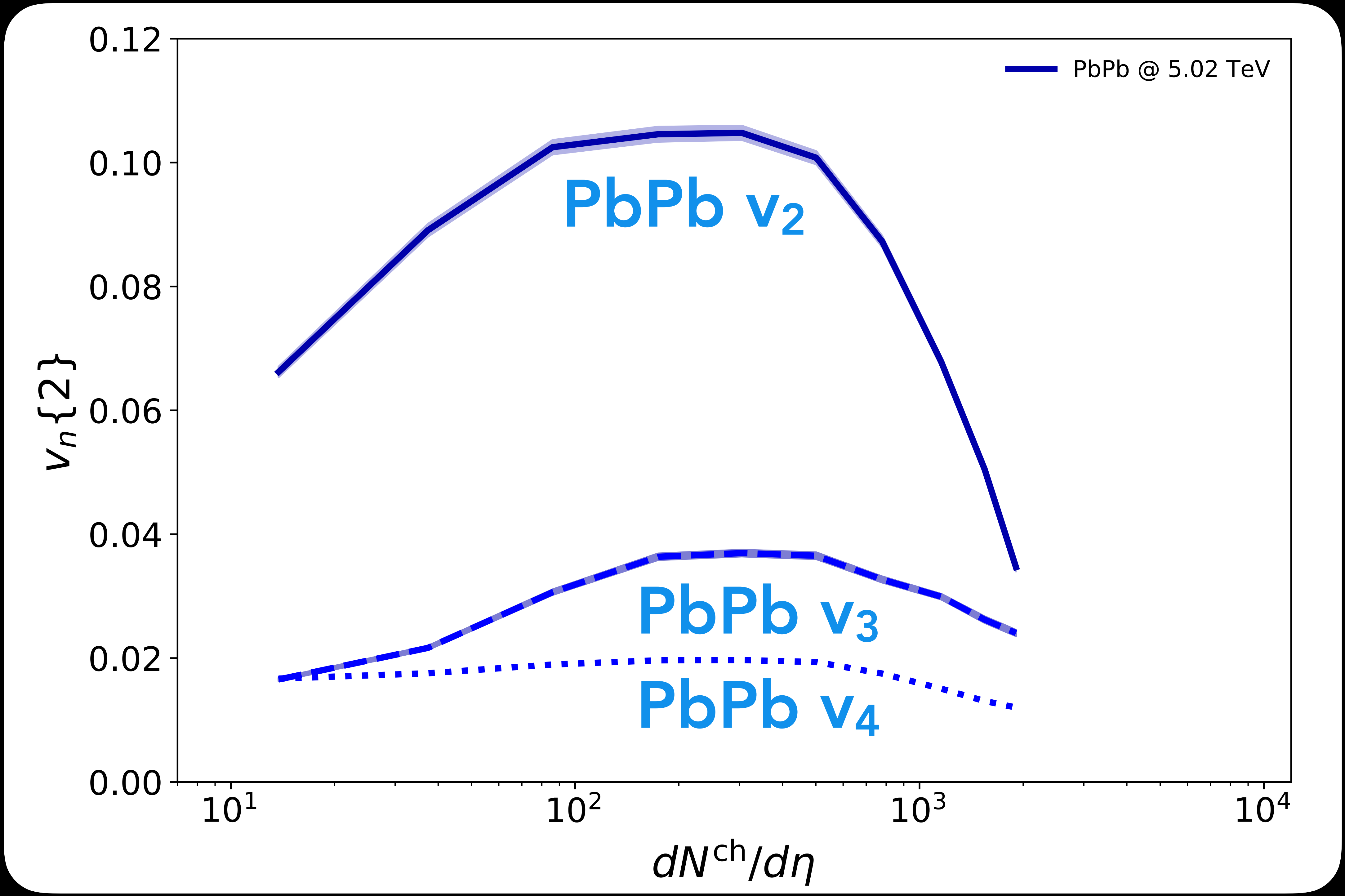
$$\frac{1}{N_{\rm trig}}\frac{dN^{\rm pair}}{d\Delta\phi} \sim 1 + 2\sum_{n=1}^{n=\infty} V_{n\Delta}(p_T^{\rm trig}, p_T^{\rm assoc})\cos(n\Delta\phi)$$

$$v_n \sim \sqrt{V_{n\Delta}}$$

CMS Collaboration, JHEP 02 (2014) 088

Anisotropy vs. multiplicity

B. Schenke, C. Shen, P. Tribedy, in preparation



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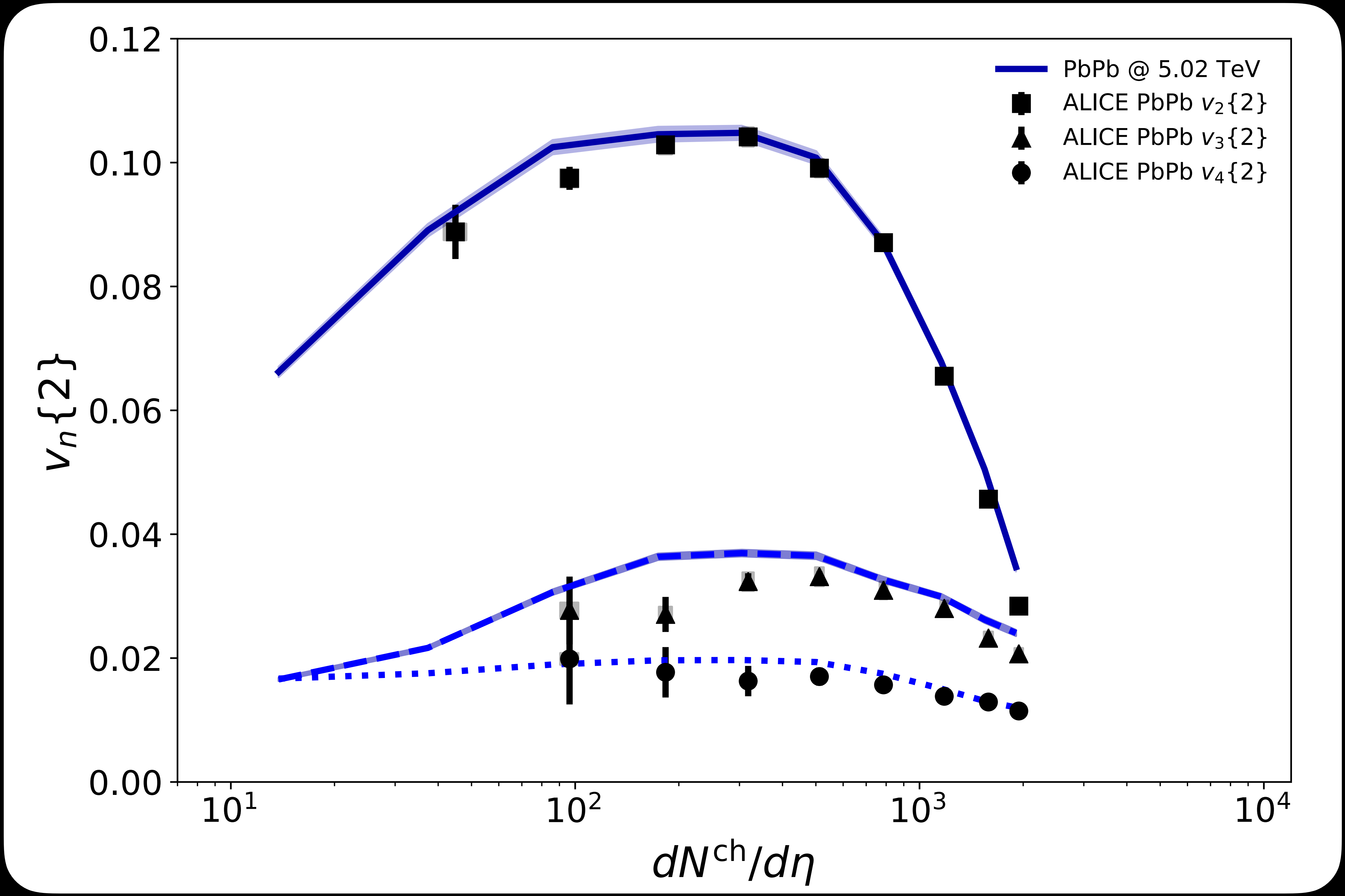

Experimental data: J. Adam et al. (ALICE), Phys. Rev. Lett. 116, 132302 (2016)
B. B. Abelev et al. (ALICE), Phys. Rev. C90, 054901 (2014)

Anisotropy vs. multiplicity

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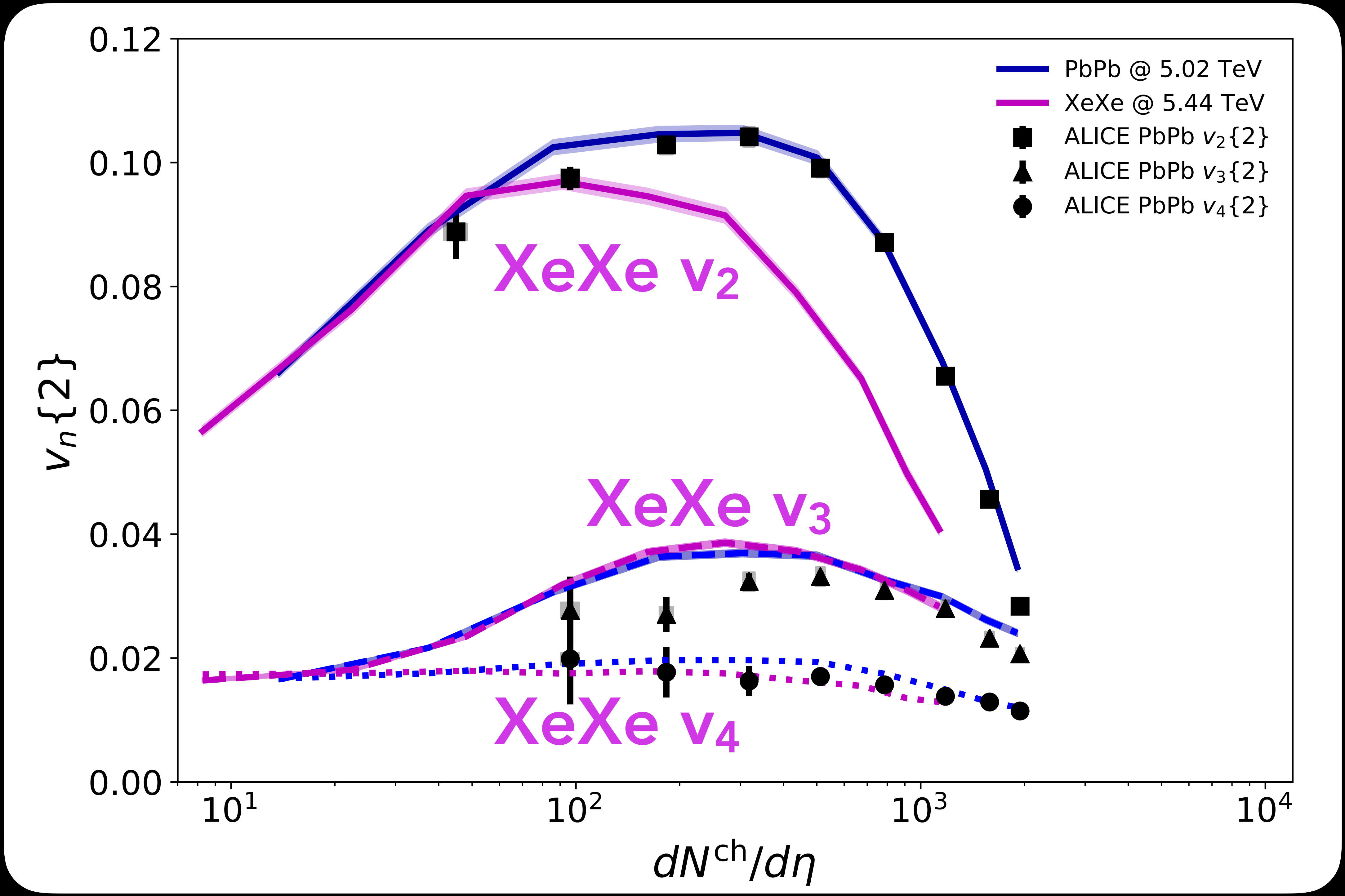

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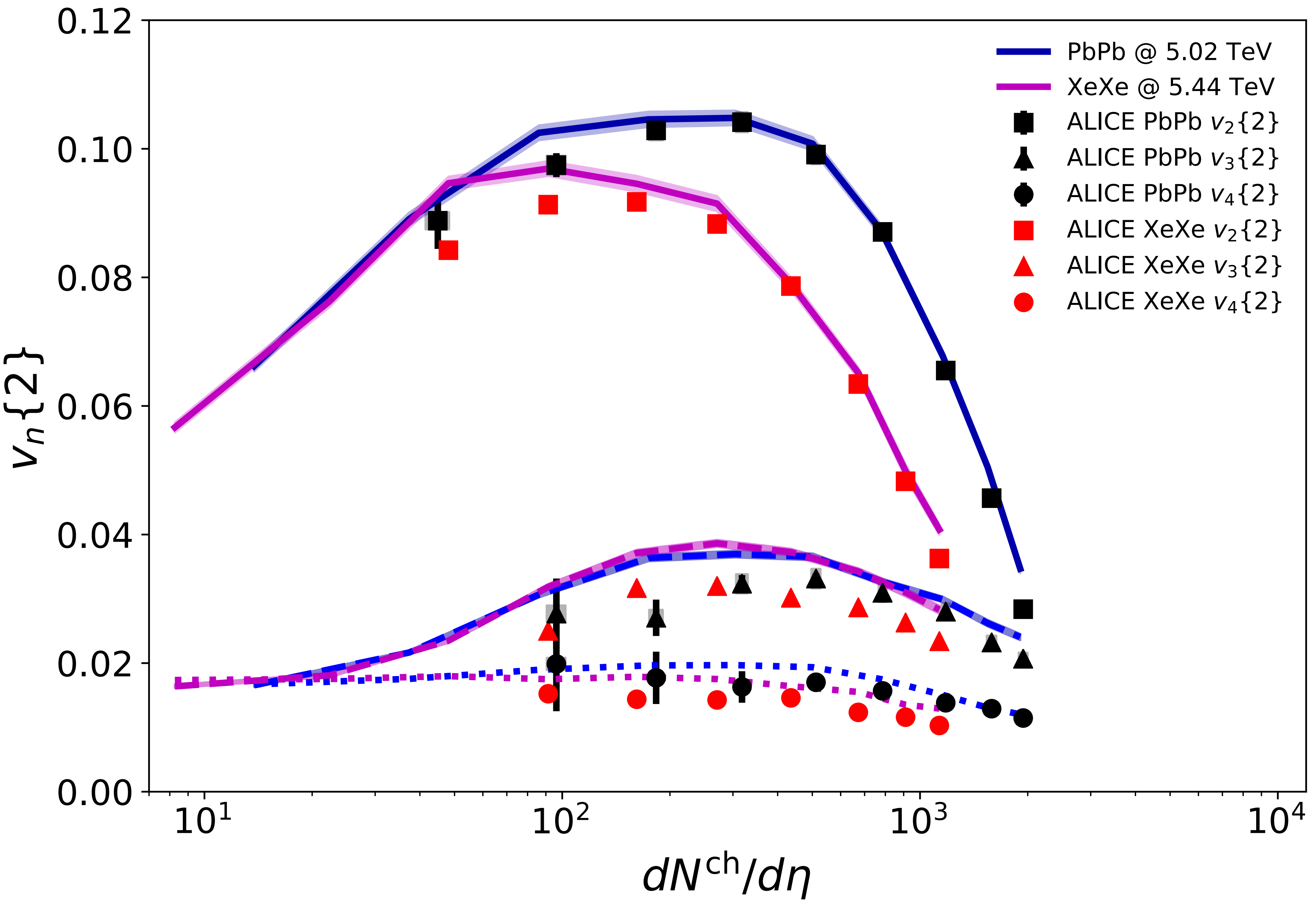
B. B. Abelev et al. (ALICE), Phys. Rev. C90, 054901 (2014)

Anisotropy vs. multiplicity

B. Schenke, C. Shen, P. Tribedy, in preparation

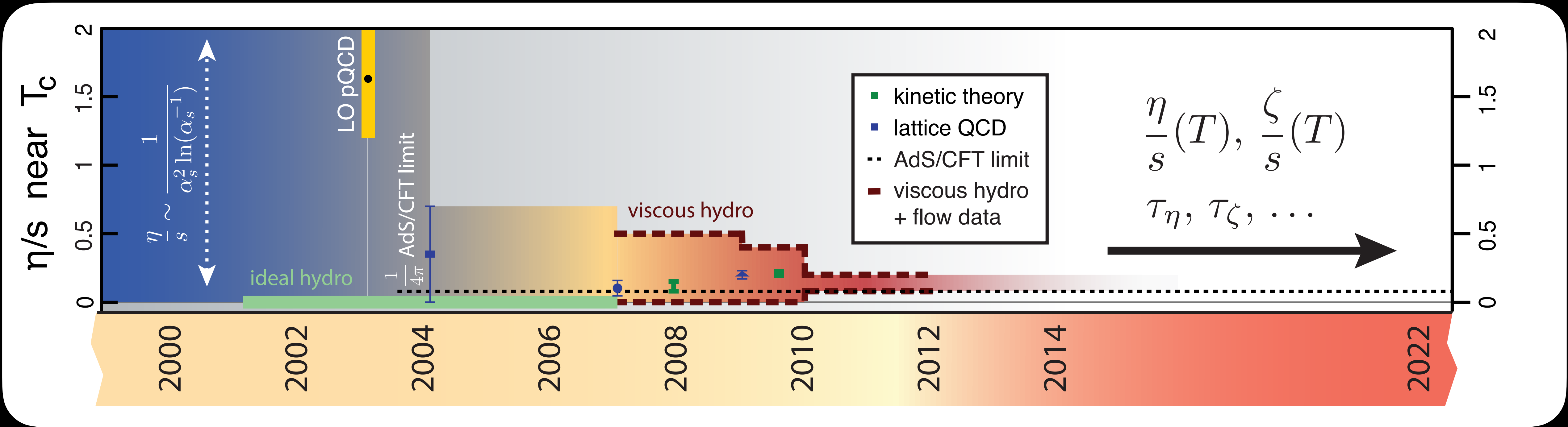


Experimental data: J. Adam et al. (ALICE), Phys. Rev. Lett. 116, 132302 (2016)

B. B. Abelev et al. (ALICE), Phys. Rev. C90, 054901 (2014), ALICE Collaboration, arXiv:1805.01832

Much progress has been made

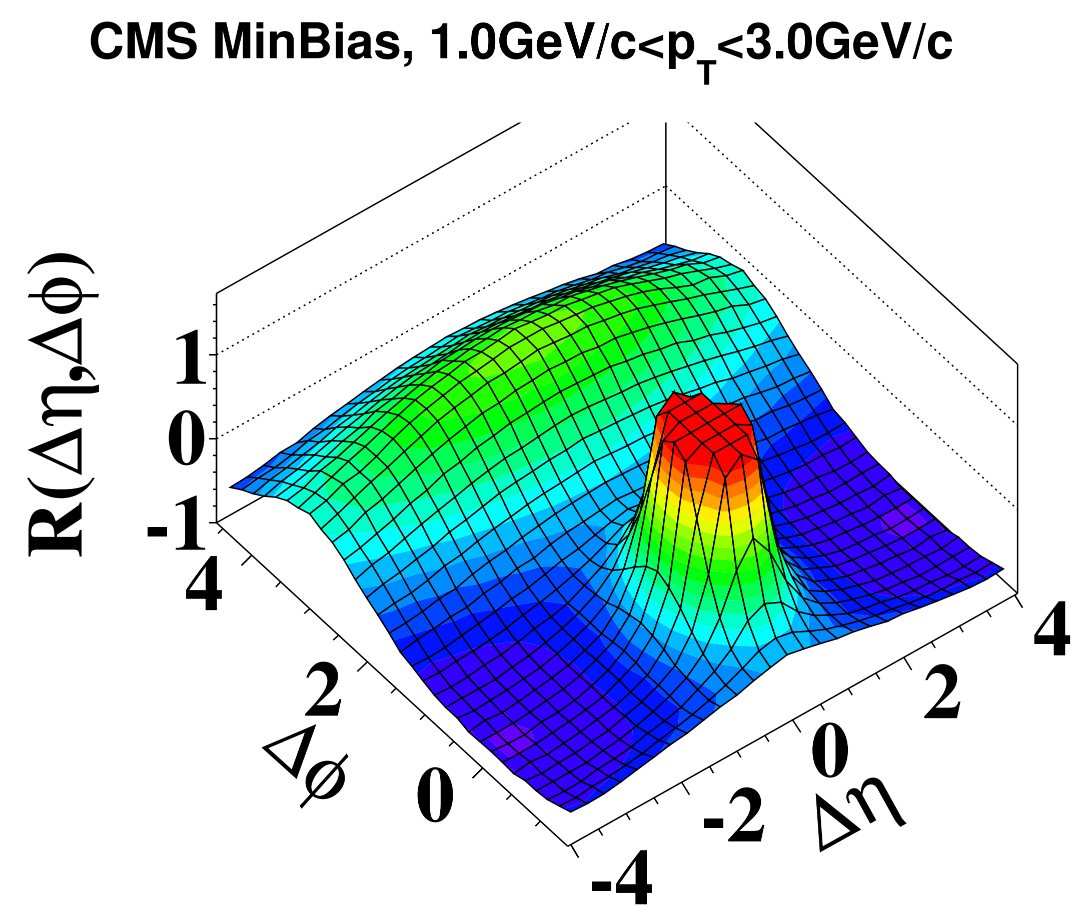
Broad theoretical efforts and experimental advances lead to increasingly precise determination of η/s



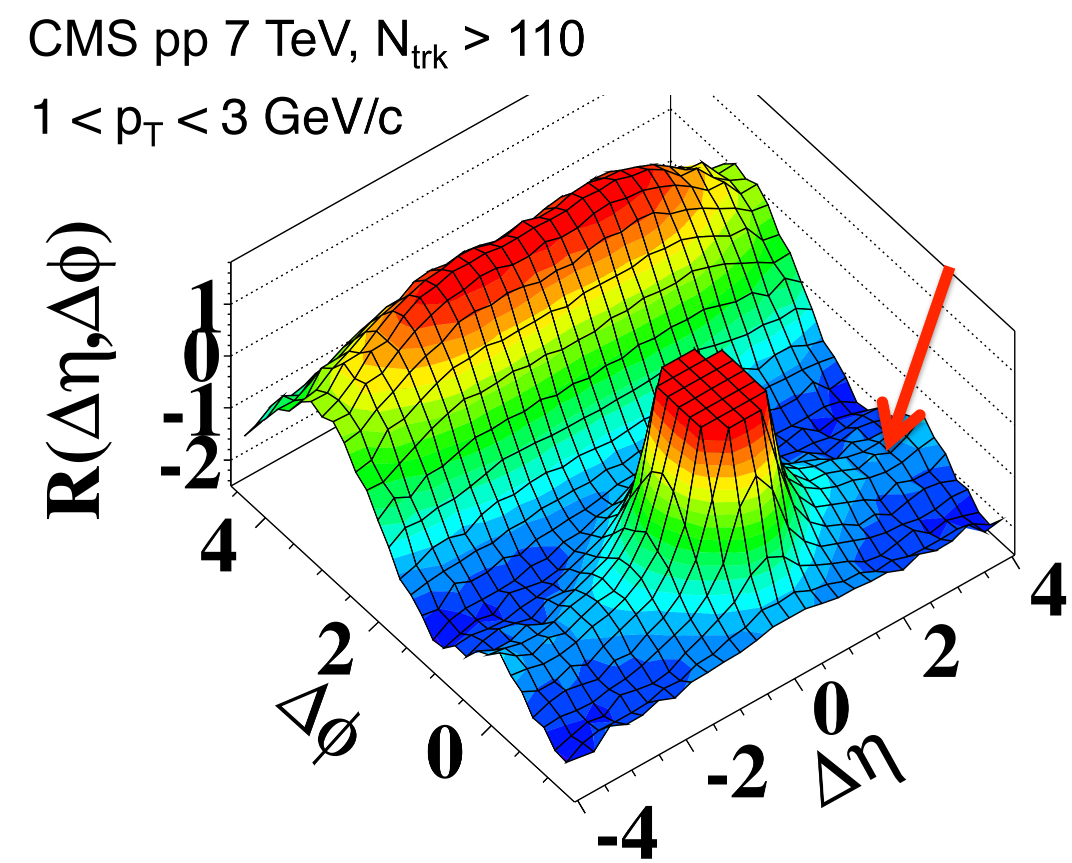
LO pQCD:	P. Arnold, G. D. Moore, L. G. Yaffe, JHEP 0305 (2003) 051
AdS/CFT:	P. Kovtun, D. T. Son, A. O. Starinets, Phys.Rev.Lett. 94 (2005) 111601
Lattice QCD:	A. Nakamura, S. Sakai, Phys.Rev.Lett. 94 (2005) 072305
	H. B. Meyer, Phys.Rev. D76 (2007) 101701; Nucl.Phys. A830 (2009) 641C-648C
Ideal hydro:	P. F. Kolb, J. Sollfrank, U. W. Heinz, Phys.Rev. C62 (2000) 054909
	P. F. Kolb, P. Huovinen, U. W. Heinz, H. Heiselberg, Phys.Lett. B500 (2001) 232-240
pQCD/kin. theory:	Z. Xu, C. Greiner, H. Stöcker, Phys.Rev.Lett. 101 (2008) 082302
	J.-W. Chen, H. Dong, K. Ohnishi, Q. Wang, Phys.Lett. B685 (2010) 277-282
Viscous hydro:	P. Romatschke, U. Romatschke, Phys.Rev.Lett. 99 (2007) 172301
	M. Luzum, P. Romatschke, Phys.Rev. C78 (2008) 034915
	H. Song, U. W. Heinz, J.Phys. G36 (2009) 064033
	H. Song, S. A. Bass, U. Heinz, T. Hirano, C. Shen, Phys.Rev.Lett. 106 (2011) 192301

RIDGE IN SMALL COLLISION SYSTEMS

minimum bias p+p



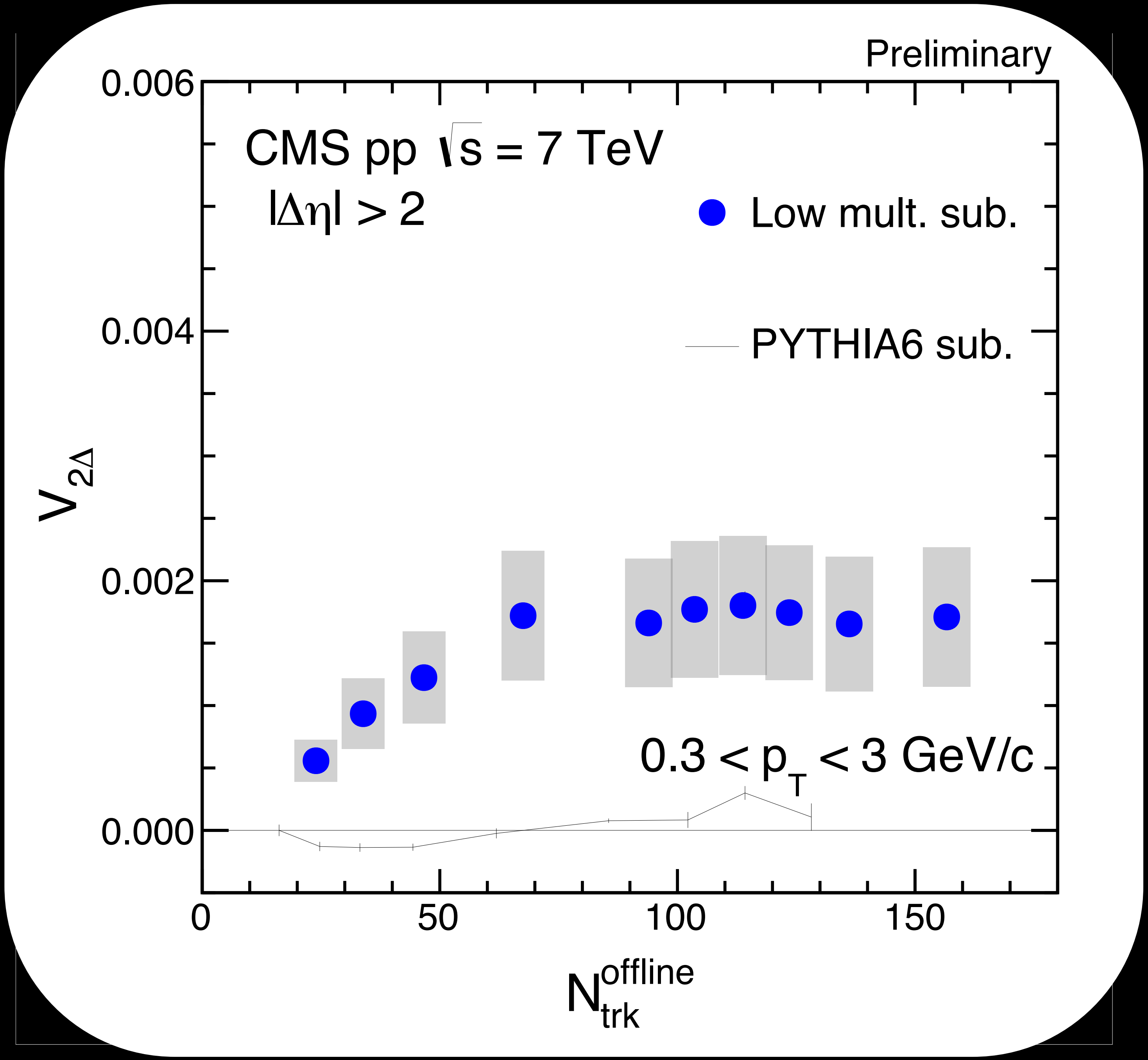
high multiplicity p+p



CMS COLLABORATION, JHEP09 (2010) 091

$v_{2\Delta}$ IN p+p COLLISIONS

Result after correcting for back-to-back jet correlations estimated from low multiplicity events



No ridge in PYTHIA

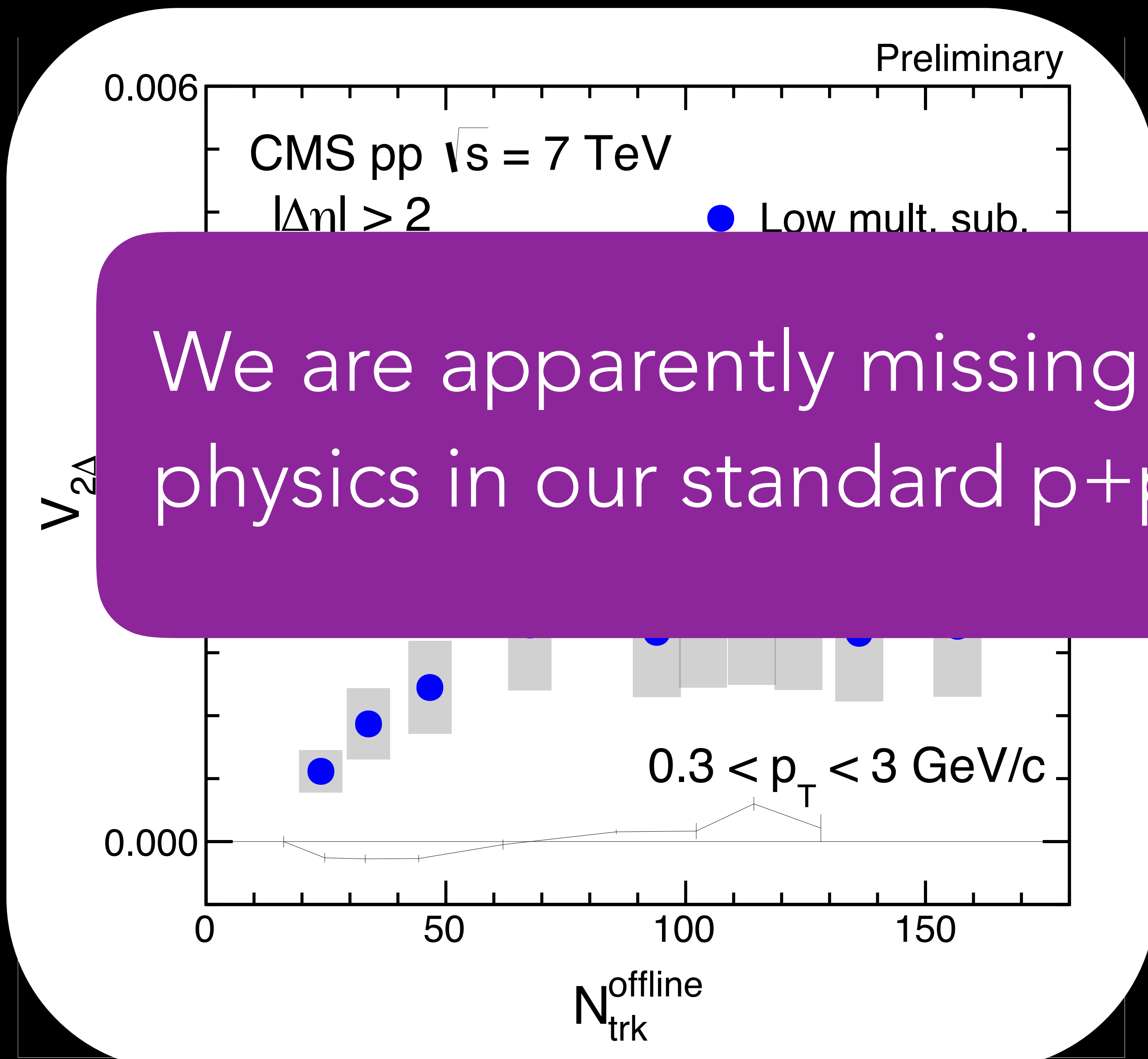
CMS PAS HIN-15-009

But progress including *final* state effects via 'string shoving'

In Pythia8 v.8.235; Bierlich, Gustafson, Lönnblad: PLB779 (2018) 58-63
Bierlich, arXiv:1606.09456 [hep-ph]
Bierlich, Gustafson, Lönnblad, Tarasov JHEP 1503 (2015) 148

$v_{2\Delta}$ IN p+p COLLISIONS

Result after correcting for back-to-back jet correlations estimated from low multiplicity events



No ridge in PYTHIA

ding

ia

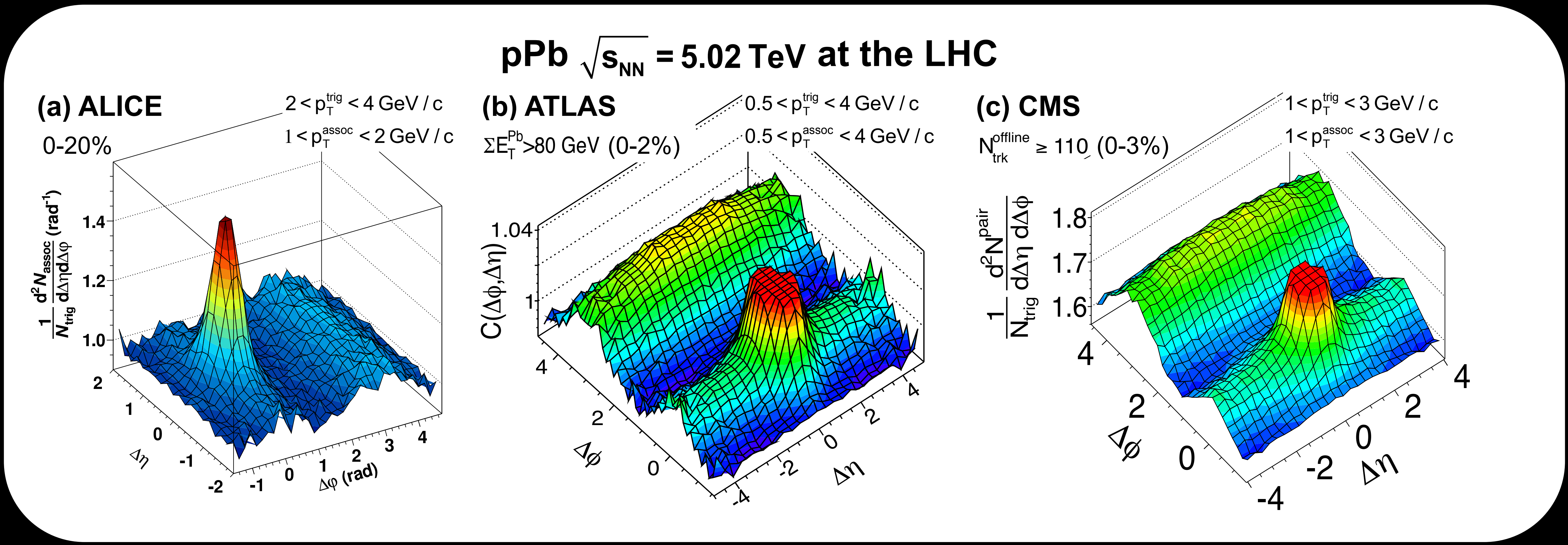
'string shoving'

We are apparently missing important physics in our standard p+p event generators!

In Pythia8 v.8.235; Bierlich, Gustafson, Lönnblad: PLB779 (2018) 58-63
Bierlich, arXiv:1606.09456 [hep-ph]
Bierlich, Gustafson, Lönnblad, Tarasov JHEP 1503 (2015) 148

RIDGE IN p+Pb COLLISIONS

high multiplicity p+Pb

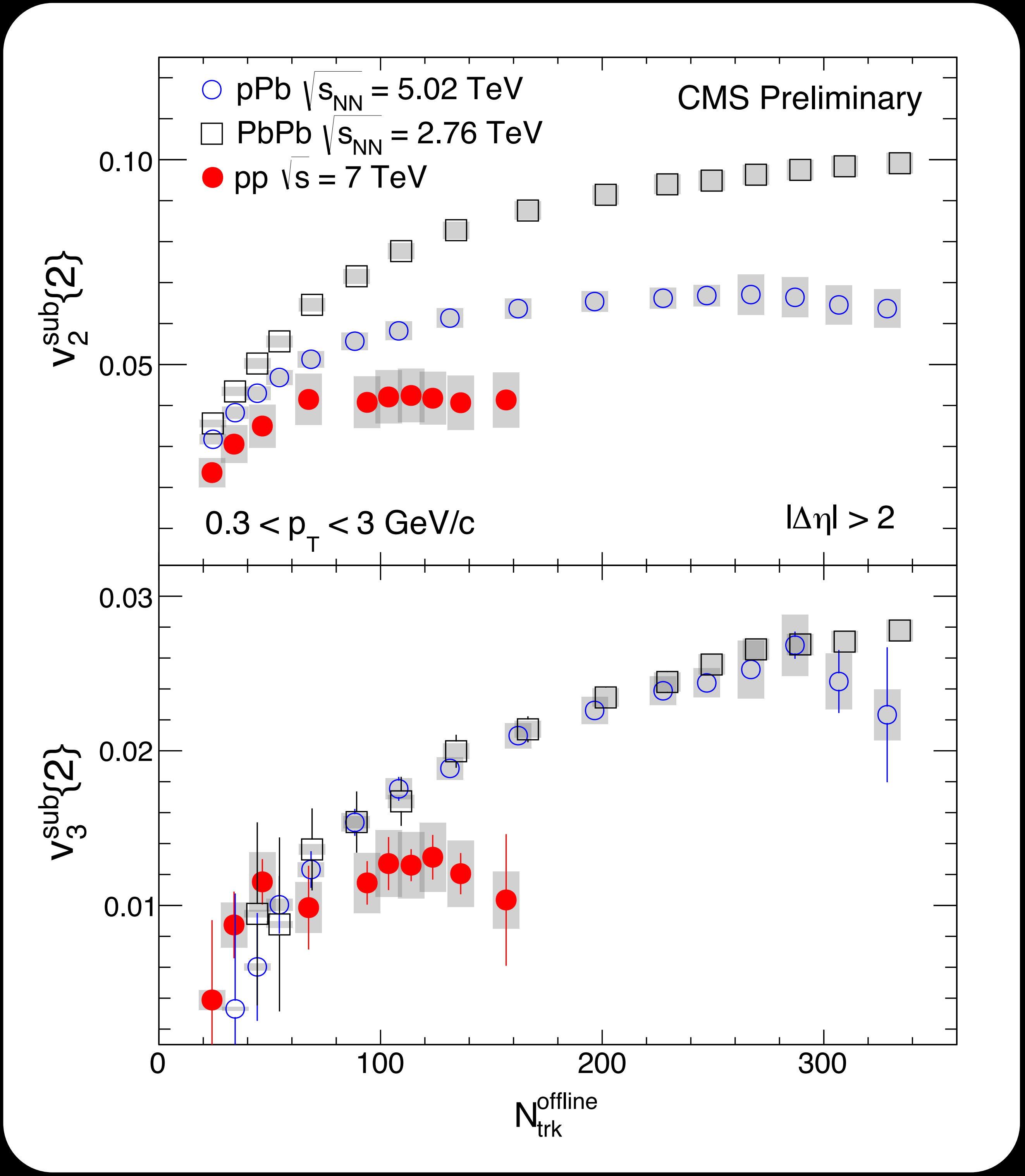


ALICE COLLABORATION, PHYS. LETT. B 719 (2013) 29

ATLAS COLLABORATION, PHYS. REV. LETT. 110 (2013) 182302

CMS COLLABORATION, PHYS. LETT. B 718 (2013) 795

v_2 IN p+p, p+Pb, Pb+Pb COLLISIONS



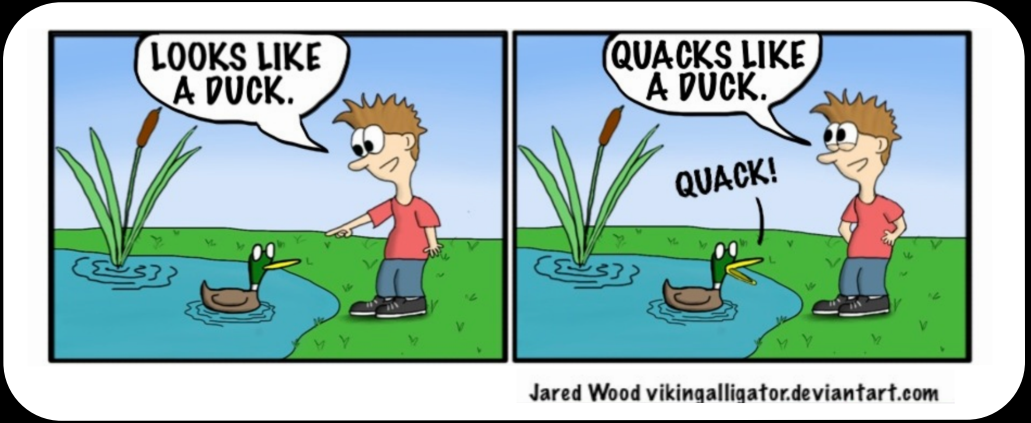
CMS PAS HIN-15-009

SEE ALSO:

ALICE COLLABORATION
PHYS. LETT. B719 (2013) 29-41;
PHYS. REV. C 90, 054901

ATLAS COLLABORATION
PHYS. REV. LETT. 110, 182302 (2013); PHYS. REV. C 90.044906 (2014)

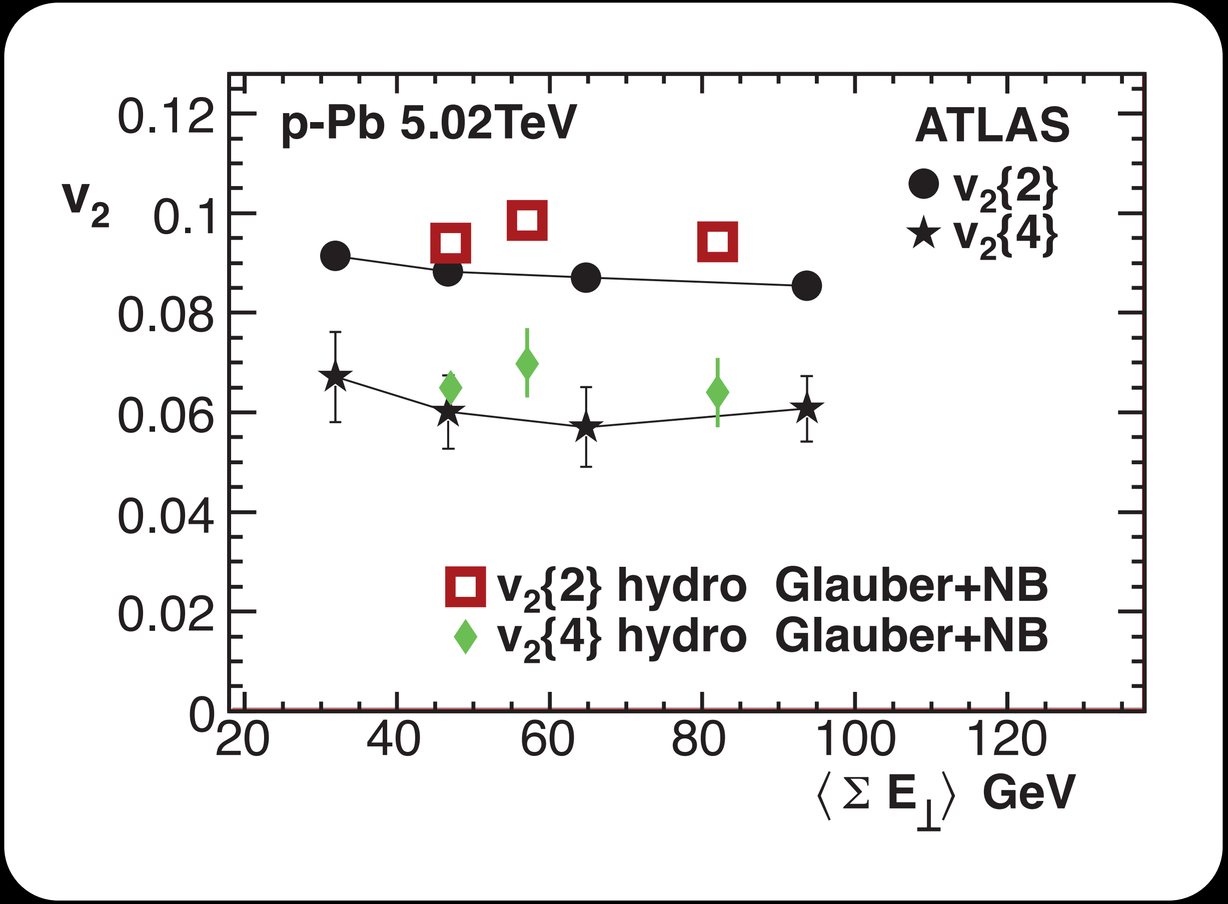
CMS COLLABORATION
PHYS.REV.LETT. 115, 012301 (2015)



HYDRO IN SMALL SYSTEMS

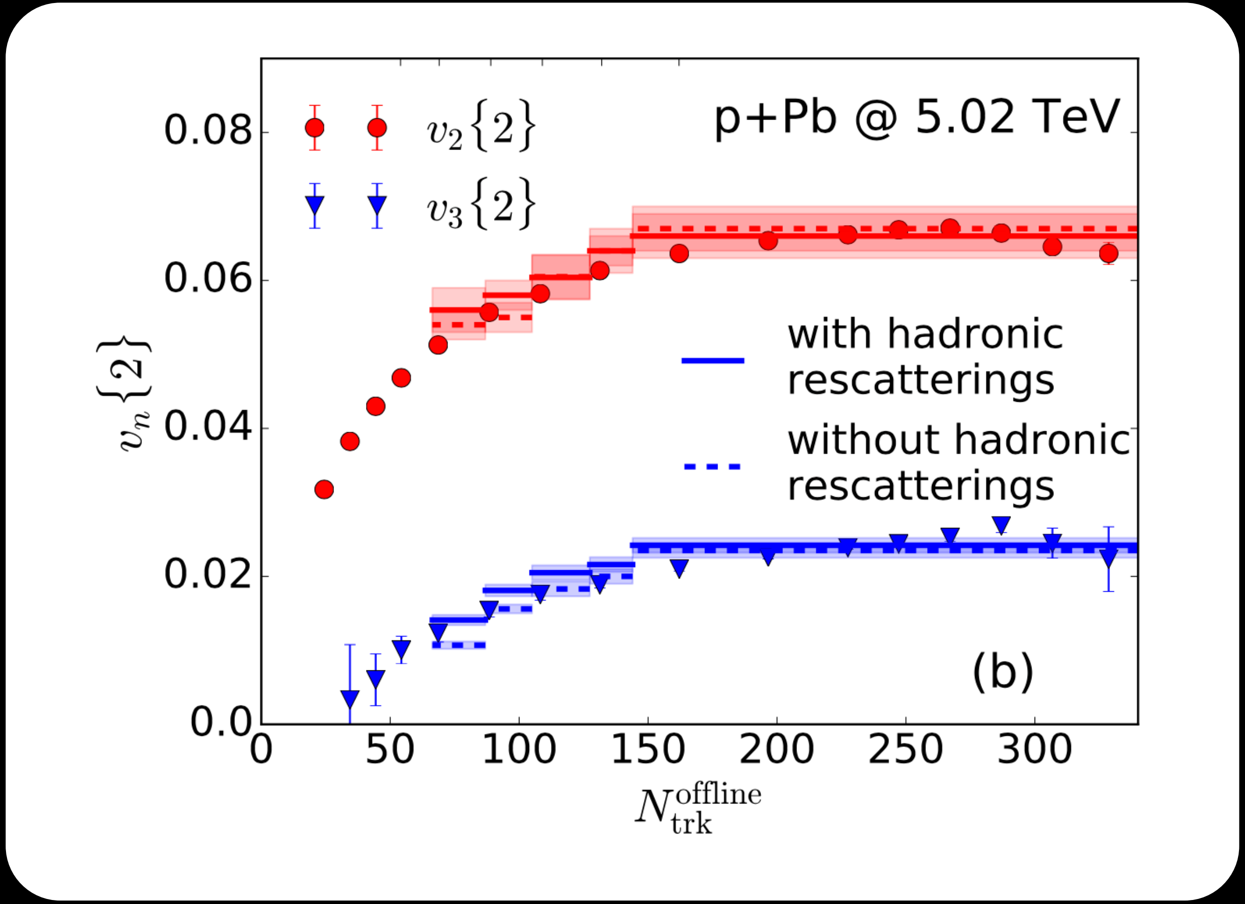
MC-Glauber initial state + viscous hydrodynamics works

ATLAS Coll. PLB725 (2013) 60-78



Bozek, Broniowski, PRC88 (2013) 014903

CMS Coll. PLB724, 213–240 (2013)



Shen, Paquet, Denicol, Jeon, Gale, PRC95 (2017) 014906

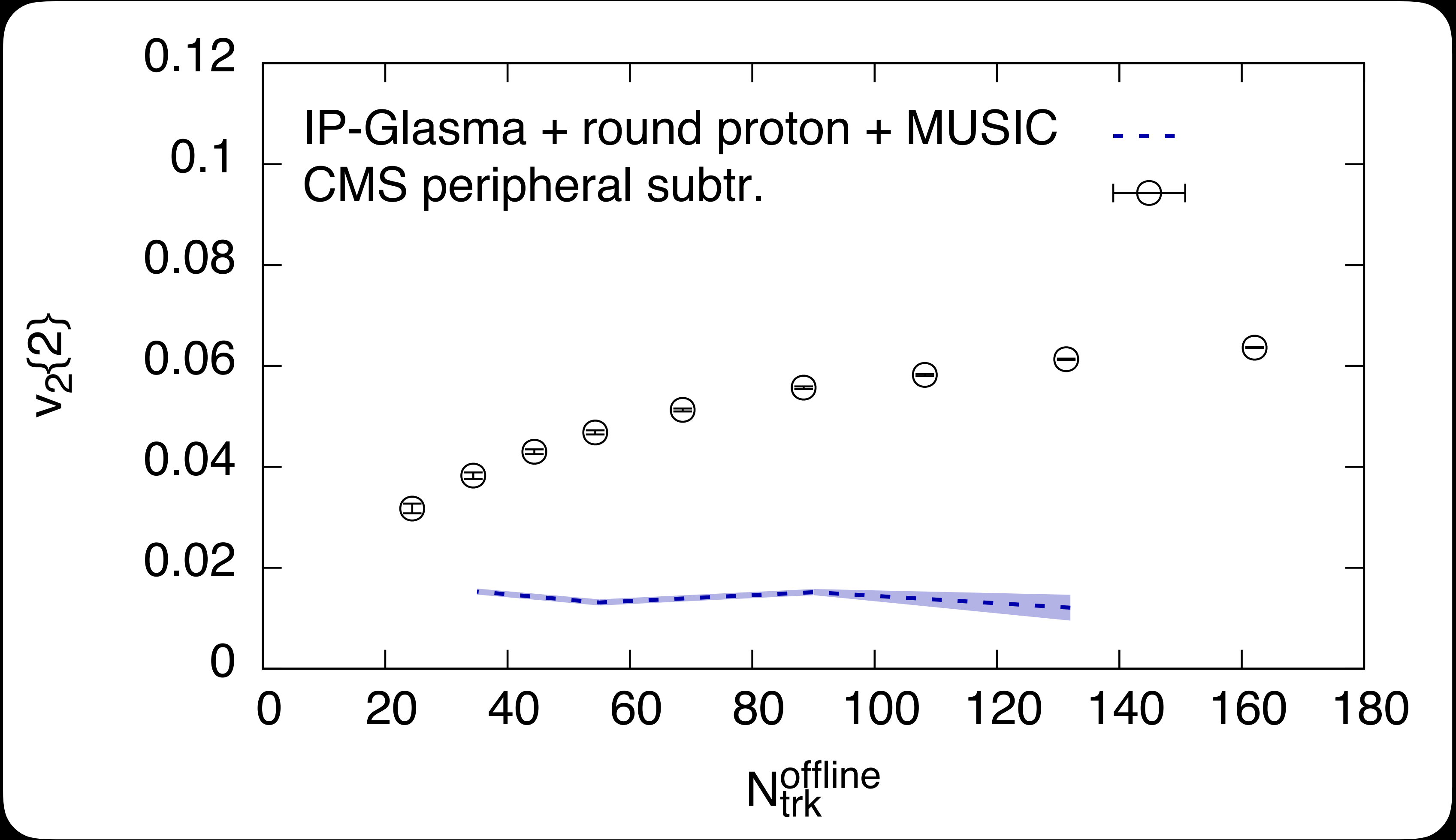
Also see: Kozlov, Luzum, Denicol, Jeon, Gale; Werner, Beicher, Guiot, Karpenko, Pierog; Romatschke; Kalaydzhyan, Shuryak, Zahed; Ghosh, Muhuri, Nayak, Varma; Qin, Mueller; Bozek, Broniowski, Torrieri; Habich, Miller, Romatschke, Xiang; T. Hirano, K. Kawaguchi, K. Murase; …

p+Pb v_2 IP-GLASMA + MUSIC

Did not work.

Not because hydro does not work.

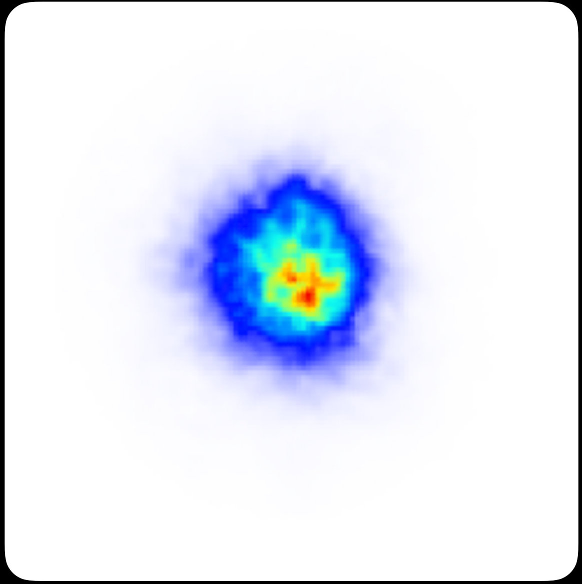
But because initial state was missing physics.



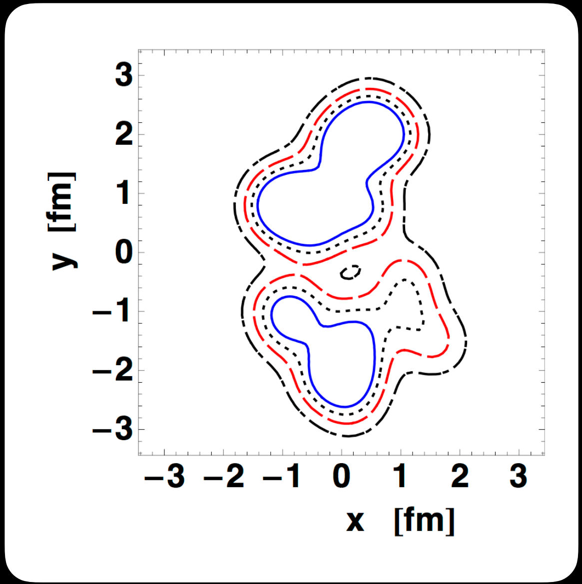
B. Schenke, R. Venugopalan, Phys. Rev. Lett. 113, 102301 (2014)

Experimental data: CMS Collaboration, Phys.Lett. B724, 213 (2013)

IP-Glasma



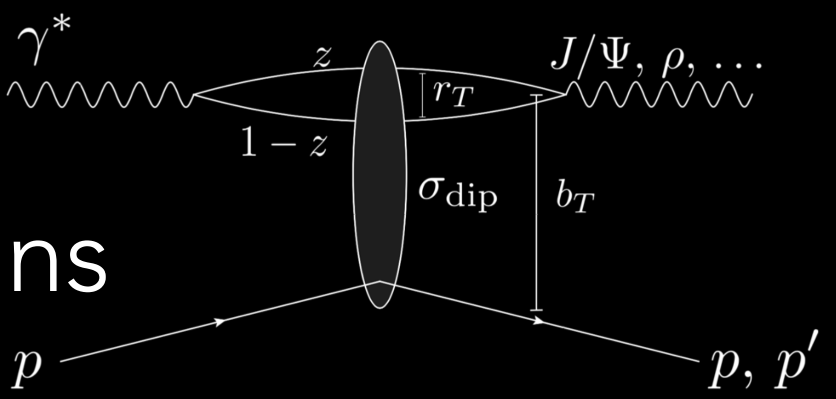
MC-Glauber



P. Bozek, Phys.Rev. C85 (2012) 014911

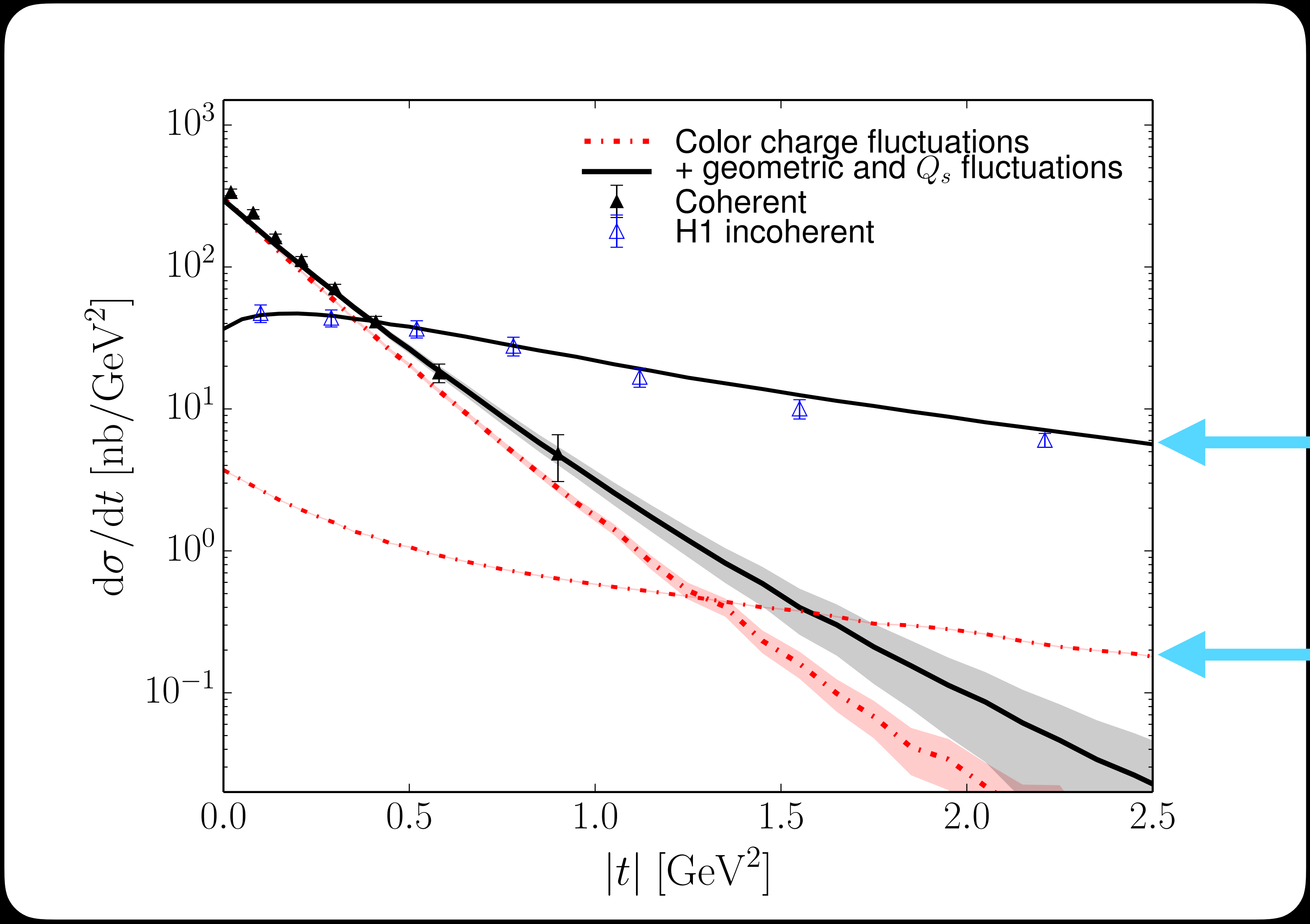
NEED PROTON SHAPE FLUCTUATIONS!

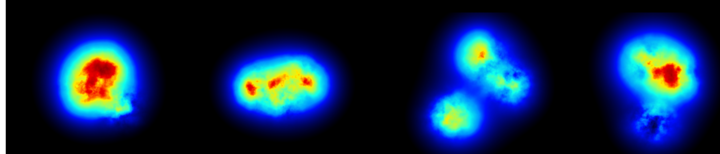
H. Mäntysaari, B. Schenke, Phys. Rev. Lett. 117 (2016) 052301; Phys.Rev. D94 (2016) 034042



Exclusive diffractive J/Ψ production:

Incoherent x-sec sensitive to fluctuations





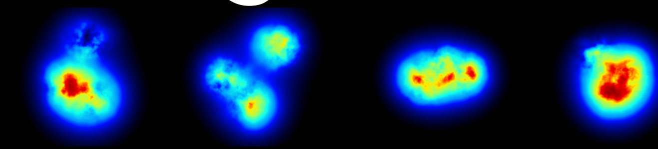
tuned shape fluctuations

round proton

H1 Collaboration, Eur. Phys. J. C73 (2013) no. 6 2466

IP-Glasma + hydro + fluctuating proton geometry

H. Mäntysaari, B. Schenke, C. Shen, P. Tribedy, Phys. Lett. B772, 681–686 (2017)

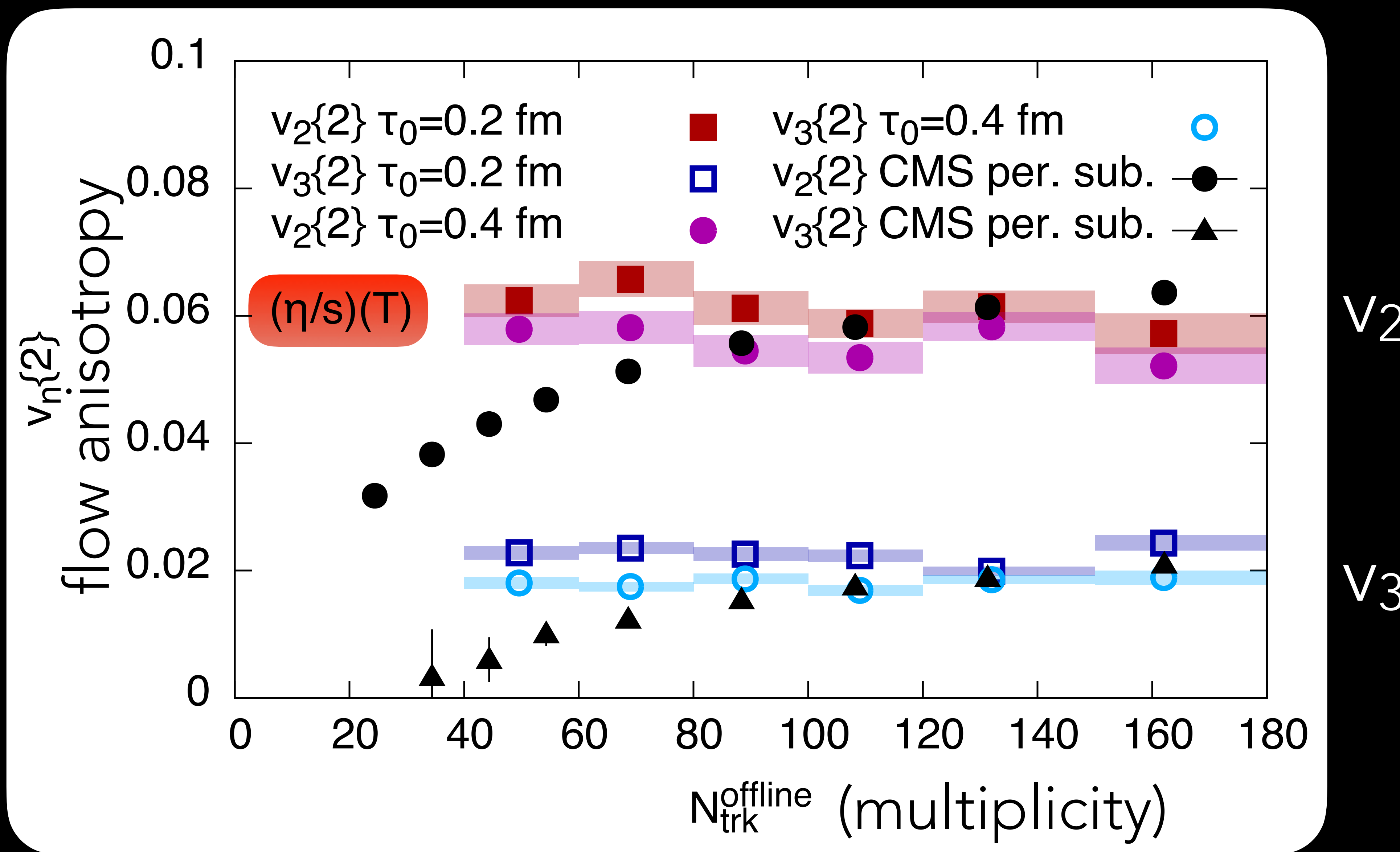


Constrain that fluctuating geometry by HERA diffractive J/Ψ prod.

H. Mäntysaari, B. Schenke, Phys. Rev. Lett. 117 (2016) 052301; Phys.Rev. D94 (2016) 034042

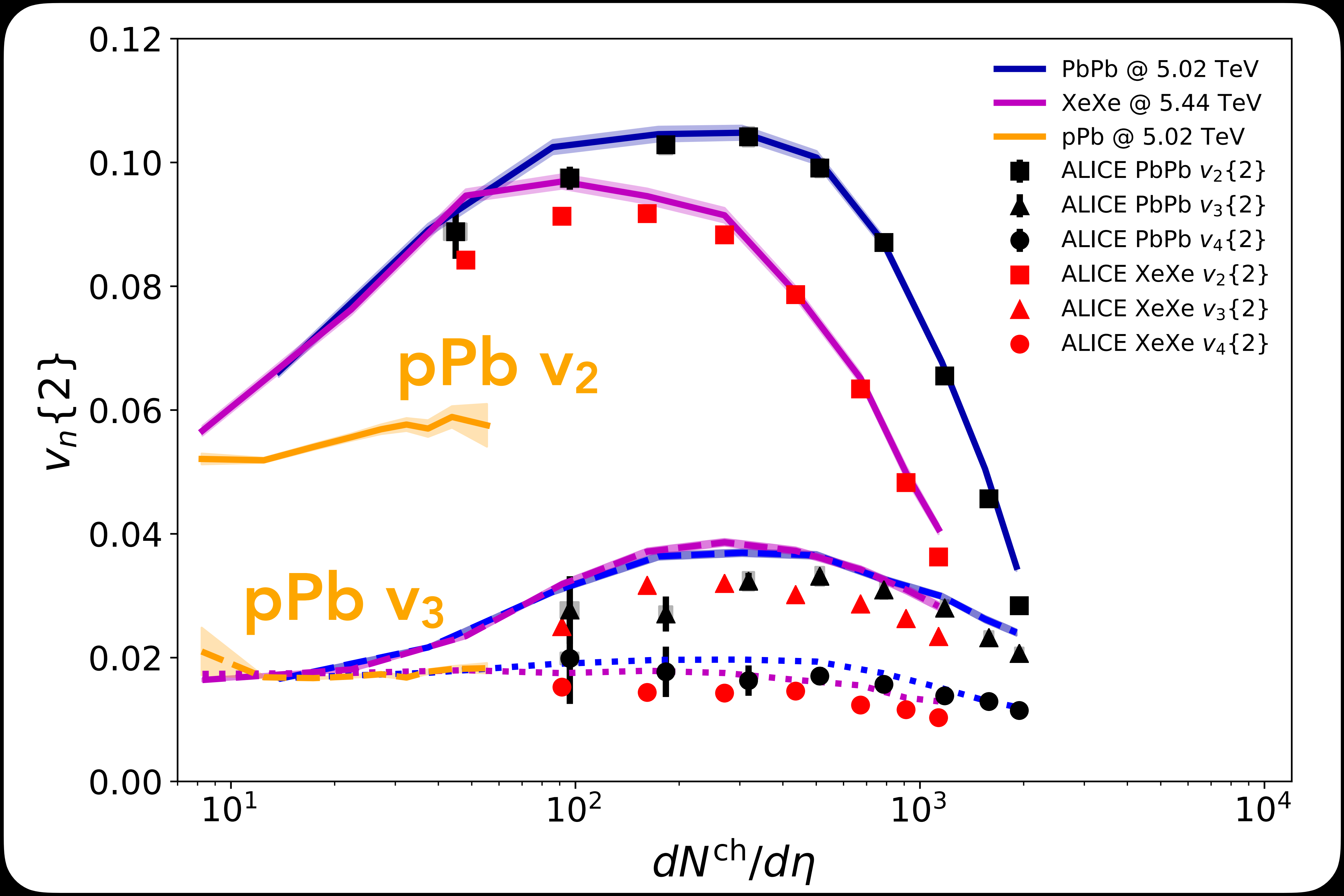
Round proton would produce much smaller v_n

Would not describe the data



Anisotropy vs. multiplicity

B. Schenke, C. Shen, P. Tribedy, in preparation

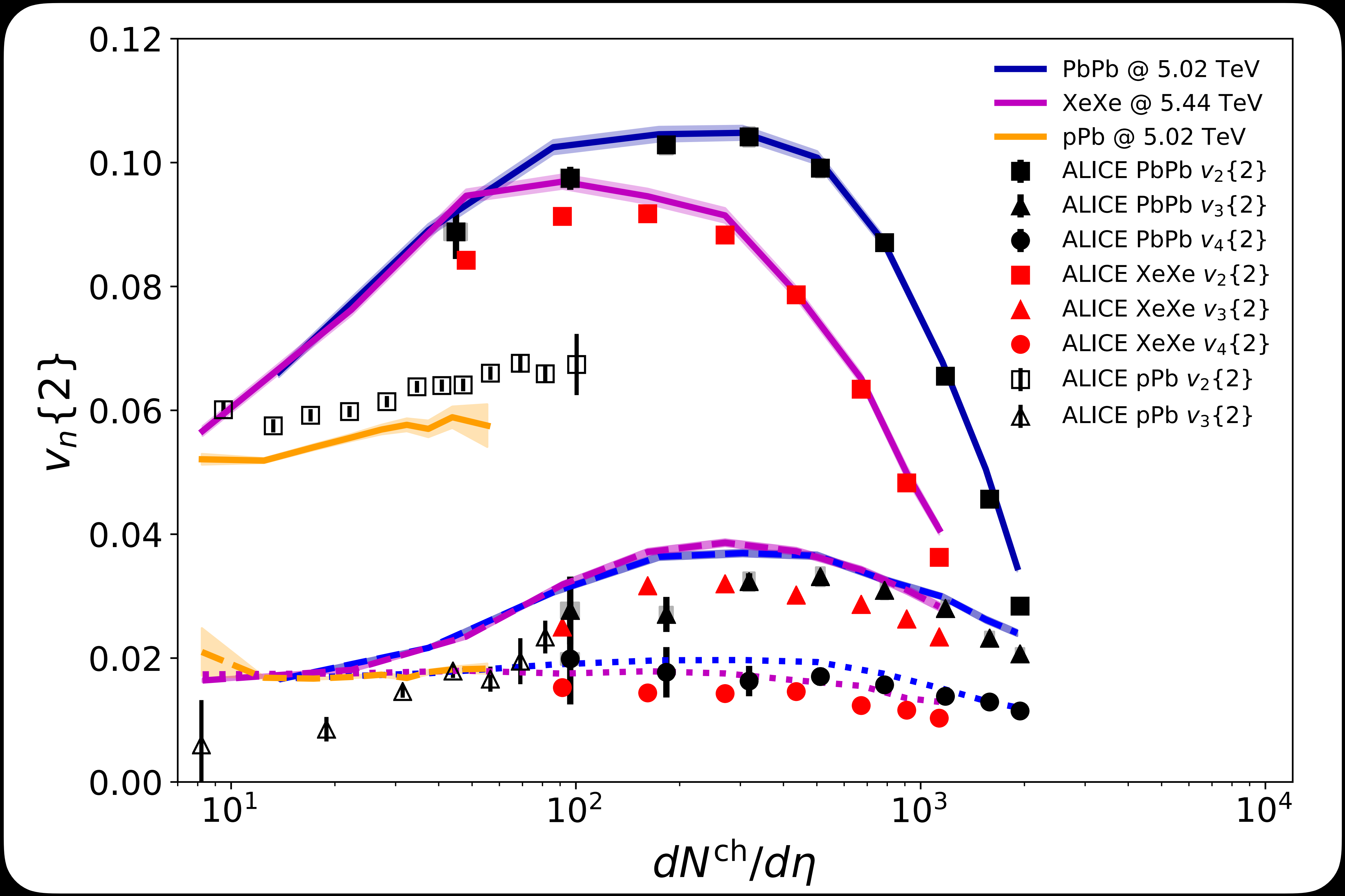


Experimental data: J. Adam et al. (ALICE), Phys. Rev. Lett. 116, 132302 (2016)

B. B. Abelev et al. (ALICE), Phys. Rev. C90, 054901 (2014), ALICE Collaboration, arXiv:1805.01832

Anisotropy vs. multiplicity

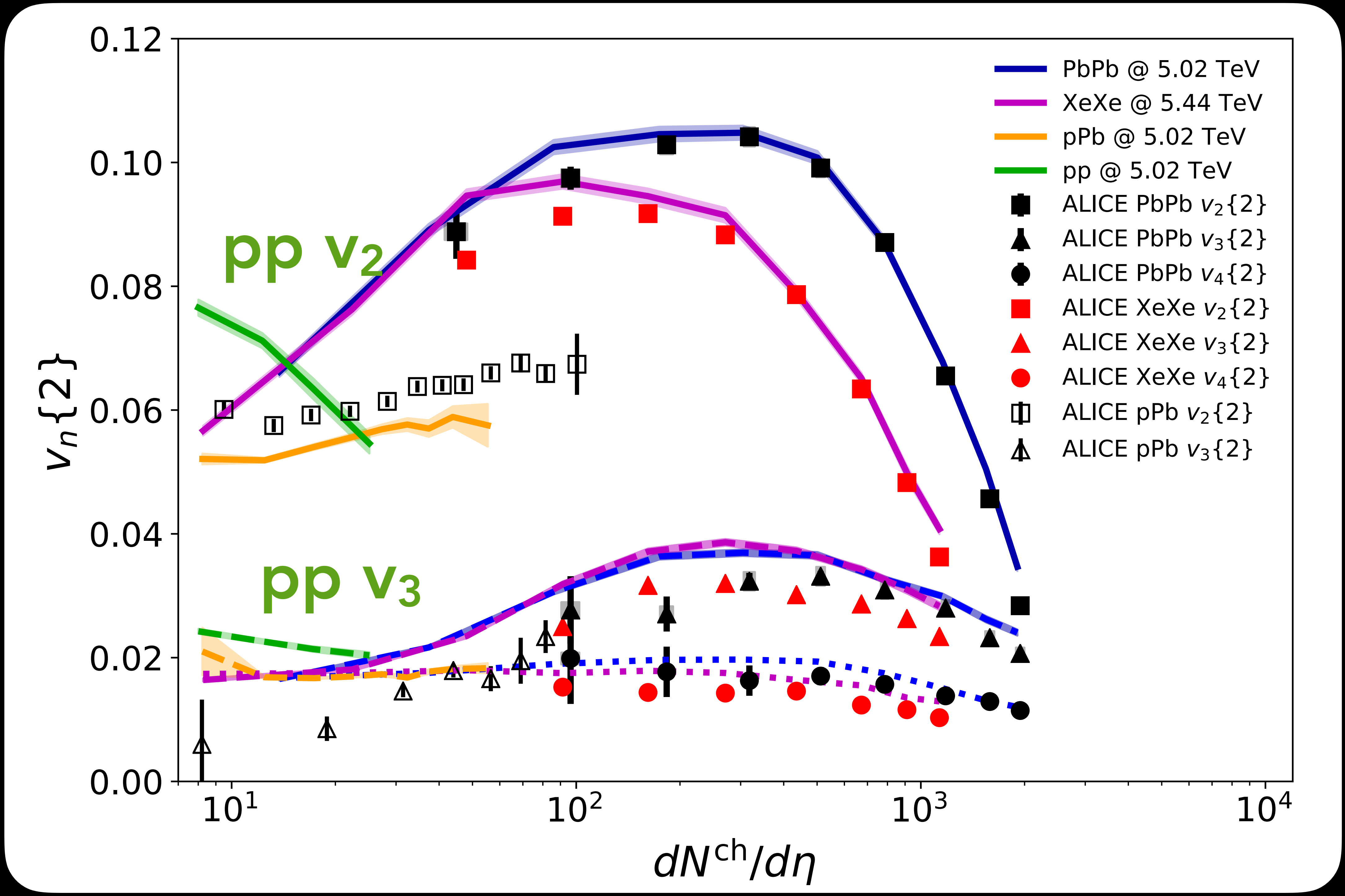
B. Schenke, C. Shen, P. Tribedy, in preparation



Experimental data: J. Adam et al. (ALICE), Phys. Rev. Lett. 116, 132302 (2016)
B. B. Abelev et al. (ALICE), Phys. Rev. C90, 054901 (2014), ALICE Collaboration, arXiv:1805.01832

Anisotropy vs. multiplicity

B. Schenke, C. Shen, P. Tribedy, in preparation

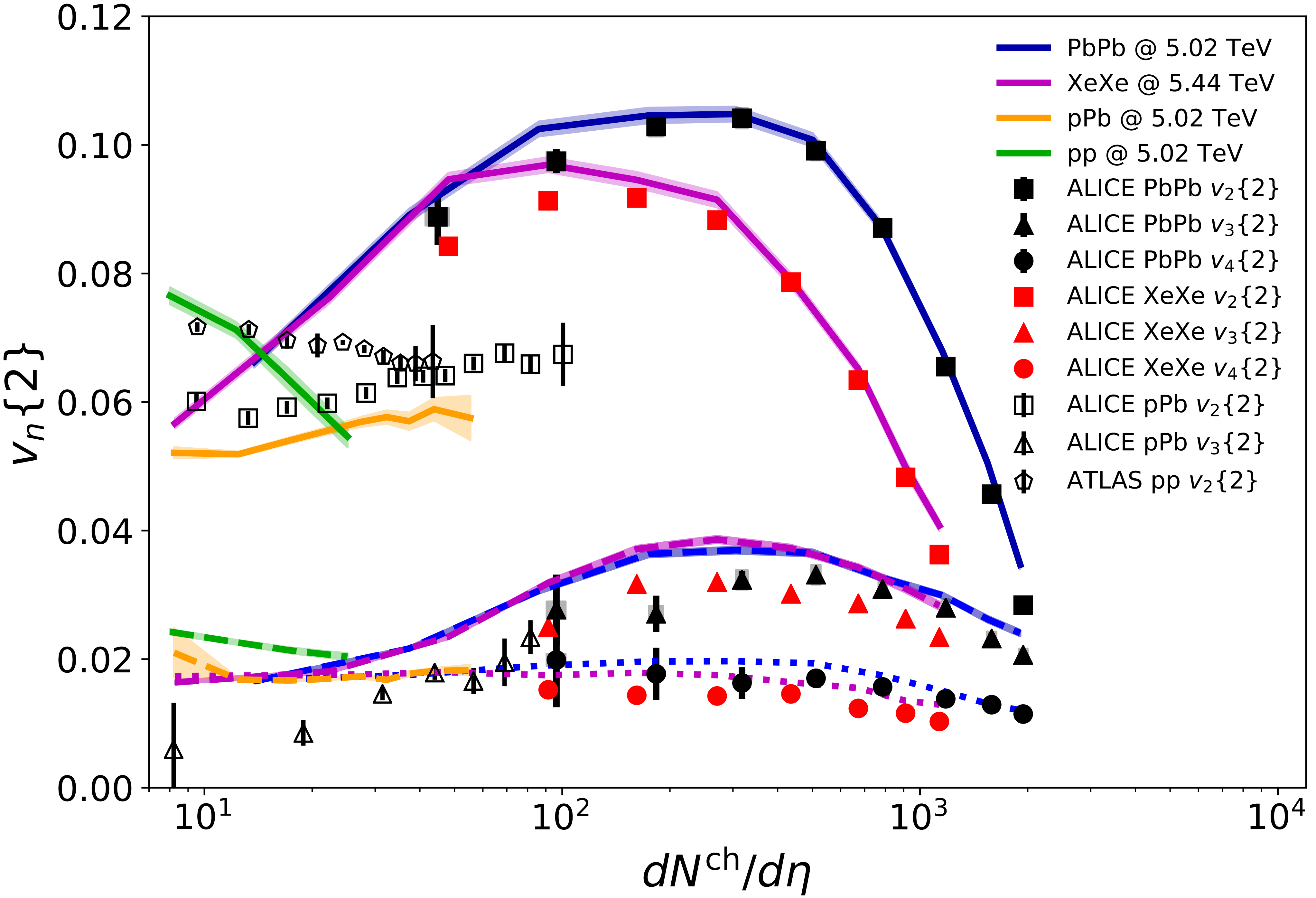


Experimental data: J. Adam et al. (ALICE), Phys. Rev. Lett. 116, 132302 (2016)

B. B. Abelev et al. (ALICE), Phys. Rev. C90, 054901 (2014), ALICE Collaboration, arXiv:1805.01832

Anisotropy vs. multiplicity

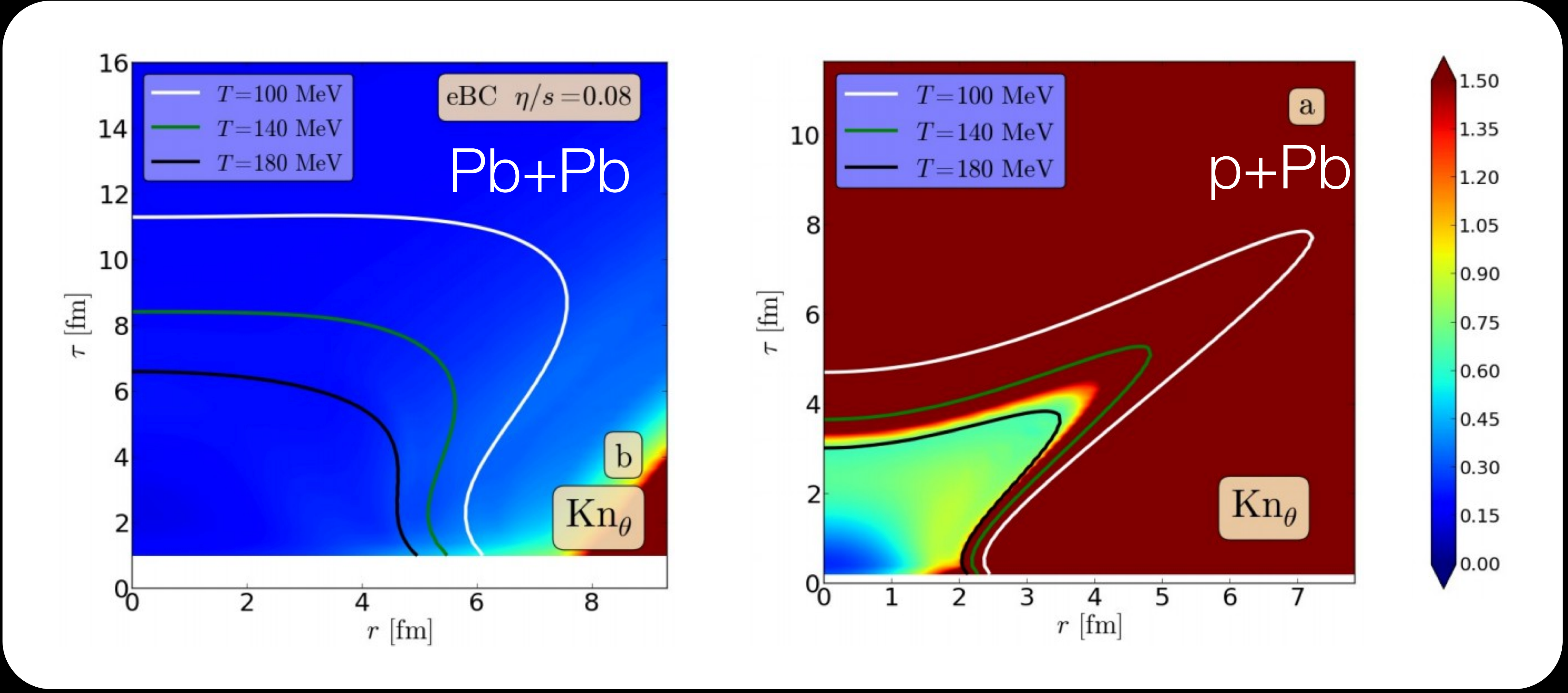
B. Schenke, C. Shen, P. Tribedy, in preparation



Experimental data: J. Adam et al. (ALICE), Phys. Rev. Lett. 116, 132302 (2016)
B. B. Abelev et al. (ALICE), Phys. Rev. C90, 054901 (2014), ALICE Collaboration, arXiv:1805.01832
ATLAS Collaboration, Eur. Phys. J. C (2017) 77:428

CAN WE TRUST HYDRODYNAMICS?

Knudsen number: ratio of a microscopic over macroscopic scale
Small Knudsen number means hydrodynamics is valid



H. NIEMI, G.S. DENICOL, E-PRINT: ARXIV:1404.7327

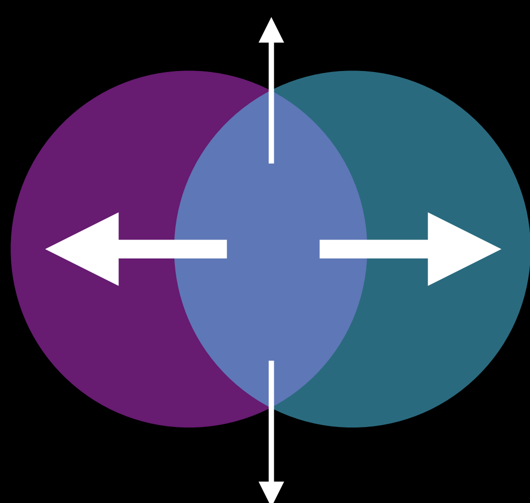
see review W. Florkowski, M. P. Heller, M. Spalinski, Rept.Prog.Phys. 81 (2018) 046001 on recent progress in understanding the validity of relativistic hydrodynamics in systems with large gradients

KINETIC THEORY "ANISOTROPIC ESCAPE"

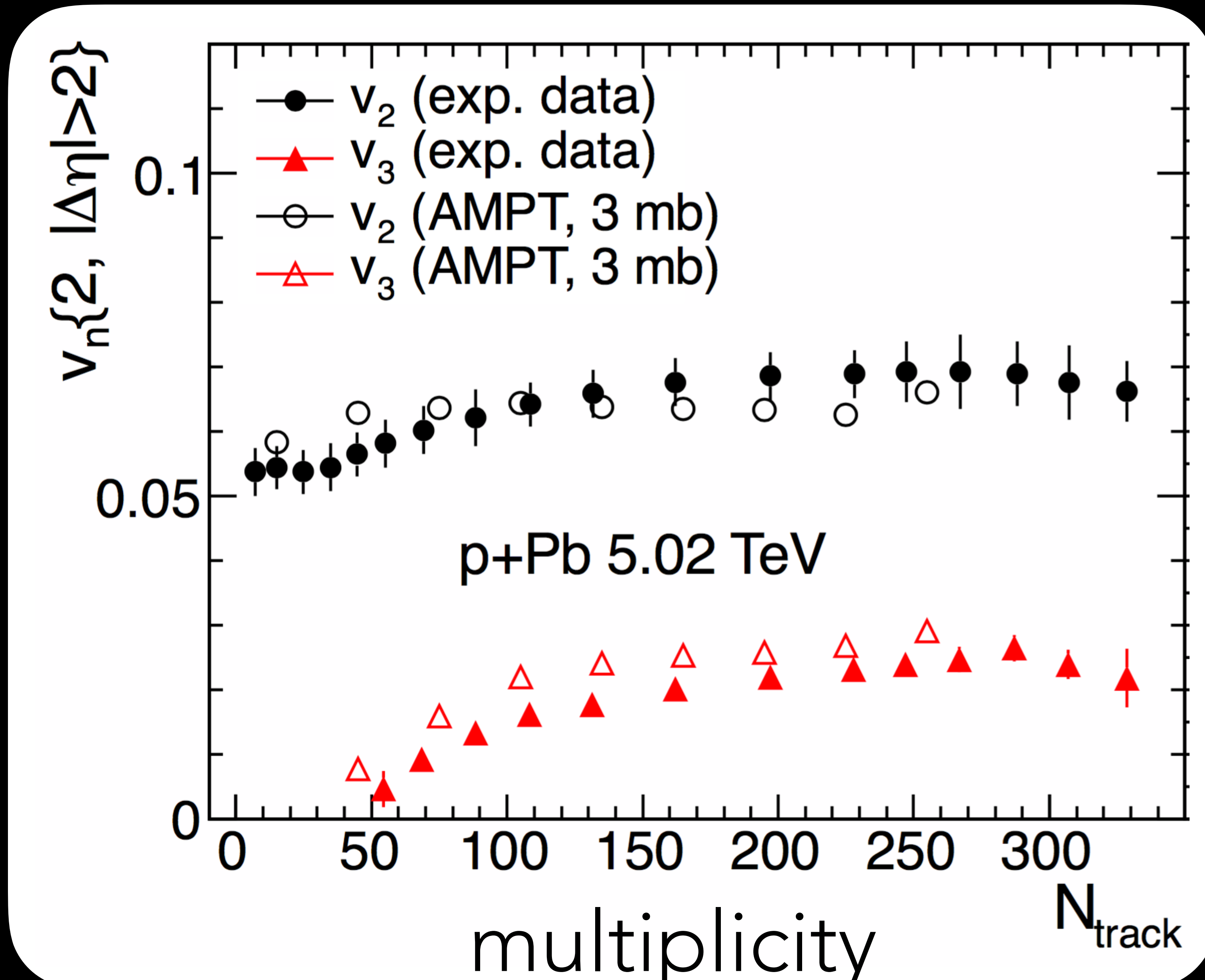
A. Bzdak, G.-L. Ma, PRL113 (2014) 252301; G.-L. Ma, A. Bzdak, PLB739 (2014) 209-213;
J.D. Orjuela Koop, A. Adare, D. McGlinchey, J.L. Nagle, PRC92 (2015) 054903; P. Bozek, A. Bzdak, G.-L. Ma, PLB748 (2015) 301-305; L. He, T. Edmonds, Z.-W. Lin, F. Liu, D. Molnar, F. Wang, PLB753 (2016)

Final state effect, but weakly interacting (3 mb x-sect.)

Described in AMPT

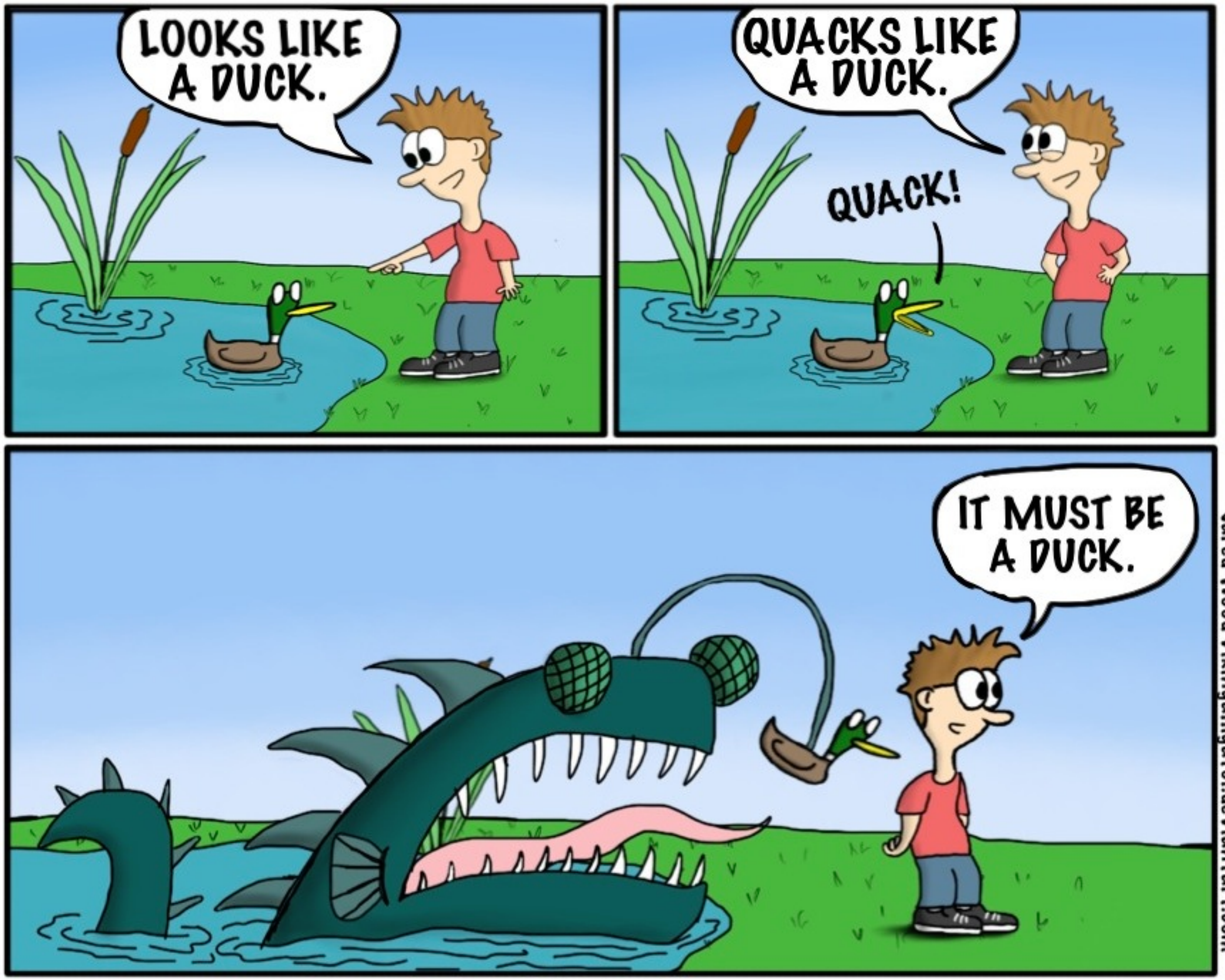


Partons are more likely to escape in the short direction → v_n



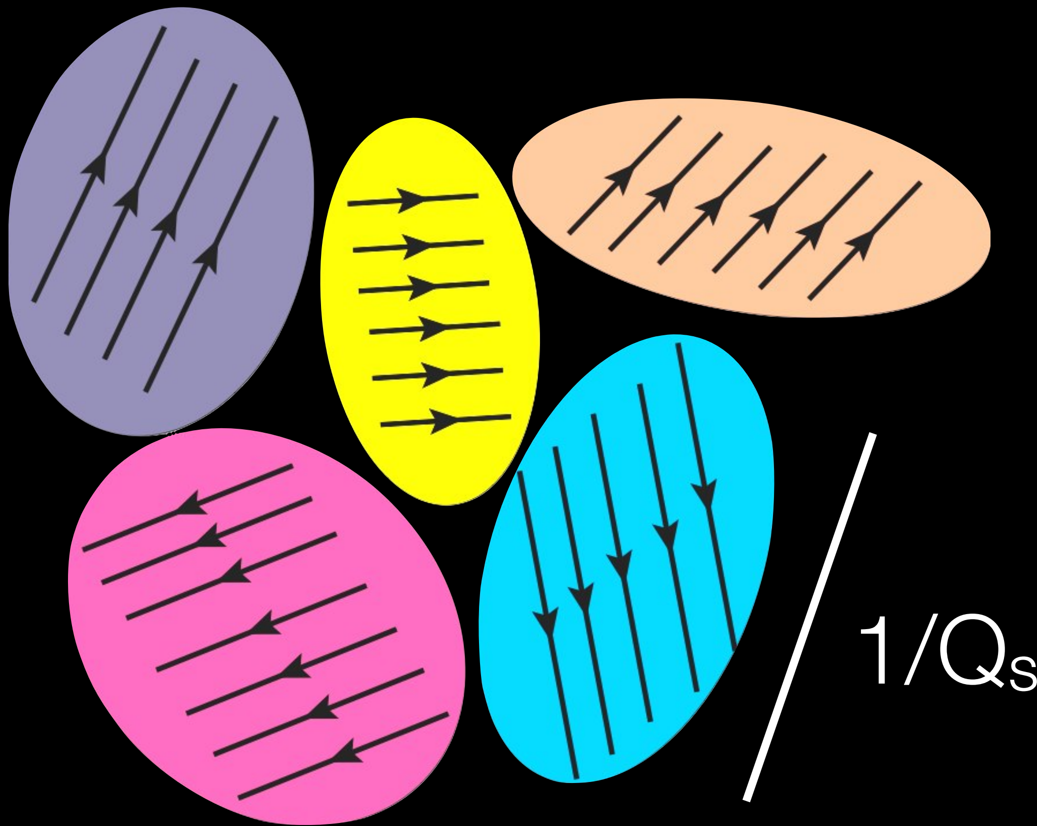
also see Kurkela, Wiedemann, Wu, arXiv:1803.02072

INITIAL STATE MOMENTUM CORRELATIONS



INITIAL STATE PICTURE

Intuitive picture:
Quarks or gluons are produced from color field domains in the Pb or p target



Particles that come from the same domain are correlated

Effect is suppressed by the number of colors and the number of domains (it is small for heavy ions)

FIGURE: T. LAPPI, B. SCHENKE, S. SCHLICHTING, R. VENUGOPALAN JHEP 1601 (2016) 061; SEE ALSO: A. DUMITRU, A.V. GIANNINI, NUCL.PHYS.A933 (2014) 212; A. DUMITRU, V. SKOKOV, PHYS.REV.D91 (2015) 074006; A. DUMITRU L. MCLERRAN, V. SKOKOV, PHYS.LETT.B743 (2015), 134; V. SKOKOV. PHYS.REV.D91 (2015) 054014

INITIAL STATE PICTURE

High-multiplicity events are rare configurations of nuclear wave-function with large number of small-x gluons

Situation described by the **Color Glass Condensate** an effective theory of QCD at high energy.
Particle production is governed by the **Yang Mills equations**

$$[D_\mu, F^{\mu\nu}] = J^\nu$$

J^ν: Combination of incoming target and projectile color currents

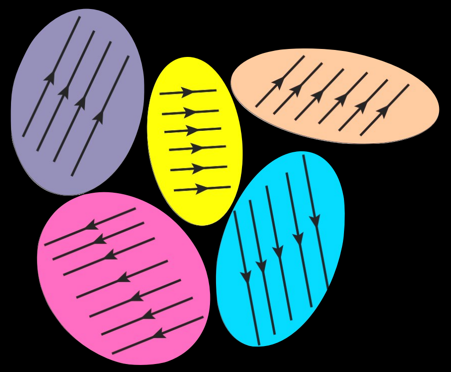
This is it.

Different approximations and assumptions on the market

APPROXIMATIONS

- **Glasma graph approximation:** two gluon exchange (not more) and Gaussian statistics of color charges (MV model)
  Gelis,Lappi Venugopalan PRD 78 054020 (2008), PRD 79 094017 (2009); Dumitru, Gelis, McLerran,Venugopalan NPA810, 91 (2008); Dumitru, Jalilian-Marian PRD 81 094015 (2010); Dusling, Venugopalan PRD 87 (2013), ...
- **Non-linear Gaussian approximation:**
  Resums multi-gluon exchanges - still Gaussian statistics
  McLerran, Venugopalan, PRD 59 (1999) 094002; Dominguez, Marquet, Wu, NPA 823 (2009) 99; Lappi, Schenke, Schlichting, Venugopalan, JHEP 1601 (2016) 061; ...
- **Numerical solution:** Solves the Yang-Mills equations exactly for any initial color source statistics and spatial configuration, includes multiple-gluon exchange, “rescattering”
  Krasnitz, Venugopalan, NPB 557 (1999) 237; Krasnitz, Nara, Venugopalan, NPA 717 (2003) 268; Lappi, PRC 67 (2003) 054903; Schenke, Tribedy, Venugopalan, PRL 108 (2012) 252301; Schenke, Schlichting, Venugopalan, PLB 747, 76-82 (2015), ...
- One can add **JIMWLK** evolution which will introduce leading quantum correction and (some) non-Gaussian correlations
  J. Jalilian-Marian, A. Kovner, A. Leonidov, and H. Weigert, NPB504, 415 (1997), PRD59, 014014 (1999)
  E. Iancu, A. Leonidov, and L. D. McLerran, NPA692, 583 (2001); A. H. Mueller, PLB523, 243 (2001)
  Lappi, PLB 744 (2015) 315-319, ...

INITIAL STATE PICTURE GENERATES ANISOTROPY



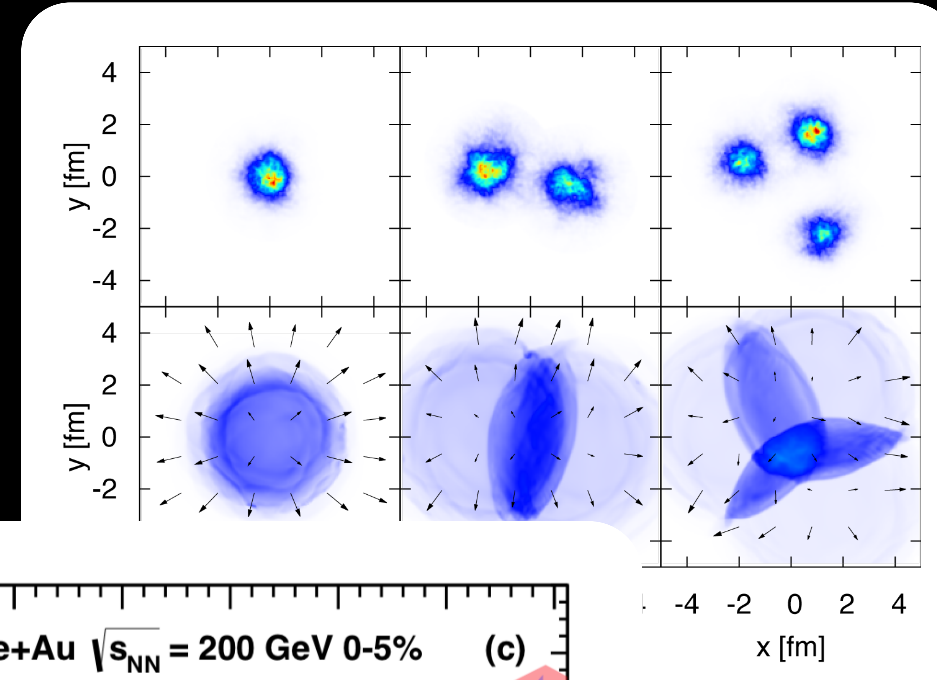
Gelis,Lappi Venugopalan PRD 78 054020 (2008), PRD 79 094017 (2009)
Dumitru, Gelis, McLerran,Venugopalan NPA810, 91 (2008); Dumitru, Jalilian-Marian PRD 81 094015 (2010);
A. Dumitru, K. Dusling, F. Gelis, J. Jalilian-Marian, T. Lappi, R. Venugopalan, PLB697 (2011) 21-25
Dusling, Venugopalan PRD 87 (2013) 5, 051502; PRD 87 (2013) 5, 054014; PRD 87 (2013) 9, 094034

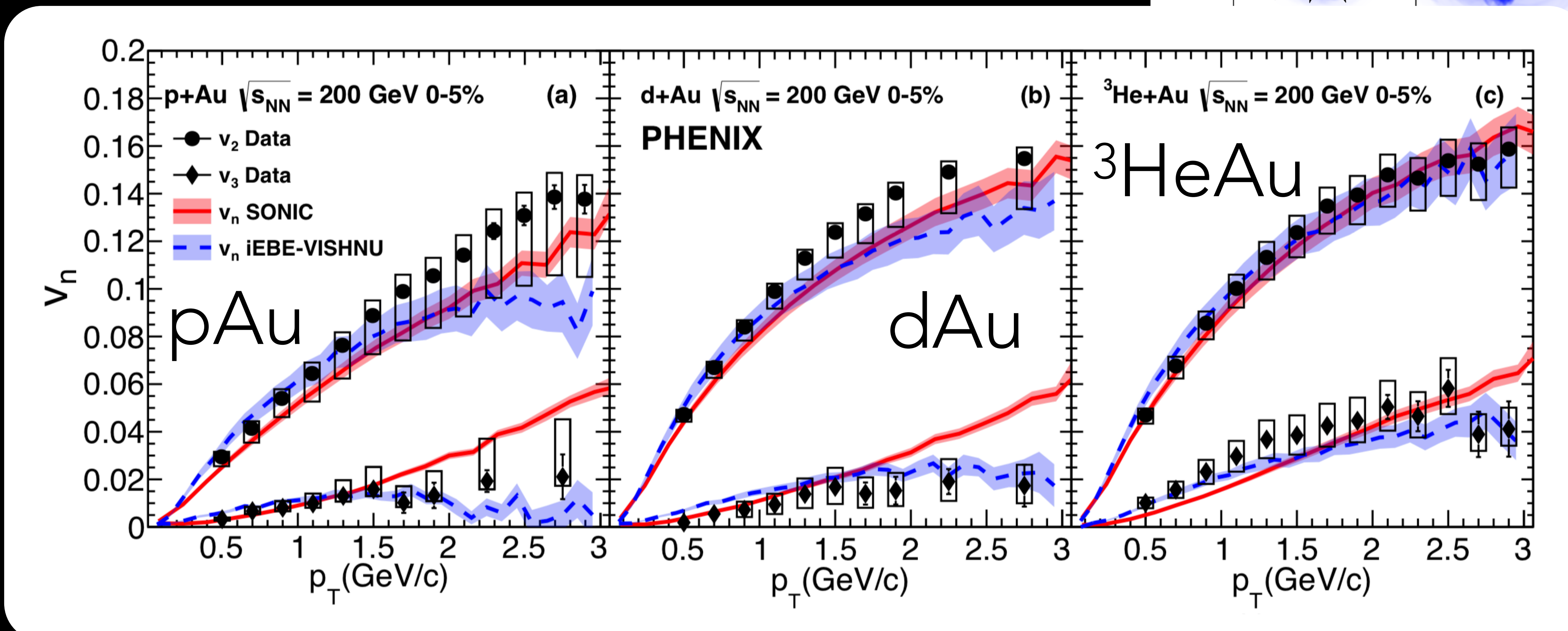
CAN WE DISTINGUISH INITIAL FROM FINAL STATE EFFECTS?

Many possibilities. Different observables.
I will focus on studying
Different collision systems: Allow to control initial geometry
They are (were?) the most promising tool

SYSTEM DEPENDENCE OF ANISOTROPIES

Hydrodynamics converts initial shape to momentum anisotropy.
At RHIC different systems with different average shapes were studied.





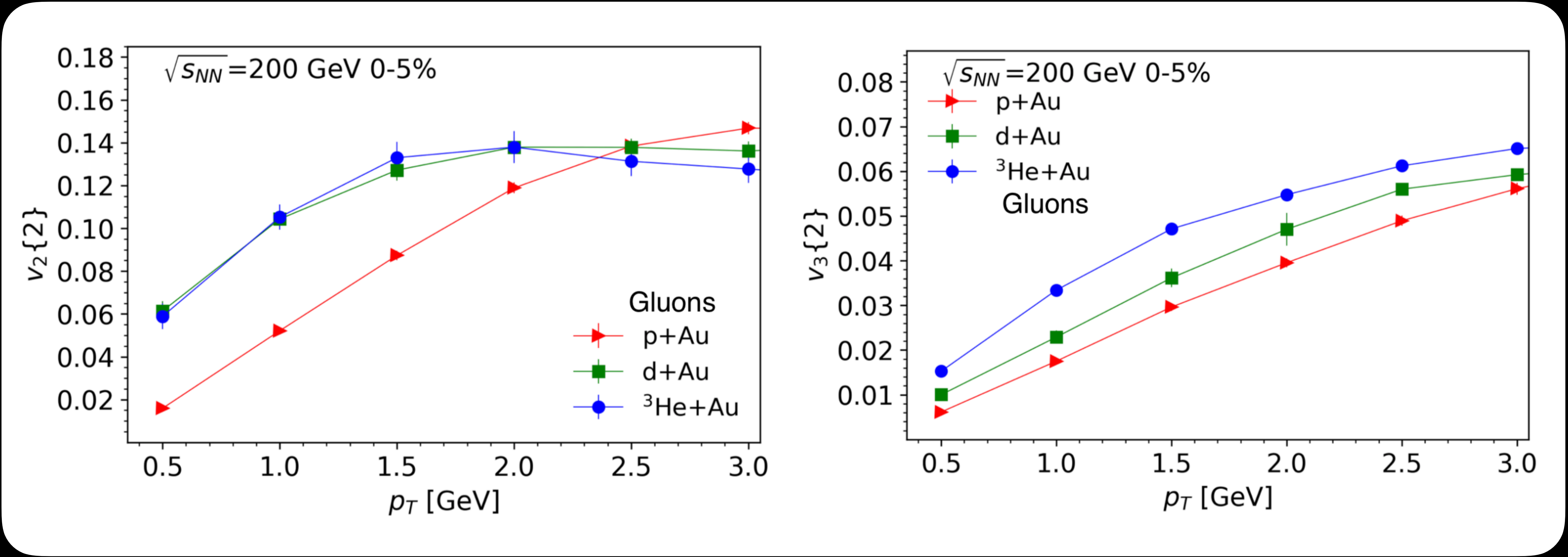
PHENIX, arXiv:1805.02973

Hydrodynamics correctly describes anisotropies in different systems

OTHER *CALCULATIONS:* BOZEK, BRONIOWSKI, PLB739 (2014) 308; NAGLE ET AL, PRL113 (2014); BOZEK, BRONIOWSKI, PLB747 (2015) 135; SCHENKE, VENUGOPALAN, NPA931 (2014) 1039; ROMATSCHKE, EUR. PHYS. J. C75 (2015) 305

SYSTEM DEPENDENCE OF ANISOTROPIES

Recent results from initial state momentum correlations:



Mark Mace, Vladimir V. Skokov, Prithwish Tribedy, Raju Venugopalan, arXiv:1805.09342

System dependence present when selecting 0-5% central events (which have different multiplicities in different systems)

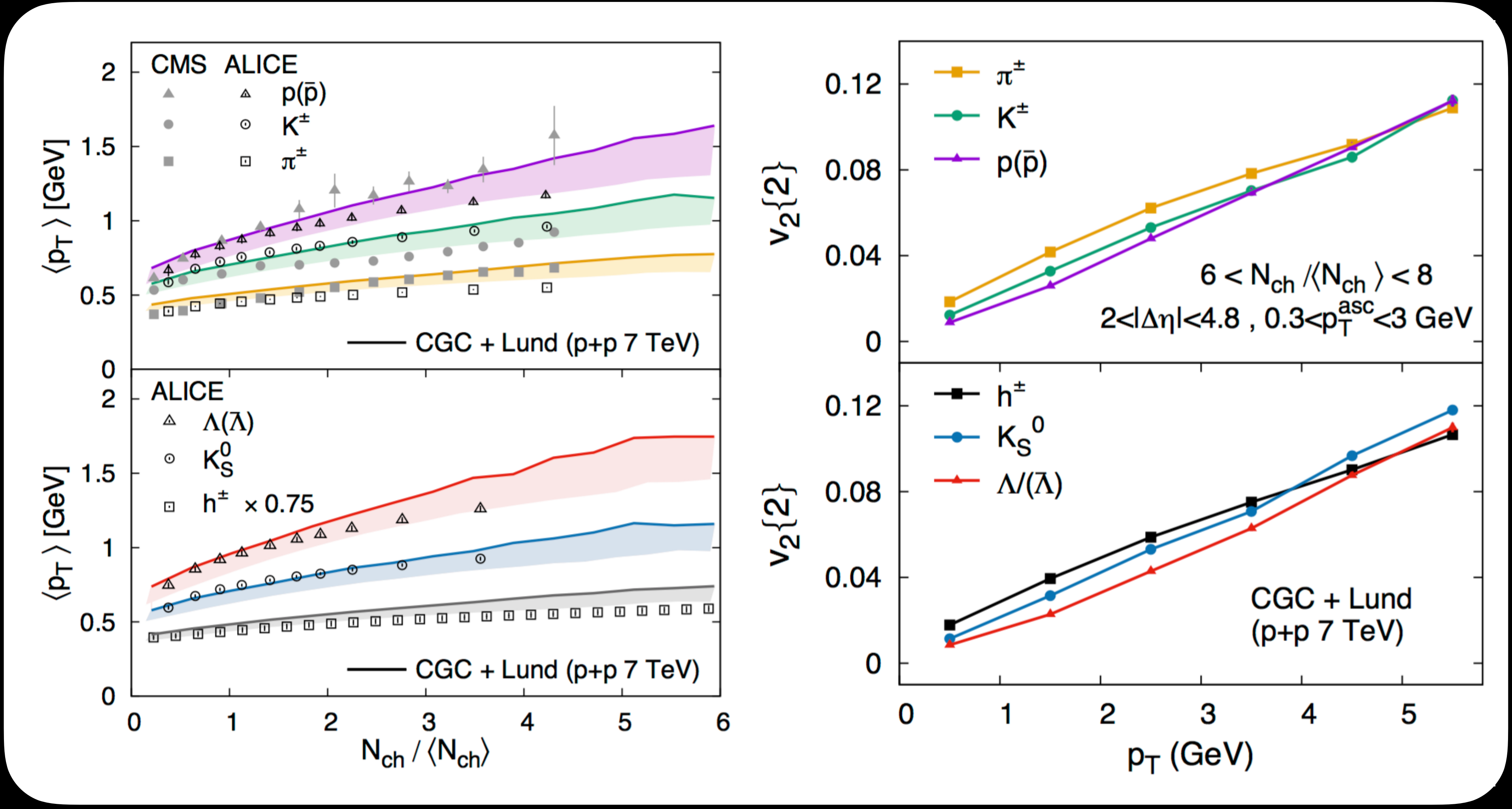
IP-GLASMA ~~+ HYDRO~~ + LUND

B. Schenke, S. Schlichting, P. Tribedy, R. Venugopalan, Phys. Rev. Lett. 117, 162301 (2016)

First step towards an event generator with dense initial gluon fields and Lund fragmentation

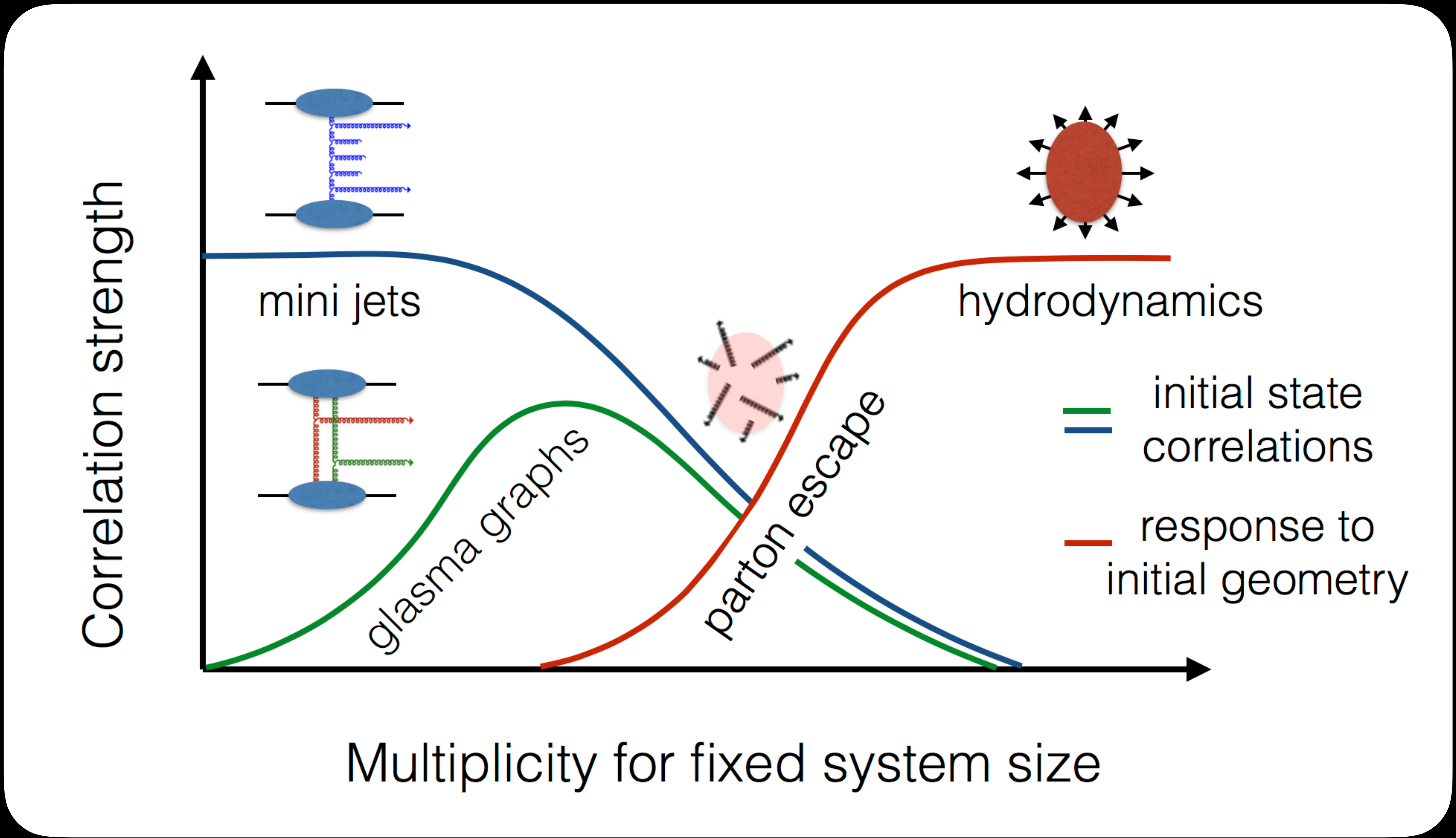
Sample gluons and connect with Lund strings

Arrange similar to what color reconnection would do



Emission from common boosted source: mass splitting

STUDY RELATIVE STRENGTH OF INITIAL AND FINAL STATE CORRELATIONS IN THEORY



S. Schlichting, Quark Matter 2015

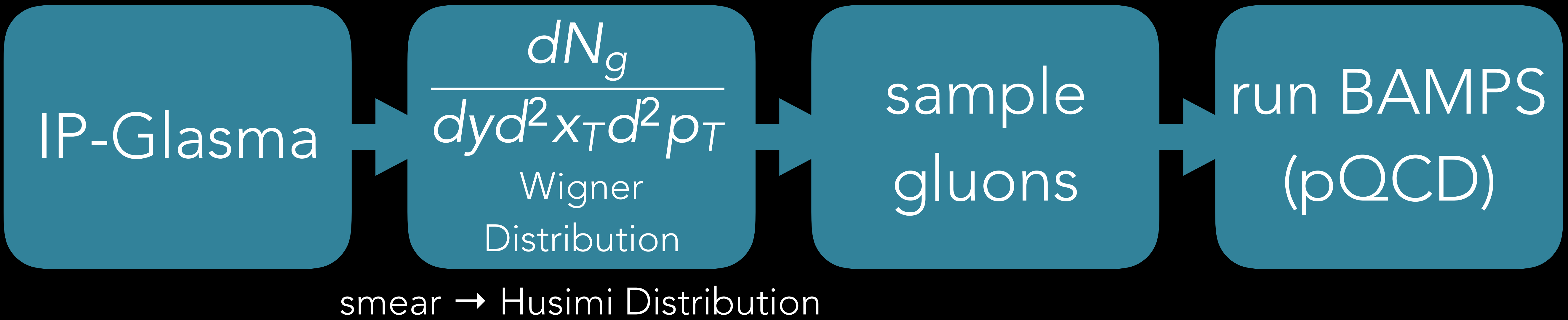
Calculate the relative contribution of "glasma graphs" and final state effects

IP-Glasma + parton cascade

M. Greif, C. Greiner, B. Schenke, S. Schlichting, Z. Xu, arXiv:1708.02076, Phys. Rev. D96, 091509(R)

To study how final state interactions affect the initial state correlations, we use a microscopic final state model, the parton cascade BAMPS

Z.Xu, C. Greiner, PRC71, 064901 (2005)



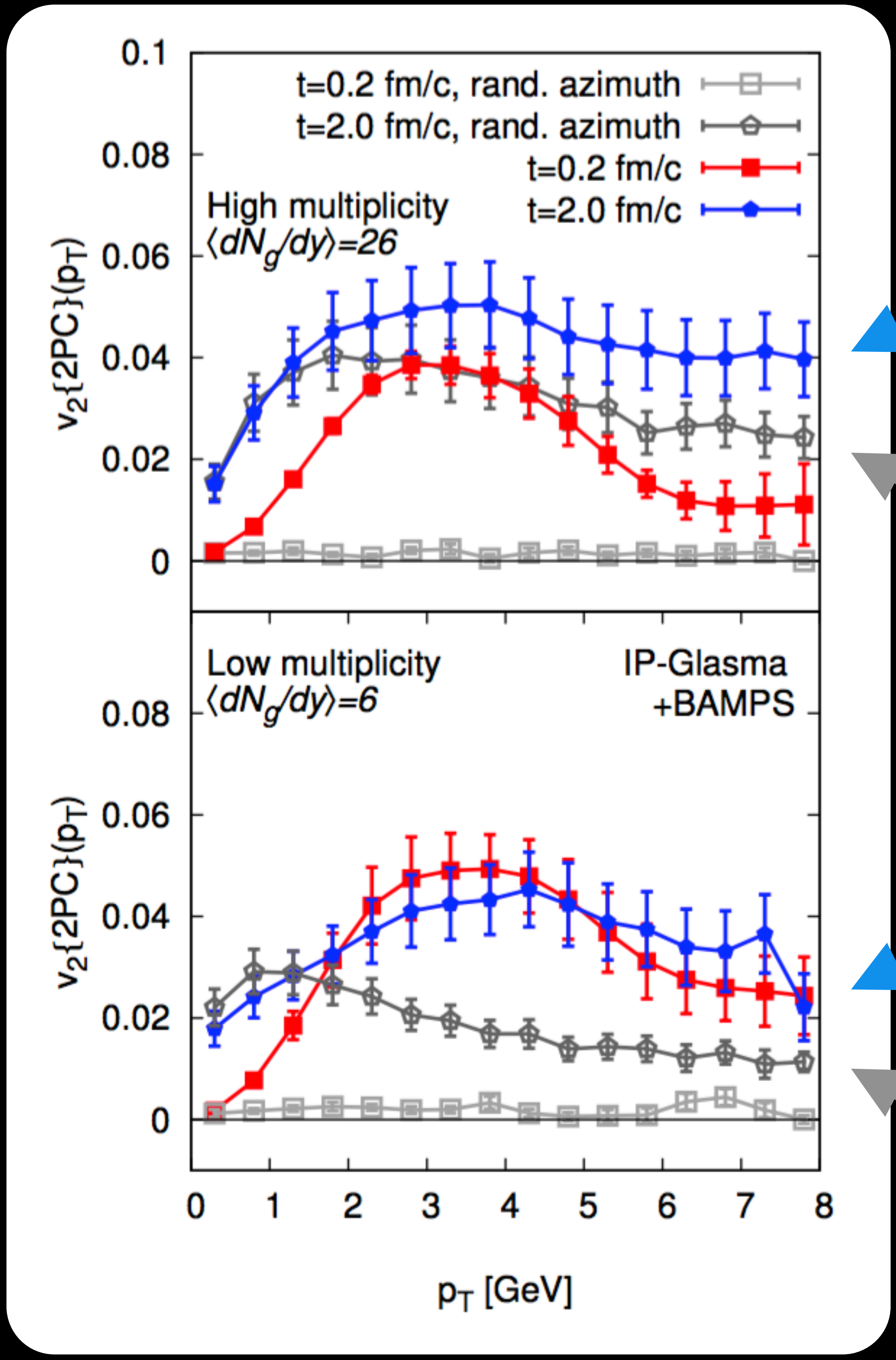
→ study momentum anisotropy as function of time

Effect of initial correlations on final v_2

M. Greif, C. Greiner, B. Schenke, S. Schlichting, Z. Xu, arXiv:1708.02076, Phys. Rev. D96, 091509(R)

high multiplicity

low multiplicity



with initial corr.

without initial corr.

with initial corr.

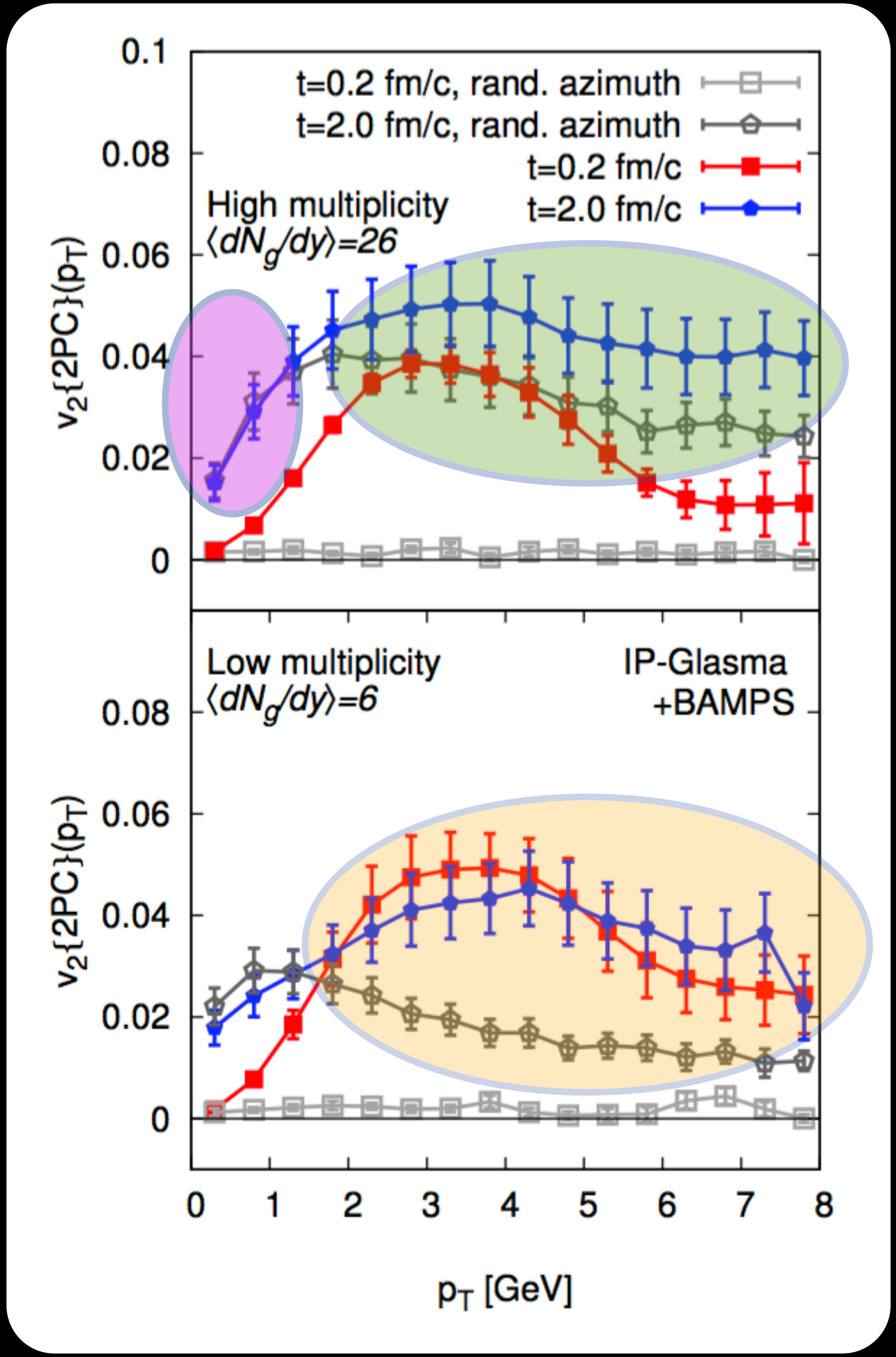
without initial corr.

Effect of initial correlations on final v_2

M. Greif, C. Greiner, B. Schenke, S. Schlichting, Z. Xu, arXiv:1708.02076, Phys. Rev. D96, 091509(R)

high multiplicity

low multiplicity



negligible effect at small p_T and high multiplicity

significant effect at p_T >2 GeV and low multiplicity

visible effect at p_T >3 GeV and high multiplicity

CONCLUSIONS & OUTLOOK

- Heavy Ion Physics is a very rich field with QCD at its heart
- Different aspects (observables) can be described with a variety of methods, ranging from perturbative QCD to thermodynamics and hydrodynamics, to thermal field theory, kinetic theory, lattice QCD, the theory of critical phenomena, quantum anomalies, etc. (many of which I could not cover)
- Many opportunities still await!

BACKUP

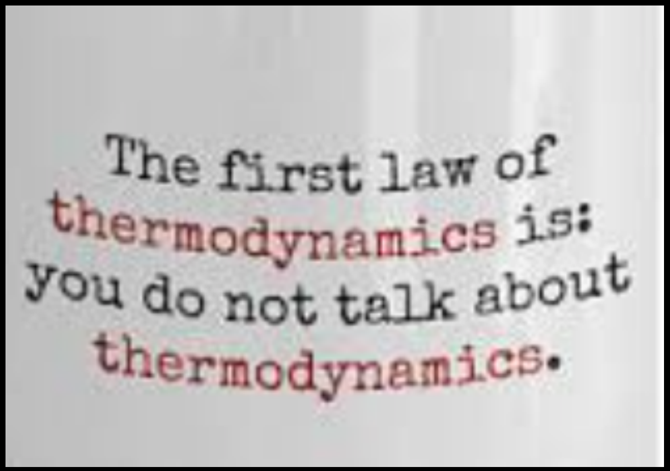
SOME THERMODYNAMICS

Before we derive the equations of hydrodynamics, let's remind ourselves of some basic thermodynamics.
The differential of the internal energy of a system is given by:

$$dU = -PdV + TdS + \mu dN$$

work and heat transferred to the system

P: pressure
V: volume
T: temperature
μ: chemical potential
S: entropy
N: non relativistic system: number of particles, rel. system: e.g. net-baryon number



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SOME THERMODYNAMICS

An extensive property is a property that changes when the size of the system changes: e.g. mass, volume, length, total charge

An intensive property is a bulk property and does not change when you take away some of the sample: e.g. temperature, refractive index, density

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SOME THERMODYNAMICS

$$dU = -PdV + TdS + \mu dN$$

For a non-viscous fluid, the mechanical work done on the system may be related to the pressure P and the volume V.
The pressure is the intensive generalized force, the volume is the extensive generalized displacement:

$$\delta W = -PdV$$

This defines the direction of work W to be the energy flow to the working system from the surroundings.
Taking the direction of heat transfer Q to be into the working fluid and assuming a reversible process, the heat is

$$\delta Q = TdS$$

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SOME THERMODYNAMICS

Energy U is an extensive function of the extensive quantities V, S, N:

$$U(\lambda V, \lambda S, \lambda N) = \lambda U(V, S, N)$$

Differentiate with respect to λ:

$$\frac{dU}{d(\lambda V)}\frac{d(\lambda V)}{d\lambda} + \frac{dU}{d(\lambda S)}\frac{d(\lambda S)}{d\lambda} + \frac{dU}{d(\lambda V)}\frac{d(\lambda V)}{d\lambda} = U$$

$$\frac{dU}{dV}V + \frac{dU}{dS}S + \frac{dU}{dN}N = U$$

λ was set to 1…

Using $dU = -PdV + TdS + \mu dN$ we get

$$U = -PV + TS + \mu N$$

SOME THERMODYNAMICS

Differentiating $U = -PV + TS + \mu N$

gives $dU = -PdV - VdP + TdS + SdT + \mu dN + Nd\mu$

Using $dU = -PdV + TdS + \mu dN$ again yields the

Gibbs-Duhem relation: $VdP = SdT + Nd\mu$

In hydrodynamics it is more useful to deal with intensive quantities

$\varepsilon = U/V$ energy density

$s = S/V$ entropy density

$n = N/V$ baryon density

We obtain $\varepsilon = -P + Ts + \mu n$ and $dP = sdT + nd\mu$

Differentiating ε and using the second equation yields

$$d\varepsilon = Tds + \mu dn$$

ENTROPY CONSERVATION

In ideal hydrodynamics entropy is conserved.
Show that from $\partial_\mu T^{\mu\nu} = 0$ and $\partial_\mu(nu^\mu) = 0$:

$$\partial_\mu T^{\mu\nu} = \partial_\mu((\varepsilon + p)u^\mu u^\nu) - \partial_\mu(pg^{\mu\nu}) = 0$$

$$\Leftrightarrow \partial_\mu((\varepsilon + p)u^\mu)u^\nu + (\varepsilon + p)u^\mu \partial_\mu u^\nu - \partial^\nu p = 0$$

$$\overset{\times u_\nu}{\Leftrightarrow} \partial_\mu((\varepsilon + p)u^\mu) + (\varepsilon + p)u^\mu \underbrace{u_\nu \partial_\mu u^\nu}_{=0} - u_\nu \partial^\nu p = 0$$

$$\Leftrightarrow u^\mu \partial_\mu(\varepsilon + p) + (\varepsilon + p)\partial_\mu u^\mu - u_\mu \partial^\mu p = 0$$

$$\Leftrightarrow u^\mu \partial_\mu \varepsilon + (\varepsilon + p)\partial_\mu u^\mu = 0$$

Also

$$u^\mu \partial_\mu(n) + n\partial_\mu u^\mu = 0 | \times \mu$$

$$u^\mu \mu \partial_\mu(n) + \mu n \partial_\mu u^\mu = 0$$

ENTROPY CONSERVATION

Now subtract

$$u^\mu \mu \partial_\mu (n) + \mu n \partial_\mu u^\mu = 0$$

from

$$u^\mu \partial_\mu \varepsilon + (\varepsilon + p)\partial_\mu u^\mu = 0$$

$$\Rightarrow u^\mu \partial_\mu \varepsilon + \underbrace{(\varepsilon + p)\partial_\mu u^\mu - \mu n \partial_\mu u^\mu}_{Ts\partial_\mu u^\mu} - u^\mu \mu \partial_\mu n = 0$$

$$\Leftrightarrow Ts\partial_\mu u^\mu + u^\mu \partial_\mu \varepsilon - u^\mu \mu \partial_\mu n = 0$$

$$\overset{d\varepsilon - \mu dn = Tds}{\Leftrightarrow} Ts\partial_\mu u^\mu + Tu^\mu \partial_\mu s = 0 | \div T$$

$$\Leftrightarrow s\partial_\mu u^\mu + u^\mu \partial_\mu s = \partial_\mu (su^\mu) = 0\,,$$

where su^μ is the entropy current.

$\Pi^{\mu\nu}$ FROM THERMODYNAMICS

We now show that

$$\partial_\mu s^\mu = \frac{1}{T}\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} = \frac{1}{2T}\pi^{\mu\nu}\nabla_{<\mu}u_{\nu>} - \frac{1}{T}\Pi\nabla_\alpha u^\alpha \geq 0$$

First

$$\begin{aligned}\frac{1}{T}\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} &= \frac{1}{2T}\Pi^{\mu\nu}\nabla_{<\mu}u_{\nu>} + \frac{1}{3T}\Pi^{\mu\nu}\Delta_{\mu\nu}\nabla_\alpha u^\alpha \\ &= \frac{1}{2T}\pi^{\mu\nu}\nabla_{<\mu}u_{\nu>} - \frac{1}{2T}\Pi\Delta^{\mu\nu}\nabla_{<\mu}u_{\nu>} \\ &\quad + \underbrace{\frac{1}{3T}\pi^{\mu\nu}\Delta_{\mu\nu}\nabla_\alpha u^\alpha}_{=0} - \frac{1}{3T}\Delta^{\mu\nu}\Delta_{\mu\nu}\Pi\nabla_\alpha u^\alpha \\ &= \frac{1}{2T}\pi^{\mu\nu}\nabla_{<\mu}u_{\nu>} - \frac{1}{2T}\Pi\Delta^{\mu\nu}\nabla_{<\mu}u_{\nu>} - \frac{1}{T}\Pi\nabla_\alpha u^\alpha\end{aligned}$$

because

$$\Delta_{\mu\nu}\Delta^{\mu\nu} = (g_{\mu\nu} - u_\mu u_\nu)(g^{\mu\nu} - u^\mu u^\nu) = 4 - 1 - 1 + 1 = 3$$

Π μν FROM THERMODYNAMICS

It remains to be shown that

$$\frac{1}{2T}\Pi\Delta^{\mu\nu}\nabla_{<\mu}u_{\nu>} = 0$$

$$\begin{aligned}
\Delta^{\mu\nu}\nabla_{<\mu}u_{\nu>} &= \Delta^{\mu\nu}(2\nabla_{(\mu}u_{\nu)} - \frac{2}{3}\Delta_{\mu\nu}\nabla_\alpha u^\alpha) \\
&= 2\Delta^{\mu\nu}\nabla_{(\mu}u_{\nu)} - 2\nabla_\alpha u^\alpha \\
&= \Delta^{\mu\nu}(\nabla_\mu u_\nu + \nabla_\nu u_\mu) - 2\nabla_\alpha u^\alpha \\
&= (g^{\mu\nu} - u^\mu u^\nu)(\nabla_\mu u_\nu + \nabla_\nu u_\mu) - 2\nabla_\alpha u^\alpha \\
&= \nabla^\nu u_\nu + \nabla^\mu u_\mu - 2\nabla_\alpha u^\alpha = 0
\end{aligned}$$

Thus

$$\partial_\mu s^\mu = \frac{1}{T}\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} = \frac{1}{2T}\pi^{\mu\nu}\nabla_{<\mu}u_{\nu>} - \frac{1}{T}\Pi\nabla_\alpha u^\alpha \geq 0$$

RELATIVISTIC NAVIER-STOKES EQUATIONS

The non-relativistic Navier-Stokes equation is of the form

$$\frac{\partial u^i}{\partial t} + u^k \frac{\partial u^i}{\partial x^k} = -\frac{1}{\rho}\frac{\partial p}{\partial x^i} - \frac{1}{\rho}\frac{\partial \Pi^{ki}}{\partial x^k}$$

$$\Pi^{ki} = -\eta\left(\frac{\partial u^i}{\partial x^k} + \frac{\partial u^k}{\partial x^i} - \frac{2}{3}\delta^{ki}\frac{\partial u^l}{\partial x^l}\right) - \zeta\delta^{ki}\frac{\partial u^l}{\partial x^l}$$

with the coefficients for shear viscosity η and bulk viscosity ζ.
So we can identify the equations

$$D\varepsilon + (\varepsilon + p)\partial_\mu u^\mu - \Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} = 0$$

$$(\varepsilon + p)Du^\alpha - \nabla^\alpha p + \Delta^\alpha_\nu \partial_\mu \Pi^{\mu\nu} = 0\,,$$

$$\Pi^{\mu\nu} = \pi^{\mu\nu} - \Delta^{\mu\nu}\Pi\,,$$

and

$$\pi^{\mu\nu} = \eta\nabla^{<\mu}u^{\nu>},\ \Pi = -\zeta\nabla_\alpha u^\alpha,\ \eta \geq 0,\ \zeta \geq 0$$

as the relativistic Navier-Stokes equations.

ACAUSALITY PROBLEM OF THE REL. NAVIER-STOKES EQUATIONS

Unlike the non-relativistic Navier-Stokes equations, the relativistic ones exhibit acausal propagation.
Consider small perturbations to an equilibrium system at rest:

$$\varepsilon = \varepsilon_0 + \delta\varepsilon(t,x), \qquad u^\mu = (1,\mathbf{0}) + \delta u^\mu(t,x)$$

(perturbation only depends on one space variable)
For $\alpha = y$ (transverse), the Navier-Stokes equation gives:

$$(\varepsilon + p)Du^y - \nabla^y p + \Delta^y{}_\nu \partial_\mu \Pi^{\mu\nu} = (\varepsilon_0 + p_0)\partial_t \delta u^y + \partial_x \Pi^{xy} + \mathcal{O}(\delta^2) = 0$$

$$\Pi^{xy} = \eta(\nabla^x u^y + \nabla^y u^x) - \left(\zeta + \frac{2}{3}\eta\right)\Delta^{xy}\nabla_\alpha u^\alpha = -\eta\partial_x \delta u^y + \mathcal{O}(\delta^2)$$

Together, that gives

$$\partial_t \delta u^y - \frac{\eta}{\varepsilon_0 + p_0}\partial_x^2 \delta u^y = \mathcal{O}(\delta^2)$$

PROBLEM WITH RELATIVISTIC NAVIER-STOKES EQUATIONS

To investigate the individual modes of the diffusion process

$$\partial_t \delta u^y - \frac{\eta}{\varepsilon_0 + p_0} \partial_x^2 \delta u^y = \mathcal{O}(\delta^2)$$

we use a mixed Laplace-Fourier wave ansatz:

$$\delta u^y(t, x) = e^{-\omega t + ikx} f_{\omega,k}$$

This leads to the “dispersion relation”

$$\omega = \frac{\eta}{\varepsilon_0 + p_0} k^2$$

We get as the speed of diffusion of a mode with wave number k:

$$v_T(k) = \frac{d\omega}{dk} = 2 \frac{\eta}{\varepsilon_0 + p_0} k$$

As k grows, v_T grows, eventually exceeding the speed of light.
Violates causality.

PROBLEM WITH RELATIVISTIC NAVIER-STOKES EQUATIONS

So what? Hydrodynamics is supposed to be an effective theory for long wavelength modes ($k \to 0$) anyway.
We could just not care about what happens at $k \gg 1$.

However, numerically the high k modes lead to instabilities:

- Modes that travel faster than light in one Lorentz frame, travel backwards in time in another.
- Hydrodynamics is an initial value problem and requires well defined set of initial conditions.
- If modes travel backwards in time, the initial conditions cannot be freely given. So one cannot solve the relativistic Navier-Stokes equations numerically.

The diffusion speed exceeding the speed of light is a hint but no proof of causality violation. See appendix in Romatschke, Int.J.Mod.Phys. E19 (2010) 1-53 for a proof.

FIXING THE RELATIVISTIC NAVIER-STOKES EQUATIONS

One way to regulate the theory is to introduce a relaxation time τ_π, yielding the "Maxwell-Cattaneo law":

$$\tau_\pi \partial_t \Pi^{xy} + \Pi^{xy} = \quad \eta \partial^x \delta u^y$$

which replaces

$$\Pi^{xy} = \quad \eta \partial^x \delta u^y$$

This is useful because it leads to the modified dispersion relation

$$\omega = \frac{\eta}{\varepsilon_0 + p_0} \frac{k^2}{1 - \omega \tau_\pi}$$

For $\omega, k \to 0$, this equals the original dispersion relation.
For $k \gg 1$ v_T is finite (show):

$$v_T^{\max} = \lim_{k \to \infty} \frac{d|\omega|}{dk} = \sqrt{\frac{\eta}{(\varepsilon_0 + p_0)\tau_\pi}} \leq 1 \text{ for all known fluids}$$

Works great. But introduced by hand. Unsatisfactory.

SECOND ORDER VISCOUS HYDRODYNAMICS

When we derived the relativistic Navier-Stokes equations using

$$\partial_\mu s^\mu \geq 0$$

we used the equilibirum entropy current $s^\mu = s u^\mu$. But there are dissipative corrections: use of the equilibrium s^μ may not be good approximation.

Including corrections to the entropy current

$$s^\mu = s u^\mu - \frac{\beta_0}{2T} u^\mu \Pi^2 - \frac{\beta_2}{2T} u^\mu \pi_{\alpha\beta} \pi^{\alpha\beta} + \mathcal{O}(\Pi^3)$$

with coefficients β_0 and β_2, one gets

$$\pi_{\alpha\beta} = \eta \left(\nabla_{<\alpha} u_{\beta>} - \pi_{\alpha\beta} T D \left(\frac{\beta_2}{T} \right) - 2\beta_2 D\pi_{\alpha\beta} - \beta_2 \pi_{\alpha\beta} \partial_\mu u^\mu \right)$$

$$\Pi = \zeta \left(\nabla_\alpha u^\alpha - \frac{1}{2} \Pi T D \left(\frac{\beta_0}{T} \right) - \beta_0 D\Pi - \frac{1}{2} \beta_0 \Pi \partial_\mu u^\mu \right)$$

from the condition $\partial_\mu s^\mu \geq 0$.

DERIVATION FROM KINETIC THEORY

Alternatively, second order viscous hydrodynamics can be derived from kinetic theory.

Short recap of kinetic theory:

The evolution of the one-particle distribution $f(\mathbf{p}, \mathbf{x}, t)$ follows from Liouville's theorem (conservation of density in phase-space):

$$\frac{df}{d\mathcal{T}} = \frac{dt}{d\mathcal{T}}\partial_t f + \frac{d\mathbf{x}}{d\mathcal{T}} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} f = 0$$

Using $m\frac{dt}{d\mathcal{T}} = m\gamma(\mathbf{v}) = p^0$ and $m\frac{d\mathbf{x}}{d\mathcal{T}} = m\mathbf{v}\gamma(\mathbf{v}) = \mathbf{p}$ we get

$$p^\mu \partial_\mu f = 0$$

with $p_\mu p^\mu = m^2$.

Now, with collisions one gets the Boltzmann equation

$$p^\mu \partial_\mu f = -\mathcal{C}[f] \leftarrow \text{functional of f}$$

DERIVATION FROM KINETIC THEORY

In equilibirum $f = f_0(\mathbf{p})$ and

$$p^\mu \partial_\mu f_0 = 0 = -\mathcal{C}[f_0]$$

so $\mathcal{C}[f_0] = 0$.

Hydrodynamics corresponds to the limit where $\mathcal{C}$ is large (short mean free path) and drives the system towards equilibrium.

Now, the relation between $T^{\mu\nu}$ and f is given by

$$T^{\mu\nu} = \int \frac{d^4p}{(2\pi)^3} p^\mu p^\nu \delta(p^\mu p_\mu - m^2) 2\theta(p^0) f(p,x)$$

DERIVATION OF IDEAL HYDRODYNAMICS FROM KINETIC THEORY

Ultrarelativistic limit: $m \to 0$

Taking the first moment of the Boltzmann equation one finds

$$\int d\chi p^\nu p^\mu \partial_\mu f(p^\mu, x^\mu) = -\int d\chi p^\nu \mathcal{C}[f]$$

$$= \partial_\mu \int d\chi p^\nu p^\mu f(p,x) = \partial_\mu T^{\mu\nu}$$

here we use $\int d\chi = \int \frac{d^4p}{(2\pi)^3}\delta(p^\mu p_\mu) 2\theta(p^0) = \int \frac{d^3p}{(2\pi)^3 E}$

When $\mathcal{C}$ conserves energy and momentum

$$\int d\chi p^\nu \mathcal{C}[f] = 0$$

If $T^{\mu\nu}$ can be interpreted as a fluid's energy-momentum tensor (like it can in equilibrium), this means that the first moment of the Boltzmann equation corresponds to the fundamental equations of fluid dynamics: $\partial_\mu T^{\mu\nu} = 0$

DERIVATION OF IDEAL HYDRODYNAMICS FROM KINETIC THEORY

In the relativistic case it is better to write $f_{\mathrm{eq}}(p^\mu u_\mu / T)$ instead of $f_0(\mathbf{p})$ (Lorentz invariance).

With that we can write

$$T_0^{\mu\nu} = \int d\chi p^\mu p^\nu f_{\mathrm{eq}}\left(\frac{p^\mu u_\mu}{T}\right) = a_{20} u^\mu u^\nu + a_{21} \Delta^{\mu\nu}$$

So a_{20} corresponds to energy density ε and $-a_{21}$ to pressure p.

They can be computed by contraction of above expression with $u^\mu u^\nu$ and $\Delta^{\mu\nu}$ respectively.

Their exact values depend on f_{eq}.

VISCOUS HYDRODYNAMICS FROM KINETIC THEORY

Small deviation from equilibrium:

$$f(p^\mu, x^\mu) = f_{\mathrm{eq}}(p^\mu u_\mu / T)(1 + \delta f(p^\mu, x^\mu))$$

with $\delta f \ll 1$. So one can identify

$$T^{\mu\nu} = T_0^{\mu\nu} + \int d\chi p^\mu p^\nu f_{\mathrm{eq}} \delta f = T_0^{\mu\nu} + \pi^{\mu\nu}$$

Momentum dependence of δf can be expressed in a Taylor series

$$\delta f(p^\mu, x^\mu) = c + p^\alpha c_\alpha + p^\alpha p^\beta c_{\alpha\beta} + \mathcal{O}(p^3)$$

and is an algebraic function of $\varepsilon, p, u^\mu, g^{\mu\nu}, \text{ and } \pi^{\mu\nu}$.
δf vanishes in equilibrium $\rightarrow$ $c = 0, c_\alpha = 0, c_{\alpha\beta} = c_2 \pi_{\alpha\beta}$ so

$$\pi^{\mu\nu} = \pi_{\alpha\beta} c_2 I^{\mu\nu\alpha\beta}$$

with $I^{\mu_1\mu_2\dots\mu_n} = \int d\chi p^{\mu_1} p^{\mu_2} \dots p^{\mu_n} f_{\mathrm{eq}}$

VISCOUS HYDRODYNAMICS FROM KINETIC THEORY

$I^{\mu\nu\alpha\beta}$ can be decomposed into Lorentz tensors:

$$I^{\mu\nu\alpha\beta} = a_{40} u^\mu u^\nu u^\alpha u^\beta + a_{41}(u^\mu u^\nu \Delta^{\alpha\beta} + \text{perm.}) + a_{42}(\Delta^{\mu\nu}\Delta^{\alpha\beta} + \Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha})$$

Because $u_\mu \pi^{\mu\nu} = 0$ and $\pi^\mu_\mu = 0,\ (\Delta_{\mu\nu}\pi^{\mu\nu} = 0)$, a_{40} and a_{41} vanish.

Contracting the indices on the RHS of

$$\pi^{\mu\nu} = \pi_{\alpha\beta} c_2 I^{\mu\nu\alpha\beta}$$

we find

$$c_2 = \frac{1}{2a_{42}}$$

and finally

$$f(p^\mu, x^\mu) = f_{\text{eq}}\left(\frac{p^\mu u_\mu}{T}\right)\left[1 + \overbrace{\frac{1}{2a_{42}} p^\alpha p^\beta \pi_{\alpha\beta}}^{\delta f}\right]$$

VISCOUS HYDRODYNAMICS FROM KINETIC THEORY

$$f(p^\mu, x^\mu) = f_{\rm eq}\left(\frac{p^\mu u_\mu}{T}\right)\left[1 + \frac{1}{2a_{42}} p^\alpha p^\beta \pi_{\alpha\beta}\right]$$

for a Boltzmann gas $f_{\rm eq}(x) = e^{-x}$ and

$$a_{42} = (\varepsilon + \mathcal{P})T^2$$

which follows from contraction of $I^{\mu\nu\alpha\beta}$ with the appropriate projector

VISCOUS HYDRODYNAMICS FROM KINETIC THEORY

The first moment of the Boltzmann equation gave conservation of $T^{\mu\nu}$.

The integral

$$\int d\chi p^\alpha p^\beta \mathcal{C}$$

does not trivially vanish (except in equilibrium).

So, the second moment of the Boltzmann equation

$$\int d\chi p^\alpha p^\beta p^\mu \partial_\mu f = - \int d\chi p^\alpha p^\beta \mathcal{C}[f]$$

will carry information on the non-equilibrium (viscous) dynamics.

From our earlier expansion of f, we find for the LHS

$$\int d\chi p^\alpha p^\beta p^\mu \partial_\mu f = \partial_\mu \left(I^{\alpha\beta\mu} + \frac{\pi_{\gamma\delta}}{2a_{42}} I^{\alpha\beta\mu\gamma\delta} \right)$$

Using the second moment was a choice by Israel and Stewart (1979). Other moments could be used, leading to some ambiguity. For analysis and improvements on this see e.g. **Denicol, Molnar, Niemi, and Rischke, arXiv:1206.1554**

VISCOUS HYDRODYNAMICS FROM KINETIC THEORY

Now, projecting on the part that is symmetric and traceless with

$$P^{\mu\nu}_{\alpha\beta} = \Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\mu_\alpha - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$$

and doing the same for the RHS (assuming Boltzmann statistics) it follows from the second moment of the Boltzmann equation that

$$\pi^{\mu\nu} + \frac{a_{52}T\eta}{a_{42}^2}\left[\Delta^\mu_\alpha \Delta^\nu_\beta D\pi^{\alpha\beta} + P^{\mu\nu}_{\alpha\beta}\pi^{\phi\beta}\nabla_\phi u^\alpha + \frac{2}{3}\pi^{\mu\nu}\partial_\alpha u^\alpha\right] = \eta\nabla^{<\mu}u^{\nu>}$$

The expression $P^{\mu\nu}_{\alpha\beta}\pi^{\phi\beta}\nabla_\phi u^\alpha$ can be rewritten when introducing the fluid vorticity

$$\Omega_{\alpha\beta} = \nabla_{[\alpha}u_{\beta]}$$

where $A_{[\mu}B_{\nu]} = \frac{1}{2}(A_\mu B_\nu - A_\nu B_\mu)$ is anti-symmetrization

VISCOUS HYDRODYNAMICS FROM KINETIC THEORY

One finds

$$P^{\mu\nu}_{\alpha\beta}\pi^{\phi\beta}\nabla_\phi u^\alpha = P^{\mu\nu}_{\alpha\beta}\Delta^{\alpha\gamma}\pi^{\phi\beta}\left[\Omega_{\phi\gamma} + \frac{1}{2}\nabla_{<\phi}u_{\gamma>} + \frac{1}{3}\Delta_{\phi\gamma}\nabla_\delta u^\delta\right]$$

$$= -2\pi^{\phi(\mu}\Omega^{\nu)}_{\phi} + \frac{\pi^{\phi<\mu}\pi^{\nu>}_{\phi}}{2\eta} + \frac{2}{3}\pi^{\mu\nu}\nabla_\delta u^\delta + \mathcal{O}(\delta^3)$$

Finally we get

$$\pi^{\mu\nu} + \overbrace{\frac{a_{52}T\eta}{a_{42}^2}}^{\tau_\pi}\left[\Delta^\mu_\alpha\Delta^\nu_\beta D\pi^{\alpha\beta} + \frac{4}{3}\pi^{\mu\nu}\nabla_\alpha u^\alpha - 2\pi^{\phi(\mu}\Omega^{\nu)}_{\phi} + \frac{\pi^{\phi<\mu}\pi^{\nu>}_{\phi}}{2\eta}\right] = \eta\nabla^{<\mu}u^{\nu>} + \mathcal{O}(\delta^2)$$

where τ_π is the second order transport coefficient “relaxation time”.

VISCOUS HYDRODYNAMICS FROM KINETIC THEORY

Again, for $k \gg 1$ the "dispersion relation" for the diffusion equation becomes

$$\omega \approx \frac{\eta}{\varepsilon + p} \frac{k^2}{\omega \tau_\pi}$$

We don't know τ_π if the underlying theory is unknown.
For a massless Boltzmann gas we get

$$\tau_\pi = \frac{a_{52} T \eta}{a_{42}^2} = \frac{3}{2} \pi^2 \frac{\eta}{T^4}$$

So

$$\omega \approx \sqrt{\frac{2}{3} \frac{T^4}{\pi^2 (\varepsilon + p)}}\, k = \sqrt{\frac{1}{6}}\, k$$

such that $v_T^{\max} = \sqrt{\frac{1}{6}}$.

Studying long. velocity perturbations (sound) one finds $v_L^{\max} = \sqrt{\frac{5}{9}}$.
Bose-Einstein statistics lead to only small numerical modifications.

VISCOUS HYDRODYNAMICS FROM KINETIC THEORY

The final result

$$\pi^{\mu\nu}+\tau_\pi\left[\Delta^\mu_\alpha\Delta^\nu_\beta D\pi^{\alpha\beta}+\frac{4}{3}\pi^{\mu\nu}\nabla_\alpha u^\alpha-2\pi^{\phi(\mu}\Omega^{\nu)}_\phi+\frac{\pi^{\phi<\mu}\pi^{\nu>}_\phi}{2\eta}\right]=\eta\nabla^{<\mu}u^{\nu>}$$

$$+\mathcal{O}(\delta^2)$$

Müller (1976), Israel and Stewart (1979)

is different from what we found from the second law of thermodynamics

$$\pi_{\alpha\beta}=\eta\left(\nabla_{<\alpha}u_{\beta>}-\pi_{\alpha\beta}TD\left(\frac{\beta_2}{T}\right)-2\beta_2 D\pi_{\alpha\beta}-\beta_2\pi_{\alpha\beta}\partial_\mu u^\mu\right)$$

which for a Boltzmann gas ($\beta_2=\frac{\tau_\pi}{2\eta}=\frac{3}{4p}$) reads

$$\pi^{\mu\nu}+\tau_\pi\left[D\pi^{\mu\nu}+\frac{4}{3}\pi^{\mu\nu}\nabla_\alpha u^\alpha\right]=\eta\nabla^{<\mu}u^{\nu>}+\mathcal{O}(\delta^2)$$

where we used $\tau_\pi/(\eta T)\sim T^{-5}$ and
$D\ln T=D\ln(\varepsilon^{1/4})=-\frac{1}{3}\nabla_\alpha u^\alpha+\mathcal{O}(\delta^2)$ (from Navier-Stokes).

VISCOUS HYDRODYNAMICS FROM KINETIC THEORY

Differences between the two equations vanish when contracted with $\pi_{\mu\nu}$, hence do not contribute to entropy production.

recall $s^\mu = su^\mu - \frac{\beta_0}{2T}u^\mu \Pi^2 - \frac{\beta_2}{2T}u^\mu \pi_{\alpha\beta}\pi^{\alpha\beta}$

So the entropy-wise derivation could not capture the terms.

However, the terms are important:

Contraction with u_μ gives zero for the kinetic theory result.

But leads to an unphysical constraint $u_\mu D\pi^{\mu\nu} = 0$ for the entropy res.

$\Rightarrow$ kinetic theory result is superior.

But

$$\pi^{\mu\nu} + \tau_\pi \left[\Delta^\mu_\alpha \Delta^\nu_\beta D\pi^{\alpha\beta} + \frac{4}{3}\pi^{\mu\nu}\nabla_\alpha u^\alpha - 2\pi^{\phi(\mu}\Omega^{\nu)}_{\phi} + \frac{\pi^{\phi<\mu}\pi^{\nu>}_{\phi}}{2\eta} \right] = \eta \nabla^{<\mu}u^{\nu>} + \mathcal{O}(\delta^2)$$

misses terms of second order in gradients (because we don't know the collision term).