Hadron Structure

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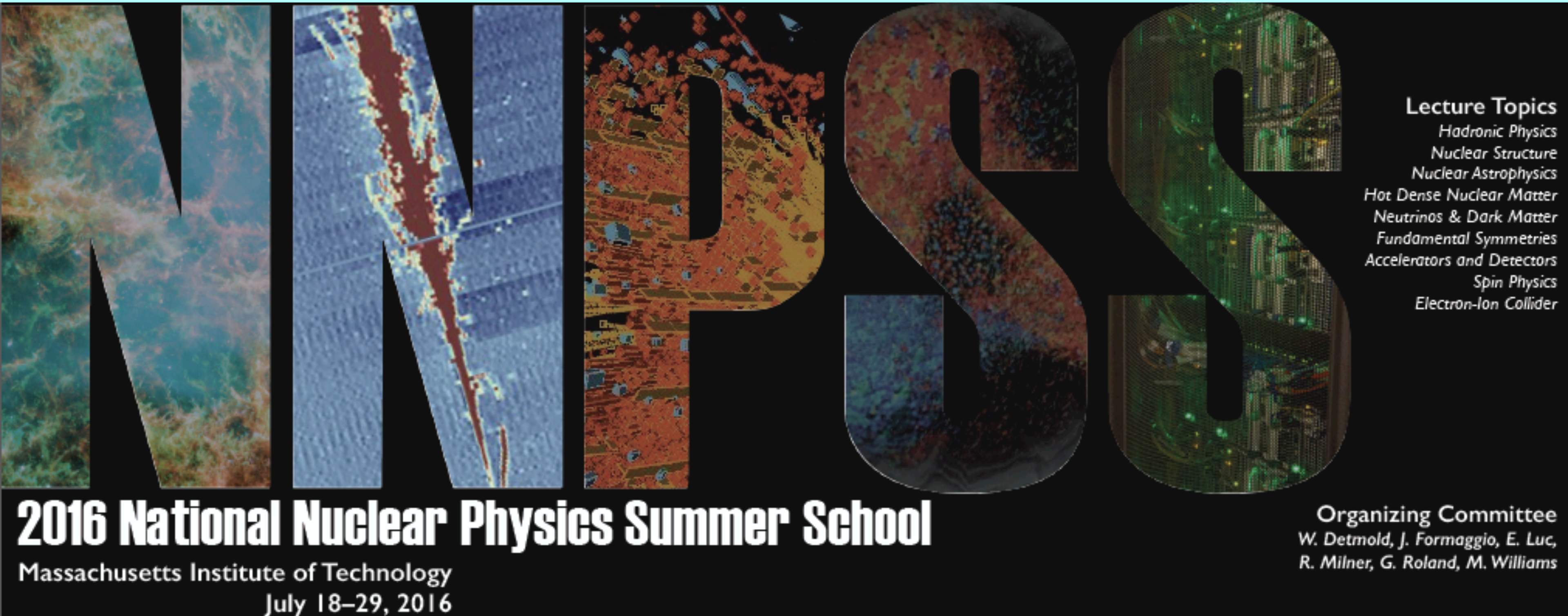


Lecture Topics
Hadronic Physics
Nuclear Structure
Nuclear Astrophysics
Hot Dense Nuclear Matter
Neutrinos & Dark Matter
Fundamental Symmetries
Accelerators and Detectors
Spin Physics
Electron-Ion Collider

2016 National Nuclear Physics Summer School
Massachusetts Institute of Technology
July 18–29, 2016

Organizing Committee
W. Detmold, J. Formaggio, E. Luc,
R. Milner, G. Roland, M. Williams

The plan for my three lectures

- ☐ **The Goal:**

  To understand the hadron structure in terms of QCD and its hadronic matrix elements of quark-gluon field operators,
  to connect these matrix elements to physical observables, and
  to calculate them from QCD (lattice QCD, inspired models, …)

- ☐ **The outline:**

  Hadrons, partons (quarks and gluons),
  and probes of hadron structure

  One lecture

  Parton Distribution Functions (PDFs) and
  Transverse Momentum Dependent PDFs (TMDs)

  One lecture

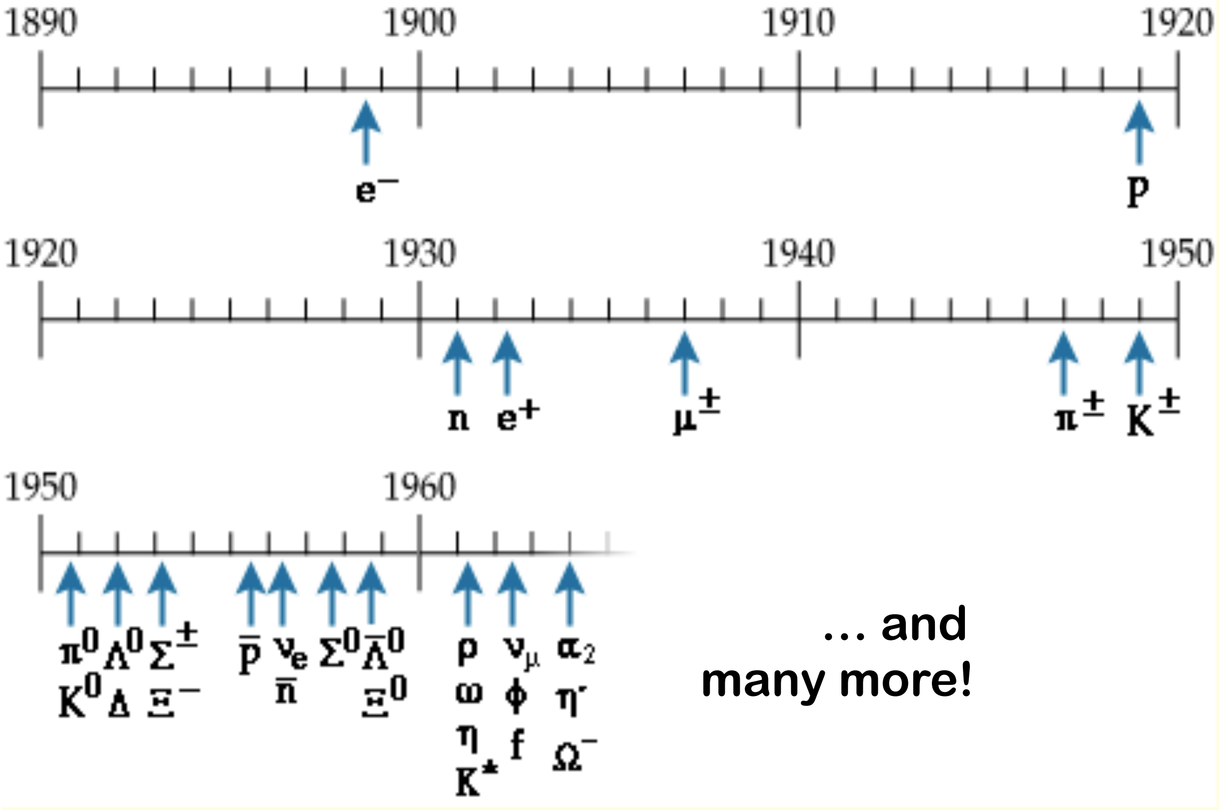
  Generalized PDFs (GPDs) and multi-parton correlation functions

  One lecture

See also
lectures by Shepard on
"Hadron Spectroscopy",
and
lectures by Deshpande on
"Electron-Ion Collider"
and
lectures by Gandolfi on
"Nuclear Structure"
and
lectures by Aschenauer on
"Accelerators & detectors"

New particles, new ideas, and new theories

- ❑ Early proliferation of new hadrons – "particle explosion":



… and many more!

Hadrons have internal structure!

❑ Nucleons cannot be point-like spin-1/2 Dirac particles:

1933: Proton's magnetic moment

Otto Stern

Nobel Prize 1943

$$\mu_p = g_p \left(\frac{e\hbar}{2m_p}\right)$$

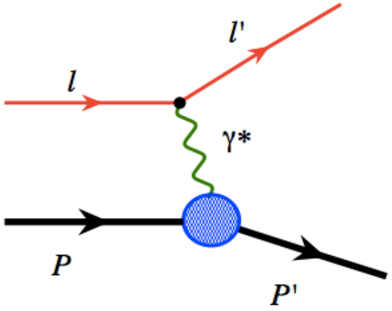
$$g_p = 2.792847356(23) \neq 2!$$

$$\mu_n = -1.913 \left(\frac{e\hbar}{2m_p}\right) \neq 0!$$

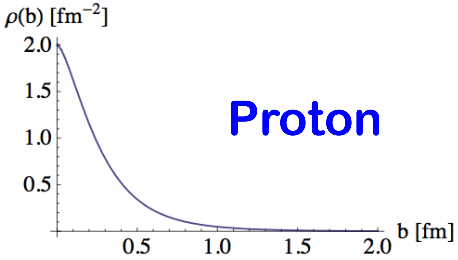
1960: Elastic e-p scattering

Robert Hofstadter

Nobel Prize 1961

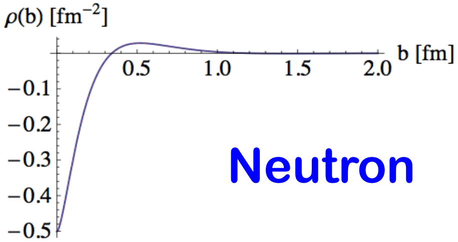


Form factors



Proton

Proton EM charge radius!

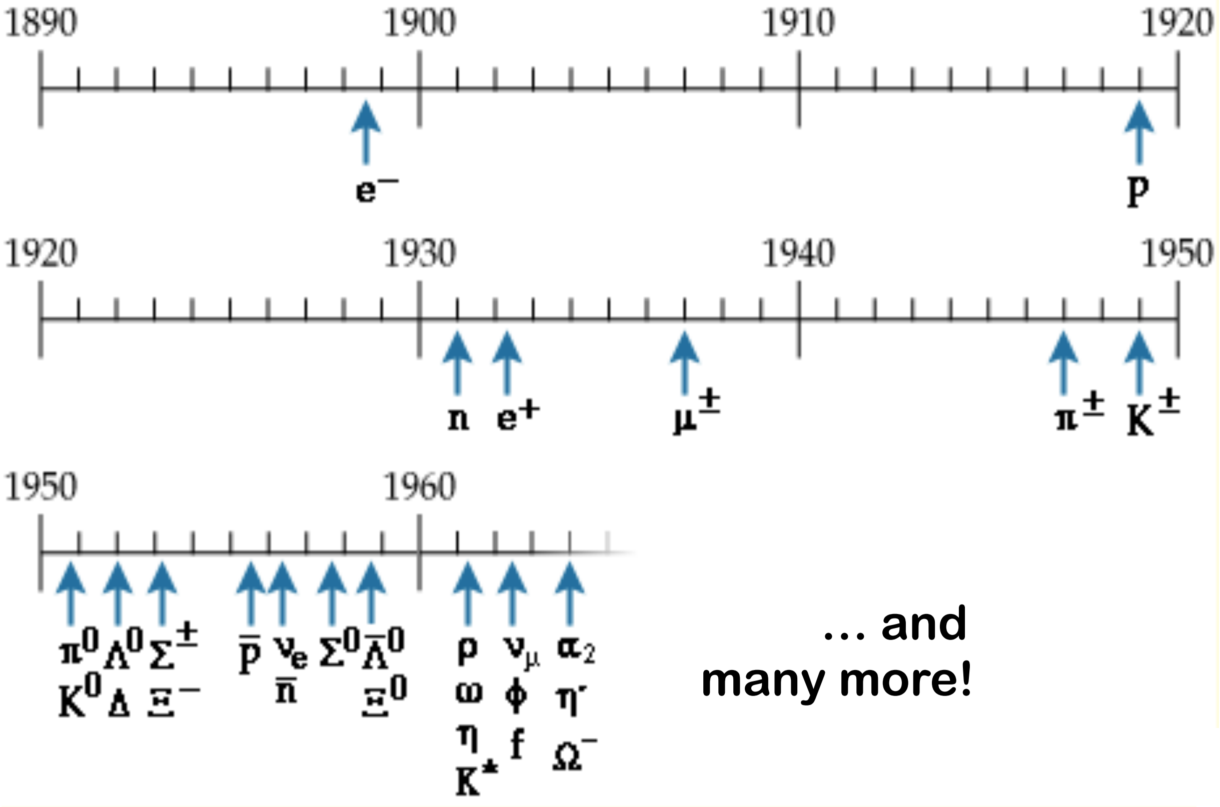


Neutron

Electric charge distribution

New particles, new ideas, and new theories

❑ Early proliferation of new particles – “particle explosion”:

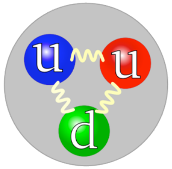


… and
many more!

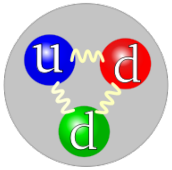
→ **Quark Model**

Nobel Prize, 1969

Proton



Neutron



The naïve Quark Model

- ❑ **Flavor SU(3) – assumption:**

  Physical states for u, d, s, neglecting any mass difference, are represented by 3-eigenstates of the fund'l rep'n of flavor SU(3)

- ❑ **Generators for the fund'l rep'n of SU(3) – 3x3 matrices:**

$$J_i = \frac{\lambda_i}{2}$$ with $\lambda_i, i = 1, 2, ..., 8$ Gell-Mann matrices

- ❑ **Good quantum numbers to label the states:**

$$J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad J_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

simultaneously diagonalized

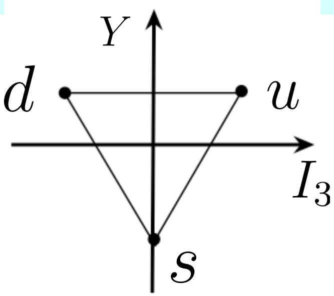
Isospin: $\hat{I}_3 \equiv J_3$, Hypercharge: $\hat{Y} \equiv \frac{2}{\sqrt{3}} J_8$

- ❑ **Basis vectors – Eigenstates:** $|I_3, Y\rangle$

$$v^1 \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow u = |\tfrac{1}{2}, \tfrac{1}{3}\rangle \qquad v^2 \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Longrightarrow d = |\text{-}\tfrac{1}{2}, \tfrac{1}{3}\rangle \qquad v^3 \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Longrightarrow s = |0, \text{-}\tfrac{2}{3}\rangle$$

The naïve Quark Model

❑ Quark states:



$$u = |\tfrac{1}{2}, \tfrac{1}{3}\rangle \quad d = |\text{-}\tfrac{1}{2}, \tfrac{1}{3}\rangle \quad s = |0,\text{-}\tfrac{2}{3}\rangle$$

Spin: **½**

Baryon #: ***B* = ⅓**

Strangeness: ***S = Y – B*** **Electric charge:** $Q \equiv I_3 + \dfrac{Y}{2}$

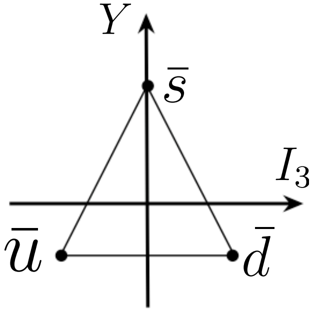
$$u\begin{cases} Q = 2/3\,e \\ s = 1/2 \\ I_3 = 1 \\ Y = 1/3 \\ B = 1/3 \\ S = 0 \end{cases} \qquad d\begin{cases} Q = -1/3\,e \\ s = 1/2 \\ I_3 = -1 \\ Y = 1/3 \\ B = 1/3 \\ S = 0 \end{cases} \qquad s\begin{cases} Q = -1/3\,e \\ s = 1/2 \\ I_3 = 0 \\ Y = -2/3 \\ B = 1/3 \\ S = -1 \end{cases}$$

❑ Antiquark states: $v_i \equiv \epsilon_{ijk}\, v^j v^k$

$$\hat{I}_3 v_1 = \epsilon_{123}[(\hat{I}_3 v^2)v^3 + v^2(\hat{I}_3 v^3)] + \epsilon_{132}[(\hat{I}_3 v^3)v^2 + v^3(\hat{I}_3 v^2)] = -\frac{1}{2}v_1$$

$$\hat{Y} v_1 = \epsilon_{123}[(\hat{Y} v^2)v^3 + v^2(\hat{Y} v^3)] + \epsilon_{132}[(\hat{Y} v^3)v^2 + v^3(\hat{Y} v^2)] = -\frac{1}{3}v_1$$

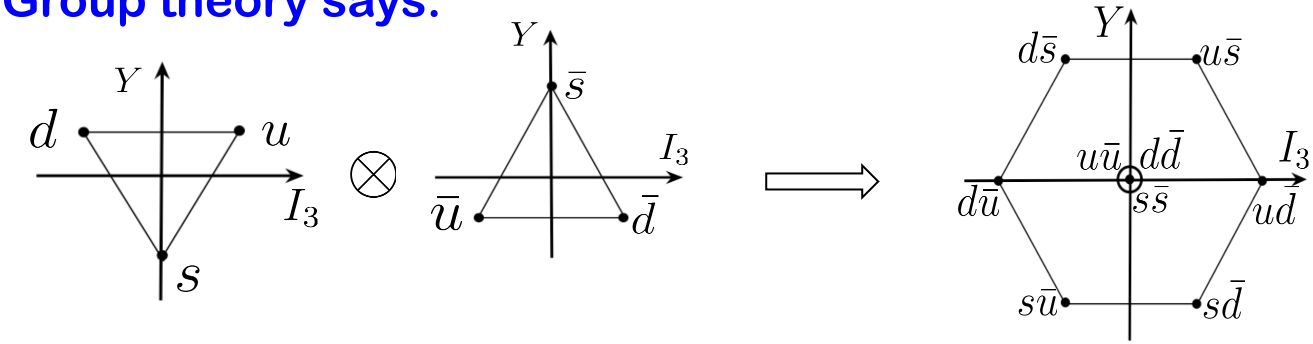
$$u \longrightarrow \bar{u} = |\text{-}\tfrac{1}{2},\text{-}\tfrac{1}{3}\rangle$$



The naïve Quark Model

❑ **Mesons = quark-antiquark $q\bar{q}$ flavor states:** $B = 0$

✧ **Group theory says:**



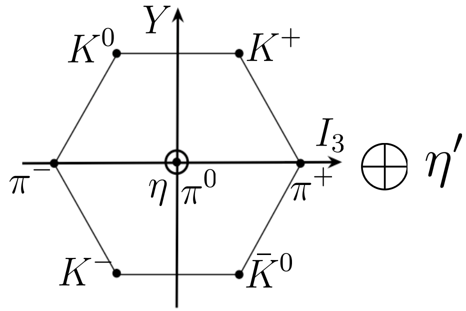
There are three states with $I_3 = 0, Y = 0$: $u\bar{u}, d\bar{d}, s\bar{s}$

✧ **Physical meson states (L=0, S=0):**

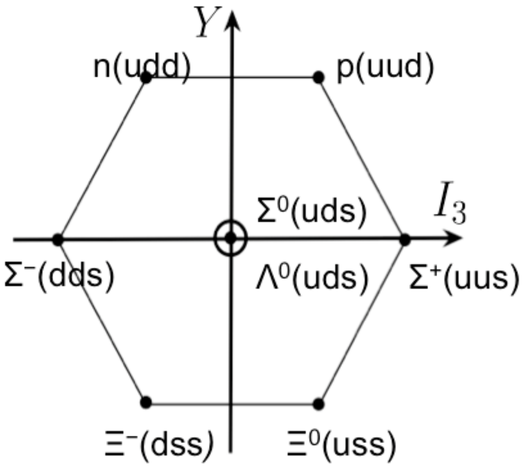
$$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \qquad \eta_8 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$

$$\eta_1 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

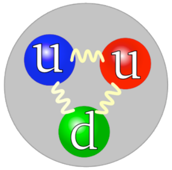
$(\eta_8, \eta_1) \rightarrow (\eta, \eta')$



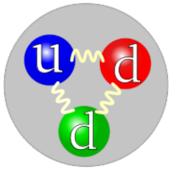
❑ **Baryon states = 3 quark qqq states:** $B = 1$



Proton



Neutron



The naïve Quark Model

- **A complete example: Proton**

$$|p\uparrow\rangle = \frac{1}{\sqrt{18}}\left[uud(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow) + udu(\uparrow\uparrow\downarrow + \downarrow\uparrow\uparrow - 2\uparrow\downarrow\uparrow) + duu(\uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow - 2\downarrow\uparrow\uparrow)\right]$$

- **Normalization:**

$$\langle p\uparrow|p\uparrow\rangle = \frac{1}{18}[(1+1+(-2)^2)+(1+1+(-2)^2)+(1+1+(-2)^2)] = 1$$

- **Charge:**

$$\hat{Q} = \sum_{i=1}^{3}\hat{Q}_i$$

$$\langle p\uparrow|\hat{Q}|p\uparrow\rangle = \frac{1}{18}[(\tfrac{2}{3}+\tfrac{2}{3}-\tfrac{1}{3})(1+1+(-2)^2)+(\tfrac{2}{3}-\tfrac{1}{3}+\tfrac{2}{3})(1+1+(-2)^2) + (-\tfrac{1}{3}+\tfrac{2}{3}+\tfrac{2}{3})(1+1+(-2)^2)] = 1$$

- **Spin:**

$$\hat{S} = \sum_{i=1}^{3}\hat{s}_i$$

$$\langle p\uparrow|\hat{S}|p\uparrow\rangle = \frac{1}{18}\{[(\tfrac{1}{2}-\tfrac{1}{2}+\tfrac{1}{2})+(-\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2})+4(\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{2})] + [\tfrac{1}{2}+\tfrac{1}{2}+4\tfrac{1}{2}]+[\tfrac{1}{2}+\tfrac{1}{2}+4\tfrac{1}{2}]\} = \frac{1}{2}$$

- **Magnetic moment:**

$$\mu_p = \langle p\uparrow|\sum_{i=1}^{3}\hat{\mu}_i(\hat{\sigma}_3)_i|p\uparrow\rangle = \frac{1}{3}[4\mu_u - \mu_d]$$

$$\mu_n = \frac{1}{3}[4\mu_d - \mu_u]$$

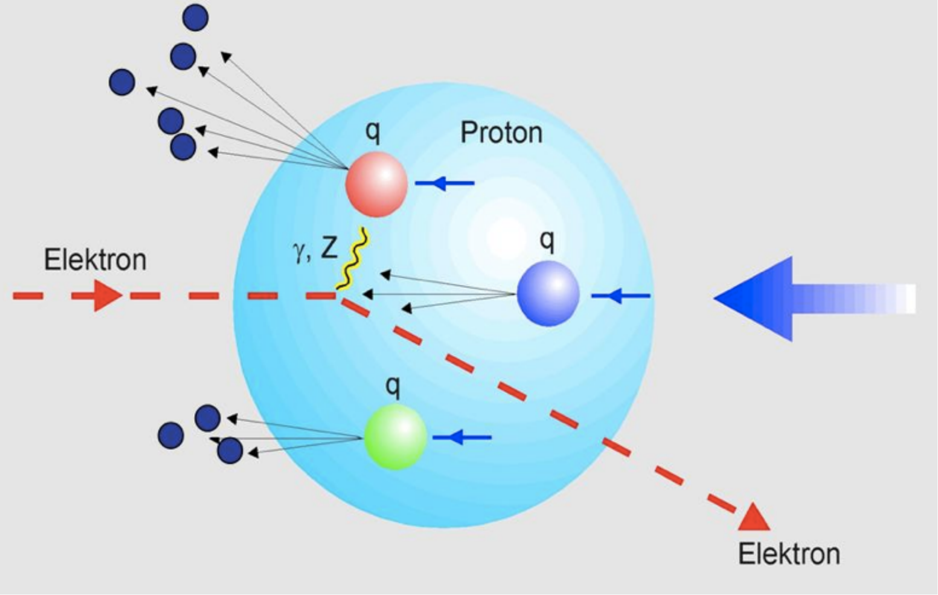
$$\frac{\mu_u}{\mu_d} \approx \frac{2/3}{-1/3} = -2$$

$$\Rightarrow \begin{cases} \left(\dfrac{\mu_n}{\mu_p}\right)_{\mathrm{QM}} = -\dfrac{2}{3} \\ \left(\dfrac{\mu_n}{\mu_p}\right)_{\mathrm{Exp}} = -0.68497945(58) \end{cases}$$

Deep inelastic scattering (DIS)

❑ Modern Rutherford experiment – DIS (SLAC 1968)

$$e(p) + h(P) \to e'(p') + X$$



✧ **Localized probe:**

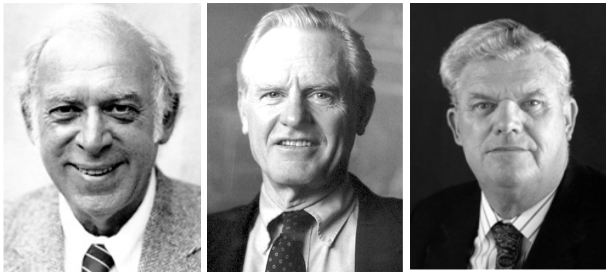
$$Q^2 = -(p-p')^2 \gg 1\ \mathrm{fm}^{-2}$$

➡ $$\frac{1}{Q} \ll 1\ \mathrm{fm}$$

✧ **Two variables:**

$$Q^2 = 4EE'\sin^2(\theta/2)$$
$$x_B = \frac{Q^2}{2m_N\nu}$$
$$\nu = E - E'$$

⟶ **Discovery of spin ½ quarks, and partonic structure!**



Nobel Prize, 1990

➡ **The birth of QCD (1973)**

– Quark Model + Yang-Mill gauge theory

Quantum Chromo-dynamics (QCD)

= A quantum field theory of quarks and gluons =

- ☐ **Fields:**

$\psi_i^f(x)$ Quark fields: spin-½ Dirac fermion (like electron)

Color triplet: $i = 1, 2, 3 = N_c$

Flavor: $f = u, d, s, c, b, t$

$A_{\mu,a}(x)$ Gluon fields: spin-1 vector field (like photon)

Color octet: $a = 1, 2, ..., 8 = N_c^2 - 1$

- ☐ **QCD Lagrangian density:**

$$\mathcal{L}_{QCD}(\psi, A) = \sum_f \overline{\psi}_i^f \left[(i\partial_\mu \delta_{ij} - gA_{\mu,a}(t_a)_{ij})\gamma^\mu - m_f \delta_{ij} \right] \psi_j^f$$
$$-\frac{1}{4}\left[\partial_\mu A_{\nu,a} - \partial_\nu A_{\mu,a} - gC_{abc}A_{\mu,b}A_{\nu,c}\right]^2$$
$$+ \text{ gauge fixing} + \text{ghost terms}$$

- ☐ **QED – force to hold atoms together:**

$$\mathcal{L}_{QED}(\phi, A) = \sum_f \overline{\psi}^f \left[(i\partial_\mu - eA_\mu)\gamma^\mu - m_f \right] \psi^f - \frac{1}{4}\left[\partial_\mu A_\nu - \partial_\nu A_\mu\right]^2$$

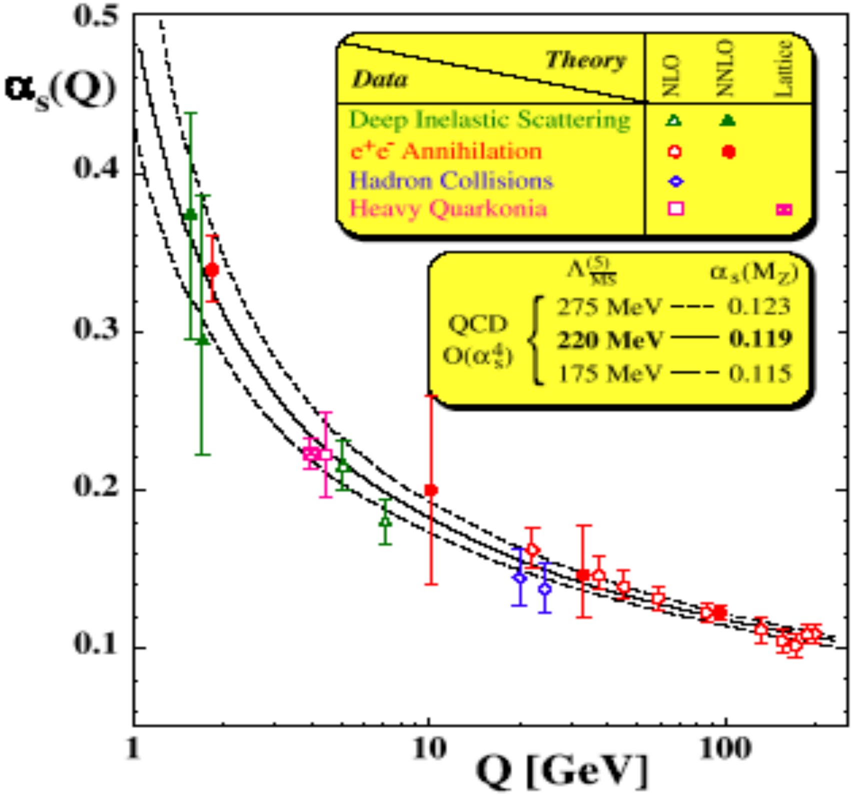
- ☐ **QCD Color confinement:**

Gluons are dark, No free quarks or gluons ever been detected!

QCD Asymptotic Freedom

☐ **Interaction strength:** $$\alpha_s(\mu_2) = \frac{\alpha_s(\mu_1)}{1-\frac{\beta_1}{4\pi}\alpha_s(\mu_1)\ln\left(\frac{\mu_2^2}{\mu_1^2}\right)} \equiv \frac{4\pi}{-\beta_1\ln\left(\frac{\mu_2^2}{\Lambda^2_{\rm QCD}}\right)}$$

μ_2 **and** μ_1 **not independent**



Asymptotic Freedom ⇔ antiscreening

QCD: $$\frac{\partial\alpha_s(Q^2)}{\partial\ln Q^2} = \beta(\alpha_s) < 0$$

Compare

QED: $$\frac{\partial\alpha_{EM}(Q^2)}{\partial\ln Q^2} = \beta(\alpha_{EM}) > 0$$

D.Gross, F.Willczek, Phys.Rev.Lett 30, (1973)
H.Politzer, Phys.Rev.Lett. 30, (1973)

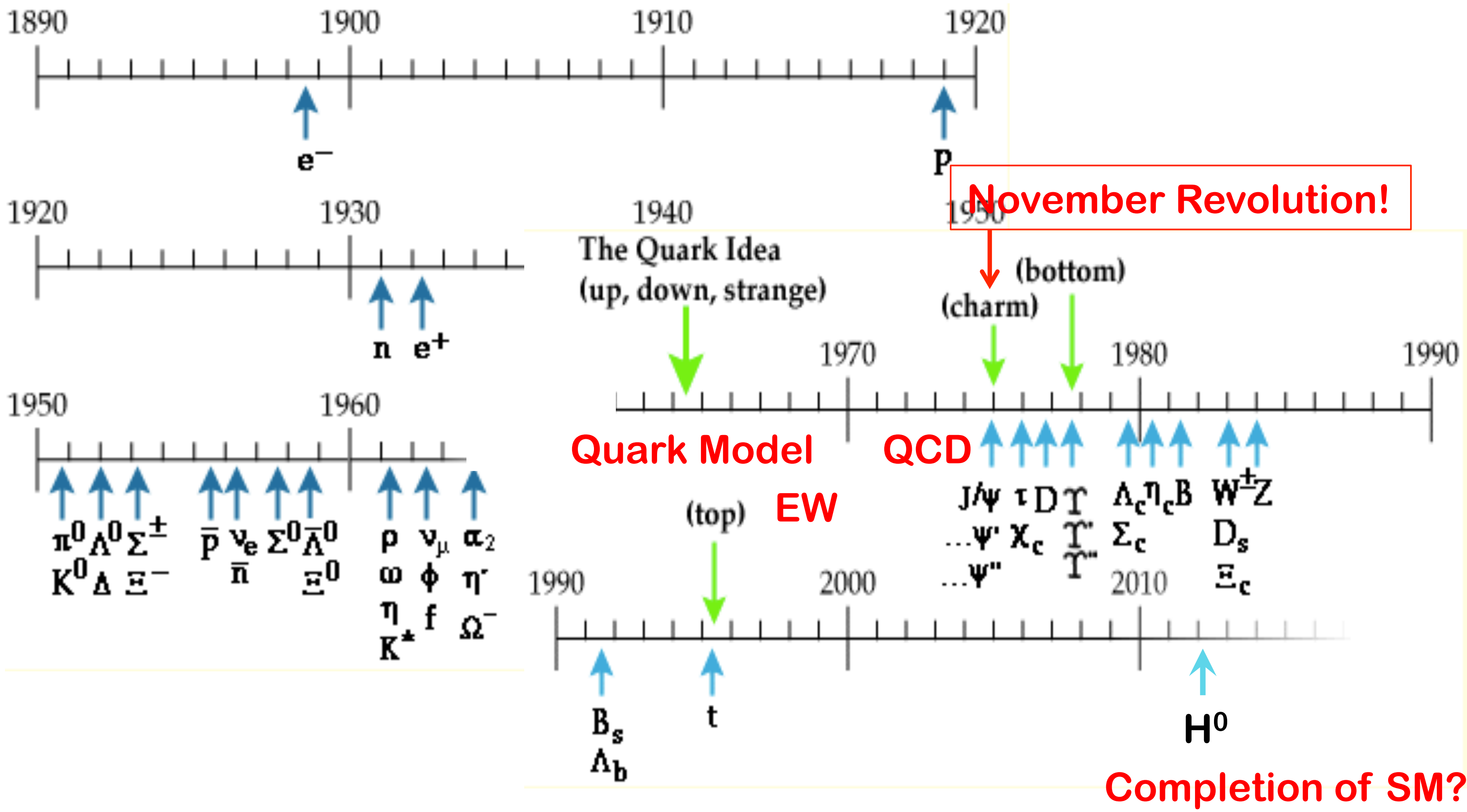
→ **Discovery of QCD Asymptotic Freedom**

➡ **Collider phenomenology**
– ***Controllable perturbative QCD calculations***

Nobel Prize, 2004

New particles, new ideas, and new theories

☐ Proliferation of new particles – “November Revolution”:



November Revolution!

Quark Model

QCD

EW

Completion of SM?

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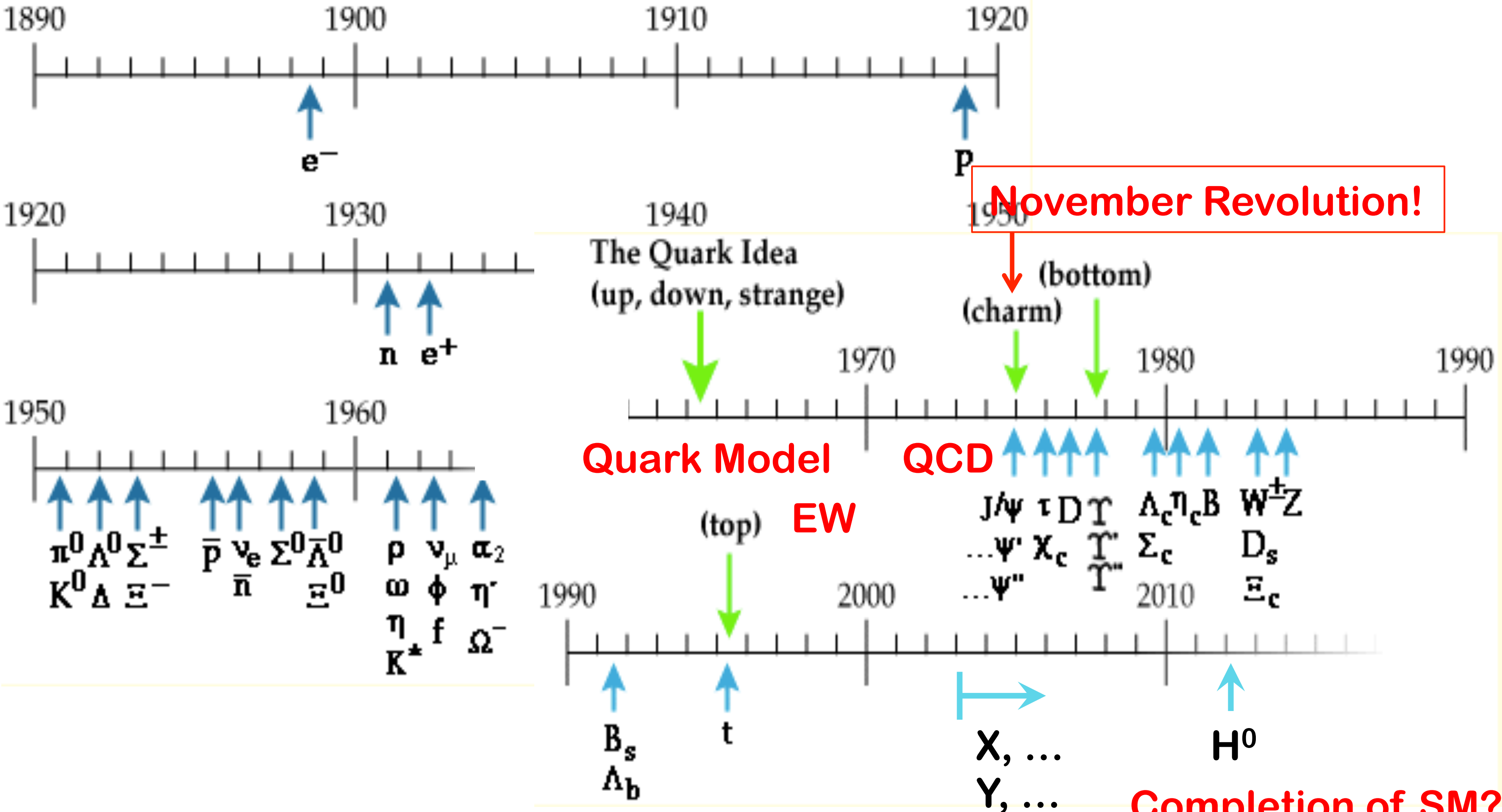


November Revolution!

Quark Model

QCD

EW

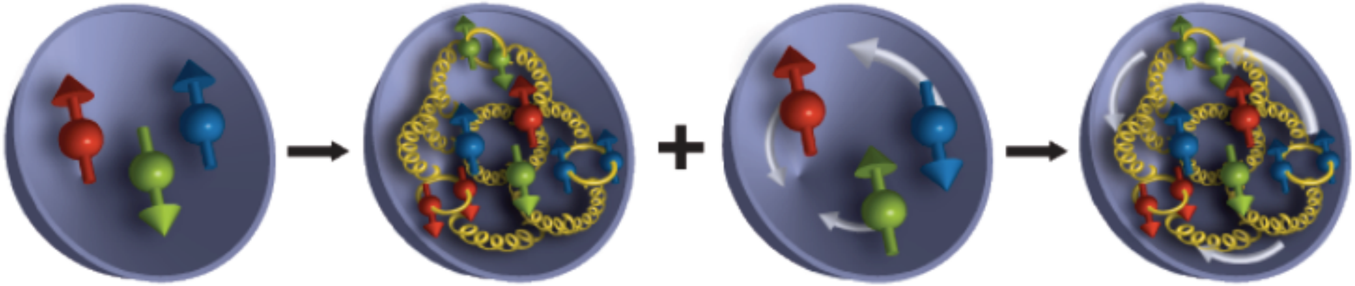
X, …
Y, …
Z, …
Pentaquark, …

Completion of SM?

A new particle explosion?

QCD and hadron internal structure

❑ Our understanding of the proton evolves



1970s 1980s/2000s Now

Hadron is a strongly interacting, relativistic bound state of quarks and gluons

❑ QCD bound states:

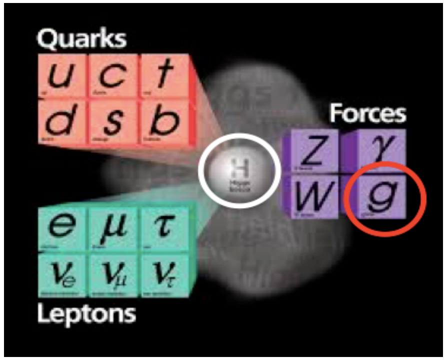
- ✧ Neither quarks nor gluons appear in isolation!
- ✧ Understanding such systems completely is still beyond the capability of the best minds in the world

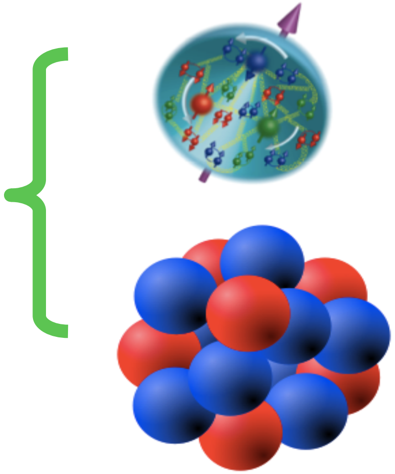
❑ The great intellectual challenge:

Probe nucleon structure without “seeing” quarks and gluons?

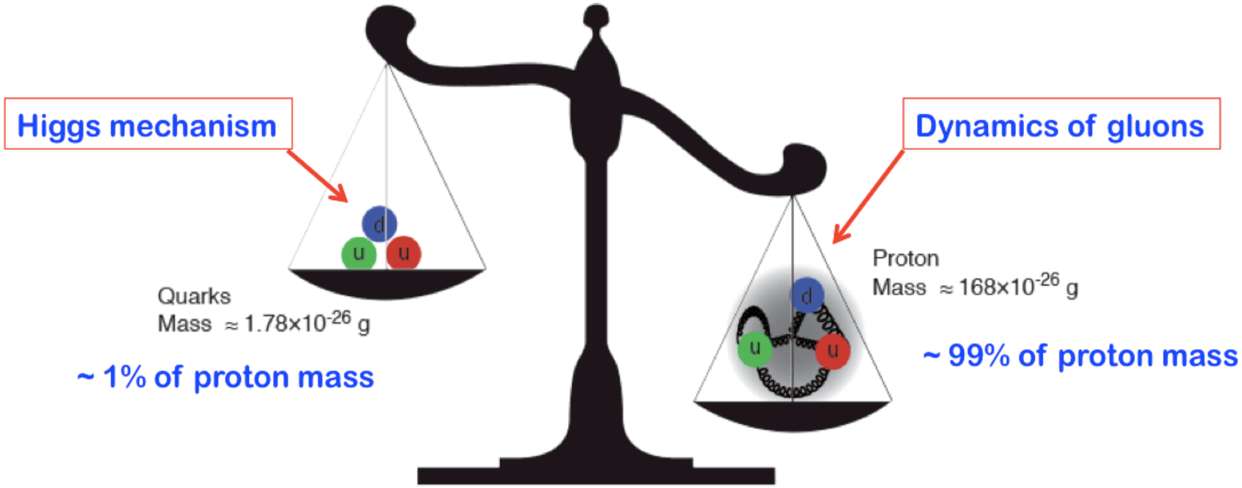
What holds hadron together – the glue?

- ❑ Understanding the glue that binds us all – the Next QCD Frontier!





- ❑ Gluons are weird particles!
  - ✧ Massless, yet, responsible for nearly all visible mass



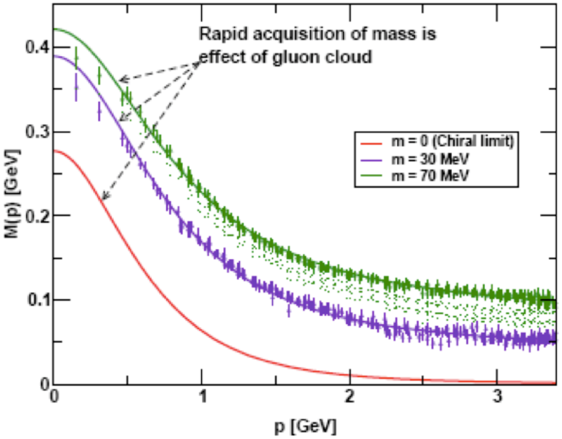
Higgs mechanism

~ 1% of proton mass

Dynamics of gluons

~ 99% of proton mass

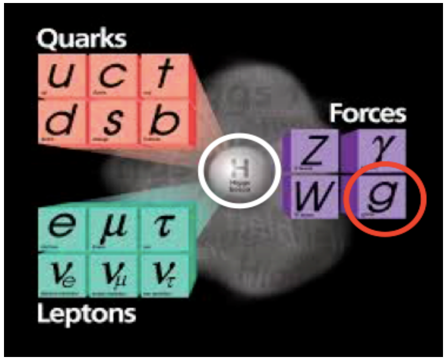
“Mass without mass!”

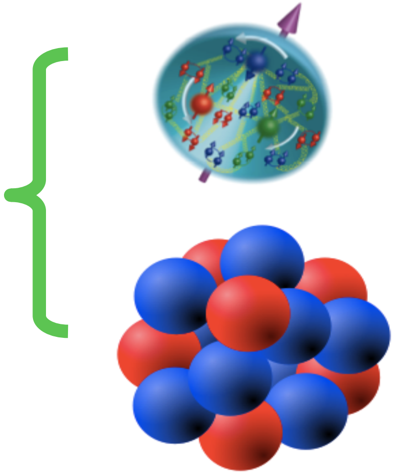


Bhagwat & Tandy/Roberts et al

What holds it together – the glue?

- ❑ **Understanding the glue that binds us all – the Next QCD Frontier!**





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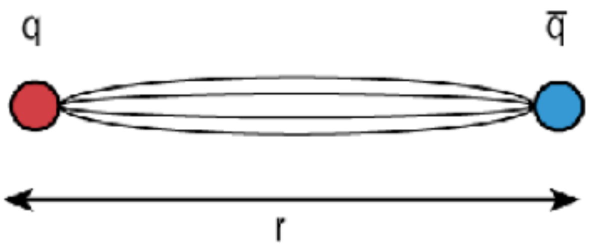
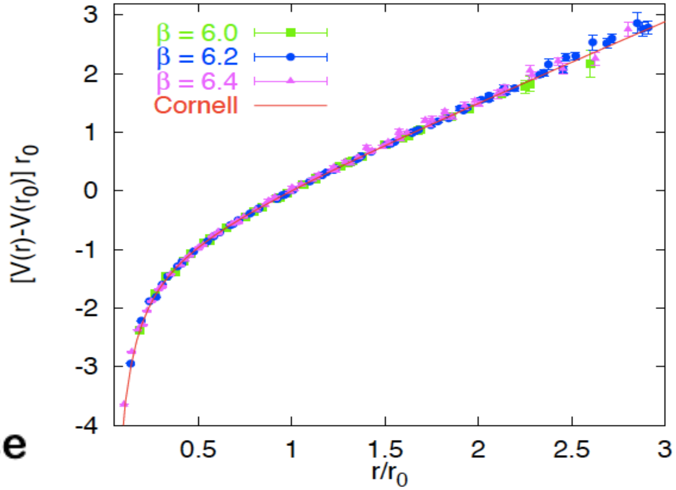
  - ✧ Carry color charge, responsible for color confinement and strong force

Force between a heavy quark pair

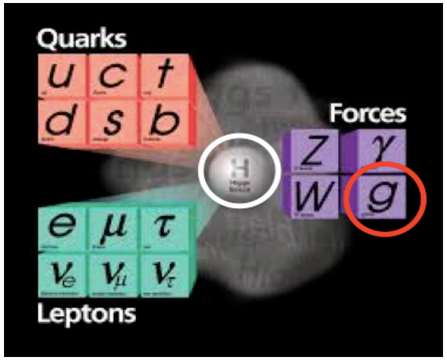


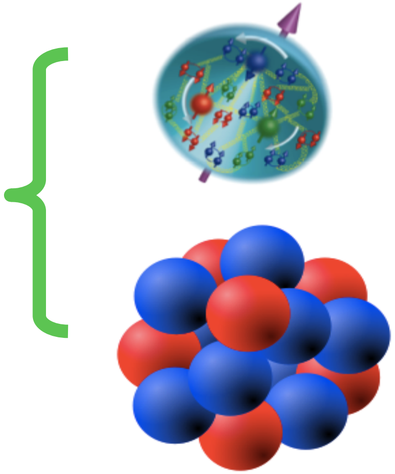
Heavy quarks experience a force of ~16 tons at ~1 Fermi (10^{-15} m) distance



What holds it together – the glue?

- ☐ **Understanding the glue that binds us all – the Next QCD Frontier!**





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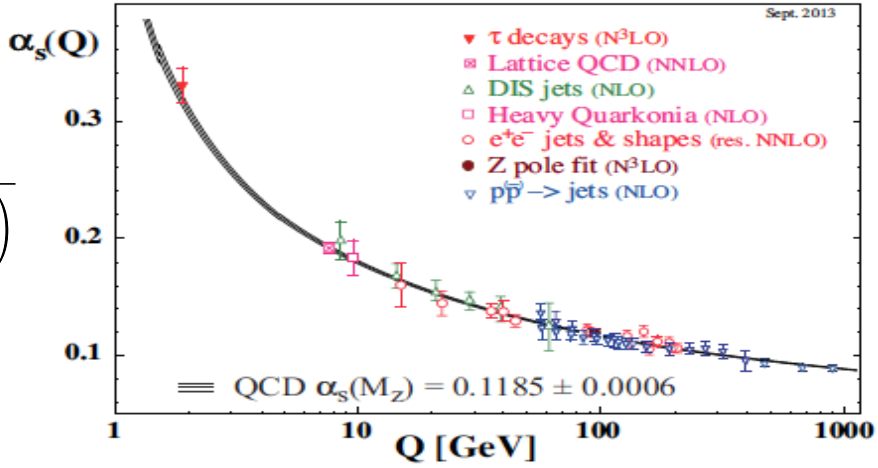
    but, also for asymptotic freedom

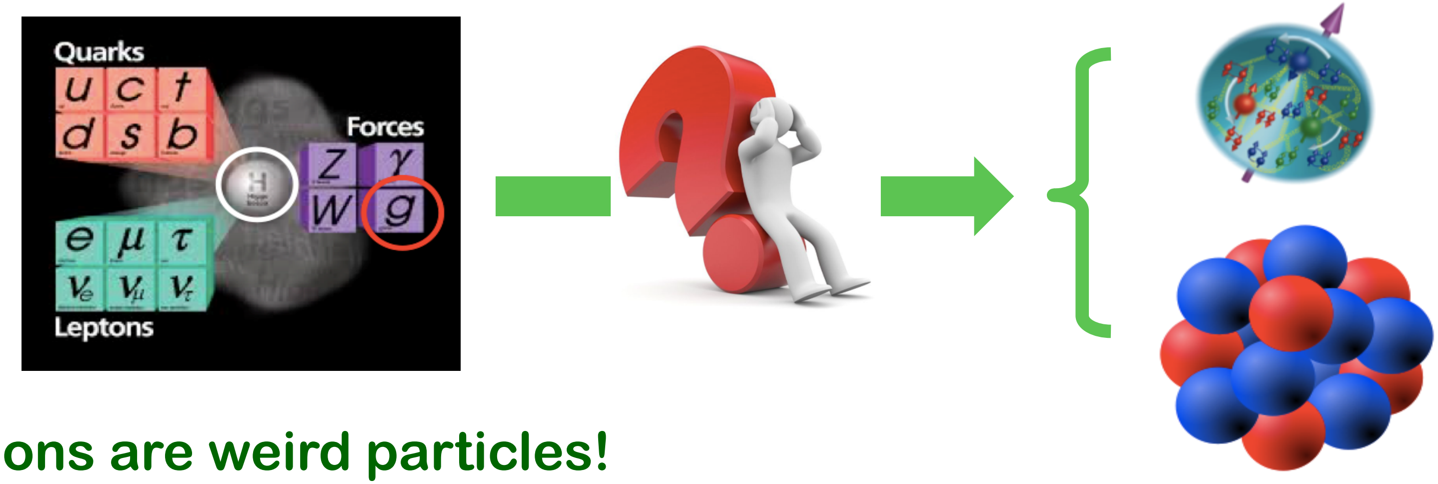
$$\alpha_s(\mu_2) = \frac{\alpha_s(\mu_1)}{1 - \frac{\beta_1}{4\pi}\alpha_s(\mu_1)\ln\left(\frac{\mu_2^2}{\mu_1^2}\right)} \equiv \frac{4\pi}{-\beta_1 \ln\left(\frac{\mu_2^2}{\Lambda_{\rm QCD}^2}\right)}$$



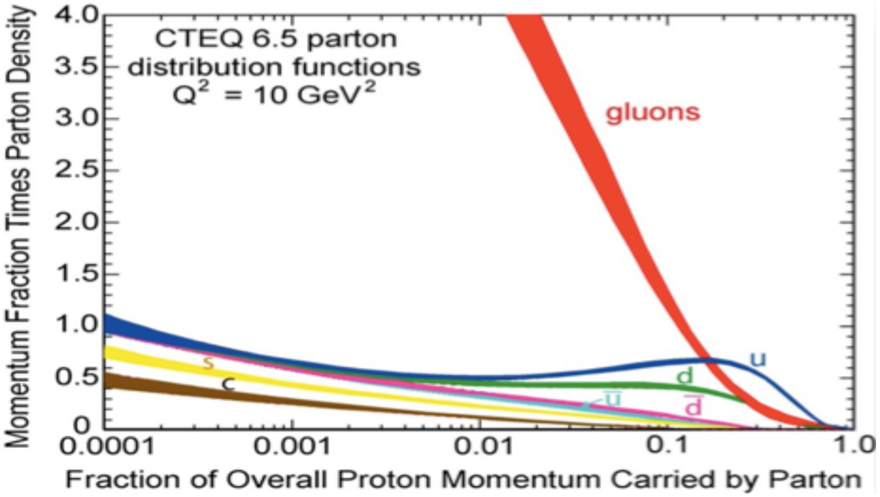
→ QCD perturbation theory

What holds it together – the glue?

- ☐ **Understanding the glue that binds us all – the Next QCD Frontier!**

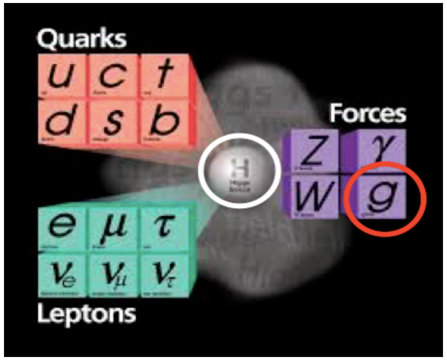


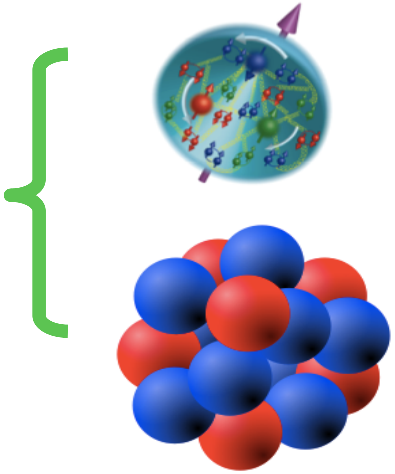
- ☐ **Gluons are weird particles!**
  - ✧ Massless, yet, responsible for nearly all visible mass
  - ✧ Carry color charge, responsible for color confinement and strong force but, also for asymptotic freedom, as well as the abundance of glue



What holds it together – the glue?

- ☐ Understanding the glue that binds us all – the Next QCD Frontier!





- ☐ Gluons are wired particles!
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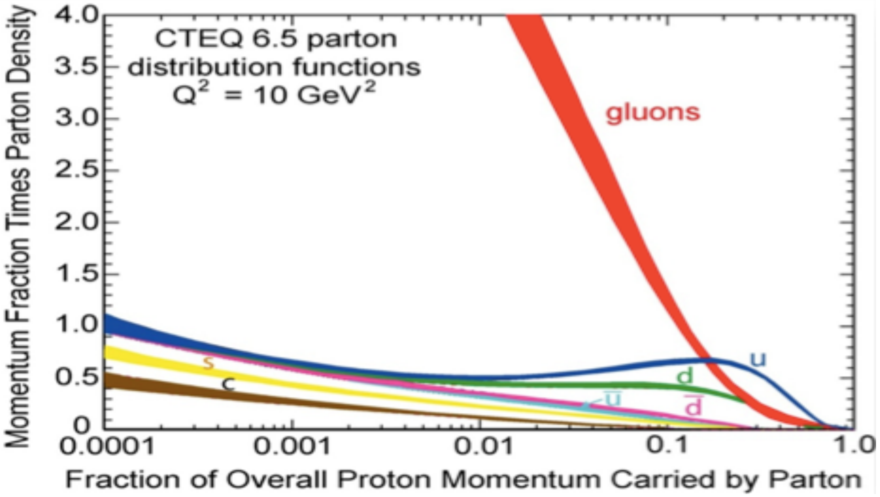
  - ✧ Carry color charge, responsible for color confinement and strong force

    but, also for asymptotic freedom,

    as well as the abundance of glue



Without gluons, there would be

NO nucleons, NO atomic nuclei…

NO visible world!

See A. Deshpande's talk on EIC

Hadron properties in QCD

❑ Mass – intrinsic to a particle:

= Energy of the particle when it is at the rest

- QCD energy-momentum tensor in terms of quarks and gluons

$$T^{\mu\nu} = \frac{1}{2}\overline{\psi} i \overleftrightarrow{D}^{(\mu}\gamma^{\nu)}\psi + \frac{1}{4} g^{\mu\nu}F^2 - F^{\mu\alpha}F^{\nu}{}_{\alpha}$$

- Proton mass:

$$m = \left.\frac{\langle p|\int d^3x\, T^{00}\,|p\rangle}{\langle p|p\rangle}\right|_{\text{Rest frame}} \sim \text{GeV}$$

X. Ji, PRL (1995)

❑ Spin – intrinsic to a particle:

= Angular momentum of the particle when it is at the rest

- QCD angular momentum density in terms of energy-momentum tensor

$$M^{\alpha\mu\nu} = T^{\alpha\nu}x^{\mu} - T^{\alpha\mu}x^{\nu} \qquad J^i = \frac{1}{2}\epsilon^{ijk}\int d^3x M^{0jk}$$

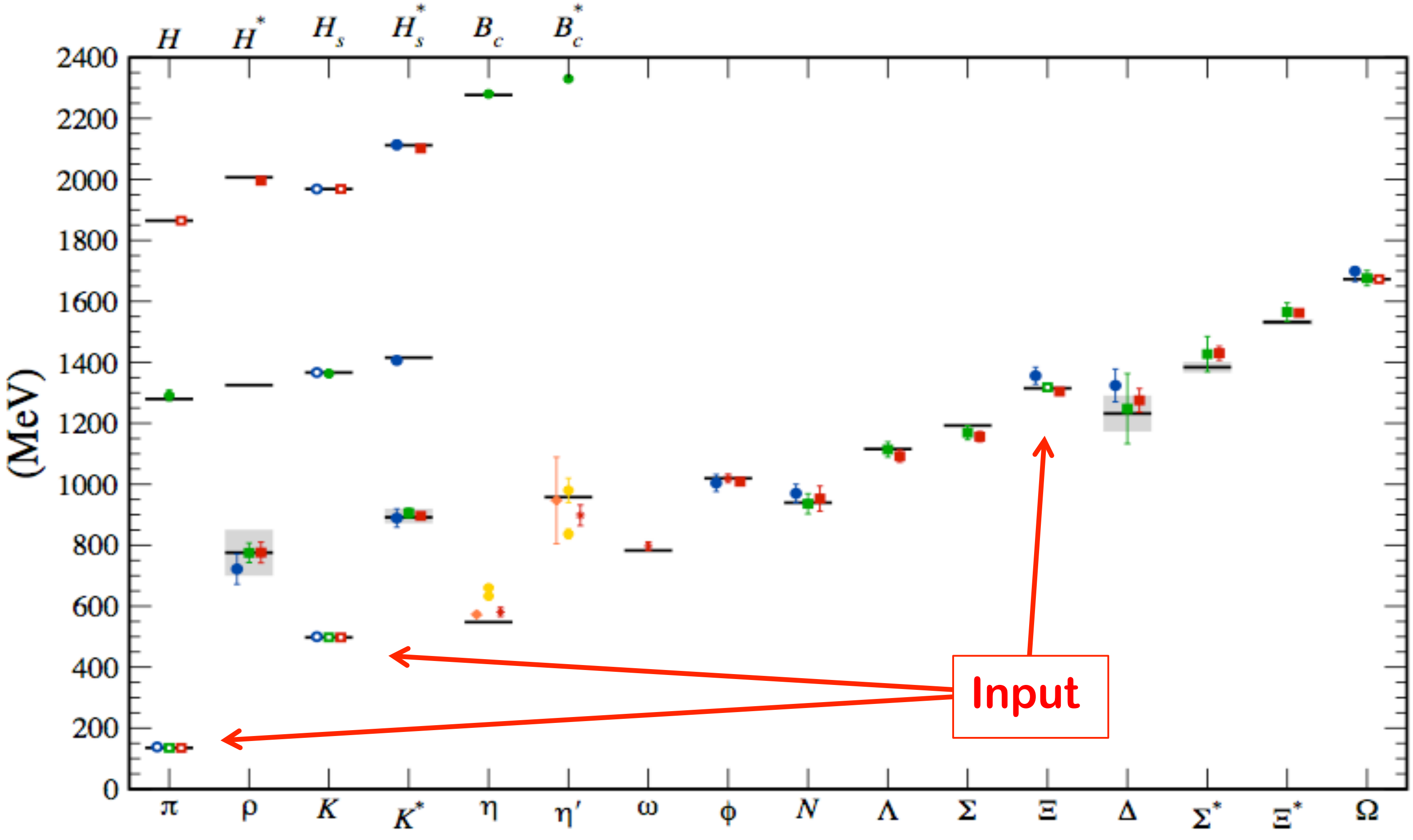
- Proton spin:

$$S(\mu) = \sum \langle P, S|\hat{J}^z_f(\mu)|P, S\rangle = \frac{1}{2}$$

We do NOT know the state, $|P, S\rangle$, in terms of quarks and gluons!

Hadron properties in QCD

❑ **Hadron mass from Lattice QCD calculation:**



A major success of QCD – is the right theory for the Strong Force!

How does QCD generate this? The role of quarks vs that of gluons?

Hadron properties in QCD

☐ Role of quarks and gluons – sum rules:

✧ Invariant hadron mass (in any frame):

$\langle p| T^{\mu\nu} |p\rangle \propto p^\mu p^\nu$ ➡ $m^2 \propto \langle p|T^{\alpha}_{\ \alpha}|p\rangle$

$$T^{\alpha}_{\alpha} = \underbrace{\frac{\beta(g)}{2g} F^{\mu\nu,a} F^{a}_{\mu\nu}}_{\text{QCD trace anomaly}} + \sum_{q=u,d,s} m_q (1+\gamma_m) \overline{\psi}_q \psi_q$$

$$\beta(g) = -(11 - 2n_f/3)\, g^3/(4\pi)^2 + \ldots$$

➡ $\dfrac{\beta(g)}{2g} \langle p|F^2|p\rangle$

Kharzeev @ Temple workshop

➡ *At the chiral limit, the entire proton mass is from gluons!*

✧ Hadron mass in the rest frame – decomposition (sum rule):

$$M = \frac{\langle P|H_{\rm QCD}|P\rangle}{\langle P|P\rangle}\Big|_{\rm rest\ \ frame} = H_q + H_m + H_g + H_a$$

where $H_q = \sum_q \psi_q^\dagger (-i\mathbf{D}\cdot\alpha)\psi_q$ – **quark energy**

$H_m = \sum_q \overline{\psi}_q m_q \psi_q$ – **quark mass**

$H_g = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$ – **gluon energy**

$H_a = \dfrac{9\alpha_s}{16\pi}(\mathbf{E}^2 - \mathbf{B}^2)$ – **trace anomaly**

Hadron properties in QCD

❑ Role of quarks and gluons – sum rules:

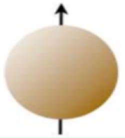
- ✧ Partonic angular momenta, the contributions to hadron spin:

$$S(\mu) = \sum_f \langle P, S|\hat{J}^z_f(\mu)|P, S\rangle = \frac{1}{2} \equiv J_q(\mu) + J_g(\mu)$$

where $\vec{J}_q = \int d^3x \left[\psi^\dagger_q \vec{\gamma}\gamma_5\psi_q + \psi^\dagger_q(\vec{x} \times (-i\vec{D}))\,\psi_q\right]$ **– quark angular mom.**

$\vec{J}_g = \int d^3x \left[\vec{x} \times (\vec{E} \times \vec{B})\right]$ **– gluon angular mom.**

- ✧ Spin decomposition (sum rule) – an incomplete story:



Proton Spin

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + (L_q + L_g)$$

Jaffe-Manohar, 90
Ji, 96, …

Different from QM!

Quark helicity: $\frac{1}{2}\int dx \left(\Delta u + \Delta\bar{u} + \Delta d + \Delta\bar{d} + \Delta s + \Delta\bar{s}\right) \sim 30\%$ **– Best known**

Quark helicity: $\Delta G = \int dx \Delta g(x) \sim 20\%(\text{with RHIC data})$ **– Start to know**

Orbital Angular Momentum (OAM): Not uniquely defined – Little known

Gauge field is tied together with the motion of fermion – D^μ !

Hadron properties in QCD

- **Role of quarks and gluons – sum rules:**
  - **Jaffe-Manohar's quark OAM density:**

$$\mathcal{L}_q^3 = \psi_q^\dagger \left[\vec{x} \times (-i\vec{\partial})\right]^3 \psi_q$$

  - **Ji's quark OAM density:**

$$L_q^3 = \psi_q^\dagger \left[\vec{x} \times (-i\vec{D})\right]^3 \psi_q$$

Hatta, Yoshida, Burkardt, Meissner, Metz, Schlegel, …

  - **Difference – generated by a "torque" of color Lorentz force**

$$\mathcal{L}_q^3 - L_q^3 \propto \int \frac{dy^- d^2 y_T}{(2\pi)^3} \langle P'|\overline{\psi}_q(0)\frac{\gamma^+}{2}\int_{y^-}^{\infty} dz^- \Phi(0,z^-)$$

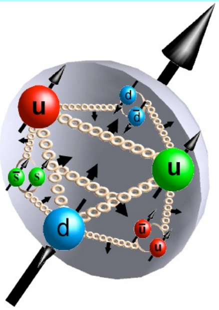
$$\times \underbrace{\sum_{i,j=1,2}\left[\epsilon^{3ij} y_T^i F^{+j}(z^-)\right]}_{\text{“Chromodynamic torque”}} \Phi(z^-,y)\psi(y)|P\rangle_{y^+=0}$$

Sum rules are NOT unique – None of these matrix elements are physical!!!

- **Value of the decomposition – a good sum rule:**

Every term of the sum rule is "independently measurable" – can be related to a physical observable with controllable approximation!

Hadron structure in QCD



- What do we need to know for the structure?
  - In theory: $\langle P,S|\mathcal{O}(\overline{\psi},\psi,A^{\mu})|P,S\rangle$ – Hadronic matrix elements

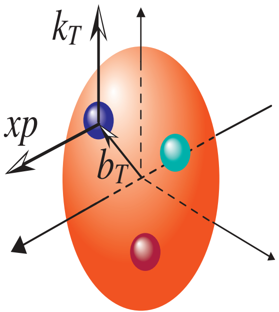
    with all possible operators: $\mathcal{O}(\overline{\psi},\psi,A^{\mu})$
  - In fact: *None of these matrix elements is a direct physical observable in QCD – color confinement!*
  - In practice: Accessible hadron structure
    = hadron matrix elements of quarks and gluons, which
    1) can be related to physical cross sections of hadrons and leptons with controllable approximation; and/or
    2) can be calculated in lattice QCD

- Single-parton structure "seen" by a short-distance probe:
  - 5D structure:

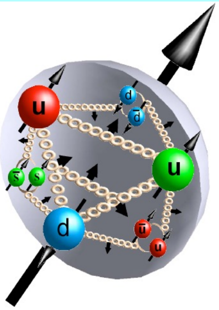
1) $\int d^2b_T$ ➡ $f(x,k_T,\mu)$ – TMDs: *2D confined motion!*

2) $\int d^2k_T$ ➡ $F(x,b_T,\mu)$ – GPDs: *2D spatial imaging!*

3) $\int d^2k_Td^2b_T$ ➡ $f(x,\mu)$ – PDFs: *Number density!*



Hadron structure in QCD



❑ **What do we need to know for the structure?**

✧ **In theory:** $\langle P,S|\mathcal{O}(\overline{\psi},\psi,A^{\mu})|P,S\rangle$ – Hadronic matrix elements

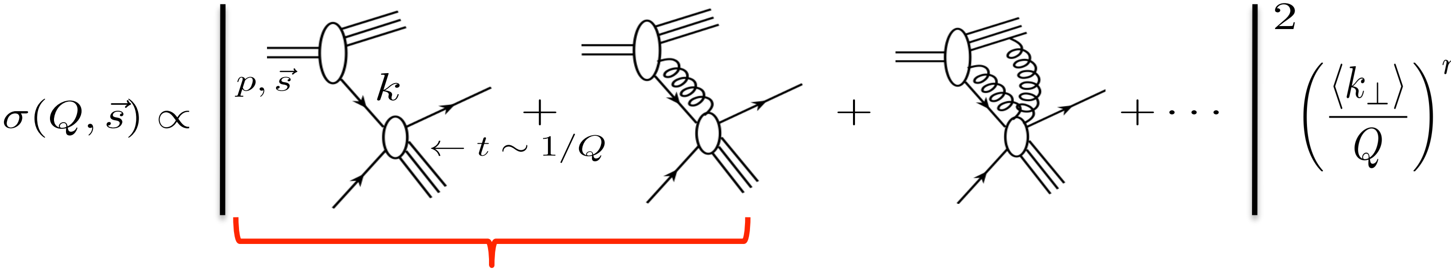
with all possible operators: $\mathcal{O}(\overline{\psi},\psi,A^{\mu})$

✧ **In fact:** *None of these matrix elements is a direct physical observable in QCD – color confinement!*

✧ **In practice:** Accessible hadron structure
= hadron matrix elements of quarks and gluons, which

1) can be related to physical cross sections of hadrons and leptons with controllable approximation; and/or
2) can be calculated in lattice QCD

❑ **Multi-parton correlations:**



– Expansion

Quantum interference ➡ 3-parton matrix element – not a probability!

Hadron structure in QCD

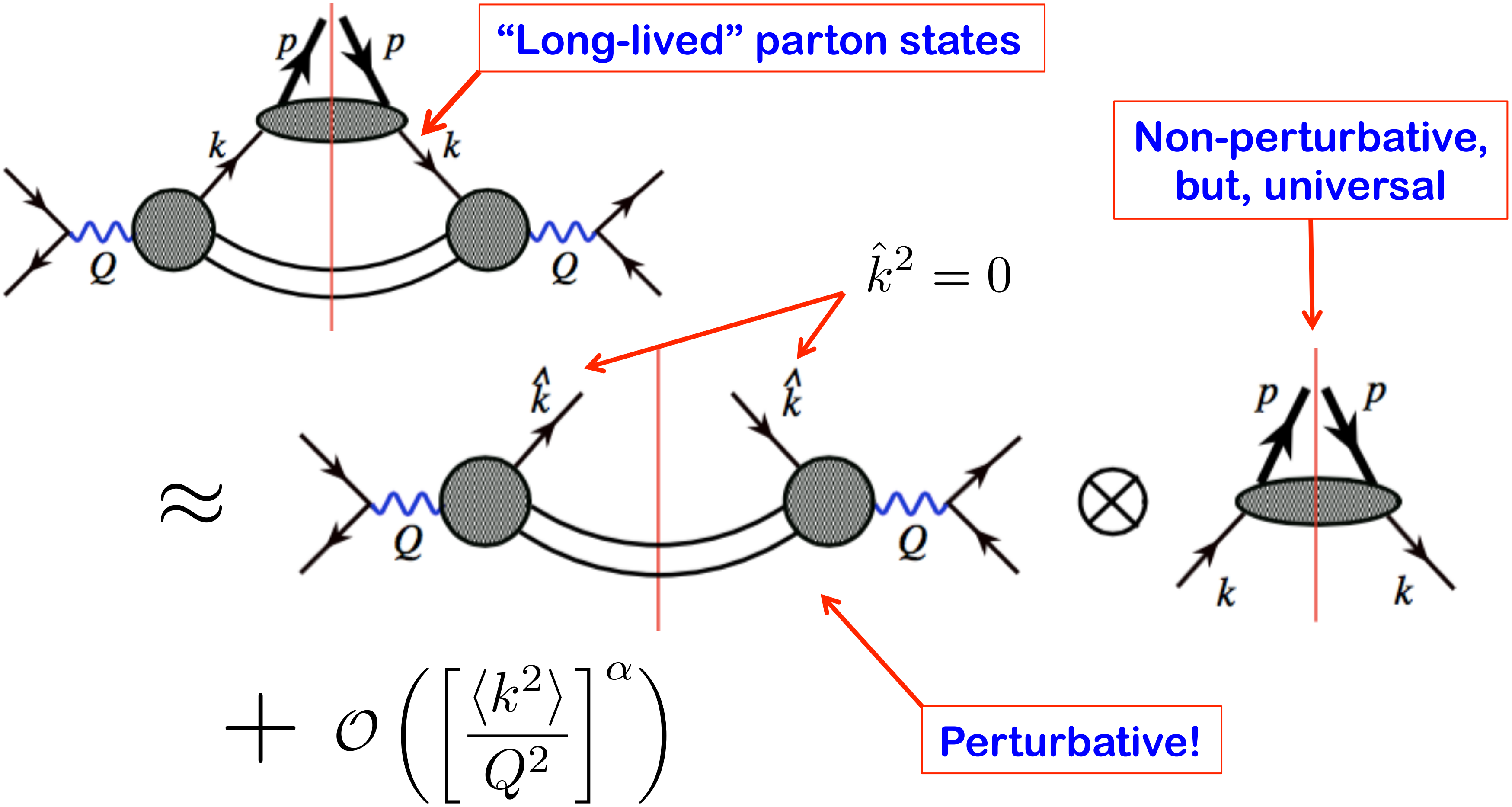
We need the probe!

How to connect QCD quarks and gluons
to observed hadrons and leptons?

Fundamentals of QCD factorization
and evolution (probing scale)

QCD factorization – approximation

- **Creation of an identified hadron – a factorizable example:**



“Long-lived” parton states

Non-perturbative, but, universal

$$\hat{k}^2 = 0$$

$$\approx \quad \otimes$$

$$+\ \mathcal{O}\left(\left[\frac{\langle k^2\rangle}{Q^2}\right]^{\alpha}\right)$$

Perturbative!

Factorization: factorized into a product of “probabilities” !

Effective quark mass

- ☐ **Running quark mass:**

$$m(\mu_2) = m(\mu_1)\exp\left[-\int_{\mu_1}^{\mu_2}\frac{d\lambda}{\lambda}[1+\gamma_m(g(\lambda))]\right]$$

Quark mass depend on the renormalization scale!

- ☐ **QCD running quark mass:**

$$m(\mu_2) \Rightarrow 0 \qquad \text{as } \mu_2 \to \infty \qquad\qquad \text{since } \gamma_m(g(\lambda)) > 0$$

- ☐ **Choice of renormalization scale:**

$\mu \sim Q$ for small logarithms in the perturbative coefficients

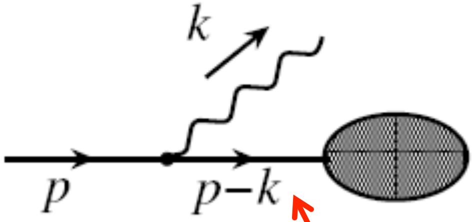
- ☐ **Light quark mass:** $m_f(\mu) \ll \Lambda_{\mathrm{QCD}} \quad \text{for } f = u, d, \text{ even } s$

QCD perturbation theory (Q>>Λ_{QCD})
is effectively a massless theory

Infrared and collinear divergences

❑ **Consider a general diagram with a *“unobserved gluon”*:**

$p^2 = 0, \quad k^2 = 0$ **for a massless theory**



Singularity

✧ $k^\mu \to 0 \ \Rightarrow \ (p-k)^2 \to p^2 = 0$

$\Longrightarrow$ **Infrared (IR) divergence**

✧ $k^\mu \ || \ p^\mu \ \Rightarrow \ k^\mu = \lambda p^\mu \quad \text{with} \ \ 0 < \lambda < 1$

$\Rightarrow \ (p-k)^2 \to (1-\lambda)^2 p^2 = 0$

$\Longrightarrow$ **Collinear (CO) divergence**

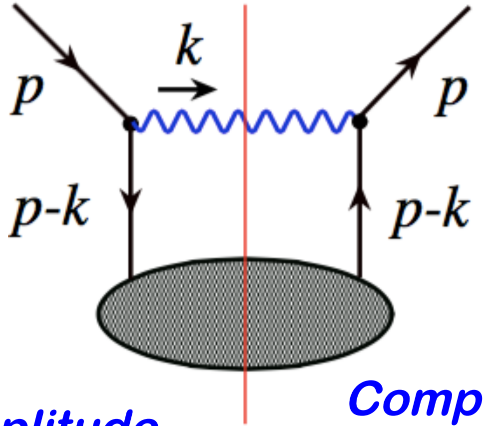
IR and CO divergences are generic problems of a massless perturbation theory

Pinch singularity and pinch surface

- "Square" of the diagram **with a *"unobserved gluon"***:

"Cut-line" – final-state

– in a "cut-diagram" notation



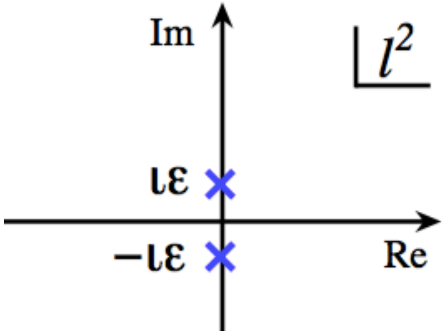
$$\propto \int \mathcal{T}(p-k,Q)\,\frac{1}{(p-k)^2+i\epsilon}\,\frac{1}{(p-k)^2-i\epsilon}\,d^4k\,\delta(k^2)_+$$

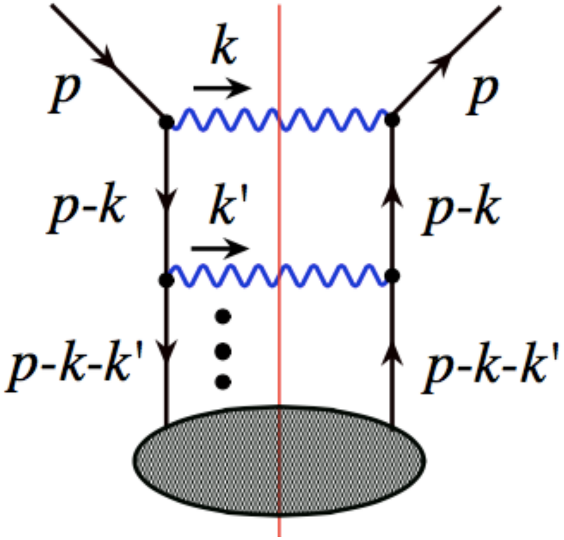
$$\propto \int \mathcal{T}(l,Q)\,\frac{1}{l^2+i\epsilon}\,\frac{1}{l^2-i\epsilon}\,dl^2$$

$$\Rightarrow \infty$$

Amplitude

Complex conjugate of the Amplitude





Pinch surfaces

Pinch singularities "perturbatively"

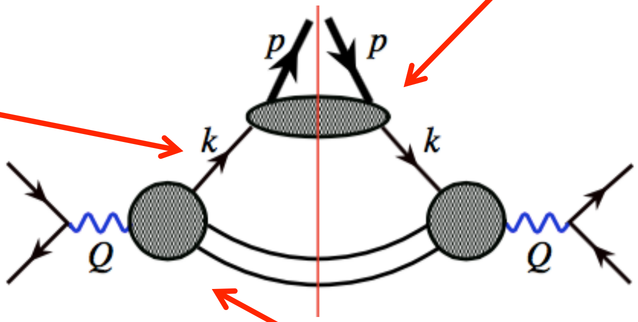
= "surfaces" in k, k', …

determined by $(p\text{-}k)^2=0$, $(p\text{-}k\text{-}k')^2=0$, …

"perturbatively"

Hard collisions with identified hadron(s)

- Creation of an identified hadron:

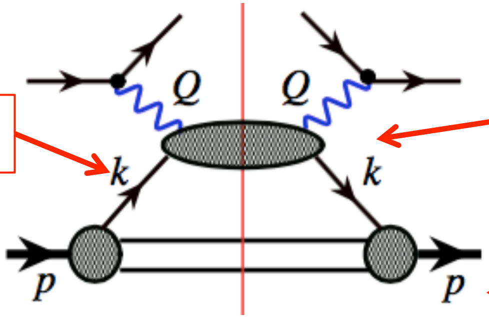


Non-perturbative!

Pinch in k^2

Perturbative!

- Identified initial hadron:

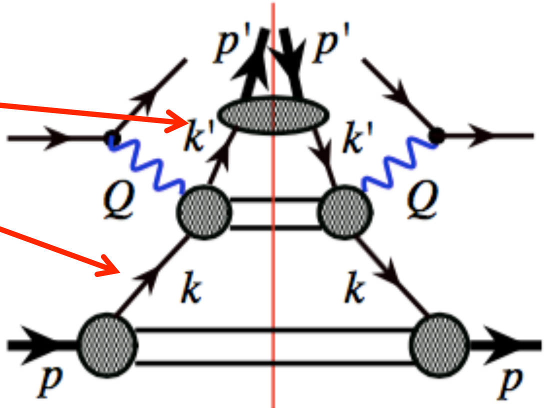


Pinch in k^2

Perturbative!

Non-perturbative!

- Initial + created identified hadron(s):



Pinch in both k^2 and k'^2

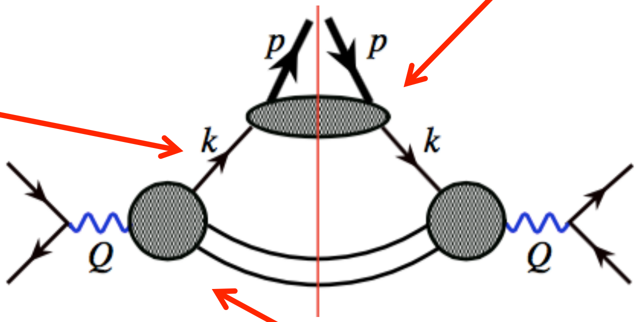
Cross section with identified hadron(s) is NOT perturbatively calculable

Hard collisions with identified hadron(s)

- **Creation of an identified hadron:**

Non-perturbative!

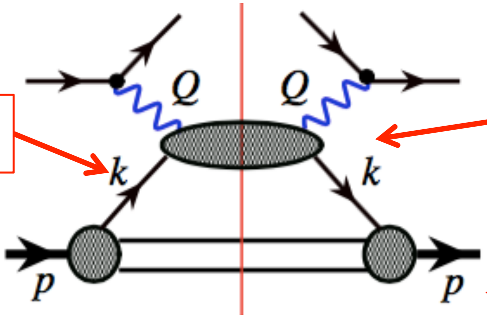
Pinch in k^2



Perturbative!

- **Identified initial hadron:**

Pinch in k^2

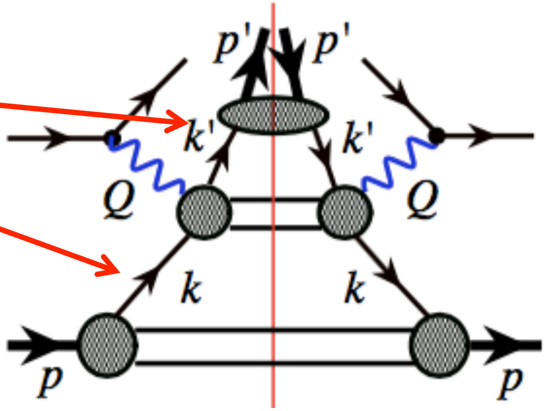


Perturbative!

Non-perturbative!

- **Initial + created identified hadron(s):**

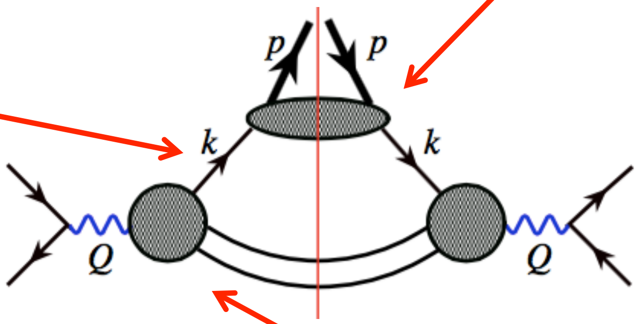
Pinch in both k^2 and k'^2



Dynamics at a HARD scale is linked by partons almost on Mass-Shell

Hard collisions with identified hadron(s)

☐ **Creation of an identified hadron:**

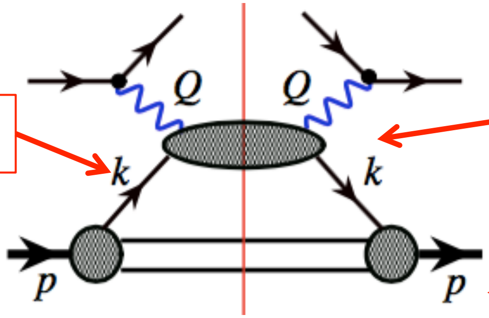


Non-perturbative!

Pinch in k^2

Perturbative!

☐ **Identified initial hadron:**

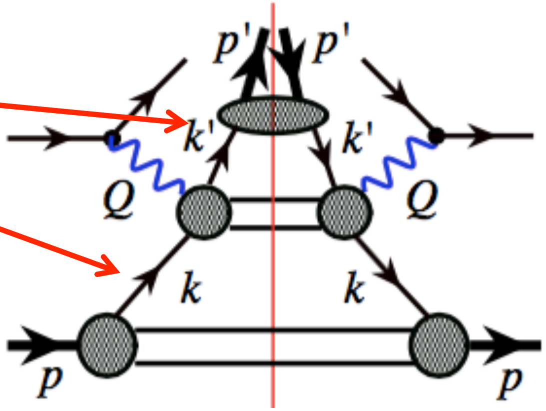


Pinch in k^2

Perturbative!

Non-perturbative!

☐ **Initial + created identified hadron(s):**



Pinch in both k^2 and k'^2

Quantum interference between dynamics
at the HARD and hadronic scales
is powerly suppressed!

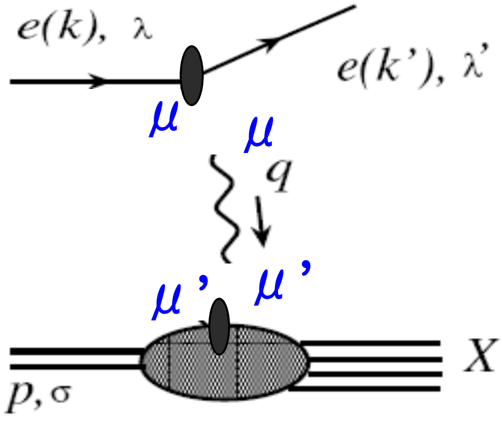
Example: Inclusive lepton-hadron DIS

❑ Scattering amplitude:

$$M(\lambda,\lambda';\sigma,q) = \bar{u}_{\lambda'}(k')\left[-ie\gamma_{\mu}\right]u_{\lambda}(k)$$

$$* \quad \left(\frac{i}{q^2}\right)\left(-g^{\mu\mu'}\right)$$

$$* \quad \left\langle X\right|eJ^{em}_{\mu'}(0)\left|p,\sigma\right\rangle$$



❑ Cross section:

$$d\sigma^{\text{DIS}} = \frac{1}{2s}\left(\frac{1}{2}\right)^2 \sum_{X}\sum_{\lambda,\lambda',\sigma}\left|M(\lambda,\lambda';\sigma,q)\right|^2 \left[\prod_{i=1}^{X}\frac{d^3 l_i}{(2\pi)^3\, 2E_i}\right]\frac{d^3k'}{(2\pi)^3\, 2E'}(2\pi)^4\,\delta^4\left(\sum_{i=1}^{X} l_i + k' - p - k\right)$$

$$\Rightarrow \quad \boxed{E'\frac{d\sigma^{\text{DIS}}}{d^3k'} = \frac{1}{2s}\left(\frac{1}{Q^2}\right)^2 L^{\mu\nu}(k,k')\,W_{\mu\nu}(q,p)}$$

❑ Leptonic tensor:

– known from QED

$$L^{\mu\nu}(k,k') = \frac{e^2}{2\pi^2}\left(k^{\mu}k'^{\nu} + k^{\nu}k'^{\mu} - k\cdot k' g^{\mu\nu}\right)$$

DIS structure functions

☐ Hadronic tensor:

$$W_{\mu\nu}(q,p,S) = \frac{1}{4\pi}\int d^4z \; \mathrm{e}^{iq\cdot z} \; \langle p,S|J_\mu^\dagger(z)J_\nu(0)|p,S\rangle$$

☐ Symmetries:

- ✧ **Parity invariance (EM current)** → $W_{\mu\nu}=W_{\nu\mu}$ sysmetric for spin avg.
- ✧ **Time-reversal invariance** → $W_{\mu\nu}=W_{\mu\nu}^*$ real
- ✧ **Current conservation** → $q^\mu W_{\mu\nu}=q^\nu W_{\mu\nu}=0$

$$W_{\mu\nu} = -\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)F_1\left(x_B,Q^2\right)+\frac{1}{p\cdot q}\left(p_\mu-q_\mu\frac{p\cdot q}{q^2}\right)\left(p_\nu-q_\nu\frac{p\cdot q}{q^2}\right)F_2\left(x_B,Q^2\right)$$

$$+iM_p\varepsilon^{\mu\nu\rho\sigma}q_\rho\left[\frac{S_\sigma}{p.q}g_1\left(x_B,Q^2\right)+\frac{(p.q)S_\sigma-(S.q)p_\sigma}{(p.q)^2}g_2\left(x_B,Q^2\right)\right]$$

$$Q^2=-q^2$$

$$x_B=\frac{Q^2}{2p\cdot q}$$

☐ Structure functions – infrared sensitive:

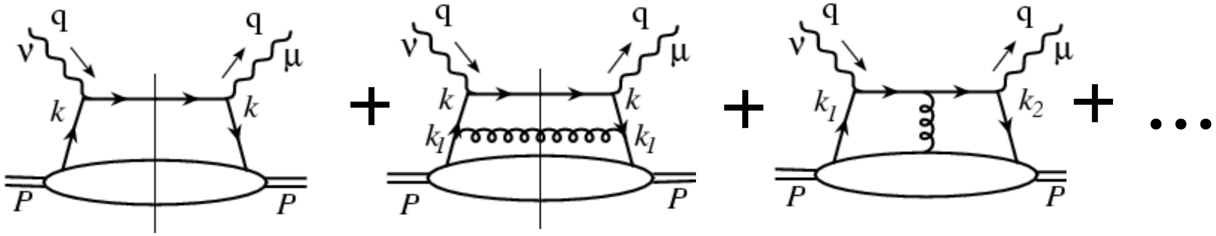
$$F_1\left(x_B,Q^2\right),F_2\left(x_B,Q^2\right),g_1\left(x_B,Q^2\right),g_2\left(x_B,Q^2\right)$$

No QCD parton dynamics used in above derivation!

Long-lived parton states

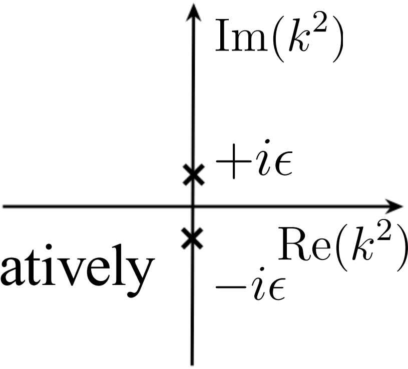
- **Feynman diagram representation of the hadronic tensor:**

$$W^{\mu\nu} \propto$$



- **Perturbative pinched poles:**

$$\int d^4k \, \mathrm{H}(Q,k) \left(\frac{1}{k^2 + i\varepsilon}\right)\left(\frac{1}{k^2 - i\varepsilon}\right) \mathrm{T}(k, \frac{1}{r_0}) \Rightarrow \infty \text{ perturbatively}$$



- **Perturbative factorization:**

$$k^\mu = xp^\mu + \frac{k^2 + k_T^2}{2xp \cdot n} n^\mu + k_T^\mu$$

Nonperturbative matrix element

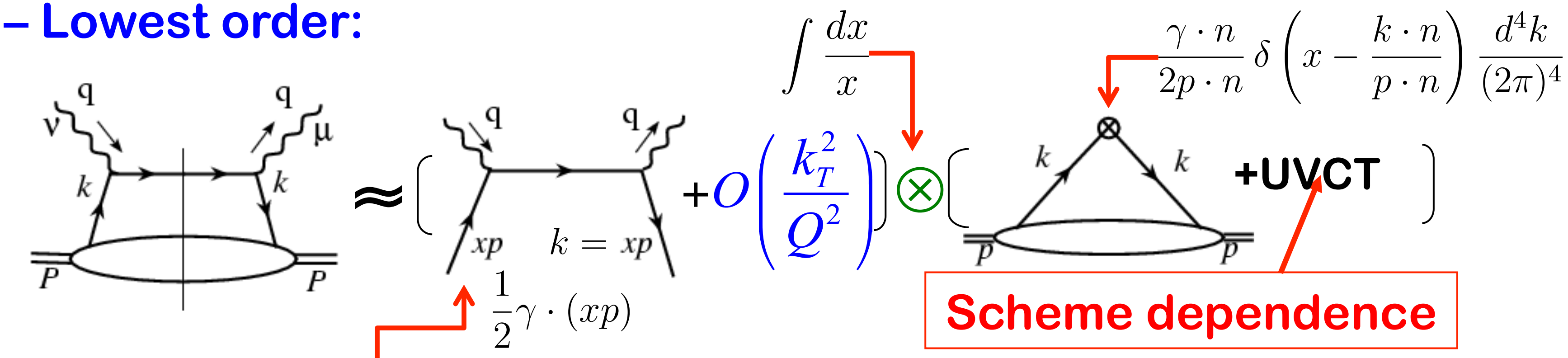
$$\int \frac{dx}{x} d^2k_T \, \mathrm{H}(Q, k^2 = 0) \boxed{\int dk^2 \left(\frac{1}{k^2 + i\varepsilon}\right)\left(\frac{1}{k^2 - i\varepsilon}\right) \mathrm{T}(k, \frac{1}{r_0})}$$

Short-distance

Collinear factorization – further approximation

❑ Collinear approximation, if $Q \sim xp\cdot n \gg k_T,\ \sqrt{k^2}$

– Lowest order:



$$\approx \left[\ldots \right. \quad k = xp \quad \left. + O\left(\frac{k_T^2}{Q^2}\right)\right] \otimes \left[\ldots \quad \text{+UVCT} \right]$$

$$\int \frac{dx}{x}$$

$$\frac{\gamma\cdot n}{2p\cdot n}\,\delta\left(x - \frac{k\cdot n}{p\cdot n}\right)\frac{d^4k}{(2\pi)^4}$$

$$\frac{1}{2}\gamma\cdot(xp)$$

Scheme dependence

Same as an elastic x-section

Parton's transverse momentum is integrated into parton distributions, and provides a scale of power corrections

❑ DIS limit: $\nu, Q^2 \longrightarrow \infty,$ while x_B fixed

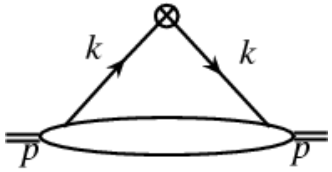
⇨ **Feynman's parton model and Bjorken scaling**

$$F_2(x_B,Q^2) = x_B \sum_f e_f^2\, \phi_f(x_B) = 2x_B F_1(x_B,Q^2)$$

Spin-½ parton!

❑ Corrections: $\mathcal{O}(\alpha_s) + \mathcal{O}\left(\langle k^2\rangle / Q^2\right)$

Parton distribution functions (PDFs)



❑ **PDFs as matrix elements of two parton fields:**

- combine the amplitude & its complex-conjugate

$$\phi_{q/h}(x,\mu^2) = \int \frac{p^+ dy^-}{2\pi}\, e^{ixp^+y^-} \langle h(p)|\bar{\psi}_q(0)\, \frac{\gamma^+}{2p^+}\, \psi_q(y^-)|h(p)\rangle \mathcal{Z}_{\mathcal{O}}(\mu^2)$$

$|h(p)\rangle$ **can be a hadron, or a nucleus, or a parton state!**

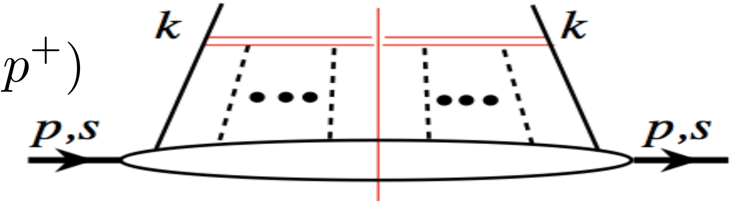
But, it is NOT gauge invariant! $\psi(x) \to e^{i\alpha_a(x)t_a}\psi(x) \quad \bar{\psi}(x) \to \bar{\psi}(x) e^{-i\alpha_a(x)t_a}$

- need a gauge link:

$$\phi_{q/h}(x,\mu^2) = \int \frac{dy^-}{2\pi}\, e^{ixp^+y^-} \langle h(p)|\bar{\psi}_q(0) \left[\mathcal{P}e^{-ig\int_0^{y^-} d\eta^- A^+(\eta^-)}\right] \frac{\gamma^+}{2p^+}\, \psi_q(y^-)|h(p)\rangle \mathcal{Z}_{\mathcal{O}}(\mu^2)$$

- corresponding diagram in momentum space:

$$\int \frac{d^4k}{(2\pi)^4}\, \delta(x - k^+/p^+)$$



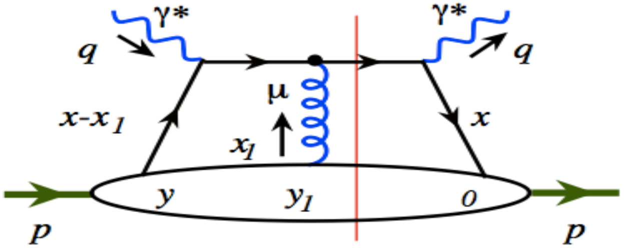
$+\,\mathrm{UVCT}(\mu^2)$

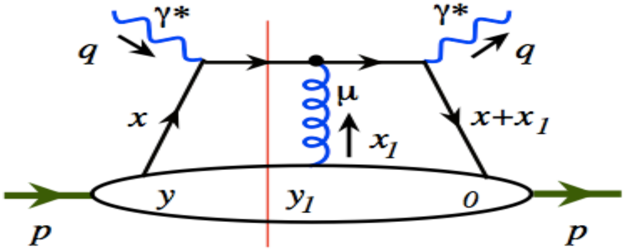
μ -dependence

Universality – process independence – predictive power

Gauge link – 1st order in coupling "g"

❑ Longitudinal gluon:





❑ Left diagram:

$$\int dx_1 \left[\int \frac{p^+ dy_1^-}{2\pi}\, \mathrm{e}^{ix_1p^+(y_1^- - y^-)} n\cdot A^a(y_1^-)\right] \mathcal{M}(-igt^a)\frac{\gamma\cdot p}{p^+}\frac{i((x-x_1-x_B)\gamma\cdot p+(Q^2/2x_Bp^+)\gamma\cdot n)}{(x-x_1-x_B)Q^2/x_B+i\epsilon}$$

$$= g\int \frac{dy_1^-}{2\pi}\, n\cdot A^a(y_1^-)t^a \left[\int dx_1\, \frac{1}{-x_1+i\epsilon}\, \mathrm{e}^{ix_1p^+(y_1^- - y^-)}\right] \mathcal{M} = -ig\int_{y^-}^{\infty} dy_1^-\, n\cdot A(y_1^-)\mathcal{M}$$

❑ Right diagram:

$$\int dx_1 \left[\int \frac{p^+ dy_1^-}{2\pi}\, \mathrm{e}^{ix_1p^+y_1^-}\, n\cdot A^a(y_1^-)\right] \frac{-i((x+x_1-x_B)\gamma\cdot p+(Q^2/2x_Bp^+)\gamma\cdot n)}{(x+x_1-x_B)Q^2/x_B-i\epsilon}(+igt^a)\,\frac{\gamma\cdot p}{p^+}\,\mathcal{M}$$

$$= g\int \frac{dy_1^-}{2\pi}\, n\cdot A^a(y_1^-)t^a \left[\int dx_1\, \frac{1}{x_1-i\epsilon}\, \mathrm{e}^{ix_1p^+y_1^-}\right] \mathcal{M} \;= ig\int_0^{\infty} dy_1^-\, n\cdot A(y_1^-)\,\mathcal{M}$$

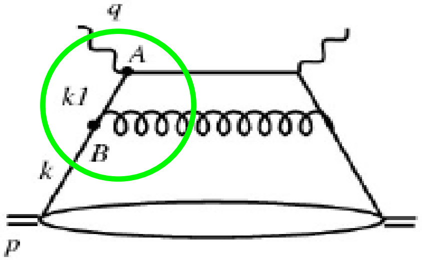
❑ Total contribution:

$$-ig\left[\int_0^{\infty} - \int_{y^-}^{\infty}\right] dy_1^-\, n\cdot A(y_1^-)\,\mathcal{M}_{\mathrm{LO}}$$

O(g)-term of the gauge link!

QCD high order corrections

- **NLO partonic diagram to structure functions:**



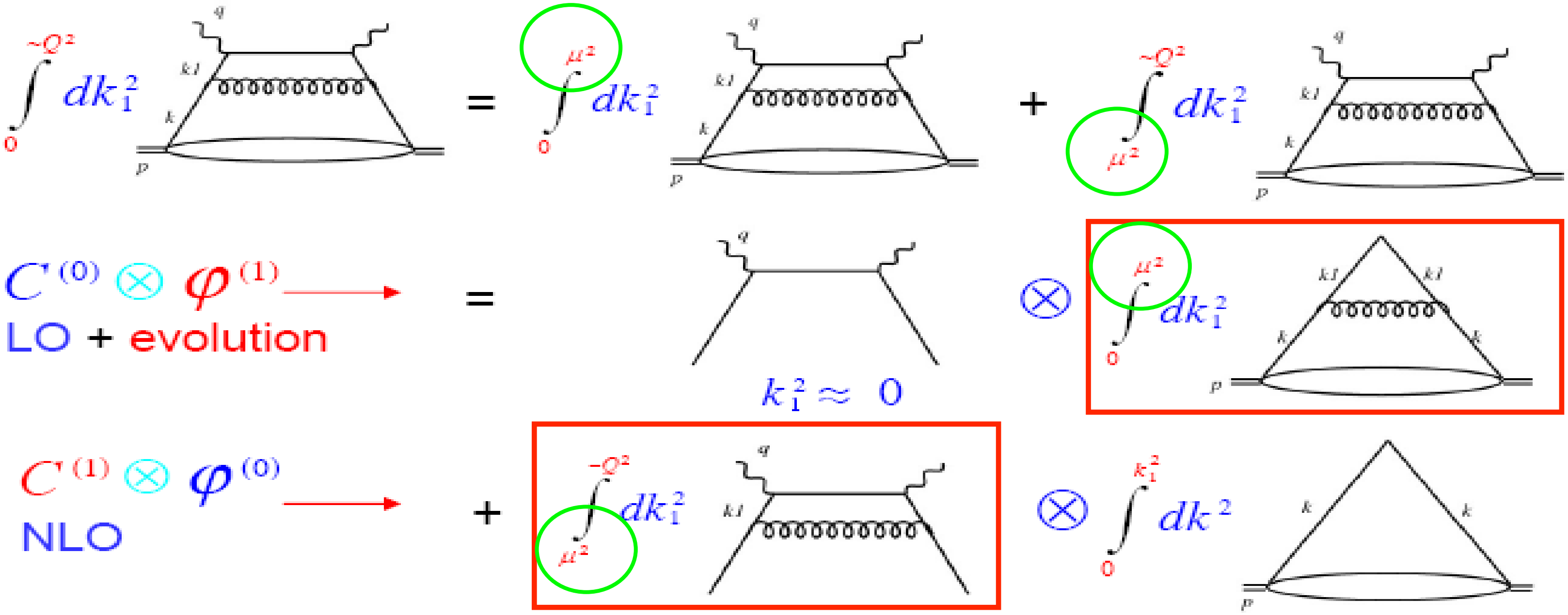
$$\propto \int_0^{-Q^2} \frac{dk_1^2}{k_1^2}$$

Dominated by

$$\begin{cases} k_1^2 \sim 0 \\ t_{AB} \rightarrow \infty \end{cases}$$

Diagram has both long- and short-distance physics

- **Factorization, separation of short- from long-distance:**



$\int_0^{\sim Q^2} dk_1^2$ = $\int_0^{\mu^2} dk_1^2$ + $\int_{\mu^2}^{\sim Q^2} dk_1^2$

$C^{(0)} \otimes \varphi^{(1)} \longrightarrow$ = $k_1^2 \approx 0$ $\otimes$ $\int_0^{\mu^2} dk_1^2$

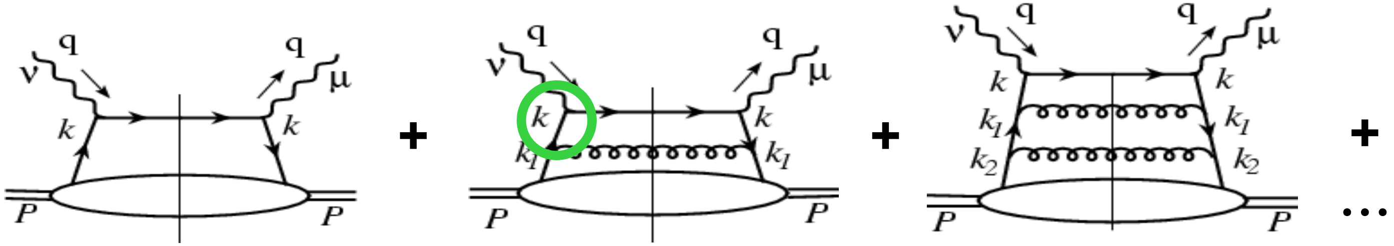
LO + evolution

$C^{(1)} \otimes \varphi^{(0)} \longrightarrow$ + $\int_{\mu^2}^{-Q^2} dk_1^2$ $\otimes$ $\int_0^{k_1^2} dk^2$

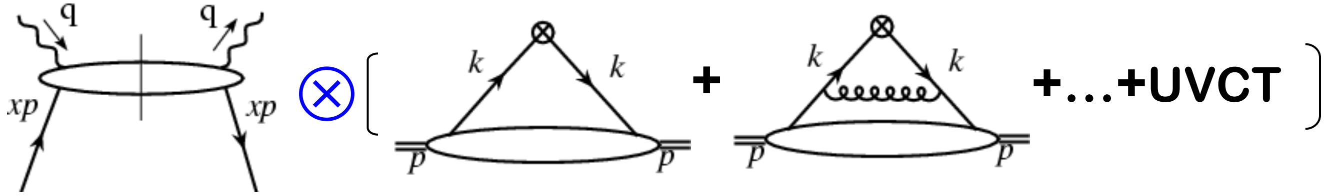
NLO

QCD high order corrections

- QCD corrections: pinch singularities in $\int d^4 k_i$



- Logarithmic contributions into parton distributions:



$$\Rightarrow\quad F_2(x_B,Q^2) = \sum_f C_f\left(\frac{x_B}{x},\frac{Q^2}{\mu_F^2},\alpha_s\right)\otimes\varphi_f\left(x,\mu_F^2\right)+O\left(\frac{\Lambda_{\rm QCD}^2}{Q^2}\right)$$

- Factorization scale: μ_F^2

→ To separate the collinear from non-collinear contribution

Recall: renormalization scale to separate local from non-local contribution

Dependence on factorization scale

- ☐ Physical cross sections should not depend on the factorization scale

$$\mu_F^2 \frac{d}{d\mu_F^2} F_2(x_B, Q^2) = 0$$

$$F_2(x_B, Q^2) = \sum_f C_f(x_B/x, Q^2/\mu_F^2, \alpha_s)\, \phi_f(x, \mu_F^2)$$

→ Evolution (differential-integral) equation for PDFs

$$\sum_f \left[\mu_F^2 \frac{d}{d\mu_F^2} C_f\left(\frac{x_B}{x}, \frac{Q^2}{\mu_F^2}, \alpha_s\right)\right] \otimes \varphi_f\left(x, \mu_F^2\right) + \sum_f C_f\left(\frac{x_B}{x}, \frac{Q^2}{\mu_F^2}, \alpha_s\right) \otimes \mu_F^2 \frac{d}{d\mu_F^2} \varphi_f\left(x, \mu_F^2\right) = 0$$

- ☐ PDFs and coefficient functions share the same logarithms

PDFs: $\log\left(\mu_F^2/\mu_0^2\right)$ or $\log\left(\mu_F^2/\Lambda_{\rm QCD}^2\right)$

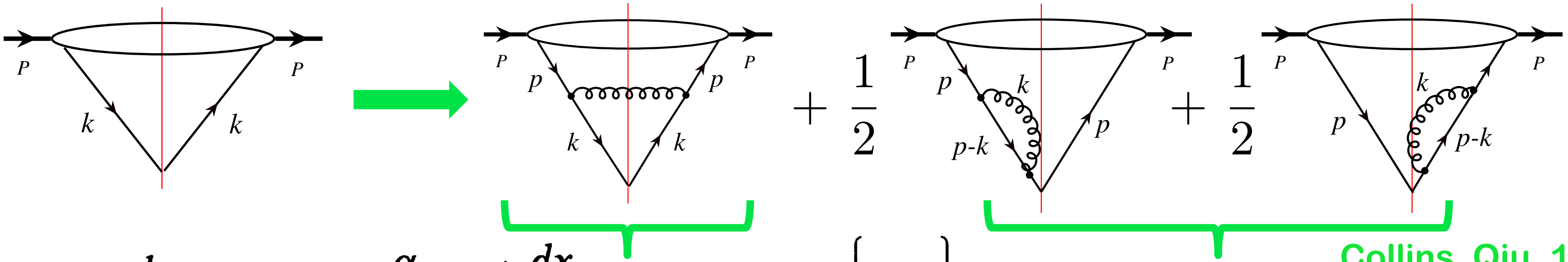
Coefficient functions: $\log\left(Q^2/\mu_F^2\right)$ or $\log\left(Q^2/\mu^2\right)$

→ DGLAP evolution equation:

$$\mu_F^2 \frac{\partial}{\partial \mu_F^2} \varphi_i(x, \mu_F^2) = \sum_j P_{i/j}\left(\frac{x}{x'}, \alpha_s\right) \otimes \varphi_j(x', \mu_F^2)$$

Calculation of evolution kernels

- ❑ **Evolution kernels are process independent**
  - ✧ Parton distribution functions are universal
  - ✧ Could be derived in many different ways
- ❑ **Extract from calculating parton PDFs’ scale dependence**



Collins, Qiu, 1989

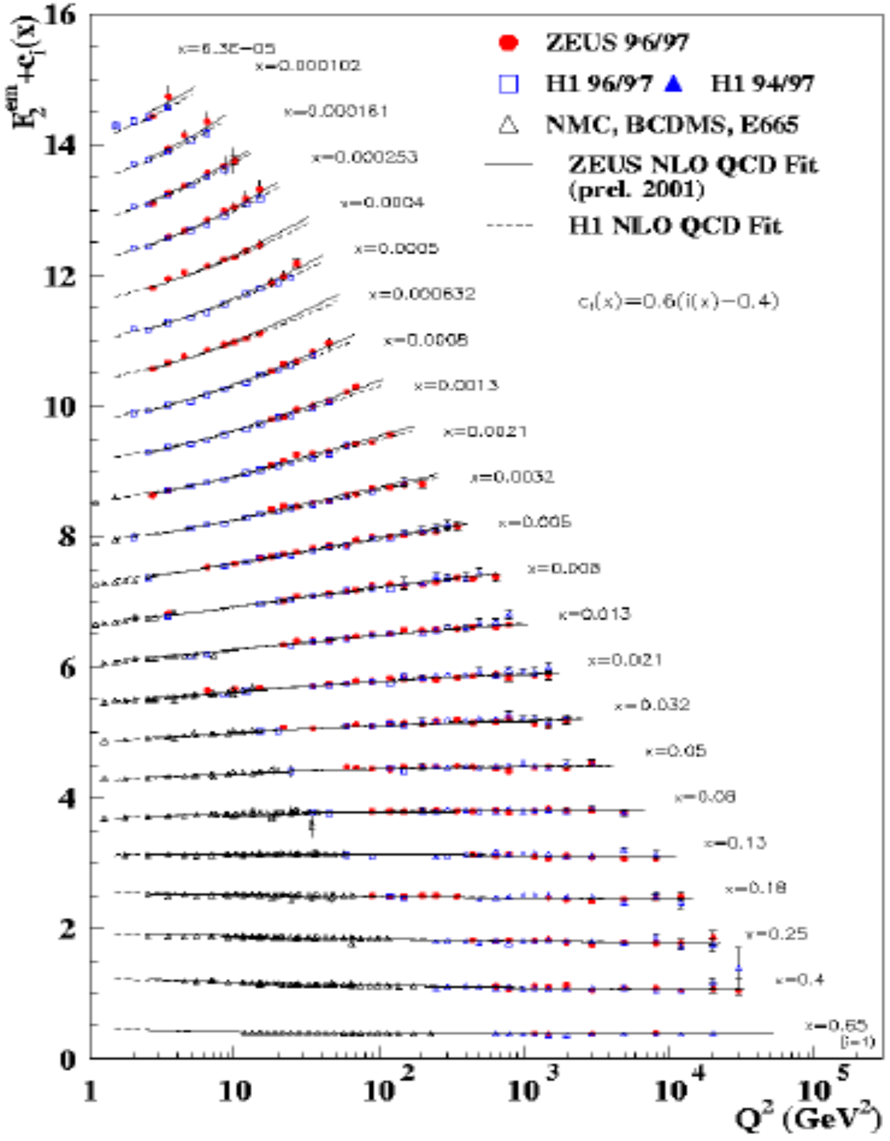
$$Q^2\frac{d}{dQ^2}q_i(x,Q^2)=\frac{\alpha_s}{2\pi}\int_x^1\frac{dx_1}{x_1}q_i(x_1,Q^2)\gamma_{qq}\left(\frac{x}{x_1}\right)-\frac{\alpha_s}{2\pi}q_i(x,Q^2)\int_0^1 dz\,\gamma_{qq}(z)$$

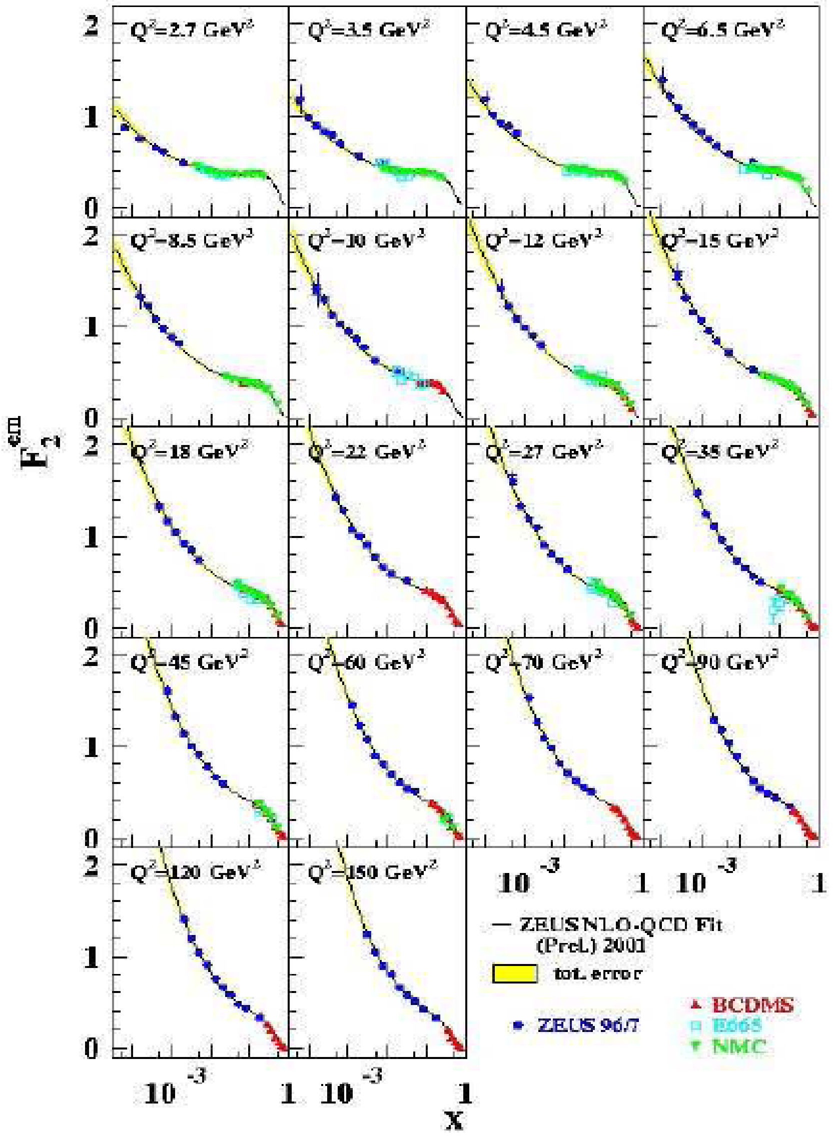
Change “Gain” “Loss”

- ✧ Same is true for gluon evolution, and mixing flavor terms

- ❑ **One can also extract the kernels from the CO divergence of partonic cross sections**

Scaling and scaling violation

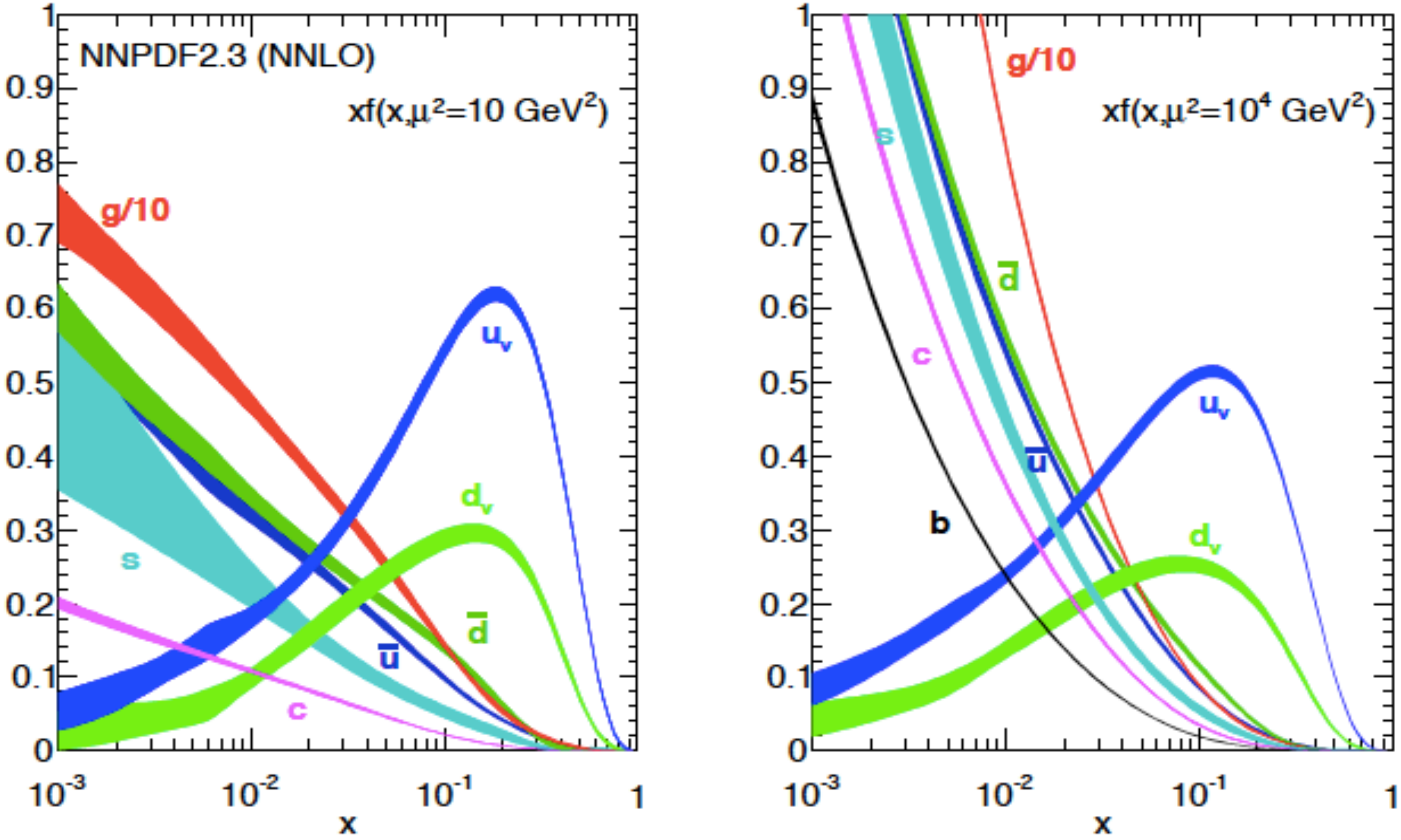




Q^2-dependence is a prediction of pQCD calculation

PDFs of a spin-averaged proton

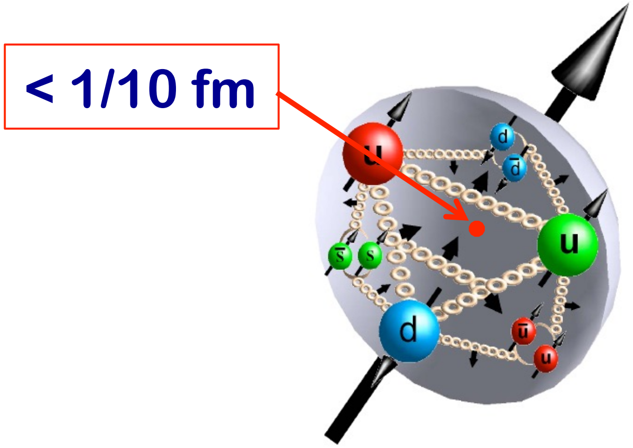
- ☐ **Modern sets of PDFs @NNLO with uncertainties:**



K.A. Olive et al. (Particle Data Group), Chin. Phys. C, 38, 090001 (2014)

Consistently fit almost all data with Q > 2GeV

Summary of lecture one



- QCD has been extremely successful in interpreting and predicting high energy experimental data!

- But, we still do not know much about hadron structure – work just started!

- Cross sections with large momentum transfer(s) and identified hadron(s) are the source of structure information

- TMDs and GPDs, accessible by high energy scattering, encode important information on hadron’s 3D structure – distributions as well as motions of quarks and gluons

- QCD factorization is necessary for any controllable “probe” for hadron’s quark-gluon structure!

Thank you!

Backup slides

How to calculate the perturbative parts?

❑ **Use DIS structure function F_2 as an example:**

$$F_{2h}(x_B,Q^2)=\sum_{q,f} C_{q/f}\left(\frac{x_B}{x},\frac{Q^2}{\mu^2},\alpha_s\right)\otimes\varphi_{f/h}\left(x,\mu^2\right)+O\left(\frac{\Lambda_{\rm QCD}^2}{Q^2}\right)$$

✧ **Apply the factorized formula to parton states:** $h\rightarrow q$

$$\boxed{\text{Feynman diagrams}}\longrightarrow F_{2q}(x_B,Q^2)=\sum_{q,f} C_{q/f}\left(\frac{x_B}{x},\frac{Q^2}{\mu^2},\alpha_s\right)\otimes\varphi_{f/q}\left(x,\mu^2\right)\longleftarrow\boxed{\text{Feynman diagrams}}$$

✧ **Express both SFs and PDFs in terms of powers of** α_s**:**

0th order: $F_{2q}^{(0)}(x_B,Q^2)=C_q^{(0)}(x_B/x,Q^2/\mu^2)\otimes\varphi_{q/q}^{(0)}\left(x,\mu^2\right)$

$$\Rightarrow\quad\boxed{C_q^{(0)}(x)=F_{2q}^{(0)}(x)}\qquad\varphi_{q/q}^{(0)}\left(x\right)=\delta_{qq}\delta\left(1-x\right)$$

1th order: $F_{2q}^{(1)}(x_B,Q^2)=C_q^{(1)}(x_B/x,Q^2/\mu^2)\otimes\varphi_{q/q}^{(0)}\left(x,\mu^2\right)$

$$+C_q^{(0)}(x_B/x,Q^2/\mu^2)\otimes\varphi_{q/q}^{(1)}\left(x,\mu^2\right)$$

$$\Rightarrow\quad\boxed{C_q^{(1)}(x,Q^2/\mu^2)=F_{2q}^{(1)}(x,Q^2)-F_{2q}^{(0)}(x,Q^2)\otimes\varphi_{q/q}^{(1)}\left(x,\mu^2\right)}$$

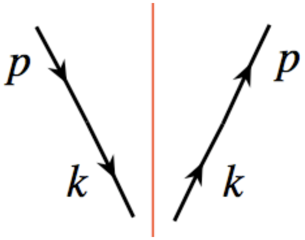
PDFs of a parton

❑ Change the state without changing the operator:

$$\phi_{q/h}(x,\mu^2) = \int \frac{dy^-}{2\pi}\, e^{ixp^+y^-} \langle h(p)|\overline{\psi}_q(0)\frac{\gamma^+}{2}\, U^n_{[0,y^-]}\, \psi_2(y^-)|h(p)\rangle$$

$|h(p)\rangle \;\Rightarrow |\mathrm{parton}(p)\rangle$ ➡ $\phi_{f/q}(x,\mu^2)$ **– given by Feynman diagrams**

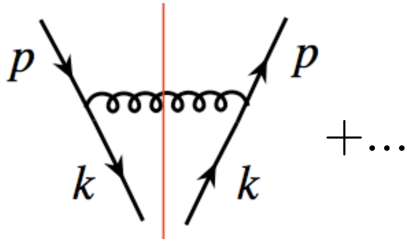
❑ Lowest order quark distribution:



✧ From the operator definition:

$$\begin{aligned}\phi^{(0)}_{q'/q}(x) &= \delta_{qq'} \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\left(\frac{1}{2}\gamma\cdot p\right)\left(\frac{\gamma^+}{2p^+}\right)\right] \delta\left(x - \frac{k^+}{p^+}\right) (2\pi)^4 \delta^4(p-k) \\ &= \delta_{qq'}\ \delta(1-x)\end{aligned}$$

❑ Leading order in α_s quark distribution:



✧ Expand to $(g_s)^2$ – logarithmic divergent:

$$\phi^{(1)}_{q/q}(x) = C_F \frac{\alpha_s}{2\pi} \int \frac{dk_T^2}{k_T^2} \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\,\delta(1-x)\right] + \mathrm{UVCT}$$

UV and CO divergence

Partonic cross sections

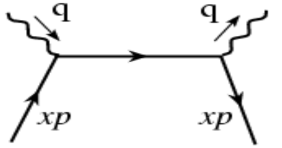
- **Projection operators for SFs:**

$$W_{\mu\nu} = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)F_1\left(x,Q^2\right) + \frac{1}{p\cdot q}\left(p_\mu - q_\mu \frac{p\cdot q}{q^2}\right)\left(p_\nu - q_\nu \frac{p\cdot q}{q^2}\right)F_2\left(x,Q^2\right)$$

$$\boxed{\begin{aligned} F_1(x,Q^2) &= \frac{1}{2}\left(-g^{\mu\nu} + \frac{4x^2}{Q^2}p^\mu p^\nu\right)W_{\mu\nu}(x,Q^2) \\ F_2(x,Q^2) &= x\left(-g^{\mu\nu} + \frac{12x^2}{Q^2}p^\mu p^\nu\right)W_{\mu\nu}(x,Q^2) \end{aligned}}$$

- **0th order:**

$$F_{2q}^{(0)}(x) = xg^{\mu\nu}W_{\mu\nu,q}^{(0)} = xg^{\mu\nu}\Big[\frac{1}{4\pi}$$



$$\Big]$$

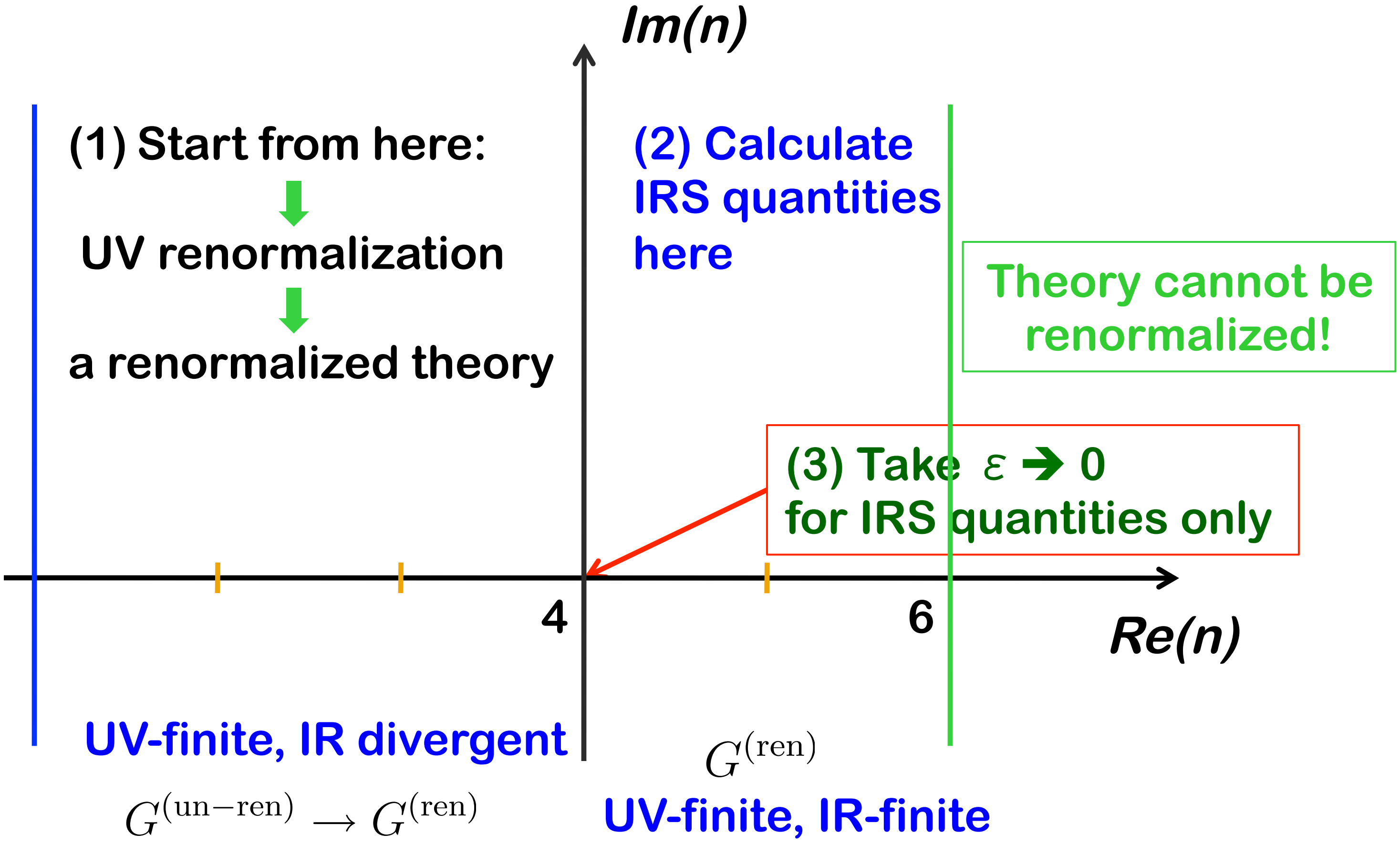
$$= \left(xg^{\mu\nu}\right)\frac{e_q^2}{4\pi}\mathrm{Tr}\left[\frac{1}{2}\gamma\cdot p\gamma_\mu\gamma\cdot\left(p+q\right)\gamma_\nu\right]2\pi\delta\left((p+q)^2\right)$$

$$= e_q^2 x\delta(1-x)$$

$$\boxed{C_q^{(0)}(x) = e_q^2 x\delta(1-x)}$$

How does dimensional regularization work?

- **Complex *n*-dimensional space:** $\int d^n k\, F(k,Q)$



(1) Start from here:

UV renormalization

a renormalized theory

(2) Calculate IRS quantities here

Theory cannot be renormalized!

(3) Take ε ➔ 0 for IRS quantities only

UV-finite, IR divergent

$G^{(\text{un}-\text{ren})} \to G^{(\text{ren})}$

$G^{(\text{ren})}$

UV-finite, IR-finite

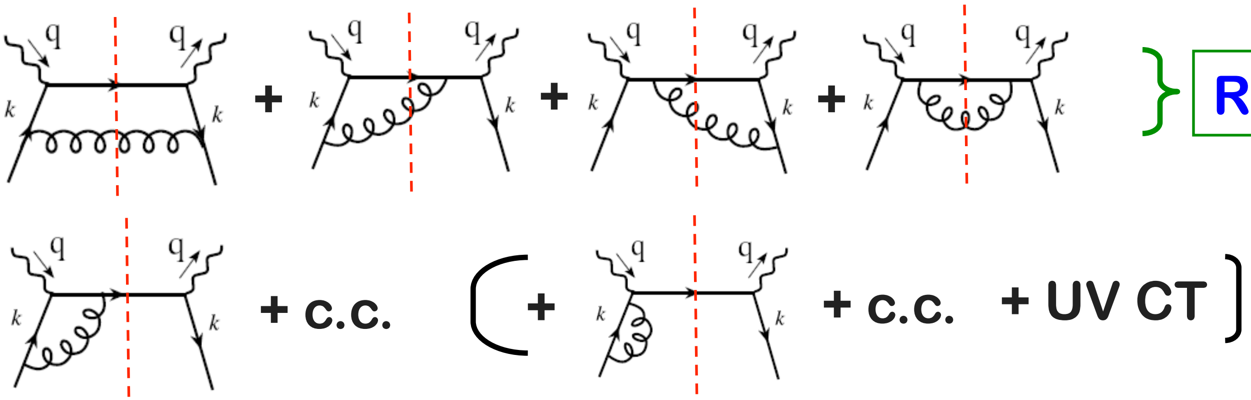
NLO coefficient function – complete example

$$\boxed{C_q^{(1)}(x,Q^2/\mu^2) = F_{2q}^{(1)}(x,Q^2) - F_{2q}^{(0)}(x,Q^2)\otimes\varphi_{q/q}^{(1)}\left(x,\mu^2\right)}$$

- ❑ **Projection operators in n-dimension:** $g_{\mu\nu}g^{\mu\nu} = n \equiv 4-2\varepsilon$

$$\boxed{\left(1-\varepsilon\right)F_2 = x\left(-g^{\mu\nu}+(3-2\varepsilon)\frac{4x^2}{Q^2}p^\mu p^\nu\right)W_{\mu\nu}}$$

- ❑ **Feynman diagrams:**



$W^{(1)}_{\mu\nu,q}$ + + + } **Real**

Virtual { + + c.c. (+ + c.c. + UV CT)

- ❑ **Calculation:** $-g^{\mu\nu}W^{(1)}_{\mu\nu,q}$ and $p^\mu p^\nu W^{(1)}_{\mu\nu,q}$

Contribution from the trace of W$_{\mu\ \nu}$

- ☐ **Lowest order in n-dimension:**

$$-g^{\mu\nu}W^{(0)}_{\mu\nu,q}=e_q^2(1-\varepsilon)\delta(1-x)$$

- ☐ **NLO virtual contribution:**

$$-g^{\mu\nu}W^{(1)V}_{\mu\nu,q}=e_q^2(1-\varepsilon)\delta(1-x)$$

$$*\left(-\frac{\alpha_s}{\pi}\right)C_F\left[\frac{4\pi\mu^2}{Q^2}\right]^{\varepsilon}\frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\left[\frac{1}{\varepsilon^2}+\frac{3}{2}\frac{1}{\varepsilon}+4\right]$$

- ☐ **NLO real contribution:**

$$-g^{\mu\nu}W^{(1)R}_{\mu\nu,q}=e_q^2(1-\varepsilon)C_F\left(-\frac{\alpha_s}{2\pi}\right)\left[\frac{4\pi\mu^2}{Q^2}\right]^{\varepsilon}\frac{\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}$$

$$*\left\{-\frac{1-\varepsilon}{\varepsilon}\left[1-x+\left(\frac{2x}{1-x}\right)\left(\frac{1}{1-2\varepsilon}\right)\right]+\frac{1-\varepsilon}{2(1-2\varepsilon)(1-x)}+\frac{2\varepsilon}{1-2\varepsilon}\right\}$$

- ❑ **The "+" distribution:**

$$\left(\frac{1}{1-x}\right)^{1+\varepsilon} = -\frac{1}{\varepsilon}\delta(1-x) + \frac{1}{(1-x)_+} + \varepsilon\left(\frac{\ln(1-x)}{1-x}\right)_+ + O\left(\varepsilon^2\right)$$

$$\int_z^1 dx\,\frac{f(x)}{(1-x)_+} \equiv \int_z^1 dx\,\frac{f(x)-f(1)}{1-x} + \ln(1-z)\,f(1)$$

- ❑ **One loop contribution to the trace of W$_{\mu\,\nu}$:**

$$-g^{\mu\nu}W^{(1)}_{\mu\nu,q} = e_q^2(1-\varepsilon)\left(\frac{\alpha_s}{2\pi}\right)\left\{-\frac{1}{\varepsilon}P_{qq}(x) + P_{qq}(x)\ln\left(\frac{Q^2}{\mu^2(4\pi e^{-\gamma_E})}\right)\right.$$

$$+C_F\left[\left(1+x^2\right)\left(\frac{\ln(1-x)}{1-x}\right)_+ - \frac{3}{2}\left(\frac{1}{1-x}\right)_+ - \frac{1+x^2}{1-x}\ln(x)\right.$$

$$\left.\left. +3-x-\left(\frac{9}{2}+\frac{\pi^2}{3}\right)\delta(1-x)\right]\right\}$$

- ❑ **Splitting function:**

$$P_{qq}(x) = C_F\left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right]$$

☐ One loop contribution to $p^\mu p^\nu$ $W_{\mu\nu}$:

$$p^\mu p^\nu W^{(1)V}_{\mu\nu,q} = 0 \qquad\qquad p^\mu p^\nu W^{(1)R}_{\mu\nu,q} = e_q^2 C_F \frac{\alpha_s}{2\pi}\frac{Q^2}{4x}$$

☐ One loop contribution to F_2 of a quark:

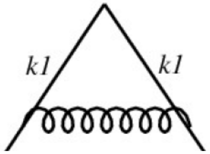
$$F^{(1)}_{2q}(x,Q^2) = e_q^2 x \frac{\alpha_s}{2\pi}\left\{\left(-\frac{1}{\varepsilon}\right)_{\rm CO} P_{qq}(x)\left(1+\varepsilon\ln(4\pi e^{-\gamma_E})\right) + P_{qq}(x)\ln\left(\frac{Q^2}{\mu^2}\right)\right.$$

$$\left. +C_F\left[(1+x^2)\left(\frac{\ln(1-x)}{1-x}\right)_+ - \frac{3}{2}\left(\frac{1}{1-x}\right)_+ - \frac{1+x^2}{1-x}\ln(x) + 3 + 2x - \left(\frac{9}{2}+\frac{\pi^2}{3}\right)\delta(1-x)\right]\right\}$$

$$\Rightarrow \quad \infty \quad \text{as} \quad \varepsilon \to 0$$

☐ One loop contribution to quark PDF of a quark:

$$\varphi^{(1)}_{q/q}(x,\mu^2) = \left(\frac{\alpha_s}{2\pi}\right) P_{qq}(x)\left\{\left(\frac{1}{\varepsilon}\right)_{\rm UV} + \left(-\frac{1}{\varepsilon}\right)_{\rm CO}\right\} + \text{UV-CT}$$



– in the dimensional regularization

Different UV-CT = different factorization scheme!

❑ Common UV-CT terms:

- ✧ **MS scheme:** $\text{UV-CT}\big|_{\text{MS}} = -\frac{\alpha_s}{2\pi} P_{qq}(x)\left(\frac{1}{\varepsilon}\right)_{\text{UV}}$

- ✧ **$\overline{\text{MS}}$ scheme:** $\text{UV-CT}\big|_{\overline{\text{MS}}} = -\frac{\alpha_s}{2\pi} P_{qq}(x)\left(\frac{1}{\varepsilon}\right)_{\text{UV}} \left(1+\varepsilon \ln(4\pi e^{-\gamma_E})\right)$

- ✧ **DIS scheme:** **choose a UV-CT, such that** $C_q^{(1)}(x, Q^2/\mu^2)\big|_{\text{DIS}} = 0$

❑ One loop coefficient function:

$$\boxed{C_q^{(1)}(x, Q^2/\mu^2) = F_{2q}^{(1)}(x, Q^2) - F_{2q}^{(0)}(x, Q^2) \otimes \varphi_{q/q}^{(1)}\left(x, \mu^2\right)}$$

$$C_q^{(1)}(x, Q^2/\mu^2) = e_q^2 x \frac{\alpha_s}{2\pi}\left\{ P_{qq}(x) \ln\left(\frac{Q^2}{\mu_{\overline{\text{MS}}}^2}\right)\right.$$

$$\left. + C_F\left[(1+x^2)\left(\frac{\ln(1-x)}{1-x}\right)_+ - \frac{3}{2}\left(\frac{1}{1-x}\right)_+ - \frac{1+x^2}{1-x}\ln(x) + 3 + 2x - \left(\frac{9}{2}+\frac{\pi^2}{3}\right)\delta(1-x)\right]\right\}$$