Non-perturbative QCD for Nuclear Physicists

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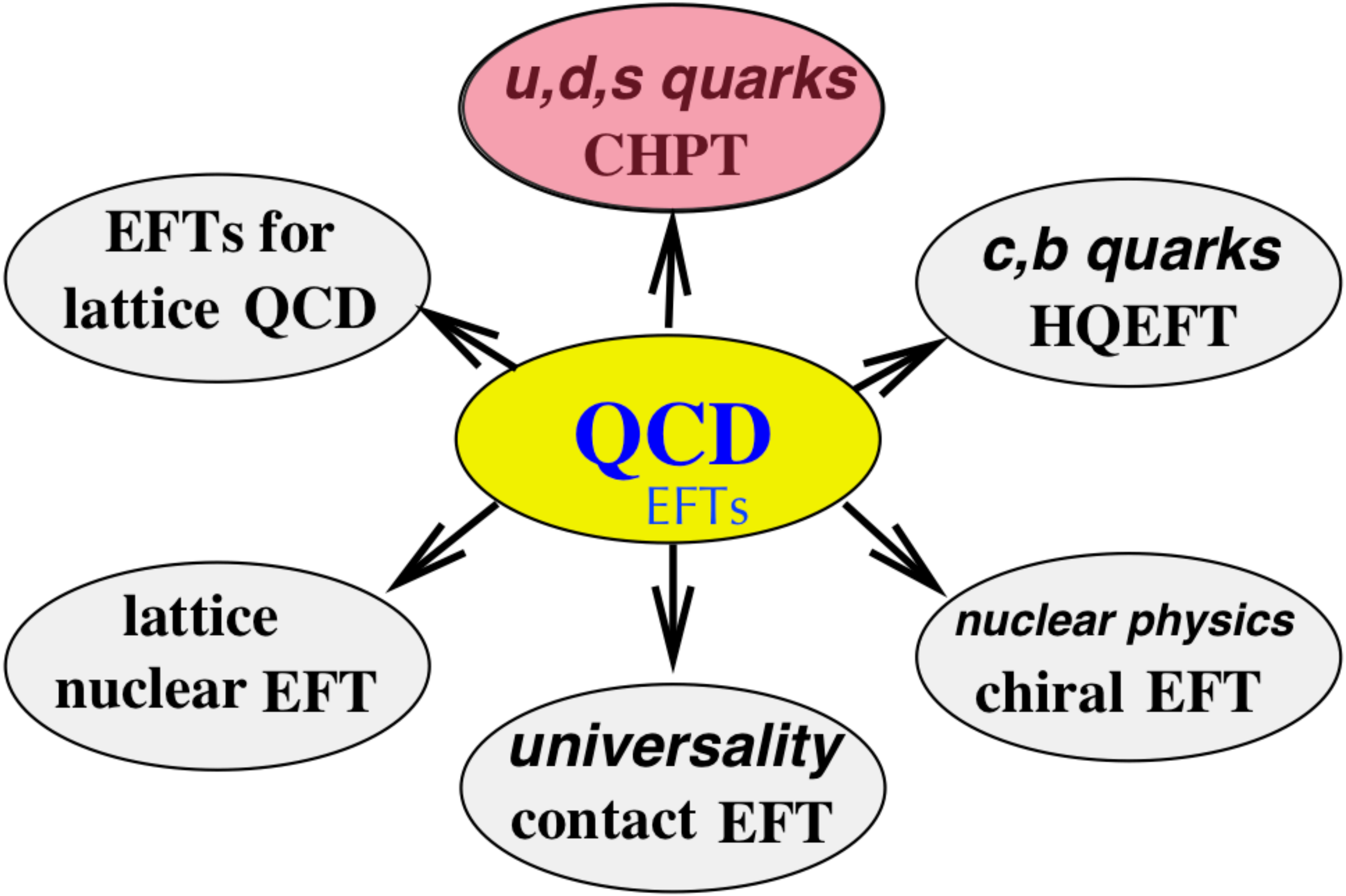
LECTURE II

Lecture II:

- ChiPT (cont)
- Lattice QCD
- $\pi\pi$


u,d,s quarks
CHPT
EFTs for
lattice QCD
c,b quarks
HQEFT
QCD
EFTs
lattice
nuclear EFT
universality
contact EFT
nuclear physics
chiral EFT

$$\mathcal{L}_{\mathrm{QCD}} = \sum_{i=1}^{3} \left(\bar{q}_i i \not{D} q_i - m_i \bar{q}_i q_i \right) - \frac{1}{2} \mathrm{Tr} G_{\mu\nu} G^{\mu\nu}$$

u,d,s active flavors

$$D_\mu = \partial_\mu + i g A_\mu \qquad\qquad A_\mu = A_\mu^a T_a$$

$$T_a \in SU(3)$$

$$U(1)_V \times SU(3)_L \times SU(3)_R$$

Baryon number

Chiral symmetry: $q_{Li} \to L_{ij} q_{Lj} \qquad q_{Rj} \to R_{ij} q_{Rj}$

Assume:

$$\langle 0|\bar{q}_{Rj} q_{Li}|0\rangle = \Lambda^3 \delta_{ij}$$

$$SU(3)_L \times SU(3)_R \rightarrow SU(3)_V$$

spontaneous symmetry breaking

Assume: $\langle 0|\bar{q}_{Rj} q_{Li}|0\rangle = \Lambda^3 \delta_{ij}$

$SU(3)_L \times SU(3)_R \to SU(3)_V$

spontaneous symmetry breaking

$q_{Li} \to L_{ij} q_{Lj}$ $\qquad$ $q_{Rj} \to R_{ij} q_{Rj}$

$$\delta_{ij} \to (LR^\dagger)_{ij} \equiv \Sigma_{ij}$$

mass scale t.b.d

Excitations of the condensate: $\Sigma \to \Sigma(x) \equiv e^{2i\boldsymbol{\pi}(x)/f}$

$$\boldsymbol{\pi}(x) = \pi_a(x) T_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \overline{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} \sim \begin{pmatrix} u\overline{u} & u\overline{d} & u\overline{s} \\ d\overline{u} & d\overline{d} & d\overline{s} \\ s\overline{u} & s\overline{d} & s\overline{s} \end{pmatrix}$$

The Goldstone bosons

Chiral Perturbation Theory ($\chi - \mathrm{PT}$)

EFT that encodes the interactions of the Goldstone modes with themselves and with other hadronic degrees of freedom.

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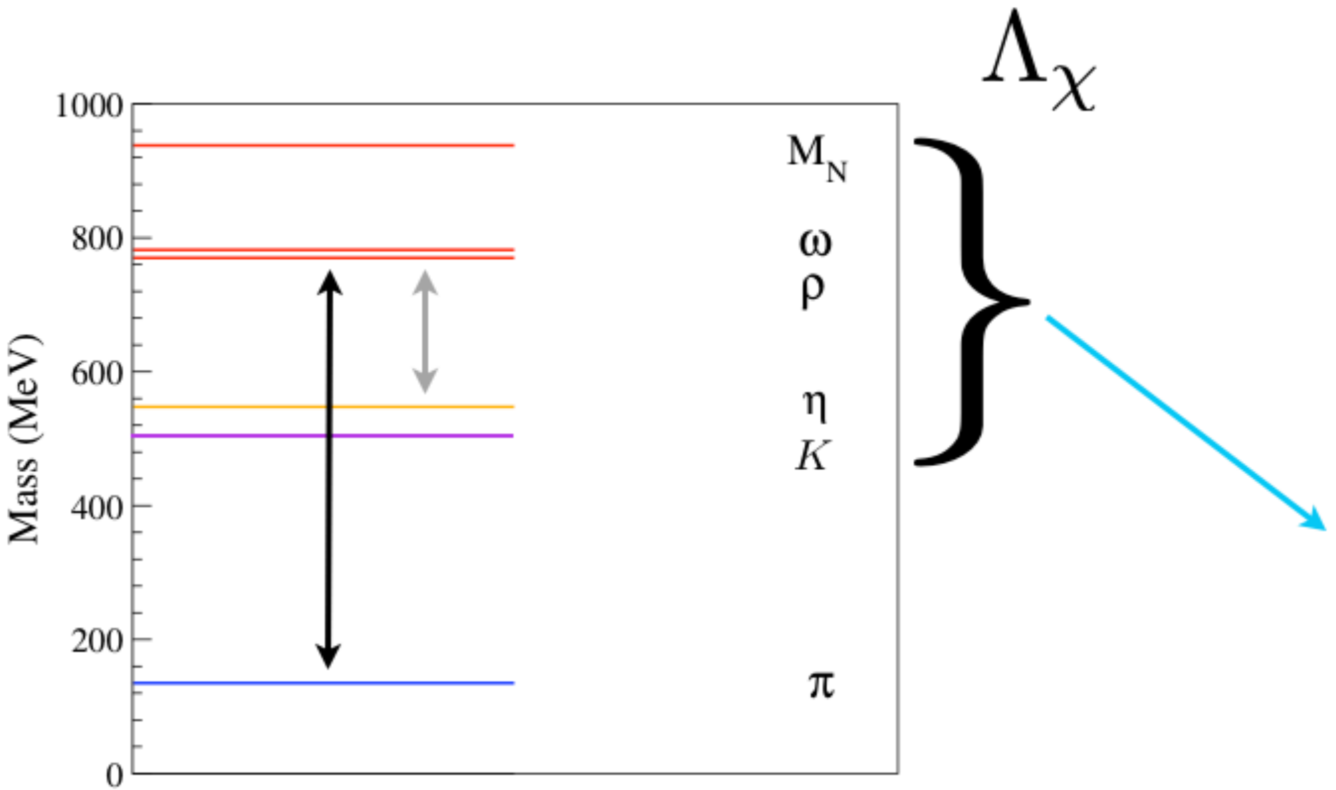
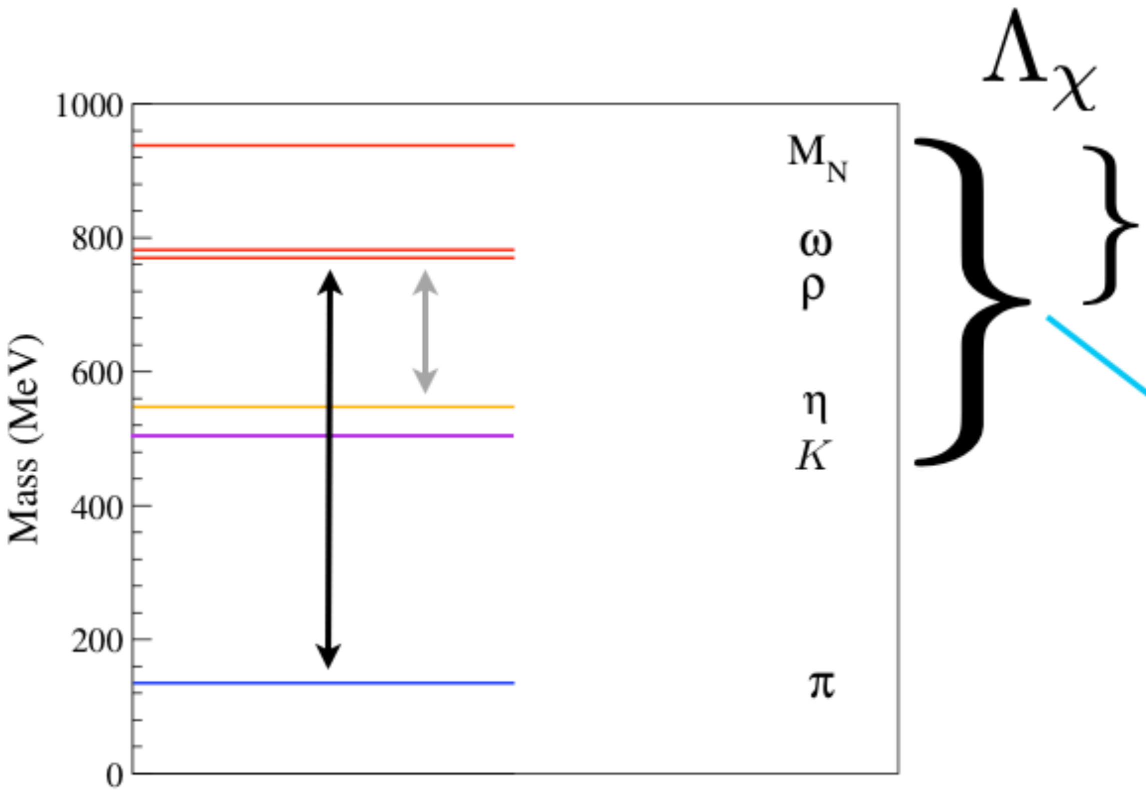


$SU(2)_L \times SU(2)_R$

$$\frac{(p\,,\, m_\pi)}{\Lambda_\chi} \ll 1$$

Chiral Perturbation Theory ($\chi - \mathrm{PT}$)

EFT that encodes the interactions of the Goldstone modes with themselves and with other hadronic degrees of freedom.



$SU(3)_L \times SU(3)_R$

$$\frac{(p\ ,\ m_\pi\ ,\ m_K\ ,\ m_\eta)}{\Lambda_\chi} \ll 1$$

$SU(2)_L \times SU(2)_R$

$$\frac{(p\ ,\ m_\pi)}{\Lambda_\chi} \ll 1$$

$$SU(3)_L \times SU(3)_R \quad : \quad \Sigma \to L\Sigma R^\dagger$$

$$\mathcal{L}_0 \;=\; \frac{f^2}{4}\,\mathrm{Tr}\partial_\mu\Sigma^\dagger\partial_\mu\Sigma = \frac{1}{2}\partial_\mu\pi_a\partial_\mu\pi_a \;+\; interactions$$

normalized kinetic term

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normalized kinetic term

$$\langle 0|\bar{u}\gamma^\mu(1-\gamma_5)d|\pi^-(p)\rangle \equiv i\sqrt{2}\, f_\pi p^\mu \qquad f_\pi = 92.4 \pm 0.25 \text{ MeV}$$

$\pi \to \mu\nu$

$$||$$

$$2\left(j^\mu_{L1} + i j^\mu_{L2}\right)$$

$$\chi - \mathrm{PT}: \quad j^\mu_{La} = -i\frac{f^2}{2}\mathrm{Tr}T_a\Sigma^\dagger\partial^\mu\Sigma = \frac{1}{2} f\partial^\mu \pi_a + \ldots$$

Matching: $f = f_\pi = 93 \text{ MeV}$

Explicit chiral symmetry breaking: $M = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix} \neq 0$

$$SU(3)_L \times SU(3)_R \quad : \quad M \to RML^\dagger$$ spurion

$$\mathcal{L}_M = \lambda_M \frac{f^2}{2} \left(\mathrm{Tr} M\Sigma + \mathrm{h.c.}\right) = \frac{1}{2} m_{\pi_a}^2 \pi_a \pi_a + \textit{interactions}$$

mass scale t.b.d

Explicit chiral symmetry breaking: $M = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix} \neq 0$

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spurion

$$\boxed{\mathcal{L}_M = \lambda_M \frac{f^2}{2} \left(\mathrm{Tr} M\Sigma + \mathrm{h.c.} \right) \; = \; \frac{1}{2} m_{\pi_a}^2 \pi_a \pi_a \; + \; interactions}$$

mass scale t.b.d (λ_M)

$$m_\pi^2 \; = \; \lambda_M (m_u + m_d) \qquad m_{K^+}^2 \; = \; \lambda_M (m_u + m_s) \qquad m_{K^0}^2 \; = \; \lambda_M (m_d + m_s)$$

$$m_\eta^2 \; = \; \frac{1}{3} \lambda_M (m_u + m_d + 4 m_s) \; + \; \mathcal{O}\left((m_u - m_d)^2\right)$$

$$m_{\pi^0}^2 \; = \; \lambda_M (m_u + m_d) \; + \; \mathcal{O}\left((m_u - m_d)^2\right)$$

$$\boxed{3 m_\eta^2 + m_\pi^2 = 4 m_K^2}$$

Gell-Mann Okubo

EXP: 1%

$$\lambda_M \ ??$$

$$\mathcal{H}_{QCD} = \ldots + \sum_i m_i \bar{q}_i q_i + \ldots$$

$$\langle 0 | \bar{q}_i q_i | 0 \rangle = \frac{\partial}{\partial m_i} \langle 0 | \mathcal{H}_{QCD} | 0 \rangle = \frac{\partial \mathcal{E}_0}{\partial m_i}$$

→ *vaccum energy density*

$$\lambda_M \; ??$$

$$\mathcal{H}_{QCD} = \ldots + \sum_i m_i \bar{q}_i q_i + \ldots$$

$$\langle 0|\bar{q}_i q_i|0\rangle = \frac{\partial}{\partial m_i}\langle 0|\mathcal{H}_{QCD}|0\rangle = \frac{\partial \mathcal{E}_0}{\partial m_i}$$

→ vaccum energy density

$\chi - \mathrm{PT}$:

$$\mathcal{E}_0 = constant - \lambda_M \frac{f^2}{2}\left(\mathrm{Tr} M\Sigma + \mathrm{h.c.}\right) + \ldots$$

Matching: $\langle 0|\bar{q}_i q_i|0\rangle = \lambda_M f^2$

$$m_\pi^2 = \frac{\langle 0|\bar{u}u|0\rangle}{f^2}(m_u + m_d)$$

Gell-Mann-Oakes-Renner

Power counting

Consider generic connected diagram:

V_d	:	vertices with d derivatives
E	:	external lines
I	:	internal lines
L	:	loops

$$p^{\sum_d dV_d - 2I + 4L}$$

Power counting

Consider generic connected diagram:

V_d	:	vertices with d derivatives
E	:	external lines
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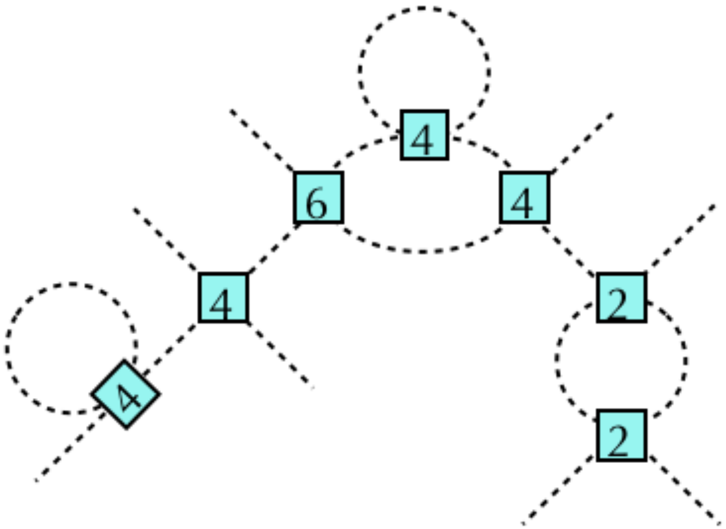


note: $p = (q, m_\pi)$

$$p^{\sum_d dV_d - 2I + 4L}$$

$$+$$

$$L = I - \sum_d V_d + 1$$

$$\downarrow$$

$$N = \sum_d \frac{1}{2}(d-2)V_d + L$$

$$f^2 p^2 \left(\frac{p^2}{f^2}\right)^N \left(\frac{1}{f}\right)^E$$

Generic *power counting*

Sub-leading Lagrangian

$SU(3)_L \times SU(3)_R$

$$\mathcal{L}_{p^4} = L_1 \left(\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right)\right)^2 + L_2 \mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma\right) \mathrm{Tr}\left(\partial^\mu \Sigma^\dagger \partial^\nu \Sigma\right)$$
$$+L_3 \mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma \partial_\nu \Sigma^\dagger \partial^\nu \Sigma\right) + L_4 \mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right) \mathrm{Tr}\left(\chi \Sigma + \mathrm{h.c.}\right)$$
$$+L_5 \mathrm{Tr}\left(\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right)\left(\chi \Sigma + \mathrm{h.c.}\right)\right) + L_6 \left(\mathrm{Tr}\left(\chi \Sigma + \mathrm{h.c.}\right)\right)^2$$
$$+L_7 \left(\mathrm{Tr}\left(\chi \Sigma - \mathrm{h.c.}\right)\right)^2 + L_8 \mathrm{Tr}\left(\chi \Sigma \chi \Sigma + \mathrm{h.c.}\right)$$

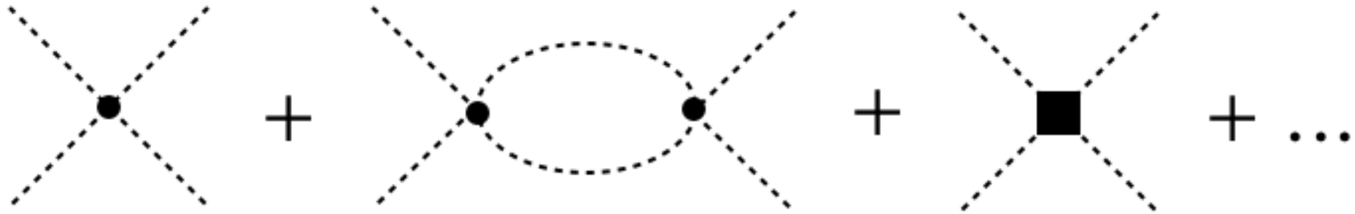
$$+WZ$$

$SU(2)_L \times SU(2)_R$

$$\mathcal{L}_{p^4} = \frac{\ell_1}{4} \left(\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right)\right)^2 + \frac{\ell_2}{4} \mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma\right) \mathrm{Tr}\left(\partial^\mu \Sigma^\dagger \partial^\nu \Sigma\right)$$
$$+\frac{\ell_4}{4} \mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right) \mathrm{Tr}\left(\chi \Sigma + \mathrm{h.c.}\right) + \frac{(\ell_3 + \ell_4)}{4} \left(\mathrm{Tr}\left(\chi \Sigma + \mathrm{h.c.}\right)\right)^2 + \frac{\ell_7}{4} \left(\mathrm{Tr}\left(\chi \Sigma - \mathrm{h.c.}\right)\right)^2$$

$$\chi \equiv 2 \lambda_M M$$

Example: $\pi\pi$ scattering



$$f^2 p^2 \left(\frac{p^2}{f^2}\right)^N \left(\frac{1}{f}\right)^E$$

$$\frac{p^2}{f^2} \qquad \frac{p^4}{f^4}\log\frac{p^2}{\mu^2} \qquad \frac{p^4}{f^4}$$

$I = 2$

s-wave scattering length

$$m_\pi a_2 = -\frac{m_\pi^2}{8\pi f_\pi^2}\left[1 + \frac{3m_\pi^2}{16\pi^2 f_\pi^2}\left(\log\frac{m_\pi^2}{\mu^2} + \bar{l}_{\pi\pi}(\mu)\right)\right]$$

$$\bar{l}_{\pi\pi} \equiv -\frac{64\pi^2}{3}\left[4(\ell_1 + \ell_2) + \ell_3 - \ell_4\right] + 1$$

Not constrained by symmetries!

But related to other processes!

Quark mass dependence of masses and decay constants!

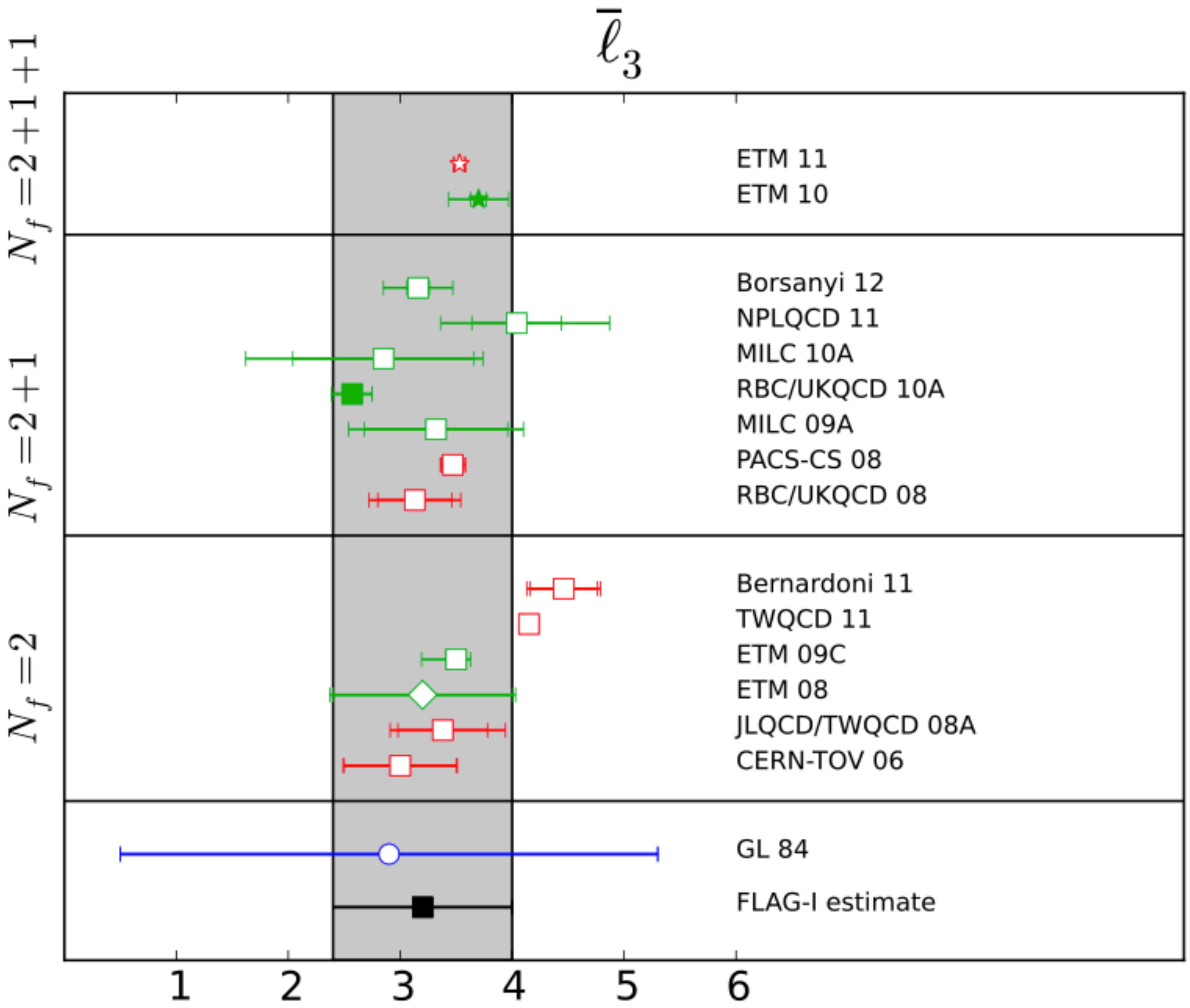
$$m_\pi^2 = 2Bm_q\left\{1 + \frac{1}{2}\xi \ln\left(\frac{\xi}{\xi^{\text{phy}}}\right) - \frac{1}{2}\xi\,\bar{l}_3\right\}$$

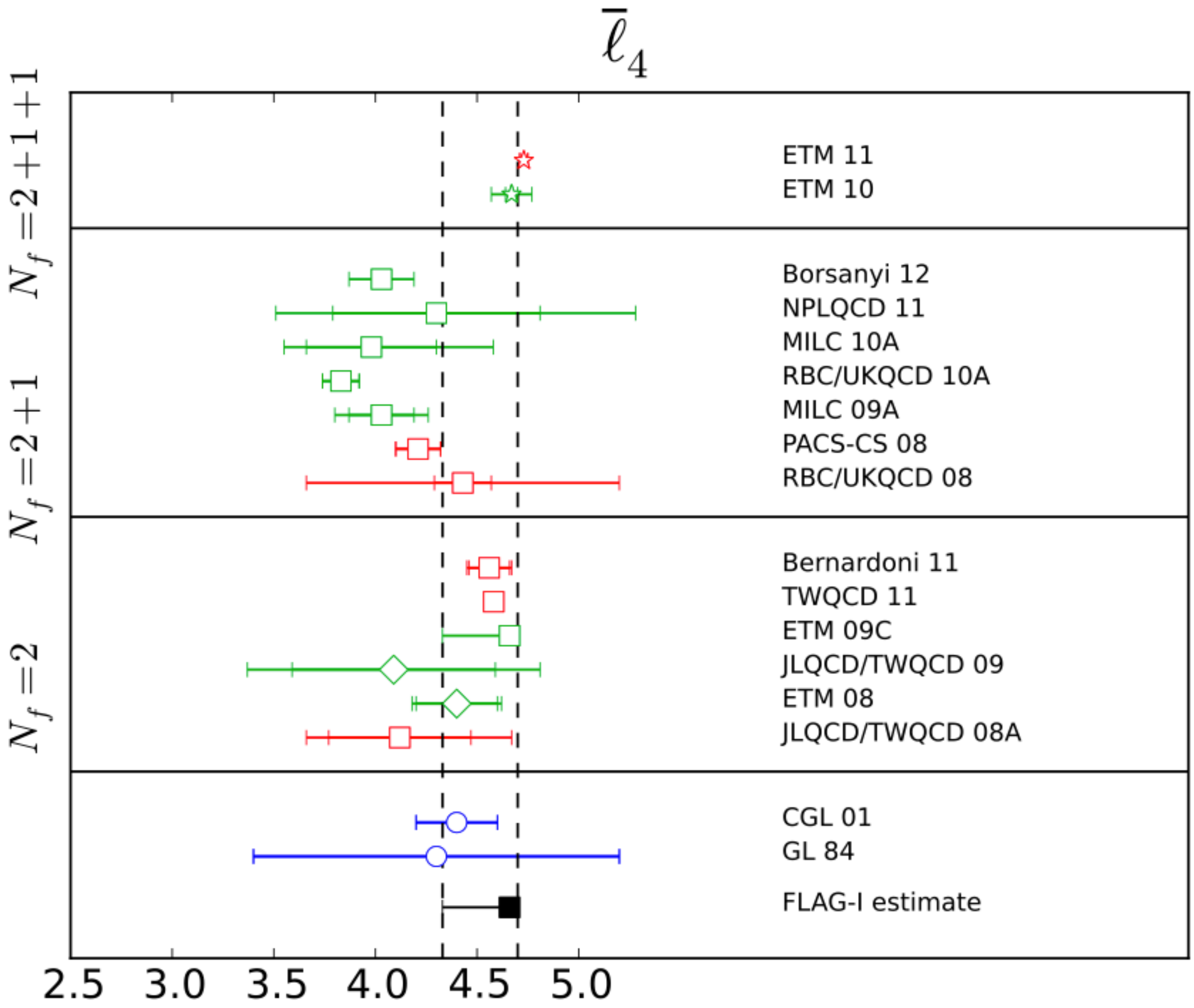
$$f_\pi = f\left\{1 - \xi \ln\left(\frac{\xi}{\xi^{\text{phy}}}\right) + \xi\bar{l}_4\right\}$$

$$\xi = \frac{m_\pi^2}{8\pi^2 f_\pi^2} \qquad \bar{l}_i = \log\frac{\Lambda_i^2}{(m_\pi^{\text{phy}})^2}$$

Calculated using Lattice QCD!

FLAG = Flavour Lattice Averaging Group





$\bar{\ell}_4$
$N_f=2+1+1$
$N_f=2+1$
$N_f=2$
ETM 11
ETM 10
Borsanyi 12
NPLQCD 11
MILC 10A
RBC/UKQCD 10A
MILC 09A
PACS-CS 08
RBC/UKQCD 08
Bernardoni 11
TWQCD 11
ETM 09C
JLQCD/TWQCD 09
ETM 08
JLQCD/TWQCD 08A
CGL 01
GL 84
FLAG-I estimate
2.5
3.0
3.5
4.0
4.5
5.0

EFT references

- D.B. Kaplan, nucl-th/9506035, nucl-th/0510023
- A. Pich, hep-ph/9806303
- D.R. Phillips, nucl-th/0203040
- H. Leutwyler, hep-ph/0212325
- M. Golterman, hep-lat/0912.4042
- *Flavianet 2008* (Benasque): E.Braaten, H.Leutwyler

QCD Euclidean Path Integral

$$\mathcal{Z} = \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{\int d^4x \left(-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \bar{\psi}[D_\mu\gamma_\mu + m]\psi \; + \; \mathcal{L}_{G.F.}\right)}$$

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \; \mathcal{O} \; e^{\int d^4x \left(-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \bar{\psi}[D_\mu\gamma_\mu + m]\psi \; + \; \mathcal{L}_{G.F.}\right)}$$

The correlation functions

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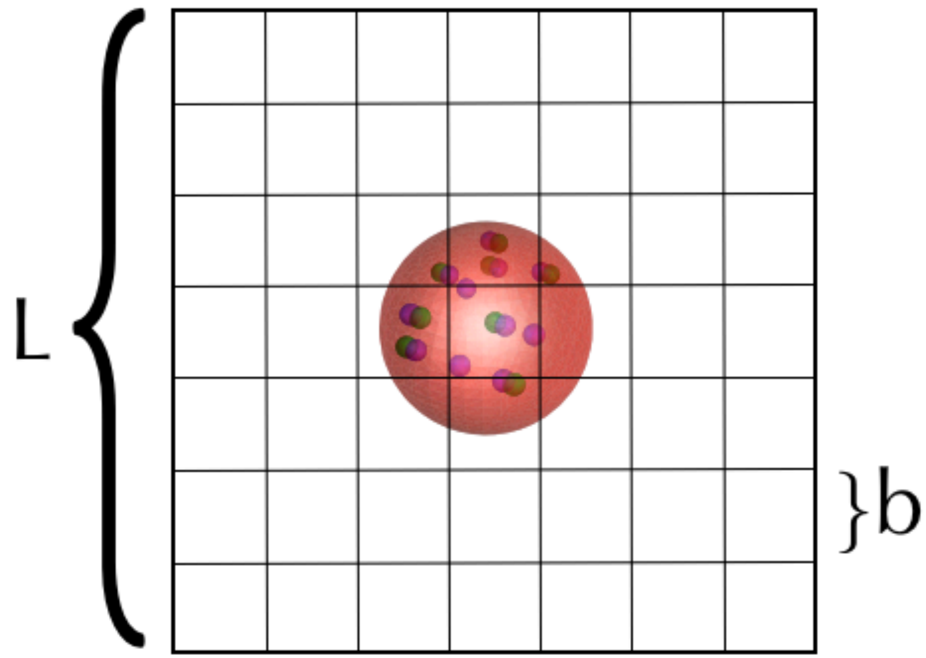
The correlation functions

Need a non-perturbative definition!

Lattice regularization

Lattice QCD = QCD on a grid or lattice



volume: $M_\pi L \gg 1$

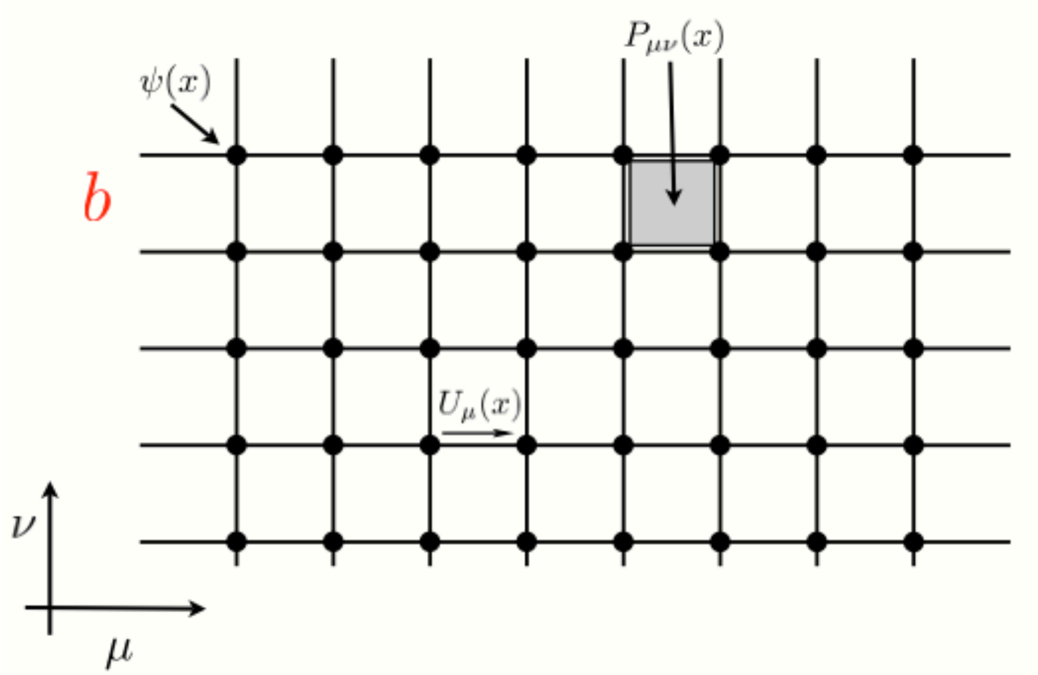
infrared cutoff

lattice spacing: $b \ll M_N^{-1}$

ultraviolet cutoff

Can use Effective Field Theory to extrapolate in L and b!

(systematic uncertainties from lattice artifacts are controlled) gauge sector: $U_\mu(x) = \exp\left(ig\int_x^{x+b\hat{\mu}} dz A_\mu(z)\right)$ special unitary matrices



$$P_{\mu\nu} = U_\mu(x)U_\nu(x+\hat{\mu})U_\mu^\dagger(x+\hat{\nu})U_\nu^\dagger(x)$$

gauge sector:

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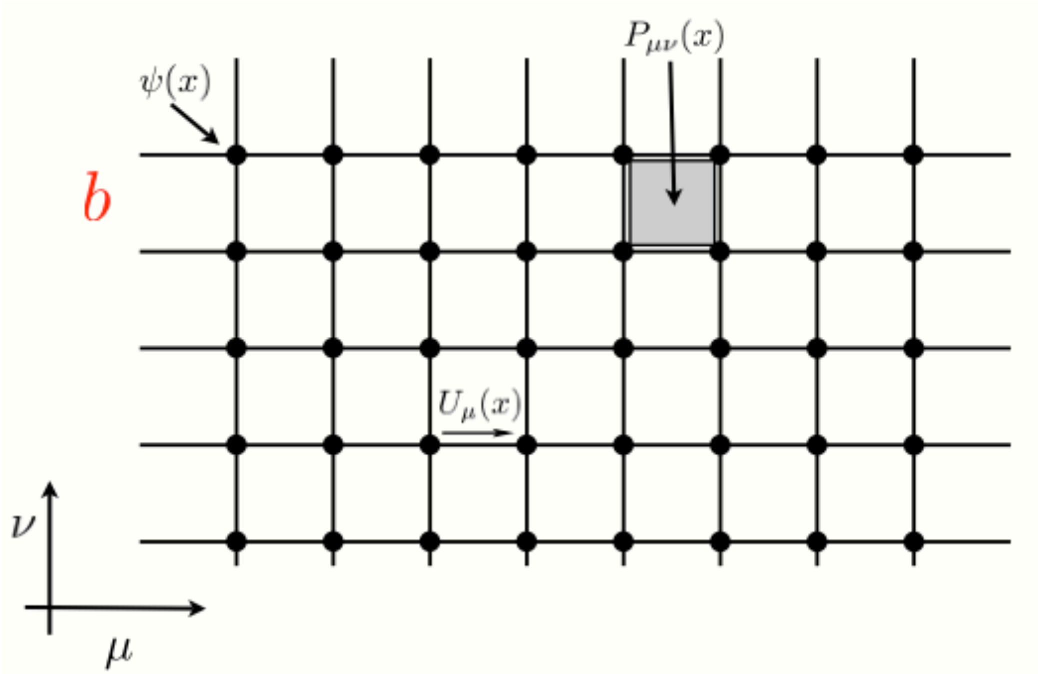
special unitary matrices



$$P_{\mu\nu} = U_\mu(x)U_\nu(x+\hat{\mu})U_\mu^\dagger(x+\hat{\nu})U_\nu^\dagger(x)$$

$$S_g(U) = \beta\sum_{x\mu\nu}\left(1-\frac{1}{3}\mathrm{ReTr}P_{\mu\nu}(x)\right) \xrightarrow[b\to 0]{} -\int d^4x\tfrac{1}{4}\left(F^a_{\mu\nu}(x)\right)^2$$

fermion sector: $S_f = \bar{\psi} D(U) \psi$

- ★ $D(U)$ is a sparse matrix (lots of zeros)
- ★ Wilson fermions
- ★ Domain wall fermions
- ★ Staggered fermions (Kogut-Susskind)
- ★ Overlap fermions

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In principle, all discretizations have the same continuum limit.

$$S^{(W)} = b^4 \sum_x \mathcal{L}^{(W)}(x) = b^4 \left(m + \frac{4}{b} \right) \sum_x \overline{\psi}(x)\psi(x)$$
$$+ \frac{b^3}{2} \sum_x \overline{\psi}(x) \left[(\gamma_\mu - 1)\ U_\mu(x)\ \psi(x + b\hat{\mu}) - (\gamma_\mu + 1)\ U_\mu^\dagger(x - b\hat{\mu})\ \psi(x - b\hat{\mu}) \right]$$

QCD Euclidean Path Integral

$$\mathcal{Z} = \int \prod_{\mu,x} dU_\mu(x) \prod_x d\bar{\psi} d\psi \;\; e^{-S_g(U) - S_f(\bar{\psi},\psi,U)}$$

$$= \int \prod_{\mu,x} dU_\mu(x) \;\; \det\left(D(U)^\dagger D(U)\right) \;\; e^{-S_g(U)}$$ (two flavors)

QCD Euclidean Path Integral

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$$= \int \prod_{\mu,x} dU_\mu(x) \; \det\left(D(U)^\dagger D(U)\right) \; e^{-S_g(U)}$$ (two flavors)

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \prod_{\mu,x} dU_\mu(x) \; \mathcal{O}\left(\frac{1}{D(U)}, U\right) \det\left(D(U)^\dagger D(U)\right) \; e^{-S_g(U)}$$

Montecarlo Integration

Matrix Inversion

$$D(U)\chi = \psi$$

Generalized Gaussian Integration

$$Z = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} dx_1 dx_2 \dots dx_N \; e^{-\vec{x} A \vec{x}^T} = \frac{\pi^{n/2}}{\sqrt{\det A}}$$

Generalized Gaussian Integration

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$$\langle x_i x_j \rangle = Z^{-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} dx_1 dx_2 \ldots dx_N \; x_i x_j \; e^{-\vec{x}A\vec{x}^T} = \frac{1}{2}[A^{-1}]_{ij}$$

Generalized Gaussian Integration

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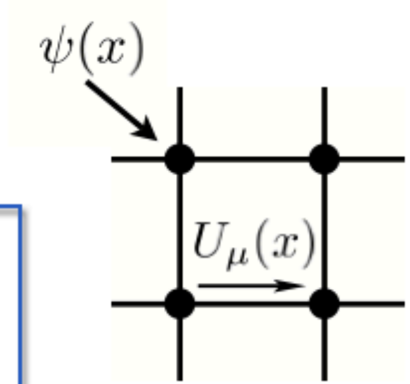
$$\langle x_i x_j \rangle = Z^{-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} dx_1 dx_2 \ldots dx_N \; x_i x_j \; e^{-\vec{x} A \vec{x}^T} = \frac{1}{2} [A^{-1}]_{ij}$$

$$\begin{aligned} \langle x_i x_j x_k x_l \rangle &= Z^{-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} dx_1 dx_2 \ldots dx_N x_i x_j x_k x_l e^{-\vec{x} A \vec{x}^T} \\ &= \frac{1}{4} ([A^{-1}]_{ij} [A^{-1}]_{kl} + [A^{-1}]_{ik} [A^{-1}]_{jl} + [A^{-1}]_{il} [A^{-1}]_{jk}) \end{aligned}$$

"Wick contractions"

QCD path integral with Montecarlo



$$\langle \mathcal{O} \rangle \sim \int dU_\mu \, d\bar{\psi} \, d\psi \, \mathcal{O}\left(U, \psi, \bar{\psi}\right) \; e^{-S_g(U) - \bar{\psi} D(U) \psi}$$

propagators (detector)

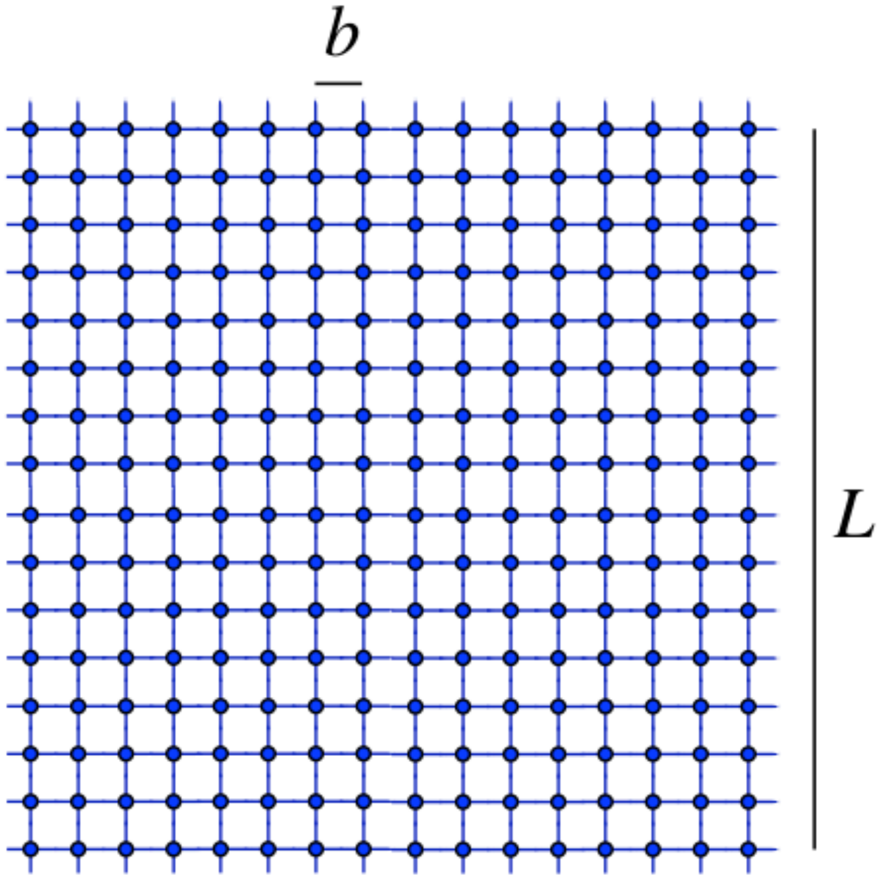
N gauge configurations (accelerator)

$$\langle \mathcal{O} \rangle \sim \int dU_\mu \, \mathcal{O}(D(U)^{-1}) \;\; \det\left(f(U)\right) \;\; e^{-S_g(U)}$$

$$= \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}(D(U_i)^{-1})$$

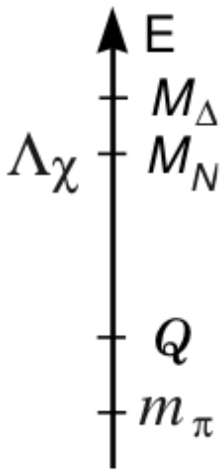
Estimate of $\mathcal{O}$ with $\sigma_{\mathcal{O}} \sim 1/\sqrt{N}$

Lattice QCD Economics



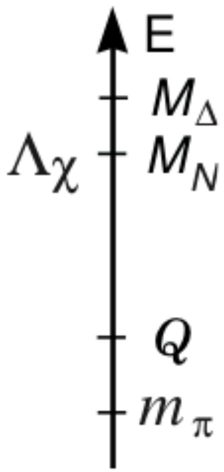
$$\mathrm{COST} \sim (L)^4 \, (b)^{-6.5} \, (M_q)^{-2.5}$$

QCD:



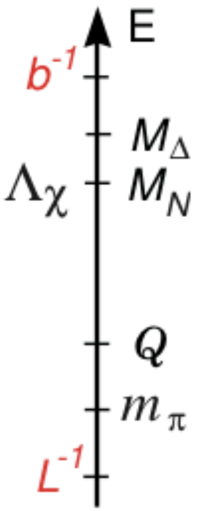
$$\frac{Q}{\Lambda_\chi}, \quad \frac{m_\pi}{\Lambda_\chi}, \quad \frac{M_\Delta - M_N}{\Lambda_\chi}, \; \cdots$$

QCD:



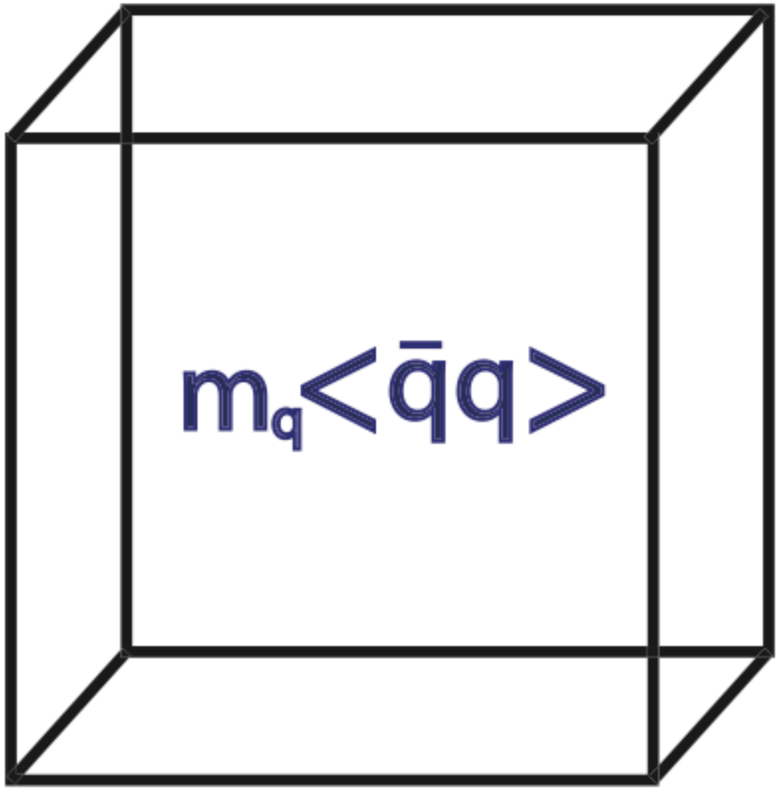
$$\frac{Q}{\Lambda_\chi}, \quad \frac{m_\pi}{\Lambda_\chi}, \quad \frac{M_\Delta - M_N}{\Lambda_\chi}, \ \ldots$$

Lattice QCD :



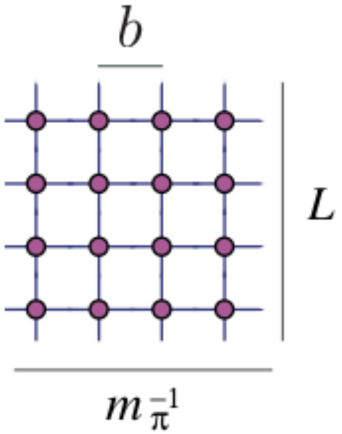
$$b\, m_\pi, \quad e^{-m_\pi L}, \quad m_\pi L, \quad \frac{1}{L\Lambda_\chi}, \ \ldots$$

What happens to chiral symmetry breaking at *finite V*?



$$|\mathbf{p}| = \frac{2\pi|\mathbf{n}|}{L} \ll \Lambda_\chi \qquad \Longrightarrow \qquad fL \gg 1$$

- $m_q \langle \bar{q} q \rangle L^4 = (m_\pi L)^2 (f L)^2 \sim p^{-2} \gg 1$:

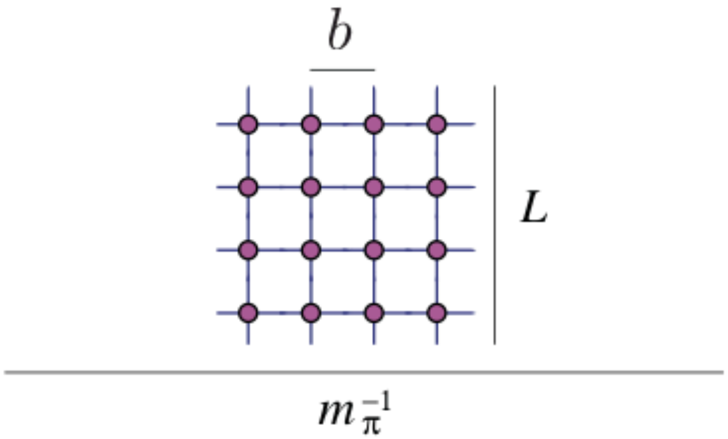


$m_\pi L \gtrsim 1$ <u>p regime</u>

$$L^{-1} \sim m_\pi \sim p$$

$(L_t \gg L_s) \quad \Longrightarrow \quad \int d^4 l \to \int dl_0 \sum_{\vec{l}}$

- $(m_\pi L)^2 (f L)^2 \sim \epsilon^0 \lesssim 1$:



$m_\pi L \ll 1$ <u>ϵ regime</u>

$$L^{-1} \sim \sqrt{m_\pi} \sim \epsilon$$

Momentum zero-modes nonperturbative

What happens to chiral symmetry breaking at *finite b*?

Symanzik action:

$$\mathcal{O}(b): \quad \mathcal{L}_{\rm QCD}^{\rm EFT} = \sum_{i=1}^{3} (\bar{q}_i i \not{D} q_i - m_i \bar{q}_i q_i) - \frac{1}{2} {\rm Tr} G_{\mu\nu} G^{\mu\nu} + b\, c_{sw} \sum_i \bar{q}_i \sigma_{\mu\nu} G^{\mu\nu} q_i \; + \; \ldots$$

Sheikholeslami-Wohlert

$$A \rightarrow RAL^{\dagger}$$

$$\mathcal{L}_{M,A} = \lambda_M \frac{f^2}{2} ({\rm Tr} M\Sigma + {\rm h.c.}) \; + \; \lambda_A \frac{f^2}{2} ({\rm Tr} A\Sigma + {\rm h.c.})$$

$$m_\pi^2 \quad = \quad \lambda_M \, (m_u + m_d) \; + \; 2\lambda_A \; b \; c_{sw}^{(V)}$$

Scattering in a finite volume

Recall NR scattering

$$\mathcal{A}_2(p) = \frac{4\pi}{M}\frac{1}{p\cot\delta(p) - ip}$$

=

sum of poles in a Finite Volume!

$$\mathcal{A}_2^{-1}(p) = 0$$

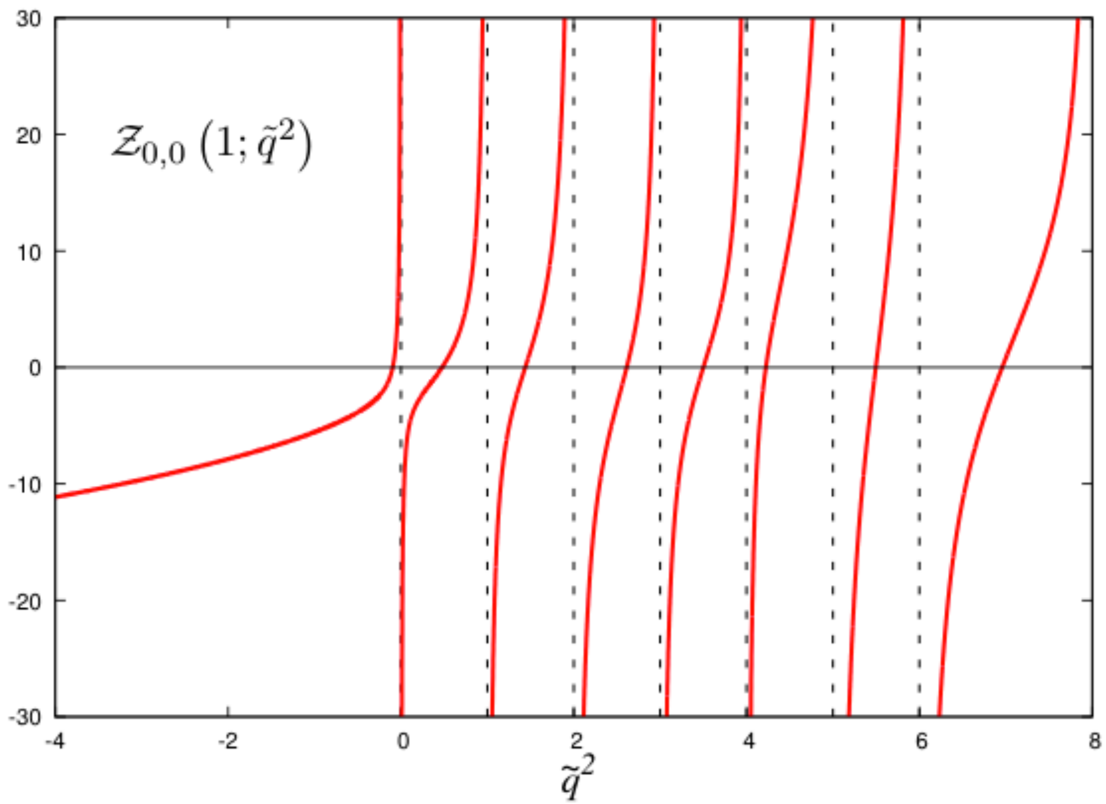
eigenvalue equation

S-wave at Finite Volume

$$q\cot\delta_0 = \frac{2}{\sqrt{\pi}L} \mathcal{Z}_{0,0}\left(1;\tilde{q}^2\right) \qquad \mathcal{Z}_{0,0}(1;\tilde{q}^2) = \frac{1}{\sqrt{4\pi}} \lim_{\Lambda_\mathbf{n}\to\infty} \left[\sum_{\mathbf{n}}^{\Lambda_\mathbf{n}} \frac{1}{|\mathbf{n}|^2 - \tilde{q}^2} - 4\pi\Lambda_\mathbf{n} \right]$$

$$+ \mathcal{O}\left(e^{-M_\pi L}\right)$$

(Luscher, 1990)



Weak coupling expansion:

$$\Delta E_0(2,L) = \frac{4\pi\, a_{\pi\pi}}{m_\pi L^3}\left\{1 - \left(\frac{a_{\pi\pi}}{\pi L}\right)\mathcal{I} + \left(\frac{a_{\pi\pi}}{\pi L}\right)^2\left[\mathcal{I}^2 - \mathcal{J}\right] + \left(\frac{a_{\pi\pi}}{\pi L}\right)^3\left[-\mathcal{I}^3 + 3\mathcal{I}\mathcal{J} - \mathcal{K}\right]\right\} + \frac{8\pi^2 {a_{\pi\pi}}^3}{m_\pi L^6}\, r_{\pi\pi} + \mathcal{O}\left(L^{-7}\right)$$

Calculated on the lattice!

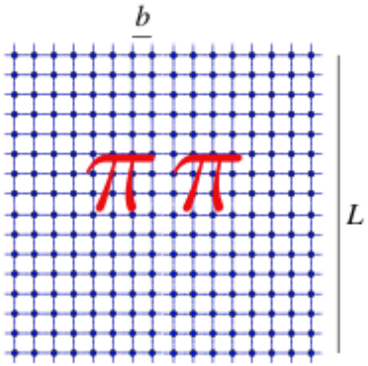
phase shift

$$\mathcal{I} = \lim_{\Lambda_j\to\infty}\sum_{\mathbf{i}\neq\mathbf{0}}^{|\mathbf{i}|\leq\Lambda_j}\frac{1}{|\mathbf{i}|^2} - 4\pi\Lambda_j = -8.91363291781$$

$$\mathcal{J} = \sum_{\mathbf{i}\neq\mathbf{0}}\frac{1}{|\mathbf{i}|^4} = 16.532315959$$

$$\mathcal{K} = \sum_{\mathbf{i}\neq\mathbf{0}}\frac{1}{|\mathbf{i}|^6} = 8.401923974433$$

$\pi\pi$ scattering in lattice QCD

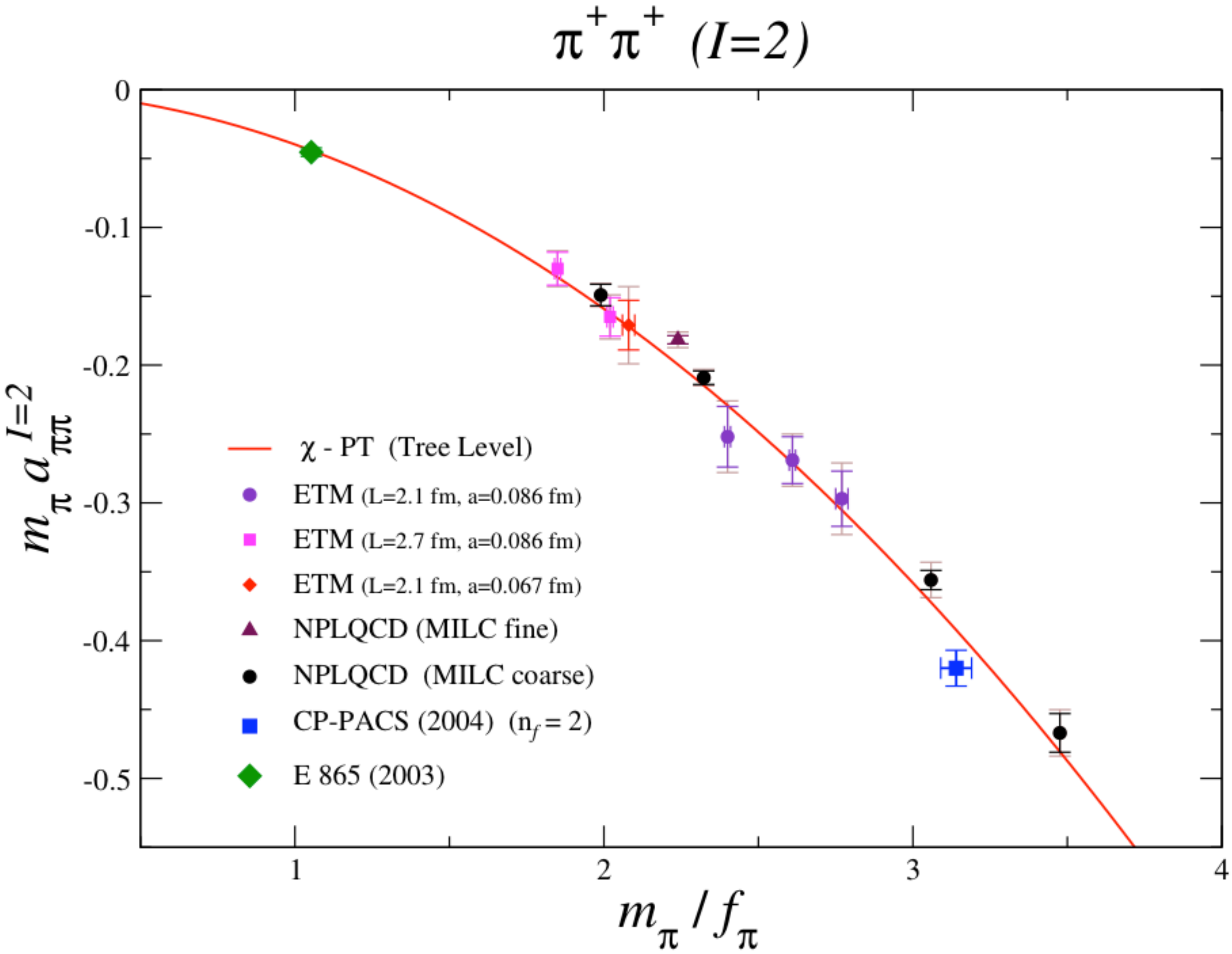


$$\mathcal{O}_{\pi^+}(t,\vec{x}) \;=\; \overline{u}(t,\vec{x})\gamma_5 d(t,\vec{x})$$

$$C_{\pi^+\pi^+}(p,t) \quad = \quad \langle 0| \sum_{|\mathbf{p}|=p} \sum_{\mathbf{x},\mathbf{y}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \mathcal{O}_{\pi^-}(t,\mathbf{x})\; \mathcal{O}_{\pi^-}(t,\mathbf{y})\; \mathcal{O}_{\pi^+}(0,\mathbf{0})\; \mathcal{O}_{\pi^+}(0,\mathbf{0})|0\rangle$$

$$\frac{C_{\pi^+\pi^+}(p,t)}{C_{\pi^+}(t)C_{\pi^+}(t)} \quad \rightarrow \sum_{n=0}^{\infty} \mathcal{A}_n \; e^{-\Delta E_n(2,L)\; t}$$

$$\Delta E_n(2,L) \;\equiv\; 2\sqrt{\vec{p}_n^{\,2} \;+\; m_\pi^2} \;-\; 2m_\pi$$



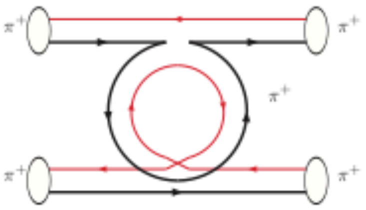
$\pi^+\pi^+$ (I=2)
$m_\pi a_{\pi\pi}^{I=2}$
m_π / f_π
0
-0.1
-0.2
-0.3
-0.4
-0.5
1
2
3
4
χ - PT (Tree Level)
ETM (L=2.1 fm, a=0.086 fm)
ETM (L=2.7 fm, a=0.086 fm)
ETM (L=2.1 fm, a=0.067 fm)
NPLQCD (MILC fine)
NPLQCD (MILC coarse)
CP-PACS (2004) (n_f = 2)
E 865 (2003)

Chiral *and* continuum extrapolation

$$m_\pi a_{\pi\pi}^{I=2}(b \neq 0) \;=\; -\frac{m_\pi^2}{8\pi f_\pi^2}\left[\,1 \;+\; \frac{m_\pi^2}{16\pi^2 f_\pi^2}\left(\,3\log\frac{m_\pi^2}{16\pi^2 f_\pi^2} \;-\; 1 \;-\; l_{\pi\pi}^{I=2}\,\right)\right]$$

Chiral *and* continuum extrapolation

$$m_\pi a_{\pi\pi}^{I=2}(b \neq 0) \;=\; -\frac{m_\pi^2}{8\pi f_\pi^2}\left[1 + \frac{m_\pi^2}{16\pi^2 f_\pi^2}\left(3\log\frac{m_\pi^2}{16\pi^2 f_\pi^2} - 1 - l_{\pi\pi}^{I=2}\right)\right]$$



MAχ–PT

$$+\frac{m_\pi^2}{8\pi f_\pi^2}\left[\frac{1}{(4\pi f_\pi)^2}\left[\frac{\tilde{\Delta}_{ju}^4}{6m_\pi^2}\right]\right]$$

$$\tilde{\Delta}_{ju}^2 \equiv \tilde{m}_{jj}^2 - m_{uu}^2 = 2B_0(m_j - m_u) + b^2\Delta_I + \ldots = 0.0769(22)$$

- Contains all $\mathcal{O}(m_\pi^2 b^2)$ and $\mathcal{O}(b^4)$ lattice artifacts.
- m_π and f_π are the lattice-physical parameters.
- Many sources of systematic error.

Error budget

- Higher-order effects in MAχPT:

$$\mathcal{O}(m_\pi^4 b^2) \sim \frac{2\pi m_\pi^4}{(4\pi f_\pi)^4} \frac{b^2 \Delta_{\mathrm{I}}}{(4\pi f_\pi)^2} < 1\%$$

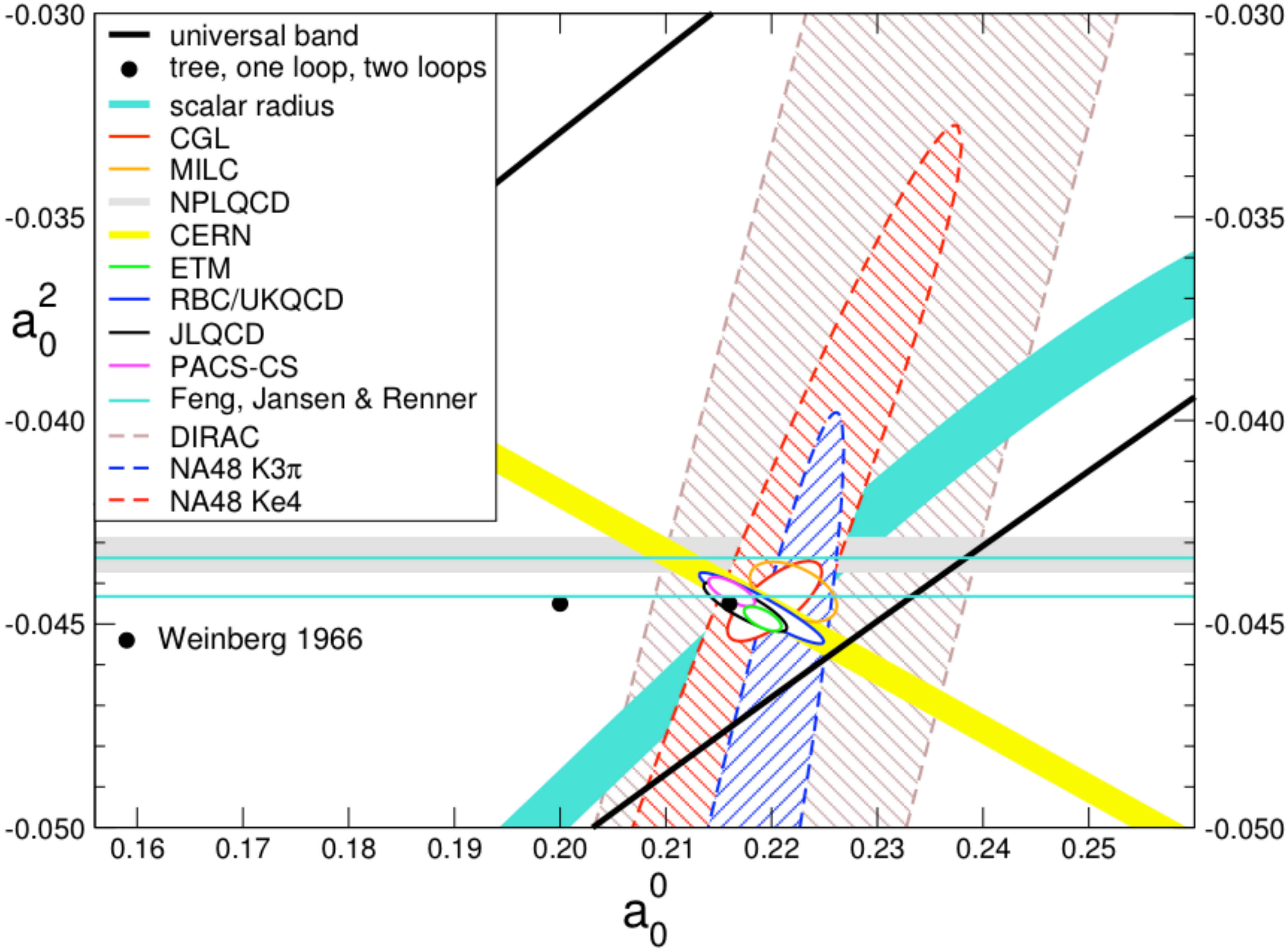
- Finite-volume effects: $\sim 4\%$ at lightest mass.
- Residual chiral symmetry breaking:

$$\frac{8\pi m_\pi^4}{(4\pi f_\pi)^4} \frac{m_{res}}{m_l} \sim 3\%$$

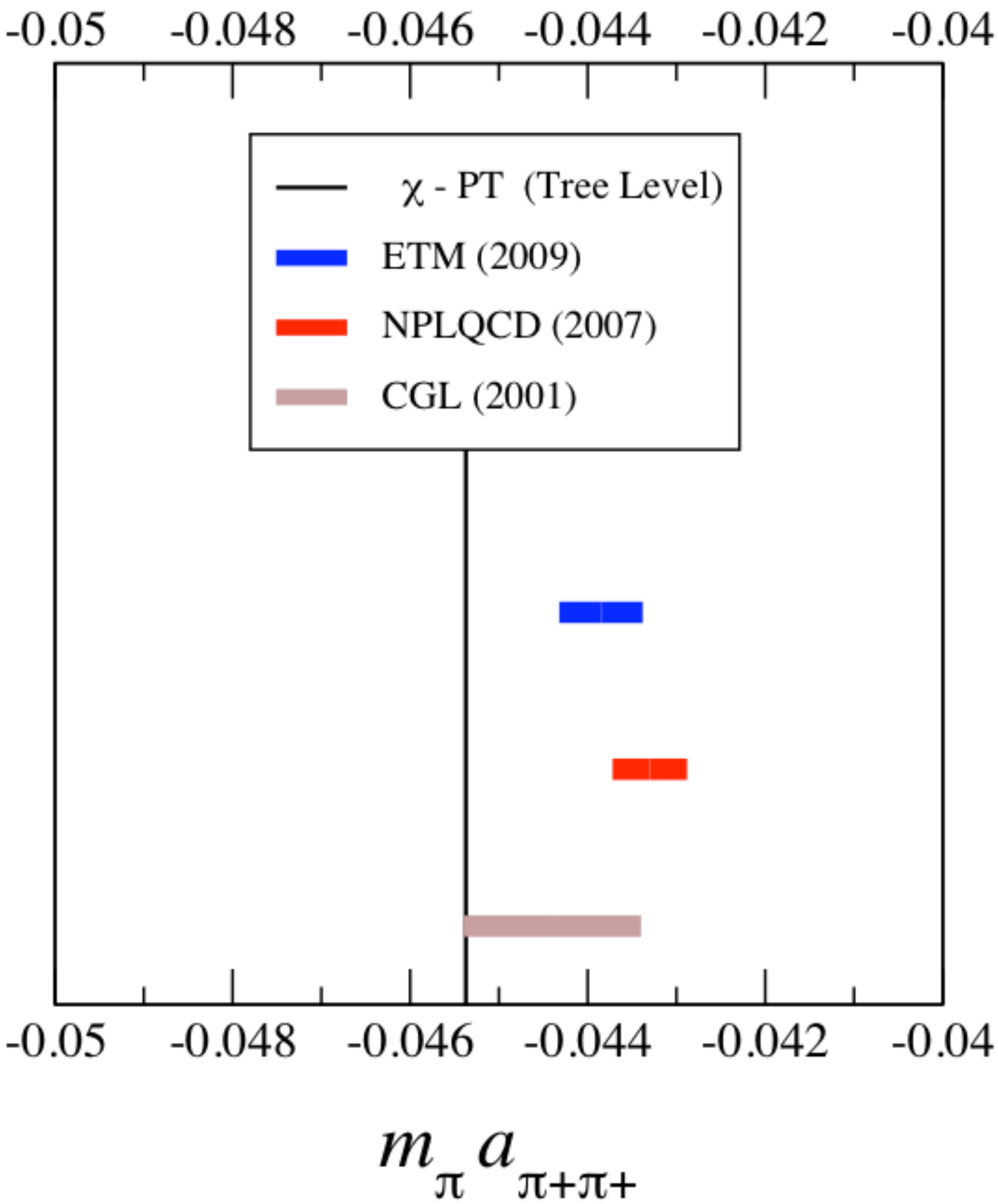
- Range corrections:

$$\frac{(m_\pi a_{\pi\pi}^{I=2})^2 p^2}{2 m_\pi^2} \sim 1\%$$

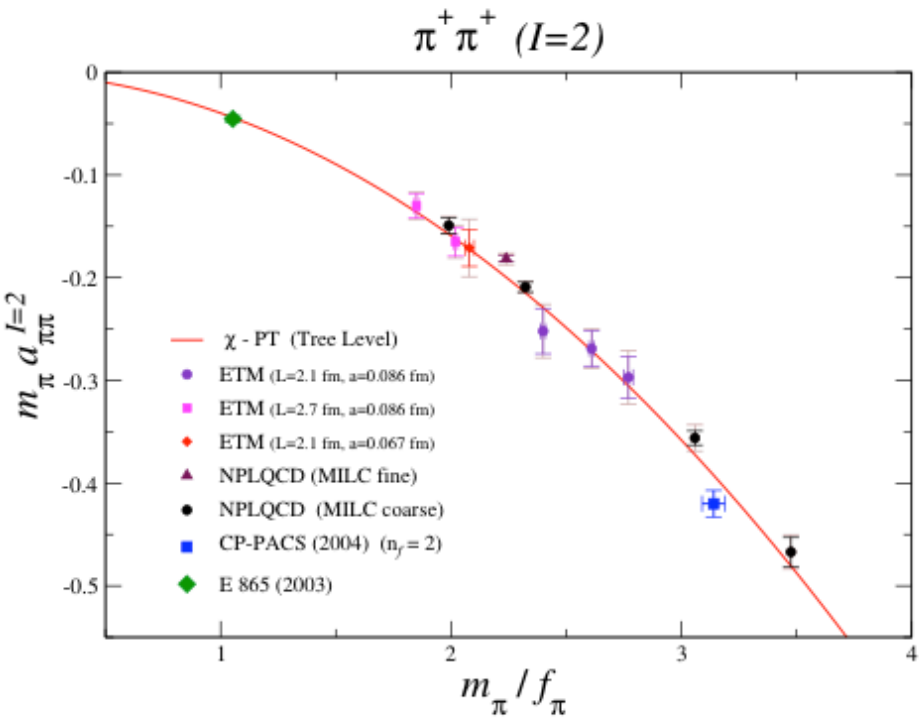
- Isospin violation: Only issue if compare to experiment!



(Courtesy of H. Leutwyler)



-0.05
-0.048
-0.046
-0.044
-0.042
-0.04
χ - PT (Tree Level)
ETM (2009)
NPLQCD (2007)
CGL (2001)
$m_\pi a_{\pi+\pi+}$



EFT works too well!!