Fermion Many-Body Systems II

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Outline

Overview of Cold Atoms

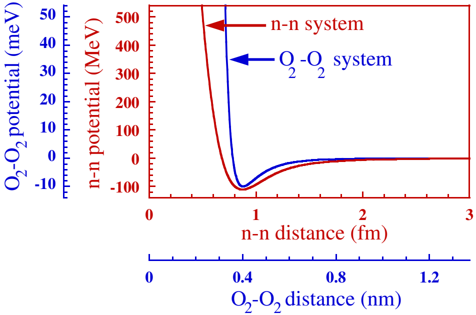
Overview of Neutron Stars

Answer: How are Cold Atoms Like Neutron Stars?

Renormalization Group Methods

Nuclear and Cold Atom Many-Body Problems

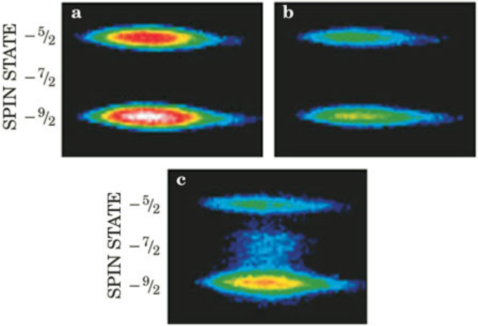
- Lennard-Jones and nucleon-nucleon potentials



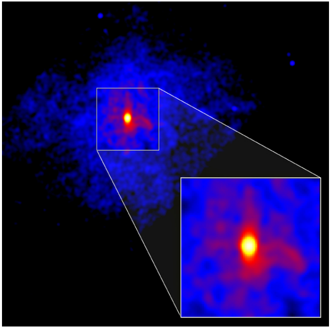
[figure borrowed from J. Dobaczewski]

- Are there universal features of such many-body systems?
- How can we deal with “hard cores” in many-body systems?

How Are Cold Atoms Like Neutron Stars?

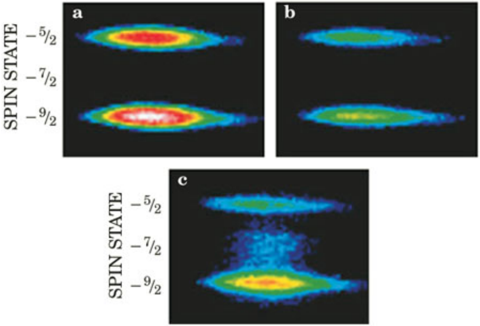


Regal et al., ultracold fermions



Chandra X-Ray Observatory image of pulsar in 3C58

How Are Cold Atoms UnLike Neutron Stars?



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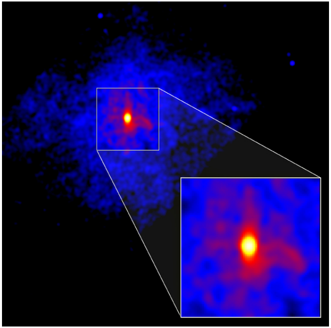

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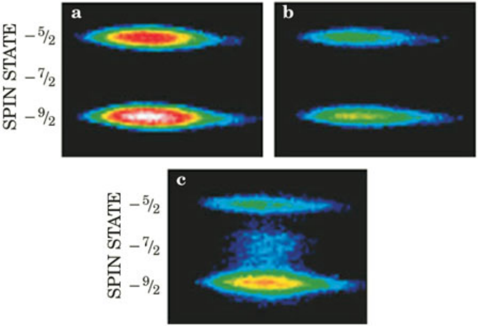

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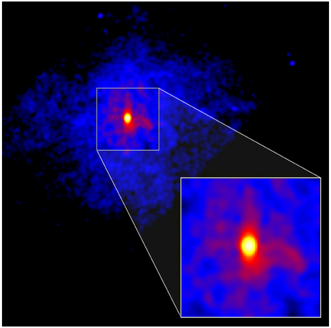

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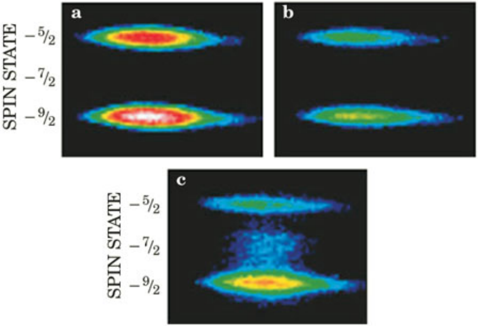

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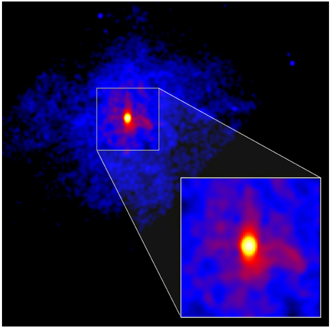
- Microscopic
- Atoms (bosons and fermions)



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- Nucleons (protons, neutrons)

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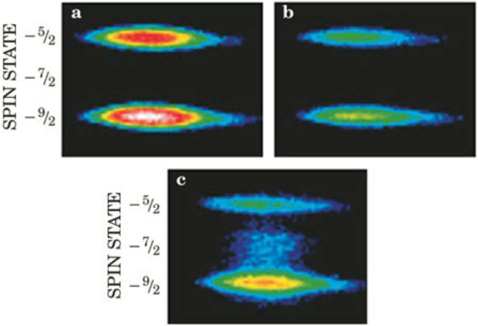
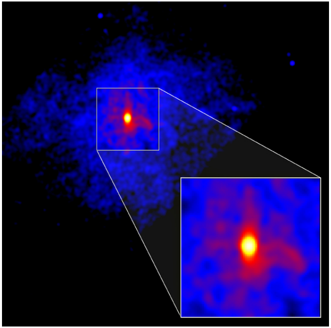


Regal et al., ultracold fermions

- Microscopic
- Atoms (bosons and fermions)
- Temperature: 0.000001 K



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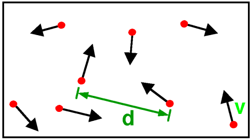
- Macroscopic
- Nucleons (protons, neutrons)
- Temperature: 1000000 K

Bosons and Fermions at Low Temperature

- Temperature of a gas of particles
  - *Distribution* of (kinetic) energies
  - Temperature as measure of average energy
  - When are quantum effects important? [Hint: $\lambda = h/p = h/mv$]
- Temperature and energy states
  - Bosons vs. fermions
- What atoms can you trap magnetically?
  - If the atom has an unpaired electron it has a magnetic moment, so external magnetic fields exert forces

What is Bose-Einstein condensation (BEC)?

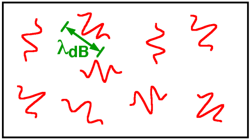


High Temperature T:

thermal velocity v

density d^{-3}

"Billiard balls"

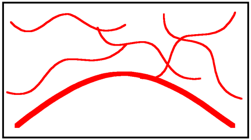


Low Temperature T:

De Broglie wavelength

$\lambda_{dB} = h/mv \propto T^{-1/2}$

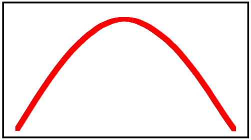
"Wave packets"



$T = T_{crit}$: Bose-Einstein Condensation

$\lambda_{dB} \approx d$

"Matter wave overlap"



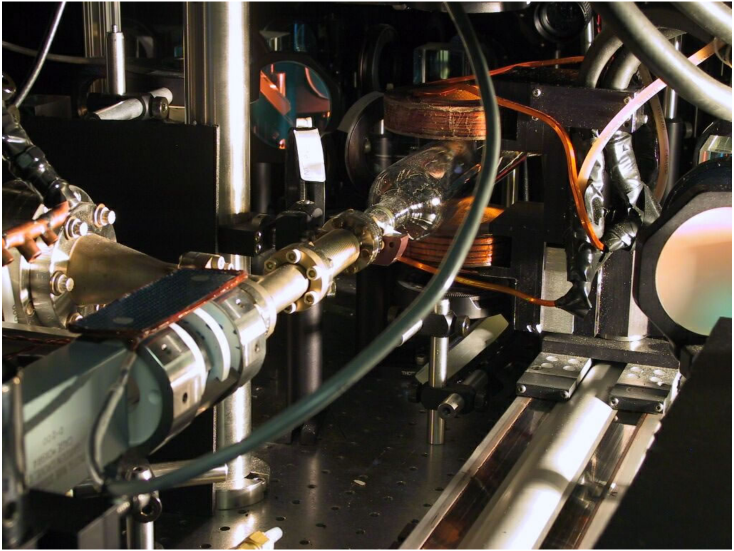
T=0: Pure Bose condensate

"Giant matter wave"

How Do You Trap and Cool Atoms?

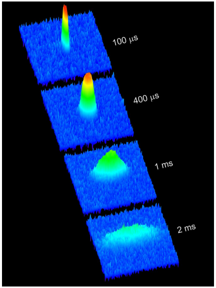
- Optical Molasses applet
  - Transfer momentum to atoms by absorbing photons from forward direction (and re-emit randomly); use Doppler shift
- Magnetic trapping applet
- Evaporative cooling applet
  - Removal of most energetic atoms by rf-induced evaporation
  - In higher magnetic field $\Longrightarrow$ in resonance with rf or microwaves, $\Longrightarrow$ spin flips and atom escapes
- Optical trapping and cooling
  - Use a single focused beam from a high-power ultra-stable CO_2 laser. Near focal point a potential well is formed.
  - Slowly reduce laser intensity to allow hottest atoms to escape.
  - Why do you need optical trapping for fermions?

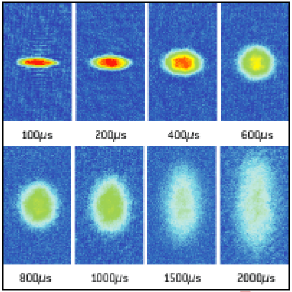
What Do These Experiments Look Like?



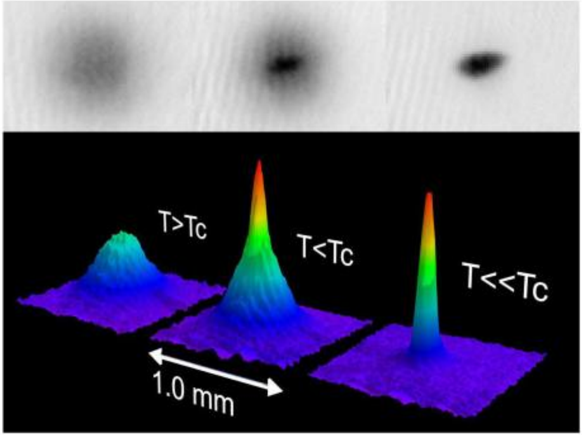
Imaging the Atoms

- Measure low temperatures by looking at extension of cloud
  $\Longrightarrow$ larger means more energetic
- Size of atomic cloud found by shadow of laser absorption
  $\Longrightarrow$ atoms' energy and temperature (since field known)
- Kinetic energy distribution by turning off trap and imaging ballistic expansion





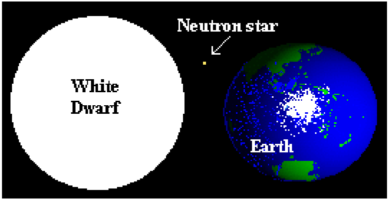
Forming a BEC



- The sharp peak is the signature of a BEC

The Fate of Stars

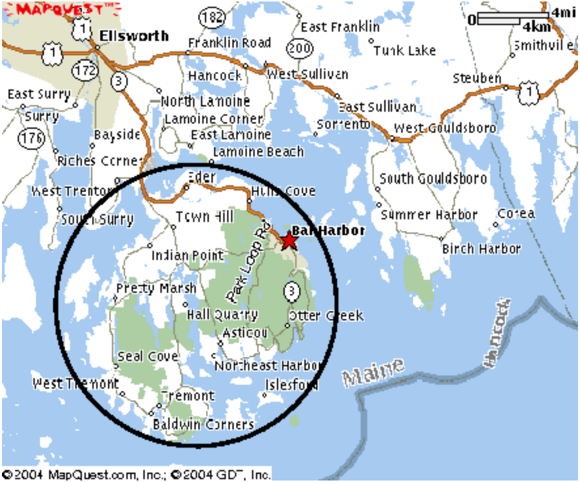
- Too light ($<$ 8 solar masses) $\Longrightarrow$ white dwarf
- Too heavy ($>$ 40 solar masses) $\Longrightarrow$ black hole
- Just right $\Longrightarrow$ star goes supernova, with neutron star left behind



The Crab Nebula in Taurus (VLT KUEYEN + FORS2)

ESO PR Photo 40f/99 (17 November 1999) © European Southern Observatory

Neutron Star vs. Bar Harbor



- Radius $\approx 10\,\text{km}$
- Mass $\approx 1.4\,M_{\text{sun}}$
- Density $\sim 10^{14}$ g/cc
- Spin rate $\leq 38,000\,\text{rpm}$
- Magnetic field $\sim 10^{12}$ Gauss

More Neutron Star Fun Facts . . .

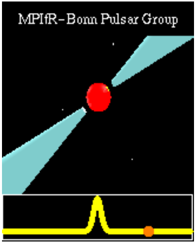
- Roughly a giant nucleus (mostly neutrons) bound by gravity (?)
- Extremely large gravitational field:

  "A marshmallow dropped from shoulder height onto the surface of a neutron star would hit the surface with the kinetic energy of an atom bomb."

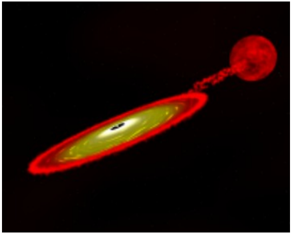
- In 10 billion year lifetime of galaxy, probably 10^8 to 10^9 neutron stars formed
- In 10^{12} gauss field, the binding energy of hydrogen is 160 eV
- Blackbody radiation peaked in X-ray (but small surface area)

Pulsating Radio Sources or Pulsars

- In 1967, Joceyln Bell observed 1.337 sec. period radio pulses
- Rotating neutron stars by process of elimination
  - $300,000\,\text{km/s} \times 0.033\,\text{s} < 10,000\,\text{km}$
  - white dwarf pulsational, rotational, orbital periods too large
- Synchrotron radiation from charged particles along field lines
  - magnetic poles don't align with rotational $\Longrightarrow$ "lighthouse effect"



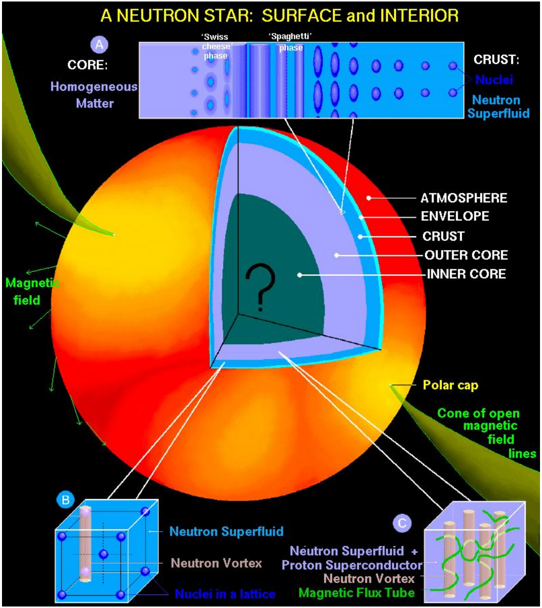
SAXJ1808.4-3658: X-Ray Millisecond Pulsar



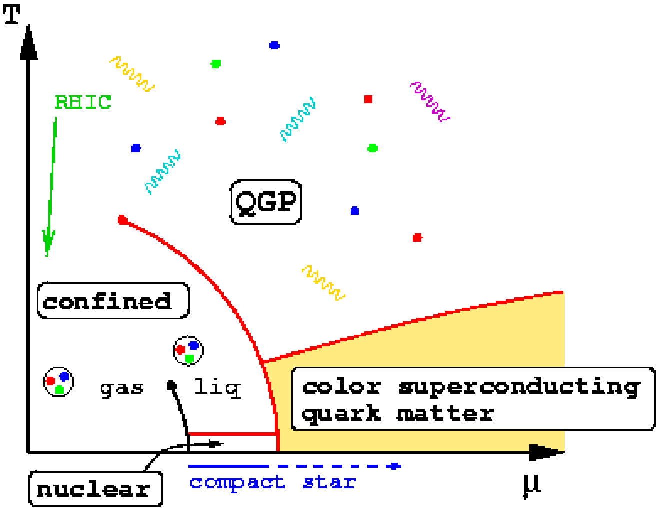
- Billion year history of binary system:
  - companion spirals into massive star
  - massive star core goes supernova $\Longrightarrow$ neutron star
  - companion loses mass $\Longrightarrow$ accretion disk
  - accretion $\Longrightarrow$ faster spin, up to few millisecond period and X-rays, which vaporize companion
  - at end, compact rapidly rotating neutron star with pair of radio beams
- NASA's Rossi X-ray Timing Explorer (RXTE)

Neutron Stars

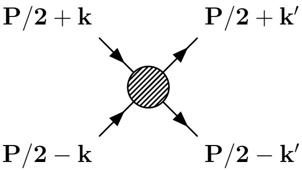
- Dilute neutrons at surface $\Longrightarrow$ neutron superfluid
- High density core: color superconductor from quark Cooper pairs (???)

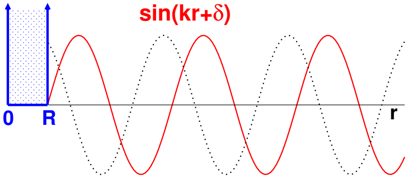


The QCD Phase Diagram



Quick Review of Scattering





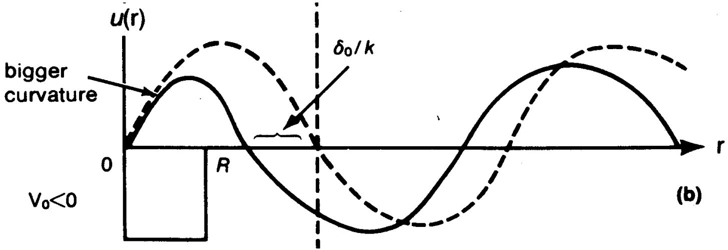
- Relative motion in frame with $P=0$: $\psi(r) \overset{r\to\infty}{\longrightarrow} e^{i\mathbf{k}\cdot\mathbf{r}} + f(k,\theta)\frac{e^{ikr}}{r}$
  where $k^2 = k'^2 = ME_k$ and $\cos\theta = \hat{k}\cdot\hat{k}'$
- Differential cross section is $d\sigma/d\Omega = |f(k,\theta)|^2$
- Central $V \Longrightarrow$ partial waves: $f(k,\theta) = \sum_l (2l+1) f_l(k) P_l(\cos\theta)$

$$\text{where} \quad f_l(k) = \frac{e^{i\delta_l(k)}\sin\delta_l(k)}{k} = \frac{1}{k\cot\delta_l(k) - ik}$$

and the S-wave phase shift is defined by

$$u_0(r) \overset{r\to\infty}{\longrightarrow} \sin[kr+\delta_0(k)] \quad \Longrightarrow \quad \delta_0(k) = -kR \text{ for hard sphere}$$

Effective Range Expansion



Total cross section: $$\sigma_{\text{total}} = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l(k)$$

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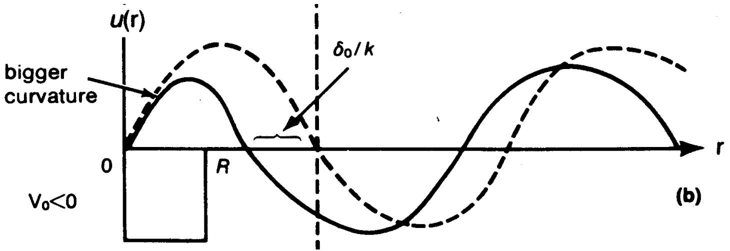


Total cross section: $\sigma_{\text{total}} = \dfrac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l(k)$

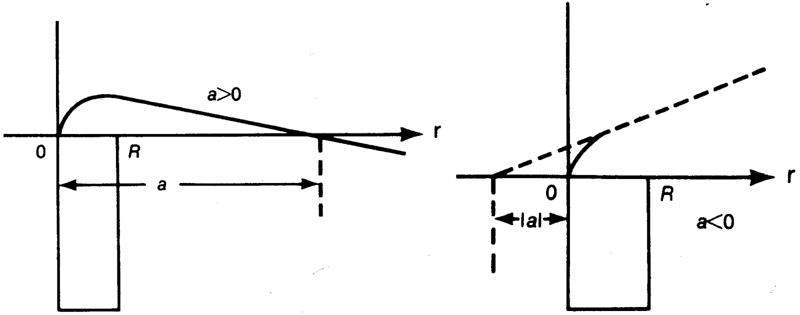
- What happens at low energy ($\lambda = 2\pi/k \gg 1/R$)?

$$k\cot\delta_0(k) \xrightarrow{k\to 0} -\frac{1}{a_0} + \frac{1}{2}r_0k^2 + \ldots$$

  - a_0 is the “scattering length” and r_0 is the “effective range”

Near-Zero-Energy Bound States

- Bound-state or near-bound state at zero energy
  $\Longrightarrow$ large scattering lengths ($a_0 \to \pm\infty$)



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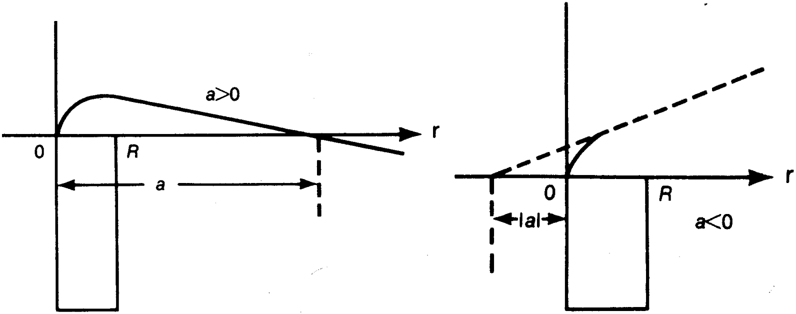
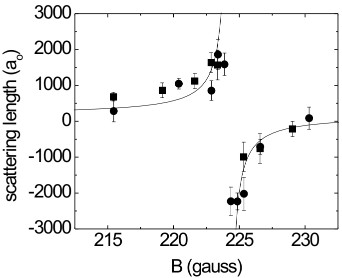


- For $kR \to 0$, the total cross section is

$$\sigma_{\text{total}} = \sigma_{l=0} = \frac{4\pi a_0^2}{1+(ka_0)^2} = \begin{cases} 4\pi a_0^2 & \text{for } ka_0 \ll 1 \\ \dfrac{4\pi}{k^2} & \text{for } ka_0 \gg 1 \text{ (unitarity limit)} \end{cases}$$

Trapped Fermion Atoms

- Low densities and temperatures $\Longrightarrow$ only a_0 enters
- Use “Feshbach resonances” to tune scattering length $a_0 \to \pm\infty$



- Superfluidity? BCS-BEC crossover? Universal behavior?

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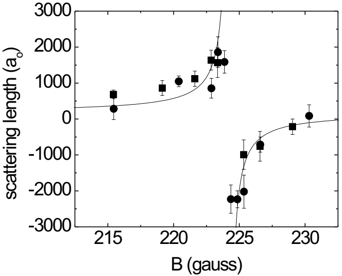

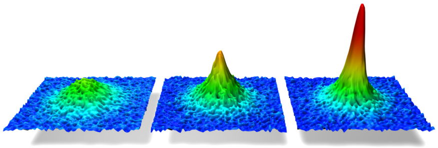

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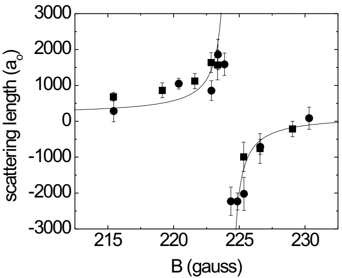

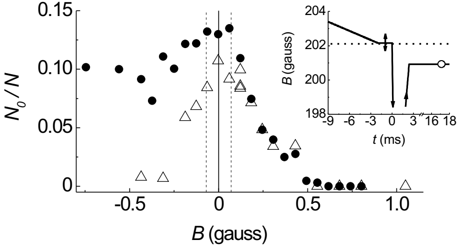

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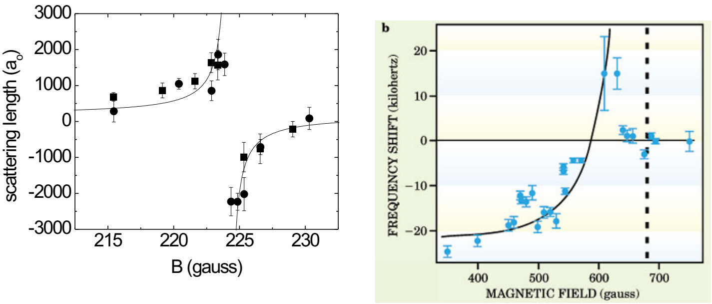


- Superfluidity? BCS-BEC crossover? Universal behavior?

Metaphor for a Fermion Condensate (JILA)

"These dancers represent our fermionic atoms. They appear to be moving independently, however, there is a subtle pairing: You can see this from the dancers eye contact and body language. These pairs act like bosons and can form a condensate. "

Metaphor for Bound Molecules (JILA)

“To see the condensation of pairs of fermionic atoms we suddenly bring together the two atoms (or dancers here) in each of these subtle pairs. When we look at the motion of these bound pairs the condensation apparent. ”

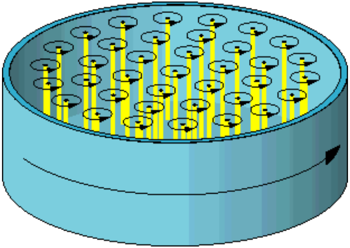
What is Superfluidity?

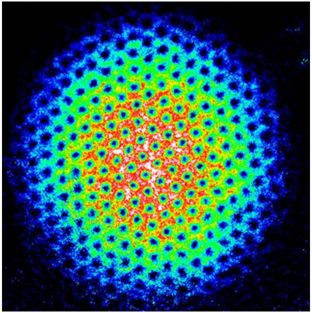
- Superfluidity is a state of matter characterized by the complete absence of viscosity
  - Superfluids in a closed loop can flow endlessly without friction
  - Discovered by Kapitsa, Allen, and Misener in 1937

What is Superfluidity?

- Superfluidity is a state of matter characterized by the complete absence of viscosity
  - Superfluids in a closed loop can flow endlessly without friction
  - Discovered by Kapitsa, Allen, and Misener in 1937
- Where is superfluidity found?
  - ^{4}He bosons $\Longrightarrow$ BEC with interactions
  - ^{3}He fermions $\Longrightarrow$ Cooper pairs Bose condense (as in superconductivity)
  - Cold atomic gases
  - Neutron stars

Superfluid Vorticity





- Quantized vorticity: velocity around each line determined by h/m (m is mass of one atom)
- Quantum phenomenon on a macroscopic scale
  - # of vortex lines $\propto h/m$: 1000 in 1 cm radius container at 1 rpm
- [For the experts: vortex lines in superfluid analogous to magnetic flux lines in type II superconductors.]

How Cold Must a Superfluid Be?

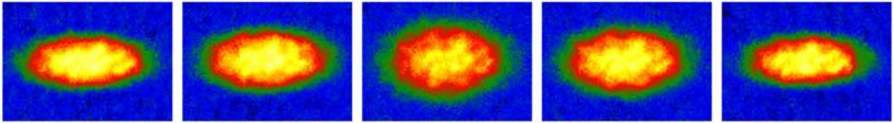
- Need coherence: large-scale quantum correlations
  - Roughly the de Broglie wavelength must be comparable to the average distance d between particles
  - Use $\lambda = h/mv$ and $mv^2/2 \approx 3kT/2$ to estimate

$$T_c \leq \frac{h^2}{3mkd^2}$$

- Transition temperatures:
  - ^{4}He transition at 2.17 K (-270.98 °C)
  - ^{3}He transition at 2.6 mK
  - neutron star can be superfluid even at much higher temperatures because of very high density (small d)

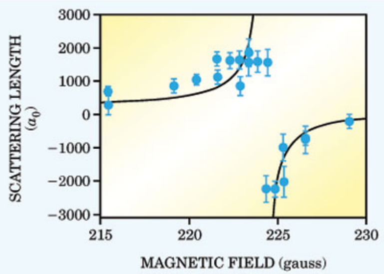
Checking for Fermion Superfluidity

- John Thomas group at Duke
- Use a focused CO_2 laser beam to trap ^{6}Li atoms (fermions) into cigar-shaped cloud and apply magnetic field
- Switch optical trap off and on to give gas a light tap
- Observe length of oscillations at different temperatures
  - normal gas: atom collisions damp oscillations
  - superfluid: unified long-lasting oscillations
- Result: Longer oscillations with lower temperatures!



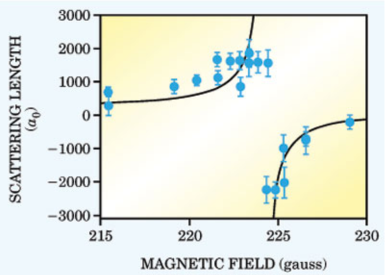
Trapped Fermion Atoms vs. QCD

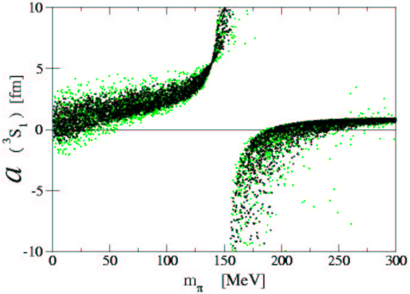
- Atoms: Change magnetic field $\Longrightarrow$ resonant scattering



Trapped Fermion Atoms vs. QCD

- Atoms: Change magnetic field $\Longrightarrow$ resonant scattering
- QCD: Adjust quark mass theoretically $\Longrightarrow$ m_π changes more

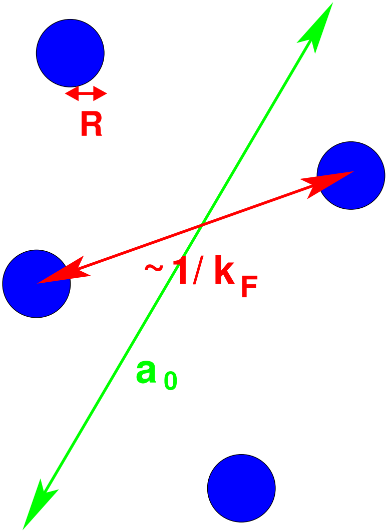




- Universal properties?

Large Scattering Length Problem

- Attractive with $a_0 \to \infty$
- If $R \ll 1/k_{\rm F} \ll a_0$, then expect scale invariance
- Energy and gap are pure numbers times $E_{\rm FG} = \frac{3}{5}\frac{k_{\rm F}^2}{2M}$



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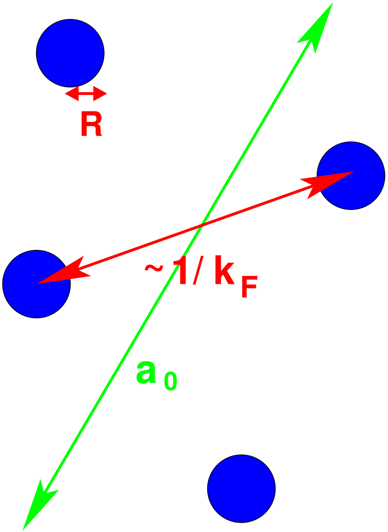
- Energy and gap are pure numbers times $E_{\rm FG} = \frac{3}{5}\frac{k_{\rm F}^2}{2M}$
- EFT power counting says: sum everything with leading vertices
  - Easy in free space $\Longrightarrow$ geometric sum of bubbles
  - Many more diagrams in the medium



Calculations Summing All Leading Diagrams

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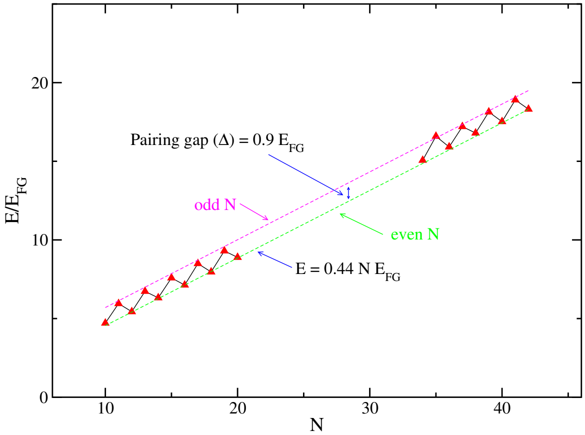
- Could there be an additional EFT expansion?

GFMC Results [J. Carlson et al.]

- Extrapolate to large numbers of fermions



- Energy per particle: $E/N = 0.44(1)E_{\mathrm{FG}}$

Diffusion MC Results [G.E. Astrakharchik et al.]

- Square-well potential tuned to $a_0 \to \infty$
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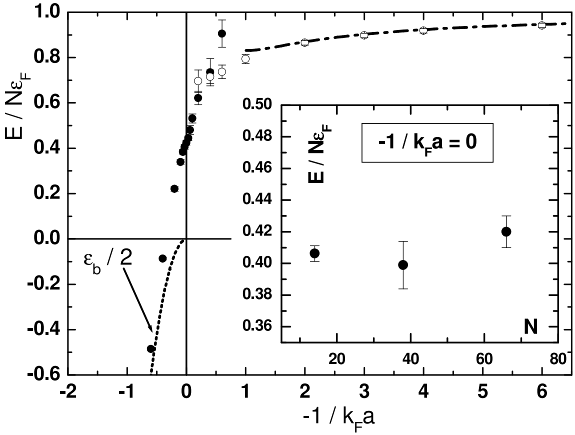

- Energy per particle: $E/N = 0.42(1)E_{\text{FG}}$

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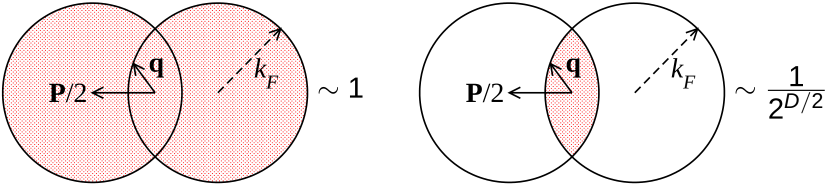

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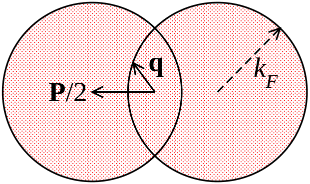

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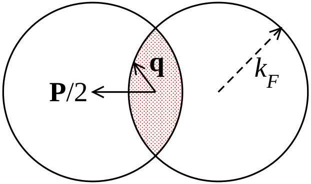

$\sim \dfrac{1}{2^{D/2}}$

- Leading $1/\mathcal{D}$ analytic expansion yields:

$$\frac{E}{A} = E_{\rm FG}\left(1 + \frac{10k_{\rm F}a_0/9}{\pi - 2k_{\rm F}a_0} + \mathcal{O}(1/\mathcal{D})\right) \xrightarrow{a_0\to\infty} \frac{4}{9}E_{\rm FG} \quad [\text{cf. } 0.44(1)\,E_{\rm FG}]$$

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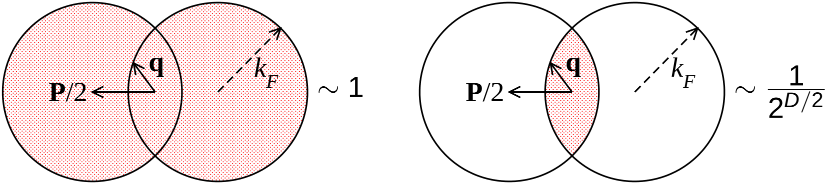

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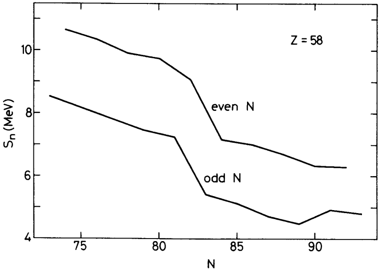
- Subleading??? Pairing gap???

Recap: How are Cold Atoms Like Neutron Stars?

- Low temperatures (on an appropriate scale)
  $\Longrightarrow$ means low energies, low resolution
  $\Longrightarrow$ quantum mechanical (wave) effects are large
- Only the scattering length a_0 is important if cold and dilute
  $\Longrightarrow$ universal physics when a_0 is large!
- Cold fermions with large scattering lengths are similar

Superfluid atoms in traps $\Longleftrightarrow$ Superfluid neutrons in neutron stars

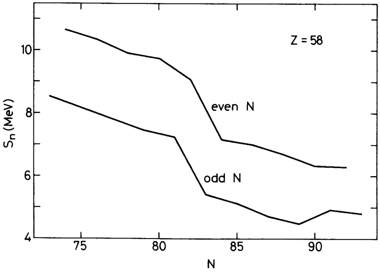
Experimental Evidence for Pairing in Nuclei



$$
\begin{aligned}
B(N,Z) = {} & \\
& (15.6\,\text{MeV}) \left[1 - 1.5 \left(\frac{N-Z}{A} \right)^2 \right] A \\
& - (17.2\,\text{MeV}) A^{2/3} - (0.70\,\text{MeV}) \frac{Z^2}{A^{1/3}} \\
& + (6\,\text{MeV})[(-1)^N + (-1)^Z]/A^{1/2}
\end{aligned}
$$

- Odd-even staggering of binding energies

Experimental Evidence for Pairing in Nuclei



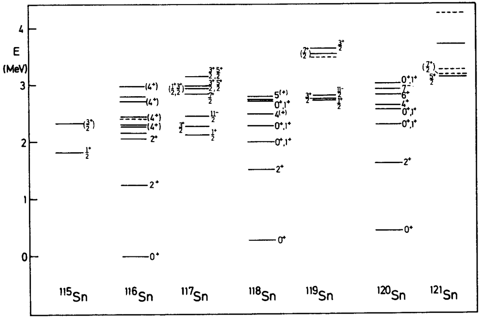


Figure 6.1. Excitation spectra of the $_{50}$Sn isotopes.

- Odd-even staggering of binding energies
- Energy gap in spectra of deformed nuclei
- Low-lying 2^+ states in even nuclei
- Deformations and moments of inertia (theory requires pairing)

1-D Molecules and Cooper Pairs

- Consider Fermions in one dimension with $\lambda\delta(x)$ attraction
- In the COM frame with relative coordinate x,

$$-\frac{1}{M}\frac{d^2\psi(x)}{dx^2} - |\lambda|\delta(x)\psi(x) = B_2\psi(x)$$

  - always a bound state with $B_2 = -M\lambda^2/4$ with $\psi(x) = Ae^{-\sqrt{M|B_2|}x}$
    (e.g., solve by integrating S-eqn across $x = 0$)
  - extent of bound state increases as $\lambda \to 0$ (cf. 3-D behavior)

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- What happens at higher density?

How Does the Fermi Sea Affect Bound States?

- Pauli blocking $\Longrightarrow$ Look in momentum space:

$$(H_0 + V)|\psi\rangle = E_2|\psi\rangle \quad \text{and} \quad \psi(x) = \sum_k C_k e^{ikx} \longrightarrow \langle k|\psi\rangle = C_k$$

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$$\Longrightarrow \quad \langle k|H_0|\psi\rangle + \sum_{k'} \langle k|V|k'\rangle\langle k'|\psi\rangle = E_2\langle k|\psi\rangle$$

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- Let $A = \sum_{k'} C_{k'}$, solve for C_k and sum over k:

$$\sum_k C_k = A = \frac{|\lambda|}{L} A \sum_k \frac{1}{2\epsilon_k^0 - E_2} \quad \text{or} \quad 1 = \frac{|\lambda|}{L} \sum_k \frac{1}{2\epsilon_k^0 - E_2}$$

- In free space, we can calculate the integral as $L \to \infty$
  - For $E_2 < 0$, we recover our previous result:

$$1 = \frac{|\lambda|}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{k^2/2M + |E_2|} = \frac{|\lambda|}{2} \sqrt{\frac{M}{|E_2|}} \quad \Longrightarrow \quad |E_2| = \frac{M\lambda^2}{4}$$

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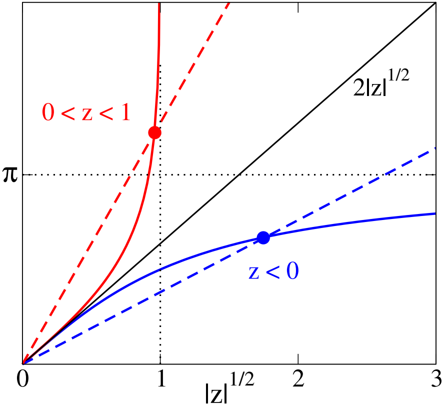
- In medium, include Pauli blocking through the integral limit:

$$1 = \frac{|\lambda|}{2\pi} 2 \int_{k_F}^{\infty} \frac{dk}{k^2/2M - E_2} \quad \xrightarrow{\epsilon_F = k_F^2/2M} \quad 1 = \frac{|\lambda|}{\pi} \sqrt{\frac{M}{2}} \int_{\epsilon_F}^{\infty} \frac{d\epsilon}{\sqrt{\epsilon}} \frac{1}{2\epsilon - E_2}$$

  - This equation has two relevant types of solutions, when

$$z \equiv \frac{E_2}{2\epsilon_F} < 0 \quad \text{and} \quad 0 < E_2 < 2\epsilon_F \quad (\text{or } 0 < z < 1)$$

Graphical Solution (Plot LHS vs. RHS)



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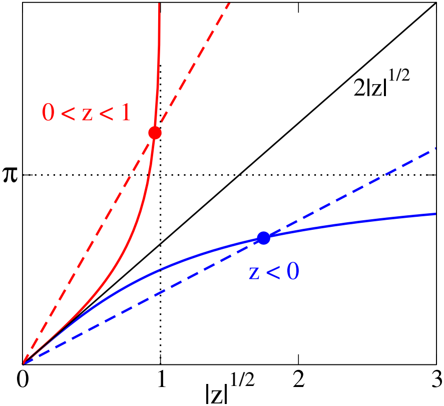

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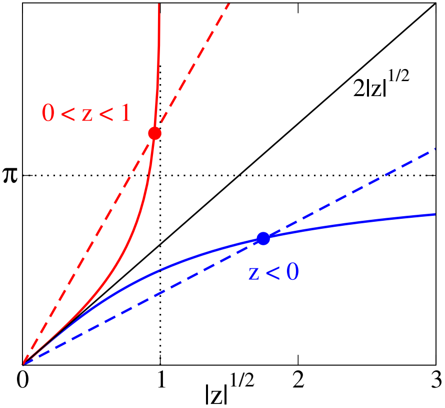

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- As $z \to 1$ (weak coupling), expand to find $\Delta = 8\epsilon_F e^{-2\pi k_F/M|\lambda|}$
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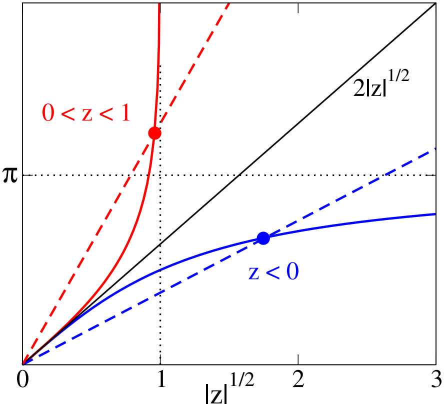

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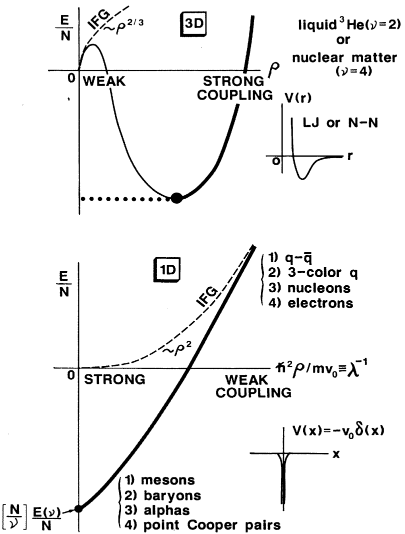
  - So the “gap” has an essential singularity in the coupling constant
  - In 3-D weak-coupling BCS, $\Delta \propto \epsilon_F e^{-\pi/2k_F|a|}$

Compare 3-D Lennard-Jones and 1-D δ–Function



- In 3-D, low-density corresponds to weak coupling (kinetic energy dominates)
- In 1-D, high-density corresponds to weak coupling (kinetic energy dominates)

Famous Laws of Theoretical Physics I

- Steven Weinberg enumerated these laws while discussing the renormalization group.

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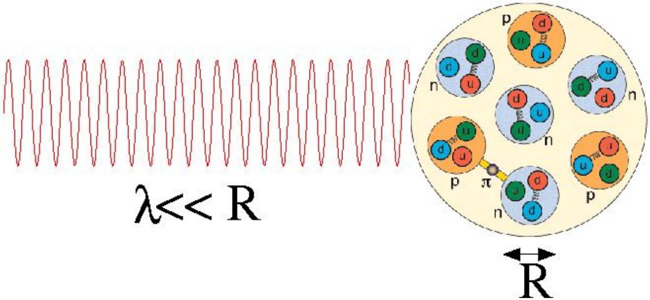
- Weinberg's Three Laws of Progress in Theoretical Physics
  1. Conservation of Information: *"You will get nowhere by churning equations."*
  2. *"Do not trust arguments based on the lowest order of perturbation theory"*
  3. *"You may use any degrees of freedom you like to describe a physical system, but if you use the wrong ones, you'll be sorry!"*

Principles of Effective Low-Energy Theories



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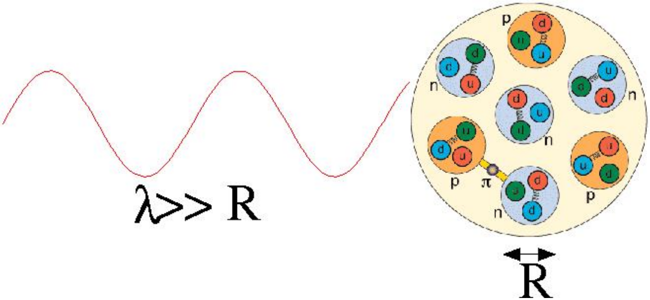

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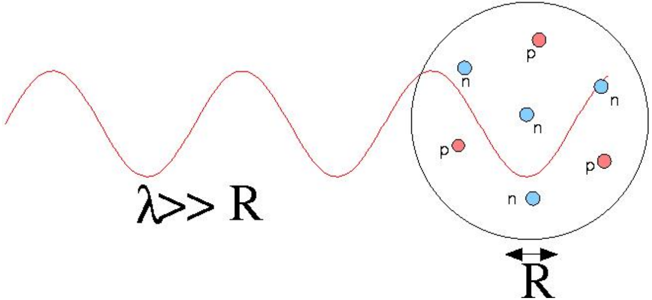
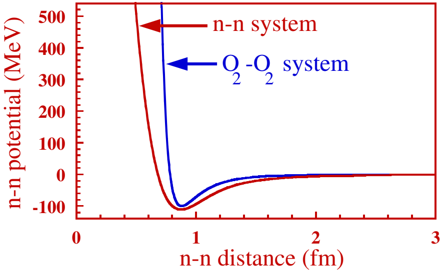
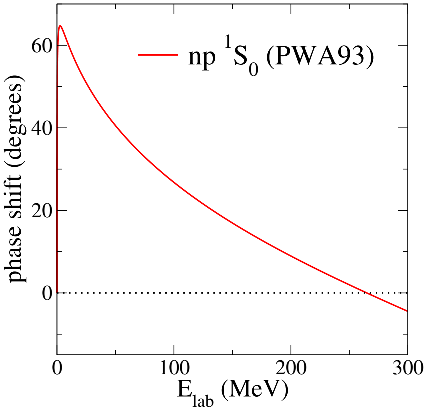


- If system is probed at low energies, fine details not resolved
  - use low-energy variables for low-energy processes
  - short-distance structure can be replaced by something simpler without distorting low-energy observables

NN Potential and Scattering Phase Shifts

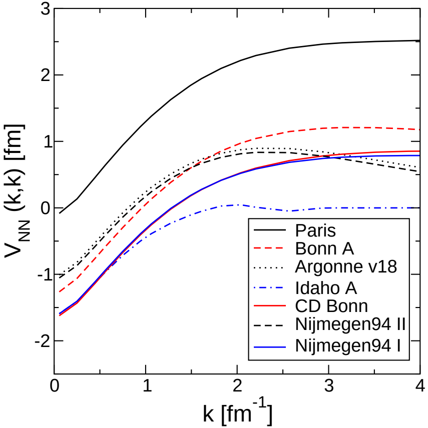


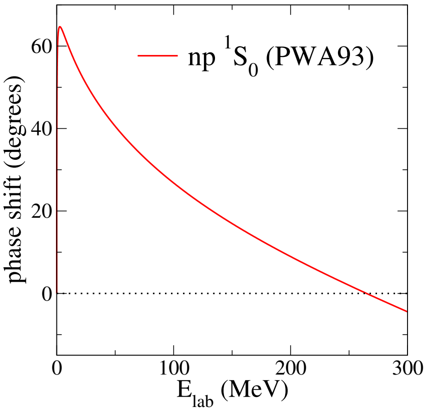


- low energies $\Longrightarrow$ large a_0
- zero crossing $\Longrightarrow$ hard core

NN Potential and Scattering Phase Shifts

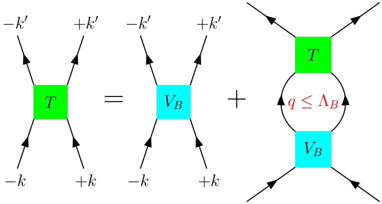
Now look at NN potentials in momentum space: s-wave $\langle k'|V|k\rangle$



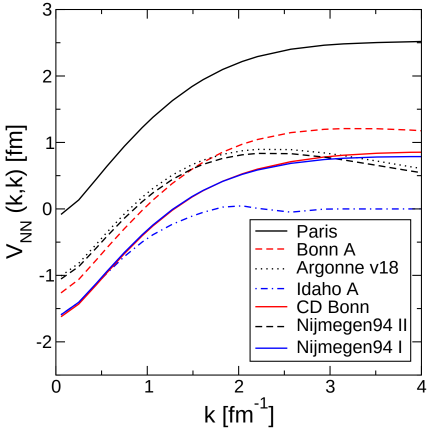


Many different potentials with same phase shifts (χ^2/dof $\approx$ 1)

NN Scattering in the COM Frame

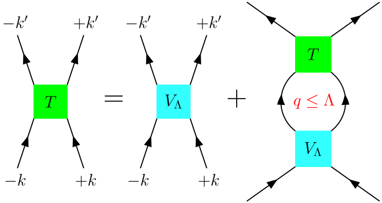


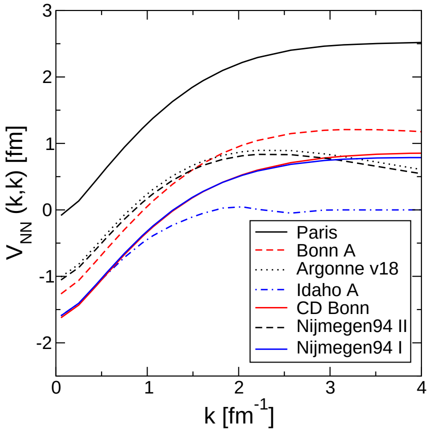
- $\chi^2/\text{dof} \approx 1$ "bare" potential V_B
- Probes intermediate states to
  $q \leq \Lambda_B = 25\,\text{fm}^{-1} \doteq 5\,\text{GeV}$
- Model dependent: $q \geq 3\,\text{fm}^{-1}$



Low-Momentum NN Potential

Bogner, Kuo, Schwenk





- Require $\dfrac{dT}{d\Lambda} = 0$
  $\Longrightarrow$ renormalization group equation for V_Λ
- Run from $\Lambda_B = 25\,\text{fm}^{-1}$ to $\Lambda = 2\,\text{fm}^{-1} \sim E_{\text{lab}} \doteq 350\,\text{MeV}$

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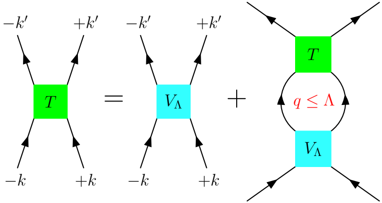

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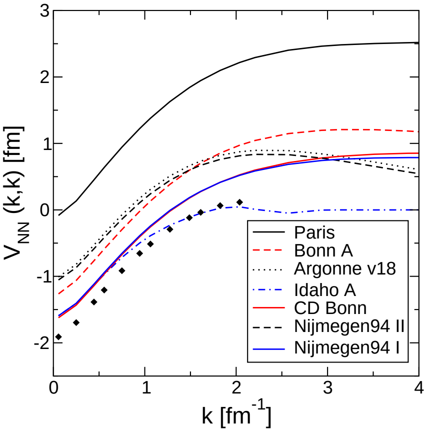

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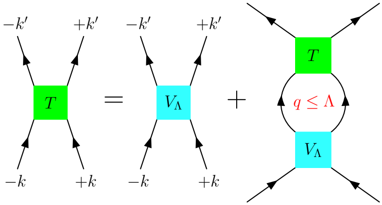

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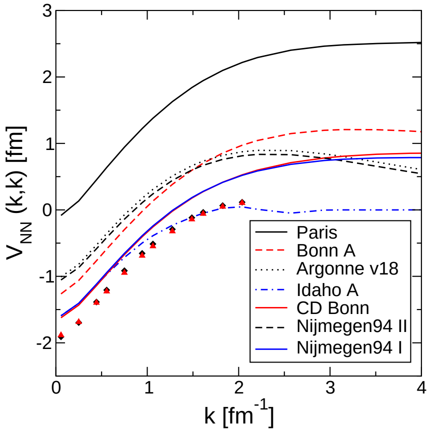

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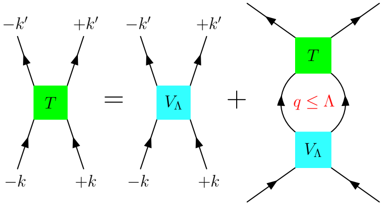

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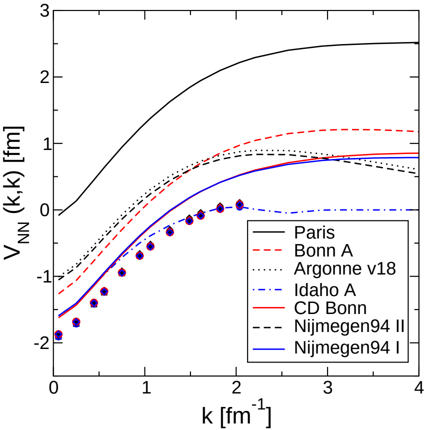
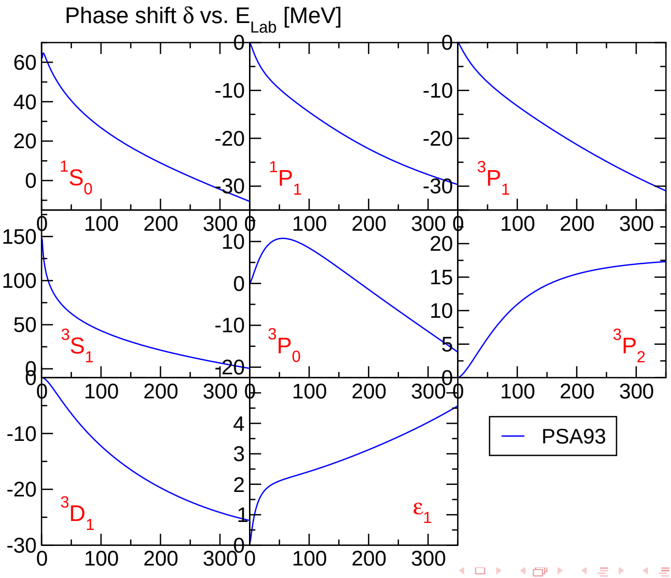
- Run from $\Lambda_B = 25\,\text{fm}^{-1}$ to $\Lambda = 2\,\text{fm}^{-1} \sim E_{\text{lab}} \doteq 350\,\text{MeV}$
- Same long distance physics
  $\Longrightarrow$ collapse!



Low-Energy Observables Unchanged



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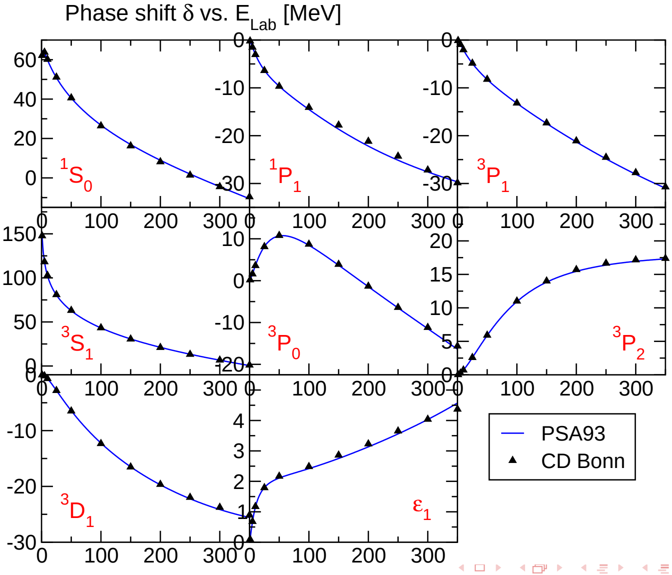
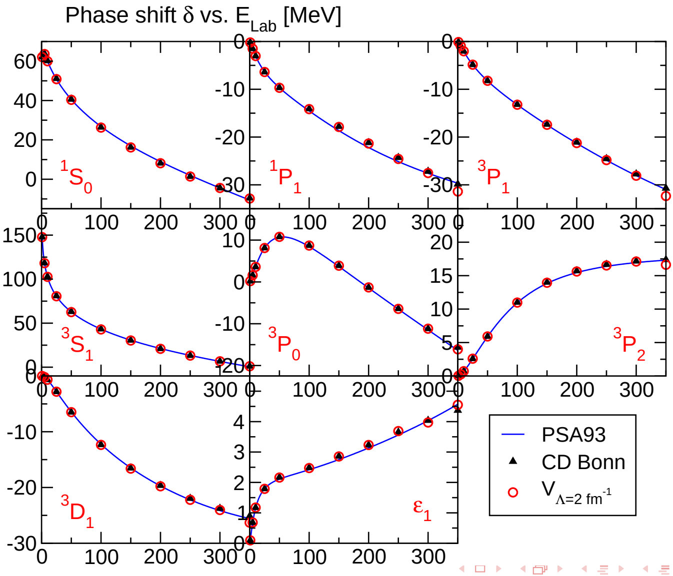
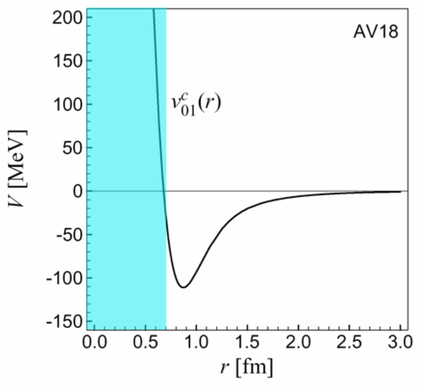


Low-Energy Observables Unchanged



Low-Momentum Pot'l in the Many-Body Problem

- Removing hard core $\Longrightarrow$ simpler many-body starting point for neutron matter [Schwenk, Friman, Brown]



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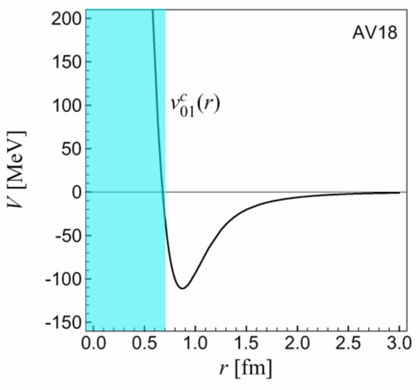
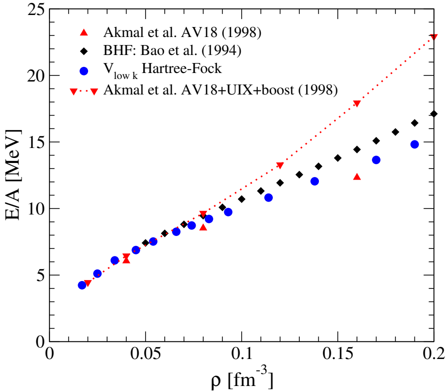
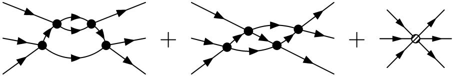


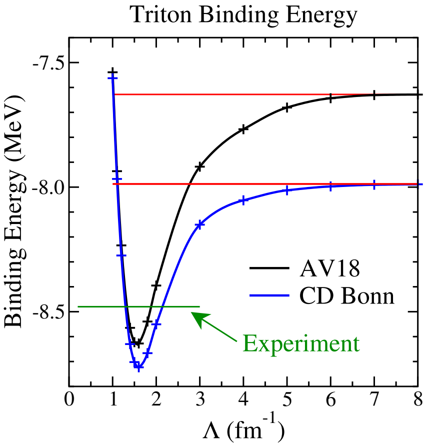


- Simple Hartree-Fock (circles) matches best calculations!

Three-Body Forces are Inevitable!

- What if we have three nucleons interacting?
- Successive two-body scatterings with short-lived high-energy intermediate states unresolved $\Longrightarrow$ must be absorbed into three-body force





[Bogner, Nogga, Schwenk]

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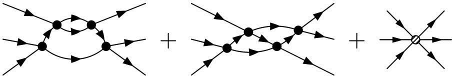
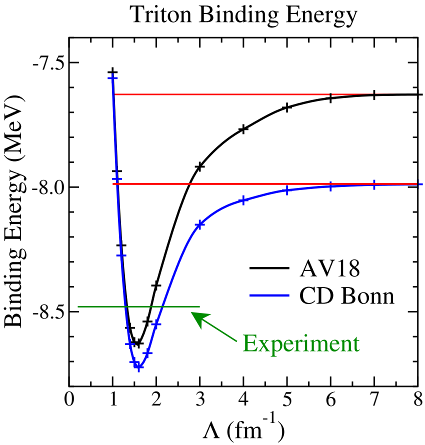
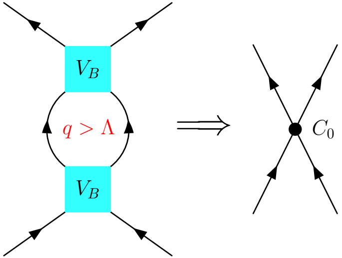


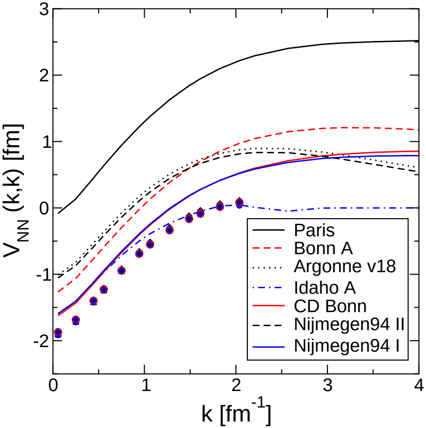
- How do we organize three-body forces?



[Bogner, Nogga, Schwenk]

Renormalization and Short-Distance Physics





- $q > \Lambda$ intermediate states $\Longrightarrow$ replace with contact term: $C_0\delta^3(\mathbf{x}-\mathbf{x}')$
- $\mathcal{L}_{\text{eft}} = \cdots + \frac{1}{2}(\psi^\dagger\psi)^2 + \cdots$

(Nuclear) Many-Body Physics: "Old" vs. "New"

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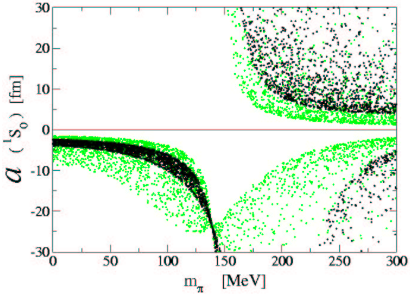
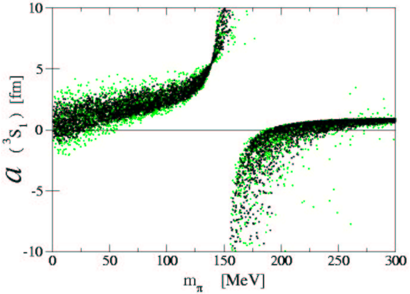
Choose diagrams by "art"	Power counting determines diagrams and truncation error

Quantum Chromodynamics and Nuclear Physics

Quark Mass (m_q) Dependence of Observables

S. Beane and M. Savage, Nucl. Phys. **A717** (2003) 91.



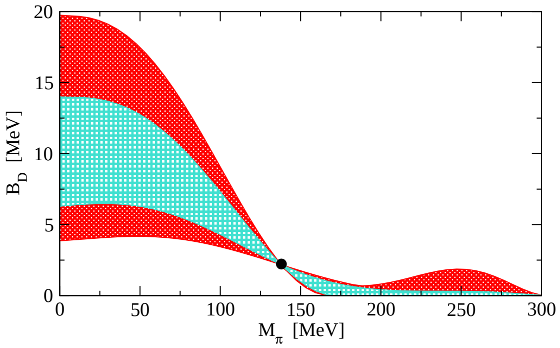


- Crude (naive?) estimate of nonrelativistic bound state:

$$E_B \approx \frac{p^2_{\rm typical}}{M} \approx \frac{f_\pi^2}{M} \approx 10\,\text{MeV} \gg E_{\rm deuteron}$$

Is the Fine-Tuning Accidental?

E. Epelbaum, U. Meißner, and W. Glöckle, Nucl. Phys. **A714** (2003) 535.



- Footnote from S. Weinberg paper [Nucl. Phys. **B363** (1991) 3]:
  E. Witten says that large N_c arguments . . . suggest that the very small binding energy of the deuteron is "a fortuitous cancellation of potential and kinetic energies for the physical value $N_c = 3$."