Fermion Many-Body Systems I

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Ohio State University

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Outline

Examples of Fermion Many-Body Systems

- Collections of “fundamental” fermions (electrons, quarks, . . .)
  - or composites of *odd* number of fermions (e.g., proton)
- Isolated atoms or molecules
  - electrons interacting via long-range (screened) Coulomb
  - find charge distribution, binding energy, bond lengths, . . .
- Bulk solid-state materials
  - metals, insulators, semiconductors, superconductors, . . .
- Liquid ^{3}He (superfluid!)
- Cold fermionic atoms in (optical) traps
- Atomic nuclei
- Neutron stars
  - color superconducting quark matter
  - neutron matter

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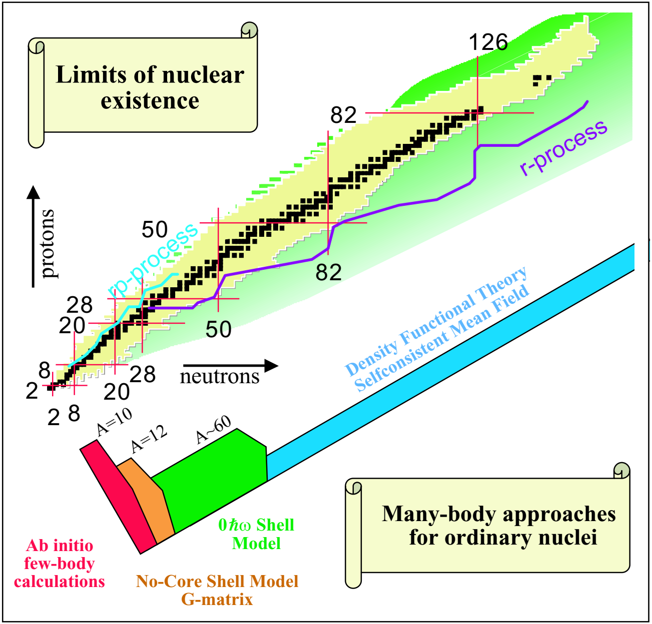
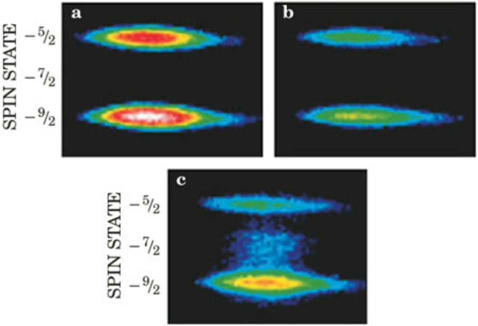
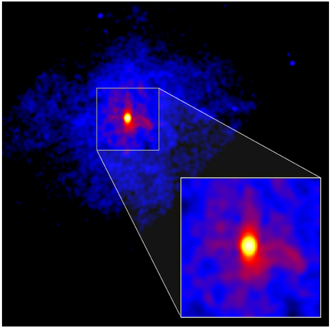

Limits of nuclear existence
126
82
50
28
20
8
2
2 8
20
28
50
82
protons
neutrons
rp-process
r-process
Density Functional Theory
Selfconsistent Mean Field
A=10
A=12
A~60
Ab initio few-body calculations
No-Core Shell Model G-matrix
0ħω Shell Model
Many-body approaches for ordinary nuclei

How Are Cold Atoms Like Neutron Stars?



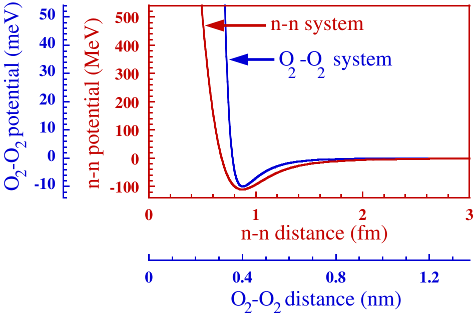
Regal et al., ultracold fermions



Chandra X-Ray Observatory image of pulsar in 3C58

Nuclear and Cold Atom Many-Body Problems

- Lennard-Jones and nucleon-nucleon potentials



[figure borrowed from J. Dobaczewski]

- Are there universal features of such many-body systems?
- How can we deal with “hard cores” in many-body systems?

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Choose diagrams by "art"	Use physical arguments (often handwaving) to justify the subset of diagrams used

Resolution and the Pointillists

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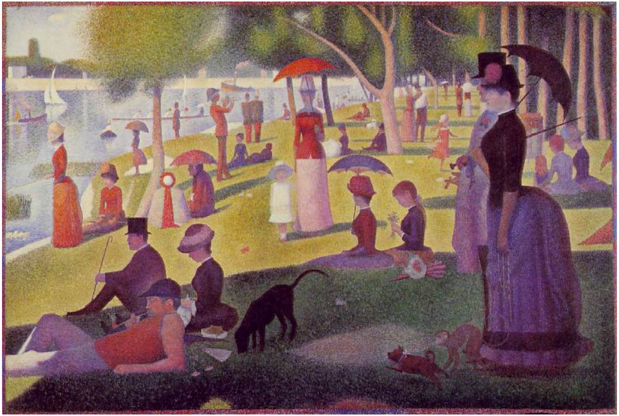
- George Seurat painted using closely spaced small dots ($\approx$ 0.4 mm wide) of pure pigment



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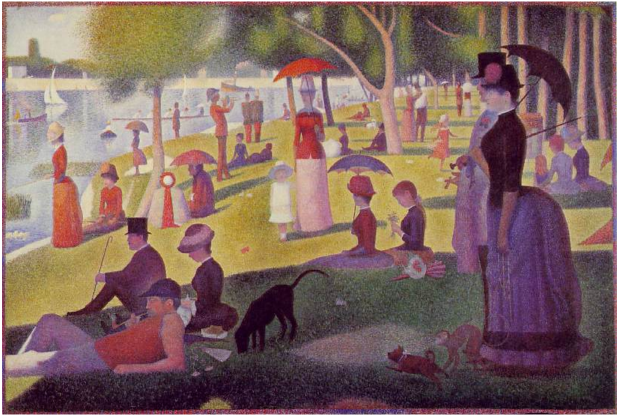
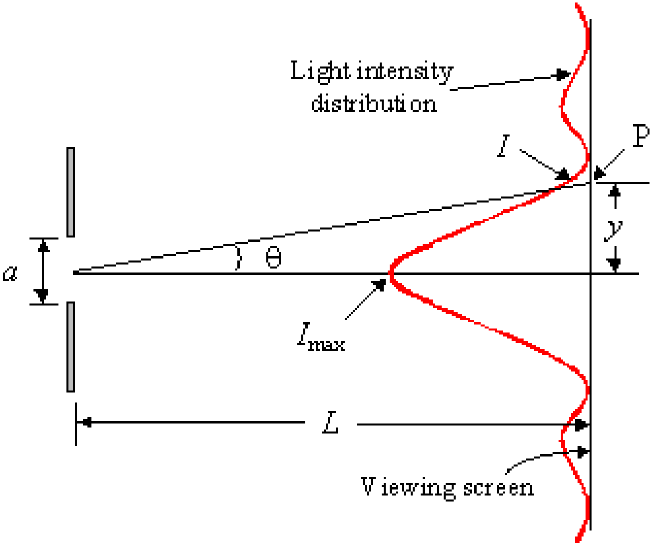


- Why do the dots blend together?

Diffraction and Resolution



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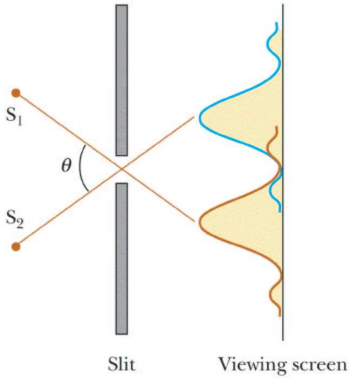

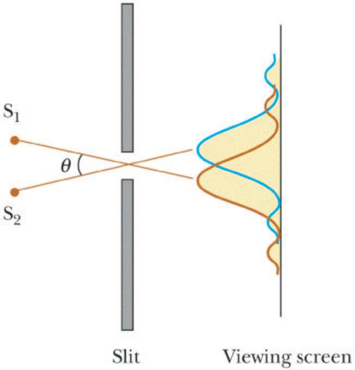

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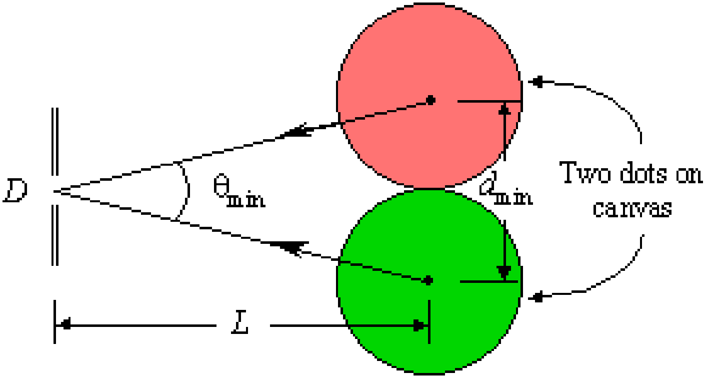

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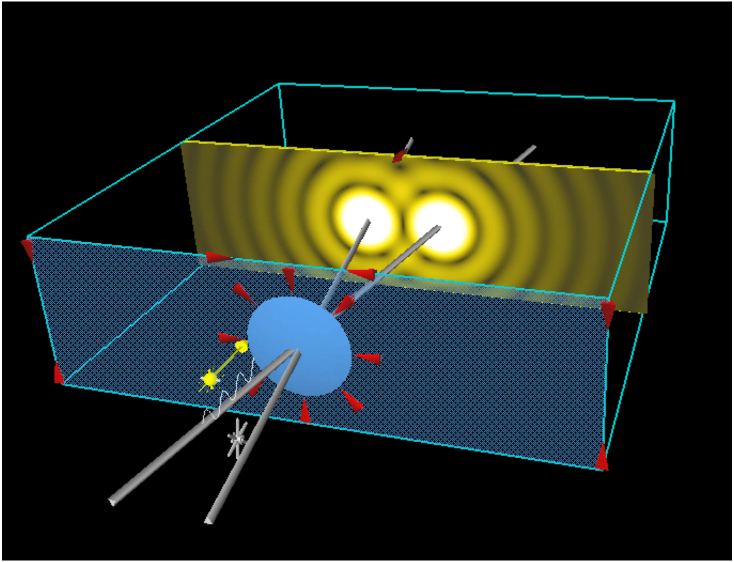

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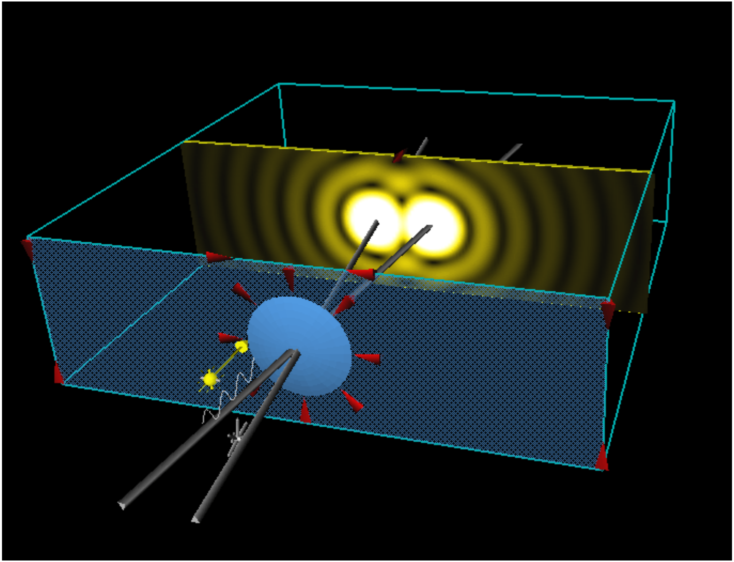

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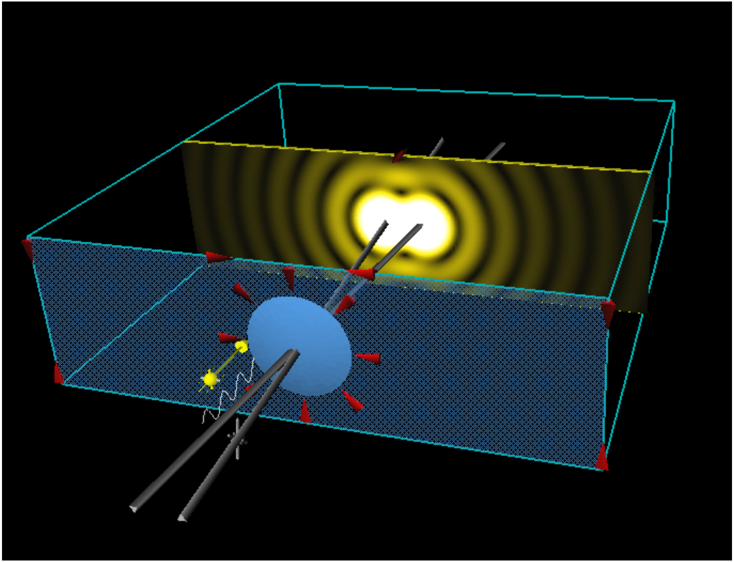

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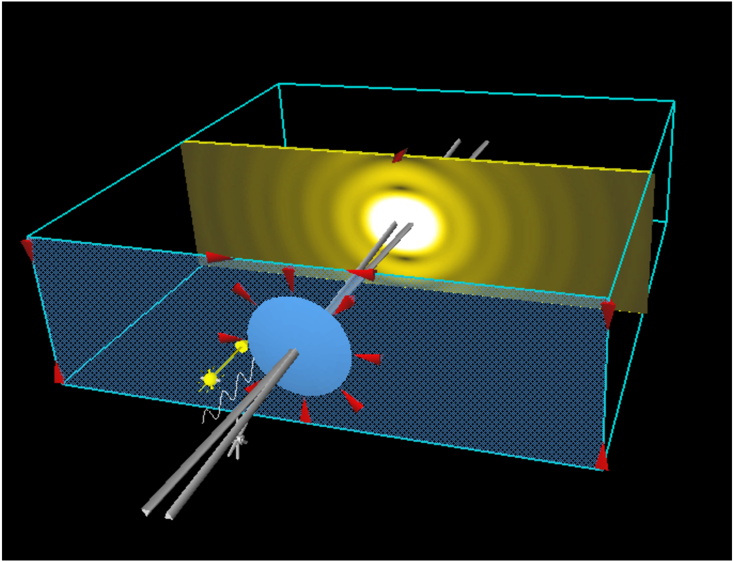
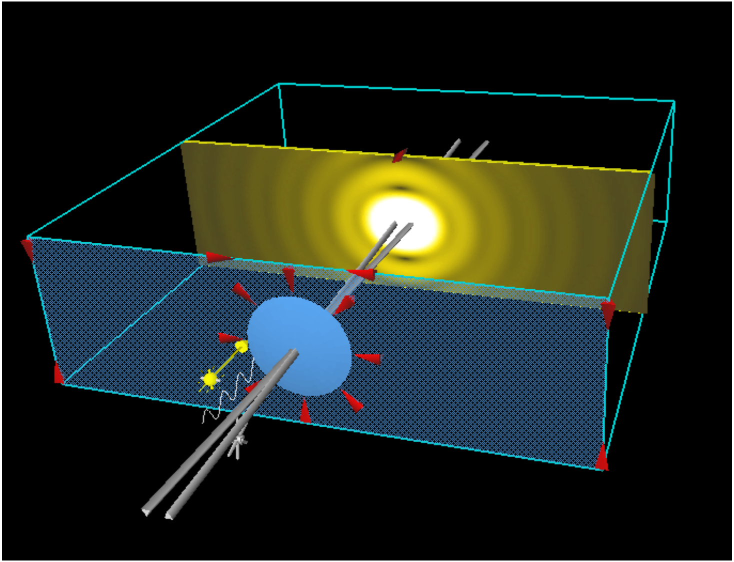
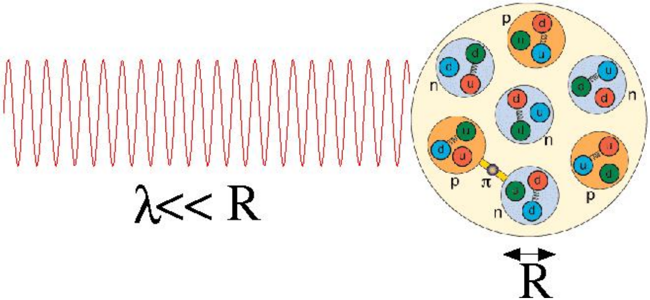


Wavelength and Resolution



Principles of Effective Low-Energy Theories



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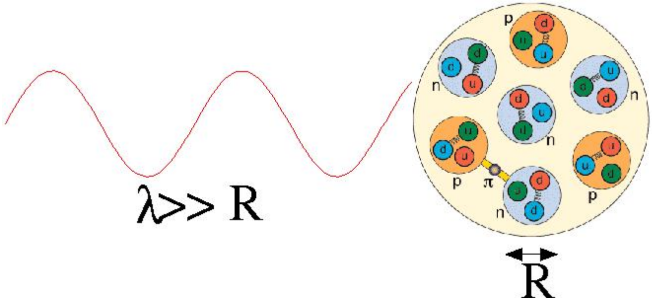

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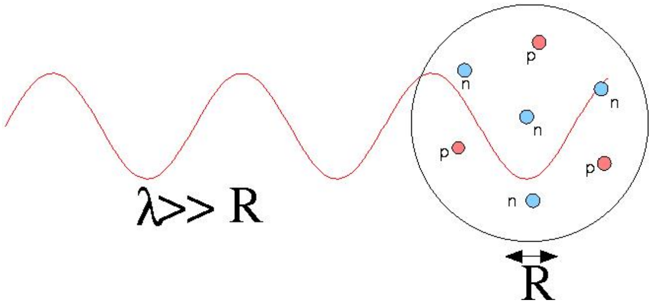
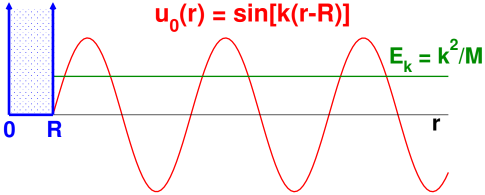


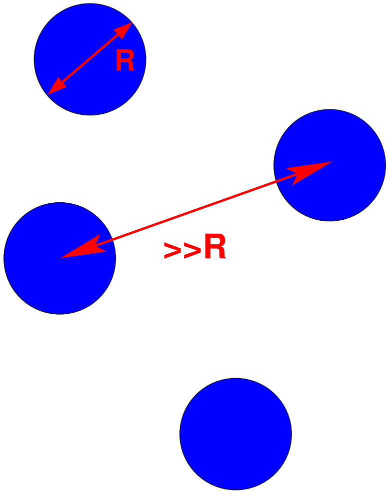
- If system is probed at low energies, fine details not resolved
  - use low-energy variables for low-energy processes
  - short-distance structure can be replaced by something simpler without distorting low-energy observables

"Simple" Many-Body Problem: Hard Spheres

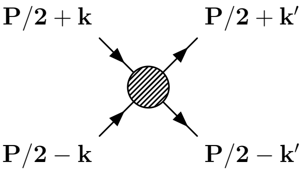
- Infinite potential at radius *R*
- Scattering solutions are simple:

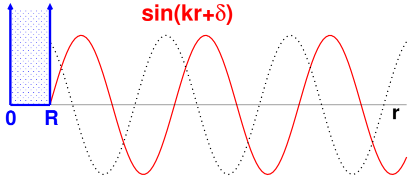


- What is the energy / particle of the many-body system at a given density?



Quick Review of Scattering





- Relative motion in frame with $P = 0$: $\psi(r) \xrightarrow{r\to\infty} e^{i\mathbf{k}\cdot\mathbf{r}} + f(k,\theta)\frac{e^{ikr}}{r}$
  where $k^2 = k'^2 = ME_k$ and $\cos\theta = \hat{k}\cdot\hat{k}'$

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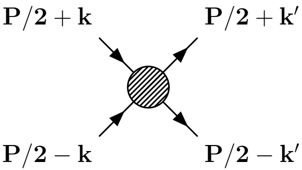

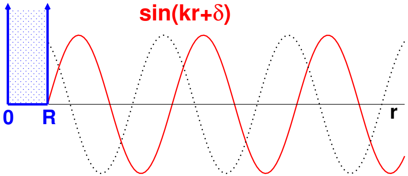

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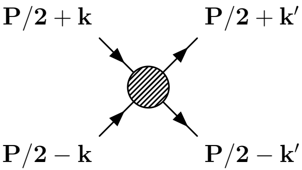

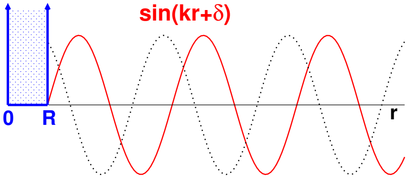

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  where $k^2 = k'^2 = ME_k$ and $\cos\theta = \hat{k}\cdot\hat{k}'$
- Differential cross section is $d\sigma/d\Omega = |f(k,\theta)|^2$
- Central $V \implies$ partial waves: $f(k,\theta) = \sum_l (2l+1) f_l(k) P_l(\cos\theta)$

$$\text{where} \quad f_l(k) = \frac{e^{i\delta_l(k)}\sin\delta_l(k)}{k} = \frac{1}{k\cot\delta_l(k) - ik}$$

and the S-wave phase shift is defined by

$$u_0(r) \xrightarrow{r\to\infty} sin[kr + \delta_0(k)] \quad \implies \quad \delta_0(k) = -kR \text{ for hard sphere}$$

At Low Energies: Effective Range Expansion

- As first shown by Schwinger, $k^{l+1}\cot\delta_l(k)$ has a power series expansion. For $l = 0$:

$$k\cot\delta_0 = -\frac{1}{a_0} + \frac{1}{2}r_0 k^2 - P r_0^3 k^4 + \cdots$$

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- The effective range expansion for hard sphere scattering is:

$$k\cot(-kR) = -\frac{1}{R} + \frac{1}{3}Rk^2 + \cdots \quad \Longrightarrow \quad a_0 = R \quad r_0 = 2R/3$$

so the low-energy effective theory is natural

In Search of a Perturbative Expansion

- If a_0 is natural, then low-energy scattering simplifies further
- For scattering at momentum $k \ll 1/R$, we should recover a perturbative expansion in kR for scattering amplitude:

$$f_0(k) \propto \frac{1}{k\cot\delta(k) - ik} \longrightarrow a_0\left[1 - ia_0 k - (a_0^2 - a_0 r_0/2)k^2 + \mathcal{O}(k^3 a_0^3)\right]$$

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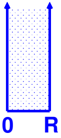
- Perturbation theory in the hard-sphere potential won't work:



$$\Longrightarrow \quad \langle \mathbf{k}|V|\mathbf{k}'\rangle \propto \int d\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, V(\mathbf{x})\, e^{-i\mathbf{k}'\cdot\mathbf{x}} \longrightarrow \infty$$

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- Easy for 2–2 scattering, but not for the many-body problem!
- EFT approach: $k \ll 1/R$ means we probe at low resolution $\Longrightarrow$ replace potential with a simpler but general interaction

EFT for “Natural” Short-Range Interaction

- A simple, general interaction is a sum of delta functions and derivatives of delta functions. In momentum space,

$$\langle \mathbf{k}|V_{\rm eft}|\mathbf{k}'\rangle = C_0 + \frac{1}{2}C_2(\mathbf{k}^2 + \mathbf{k}'^2) + C_2'\mathbf{k}\cdot\mathbf{k}' + \cdots$$

- Or, $\mathcal{L}_{\rm eft}$ has most general local (contact) interactions:

$$\mathcal{L}_{\rm eft} = \psi^\dagger\Big[i\frac{\partial}{\partial t} + \frac{\overrightarrow{\nabla}^2}{2M}\Big]\psi - \frac{C_0}{2}(\psi^\dagger\psi)^2 + \frac{C_2}{16}\Big[(\psi\psi)^\dagger(\psi\overleftrightarrow{\nabla}^2\psi) + \text{ h.c.}\Big]$$
$$+ \frac{C_2'}{8}(\psi\overleftrightarrow{\nabla}\psi)^\dagger\cdot(\psi\overleftrightarrow{\nabla}\psi) - \frac{D_0}{6}(\psi^\dagger\psi)^3 + \ldots$$

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- Dimensional analysis $\Longrightarrow$ $C_{2i} \sim \frac{4\pi}{M} R^{2i+1}$, $\quad D_{2i} \sim \frac{4\pi}{M} R^{2i+4}$

Feynman Rules for EFT Vertices

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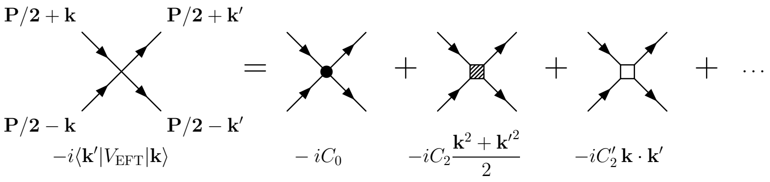
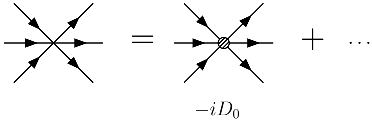




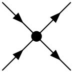
Renormalization

- Reproduce $f_0(k)$ in perturbation theory (Born series):

$$f_0(k) \propto a_0 - i a_0^2 k - (a_0^3 - a_0^2 r_0/2) k^2 + \mathcal{O}(k^3 a_0^3)$$

- Consider the leading potential $V^{(0)}_{\rm EFT}(\mathbf{x}) = C_0 \delta(\mathbf{x})$ or

$$\langle \mathbf{k} | V^{(0)}_{\rm eft} | \mathbf{k}' \rangle \Longrightarrow$$



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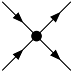

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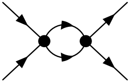


$$\implies \int \frac{d^3 q}{(2\pi)^3} \frac{1}{k^2 - q^2 + i\epsilon} \longrightarrow \infty!$$

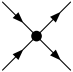
$\implies$ Linear divergence!

Renormalization

- Reproduce $f_0(k)$ in perturbation theory (Born series):

$$f_0(k) \propto a_0 - i a_0^2 k - (a_0^3 - a_0^2 r_0/2) k^2 + \mathcal{O}(k^3 a_0^3)$$

- Consider the leading potential $V^{(0)}_{\rm EFT}(\mathbf{x}) = C_0 \delta(\mathbf{x})$ or

$$\langle \mathbf{k} | V^{(0)}_{\rm eft} | \mathbf{k}' \rangle \Longrightarrow$$  $$\Longrightarrow C_0$$

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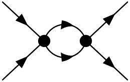

$$\Longrightarrow \int^{\Lambda_c} \frac{d^3q}{(2\pi)^3} \frac{1}{k^2 - q^2 + i\epsilon} \longrightarrow \frac{\Lambda_c}{2\pi^2} - \frac{ik}{4\pi} + \mathcal{O}(k^2/\Lambda_c)$$

$\Longrightarrow$ If cutoff at Λ_c, then can absorb into C_0, but all powers of k^2

Renormalization

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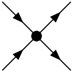

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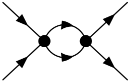
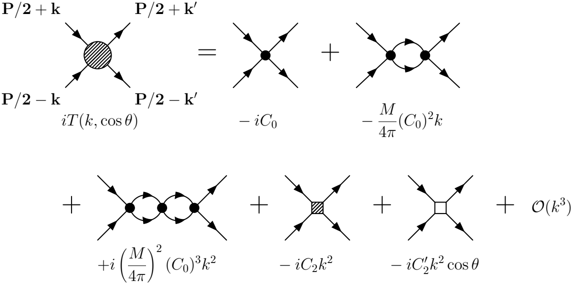


$$\Longrightarrow \int \frac{d^D q}{(2\pi)^3} \frac{1}{k^2 - q^2 + i\epsilon} \xrightarrow{D \to 3} -\frac{ik}{4\pi}$$

Dimensional regularization with minimal subtraction
$\Longrightarrow$ only one power of k!

- Dim. reg. + minimal subtraction $\Longrightarrow$ simple power counting:



- Matching: $C_0 = \dfrac{4\pi}{M} a_0 = \dfrac{4\pi}{M} R \,, \quad C_2 = \dfrac{4\pi}{M} \dfrac{a_0^2 r_0}{2} = \dfrac{4\pi}{M} \dfrac{R^3}{3} \,, \quad \cdots$

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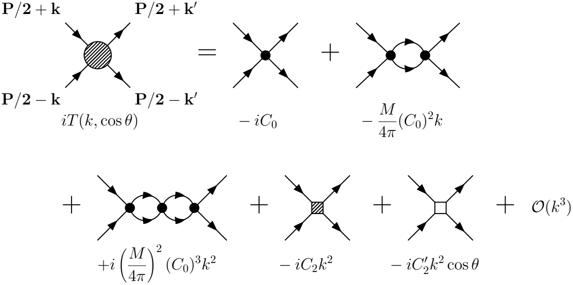

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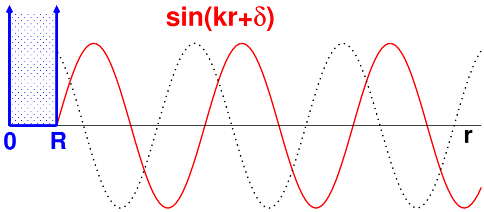
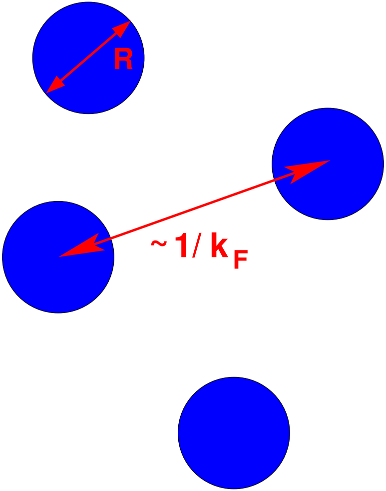
- Recovers expansion order-by-order with perturbative diagrams
  - one power of k per diagram, *natural* coefficients
  - estimate truncation error from dimensional analysis

“Simple” Many-Body Problem: Hard Spheres

- Infinite potential at radius R





- Scattering length $a_0 = R$
- Dilute $nR^3 \ll 1 \Longrightarrow k_F a_0 \ll 1$
- What is the energy / particle?

Noninteracting Fermi Sea at $T = 0$

- Put system in a large box ($V = L^3$) with periodic bc's
  - spin-isospin degeneracy ν (e.g., for nuclei, $\nu = 4$)
  - fill momentum states up to Fermi momentum $k_{\rm F}$

$$N = \nu \sum_{\mathbf{k}}^{k_{\rm F}} 1 \, , \qquad E = \nu \sum_{\mathbf{k}}^{k_{\rm F}} \frac{\hbar^2 k^2}{2M}$$

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- Note: $\int F(k)\, dk \approx \sum_i F(k_i) \Delta k_i = \sum_i F(k_i) \frac{2\pi}{L} \Delta n_i = \frac{2\pi}{L} \sum_i F(k_i)$

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$$N = \nu \frac{L}{2\pi}\int_{-k_{\rm F}}^{+k_{\rm F}} dk = \frac{\nu k_{\rm F}}{\pi} L \implies n = \frac{N}{L} = \frac{\nu k_{\rm F}}{\pi}\,; \quad \frac{E}{L} = \frac{1}{3}\frac{\hbar^2 k_{\rm F}^2}{2M} n$$

- In 3-D:

$$N = \nu \frac{V}{(2\pi)^3}\int^{k_{\rm F}} d^3k = \frac{\nu k_{\rm F}^3}{6\pi^2} V \implies n = \frac{N}{V} = \frac{\nu k_{\rm F}^3}{6\pi^2}\,; \quad \frac{E}{V} = \frac{3}{5}\frac{\hbar^2 k_{\rm F}^2}{2M} n$$

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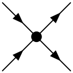
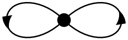
- Volume/particle $V/N = 1/n \sim 1/k_{\mathrm{F}}^3$, so spacing $\sim 1/k_{\mathrm{F}}$

Sum Over Fermions in the Fermi Sea

- Leading order $V_{\mathrm{EFT}}^{(0)}(\mathbf{x}) = C_0\delta(\mathbf{x})$

 $\Longrightarrow$ 

$$\mathcal{E}_{\mathrm{LO}} = \frac{C_0}{2}\nu(\nu-1)\left(\sum_{\mathbf{k}}^{k_{\mathrm{F}}} 1\right)^2 \propto a_0 k_{\mathrm{F}}^6$$

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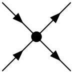

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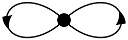

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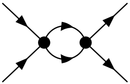

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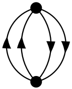

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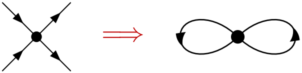

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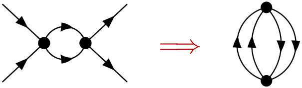


$$\mathcal{E}_{\mathrm{NLO}} \propto \int_{k_{\mathrm{F}}}^{\infty} \frac{d^3q}{(2\pi)^3}\frac{C_0^2}{k^2-q^2}$$

- Same renormalization fixes it! Particles $\longrightarrow$ holes

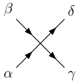
$$\int_{k_{\mathrm{F}}}^{\infty}\frac{1}{k^2-q^2} = \int_0^{\infty}\frac{1}{k^2-q^2} - \int_0^{k_{\mathrm{F}}}\frac{1}{k^2-q^2} \xrightarrow{D\to 3} -\int_0^{k_{\mathrm{F}}}\frac{1}{k^2-q^2} \propto a_0^2 k_{\mathrm{F}}^7$$

Feynman Rules for Hugenholtz Diagrams

- Ground-state energy density $\mathcal{E}$ is sum of Hugenholtz diagrams
  - same vertices as free space (same renormalization!)
- Feynman rules:

1. Each line is assigned conserved $\widetilde{k} \equiv (k_0, \vec{k})$ and

$$iG_0(\widetilde{k})_{\alpha\beta} = i\delta_{\alpha\beta}\left(\frac{\theta(k-k_{\rm F})}{k_0-\omega_{\vec{k}}+i\epsilon}+\frac{\theta(k_{\rm F}-k)}{k_0-\omega_{\vec{k}}-i\epsilon}\right) \ .$$

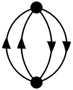
2. 

$\longrightarrow (\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma})$ (spin-independent)
3. After spin summations, $\delta_{\alpha\alpha} \to -g$ in every closed fermion loop.
4. Integrate $\int d^4k/(2\pi)^4$ with $e^{ik_0 0^+}$ for tadpoles
5. Symmetry factor $i/(S\prod_{l=2}^{l_{\rm max}}(l!)^k)$ counts vertex permutations and equivalent l–tuples of lines

Power Counting

- Power counting rules
  1. for every propagator: $M/k_{\rm F}^2$
  2. for every loop integration: $k_{\rm F}^5/M$
  3. for every n–body vertex with $2i$ derivatives: $k_{\rm F}^{2i}/M\Lambda^{2i+3n-5}$
- Diagram with V_{2i}^n n–body vertices of each type scales as $(k_{\rm F})^\beta$:

$$\beta = 5 + \sum_{n=2}^{\infty}\sum_{i=0}^{\infty}(3n+2i-5)V_{2i}^n \,.$$

- e.g.,  $\Longrightarrow V_0^2 = 2$

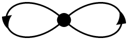
$$\Longrightarrow \beta = 5 + (3\cdot 2 + 2\cdot 0 - 5)\cdot 2 = 7 \Longrightarrow \mathcal{O}(k_{\rm F}^7)$$

$T = 0$ Energy Density from Hugenholtz Diagrams

$$\frac{E}{V} = n\frac{k_F^2}{2M}\left[\frac{3}{5}\right.$$

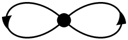
$$\left.\right]$$

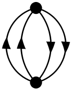
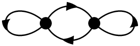
$T = 0$ Energy Density from Hugenholtz Diagrams

$\mathcal{O}(k_F^6)$: 

$$\frac{E}{V} = n \frac{k_F^2}{2M} \left[\frac{3}{5} + (\nu - 1) \frac{2}{3\pi} (k_F a_0) \right]$$

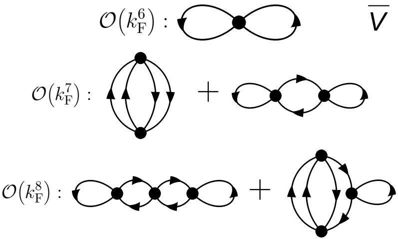
$T = 0$ Energy Density from Hugenholtz Diagrams

$\mathcal{O}(k_{\rm F}^6)$: 

$\mathcal{O}(k_{\rm F}^7)$:  + 

$$\frac{E}{V} = n\frac{k_{\rm F}^2}{2M}\left[\frac{3}{5} + (\nu-1)\frac{2}{3\pi}(k_{\rm F}a_0) + (\nu-1)\frac{4}{35\pi^2}(11-2\ln 2)(k_{\rm F}a_0)^2\right]$$

$T = 0$ Energy Density from Hugenholtz Diagrams



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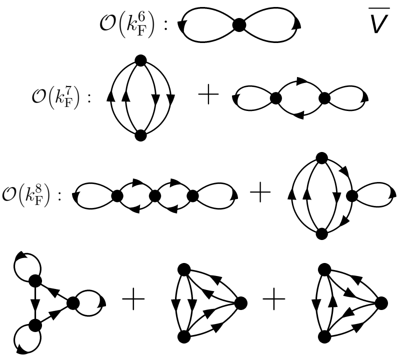

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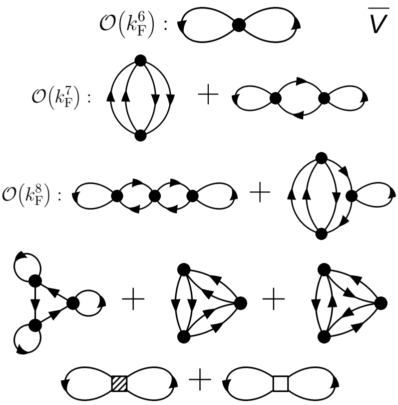

$$
\begin{aligned}
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&+ (\nu-1)\big(0.076 + 0.057(\nu-3)\big)(k_F a_0)^3 \\
&+ (\nu-1)\frac{1}{10\pi}(k_F r_0)(k_F a_0)^2 \\
&+ (\nu+1)\frac{1}{5\pi}(k_F a_p)^3 + \cdots\Big]
\end{aligned}
$$

Looks Like a Power Series in k_F! Is it?

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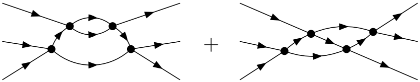
- New logarithmic divergences in 3–3 scattering



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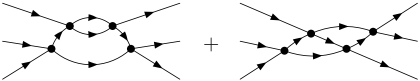

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- Changes in Λ_c must be absorbed by 3-body coupling $D_0(\Lambda_c)$
  $\Longrightarrow D_0(\Lambda_c) \propto (C_0)^4 \ln(a_0\Lambda_c) + \text{const.}$ [Braaten & Nieto]

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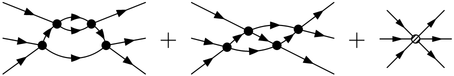

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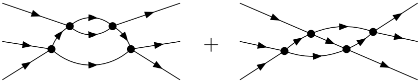

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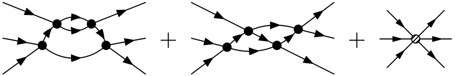

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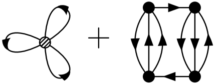
- What does this imply for the energy density?

$\mathcal{O}(k_F^9 \ln(k_F))$:



$\cdots \propto (\nu-2)(\nu-1)\,(k_F a_0)^4 \ln(k_F a_0)$

Effective Field Theory Ingredients

See “Crossing the Border” [nucl-th/0008064]

1. Use the most general $\mathcal{L}$ with low-energy dof’s consistent with global and local symmetries of underlying theory
   - $\mathcal{L}_{\rm eft} = \psi^\dagger \left[i\frac{\partial}{\partial t} + \frac{\nabla^2}{2M} \right] \psi - \frac{C_0}{2}(\psi^\dagger\psi)^2 - \frac{D_0}{6}(\psi^\dagger\psi)^3 + \ldots$

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2. Declaration of regularization and renormalization scheme
    - natural $a_0 \Longrightarrow$ dimensional regularization and minimal subtraction

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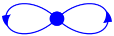
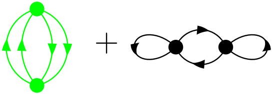
See "Crossing the Border" [nucl-th/0008064]

1. Use the most general $\mathcal{L}$ with low-energy dof's consistent with global and local symmetries of underlying theory
   - $\mathcal{L}_{\rm eft} = \psi^\dagger \left[i\frac{\partial}{\partial t} + \frac{\overrightarrow{\nabla}^2}{2M} \right] \psi - \frac{C_0}{2}(\psi^\dagger\psi)^2 - \frac{D_0}{6}(\psi^\dagger\psi)^3 + \ldots$
2. Declaration of regularization and renormalization scheme
   - natural $a_0 \Longrightarrow$ dimensional regularization and minimal subtraction
3. Well-defined power counting $\Longrightarrow$ small expansion parameters
   - use the separation of scales $\Longrightarrow \dfrac{k_{\rm F}}{\Lambda}$ with $\Lambda \sim 1/R \Longrightarrow k_{\rm F} a_0$, etc.

$\mathcal{O}(k_{\rm F}^6)$:  $\mathcal{O}(k_{\rm F}^7)$: 

$$\mathcal{E} = \rho \frac{k_{\rm F}^2}{2M} \left[\frac{3}{5} + \frac{2}{3\pi}(k_{\rm F} a_0) + \frac{4}{35\pi^2}(11 - 2\ln 2)(k_{\rm F} a_0)^2 + \cdots \right]$$

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Choose diagrams by "art"	Power counting determines diagrams and truncation error

Web Resources

- These lectures, including additional notes on each slide, are available at
  `http://www.physics.ohio-state.edu/~ntg/NPSS/`
- Class notes for a two-quarter course on “Nuclear Many-Body Physics” given by Dick Furnstahl and Achim Schwenk are available at
  `http://www.physics.ohio-state.edu/~ntg/880/`
  (username: physics, password: 880.05)

References for Many-Body Physics

- A.L. Fetter and J.D. Walecka, “Quantum Theory of Many-Particle Systems.” Classic text, but pre-path integrals. Now available in an inexpensive (about $20) Dover reprint. Get it!
- J.W. Negele and H. Orland, “Quantum Many-Particle Systems.” Detailed and careful use of path integrals. Full of good physics but most of the examples are in the problems, so it can be difficult to learn from.
- N. Nagaosa, “Quantum Field Theory in Condensed Matter Physics.” Recent text, covers path integral methods and symmetry breaking.
- A.M. Tsvelik, “Quantum Field Theory in Condensed Matter Physics.” Good on one-dimensional systems.
- M. Stone, “The Physics of Quantum Fields.” A combined introduction to quantum field theory as applied to particle physics problems and to nonrelativistic many-body problems. Some very nice explanations.
- R.D. Mattuck, “A Guide to Feynman Diagrams in the Many-Body Problems.” This is a nice, intuitive guide to the meaning and use of Feynman diagrams.
- N. Goldenfeld, “Lectures on Phase Transitions and the Renormalization Group.” The discussion of scaling, dimensional analysis, and phase transitions is wonderful.
- G.D. Mahan, “Many-Particle Physics.” Standard, encyclopedic reference for condensed matter applications.
- P. Ring and P. Schuck, “The Nuclear Many-Body Problem.” Somewhat out of date, but still a good, encyclopedic guide to the nuclear many-body problem. Doesn’t discuss Green’s function methods much and no path integrals.
- K. Huang, “Statistical Mechanics.” Excellent choice for general treatment of statistical mechanics, with good sections on many-body physics.

References for Many-Body Effective Field Theory

Fermi to Dyson in 1953 [recalled in Nature 427 (2004) 297]

Concerning a proposed pseudoscalar meson theory:

> *"There are two ways of doing calculations in theoretical physics", he said. "One way, and this is the way I prefer, is to have a clear physical picture of the process that you are calculating. The other way is to have a precise and self-consistent mathematical formalism. You have neither."*

Fermi to Dyson in 1953 [recalled in Nature 427 (2004) 297]

I was slightly stunned, but ventured to ask him why he did not consider the pseudoscalar meson theory to be a self-consistent mathematical formalism. He replied, "Quantum electrodynamics is a good theory because the forces are weak, and when the formalism is ambiguous we have a clear physical picture to guide us. With the pseudoscalar meson theory there is no physical picture, and the forces are so strong that nothing converges. To reach your calculated results, you had to introduce arbitrary cut-off procedures that are not based either on solid physics or on solid mathematics."