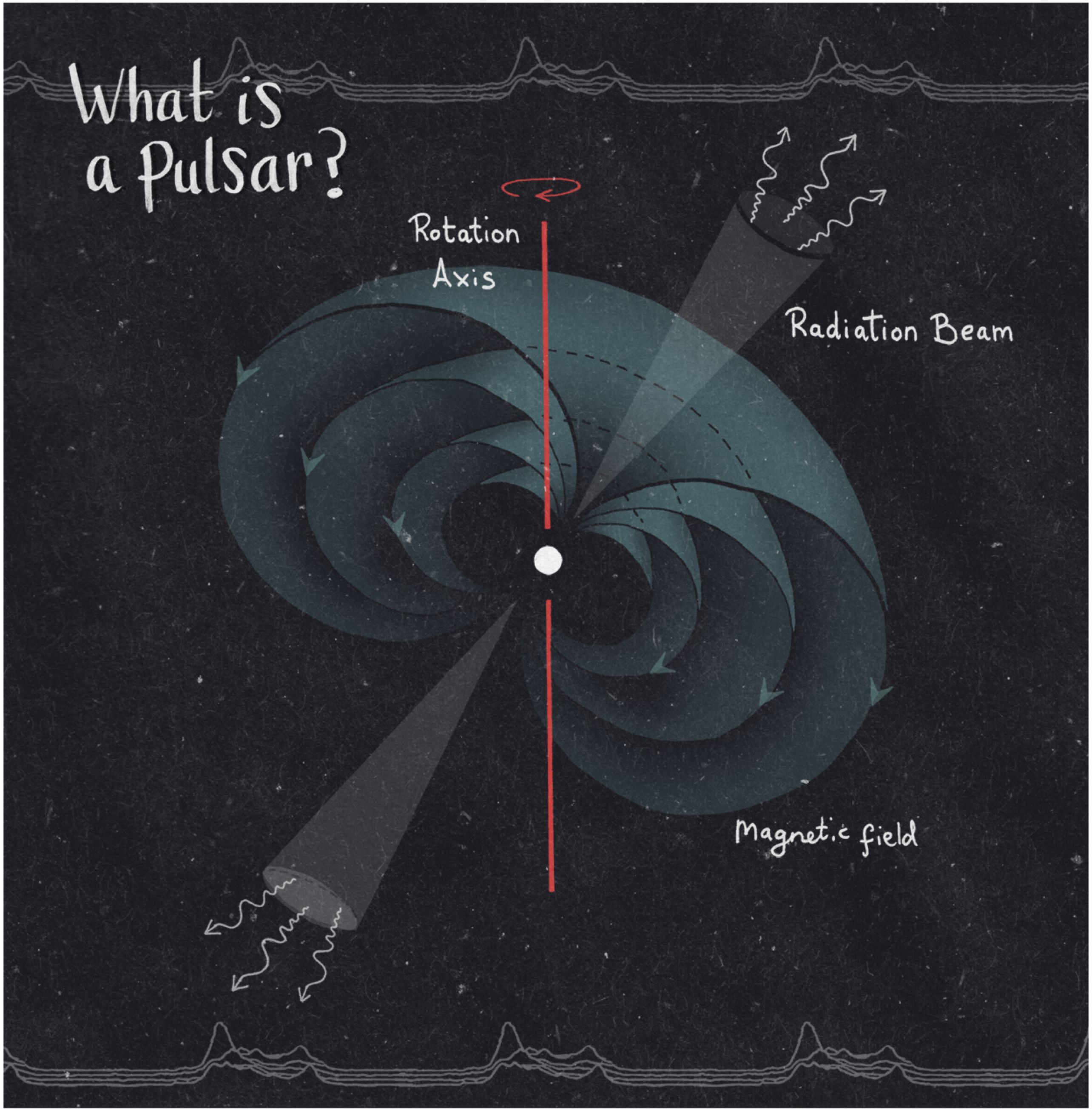
Magnetic deformations

(what is the smallest mountain?)

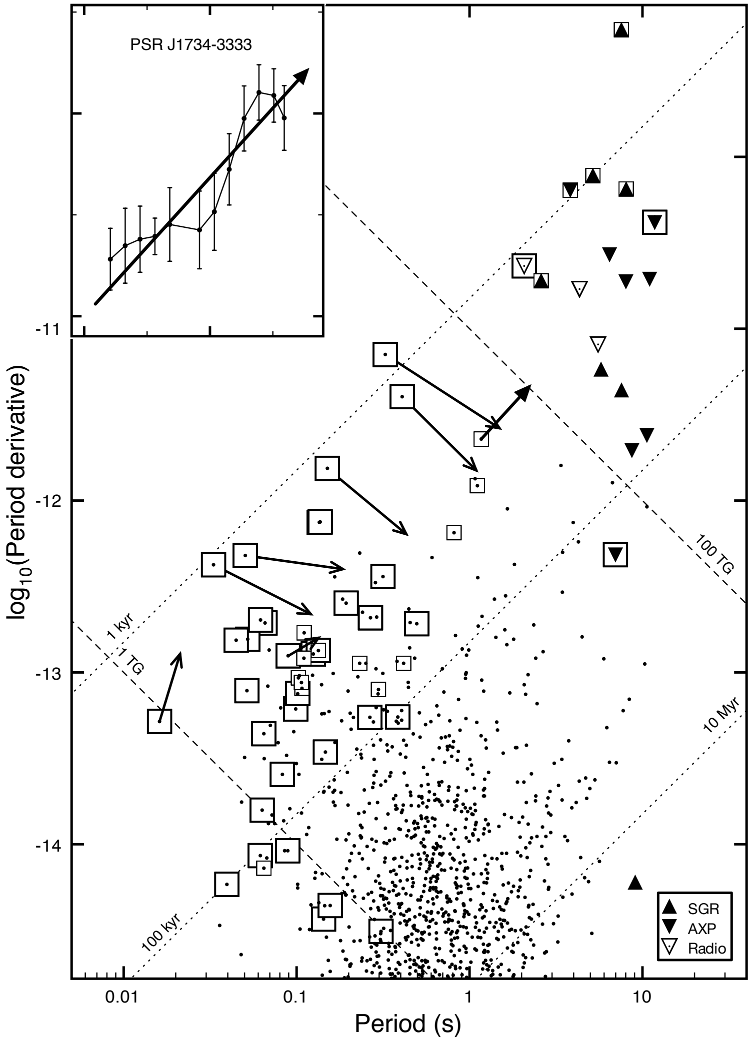
Nils Andersson
UNIVERSITY OF Southampton
na@maths.soton.ac.uk


What is a Pulsar?
Rotation Axis
Radiation Beam
Magnetic field

Sobering thought:

After 50 years we do not (quite) know why pulsars pulse.

braking index



[Espinoza et al 2011]

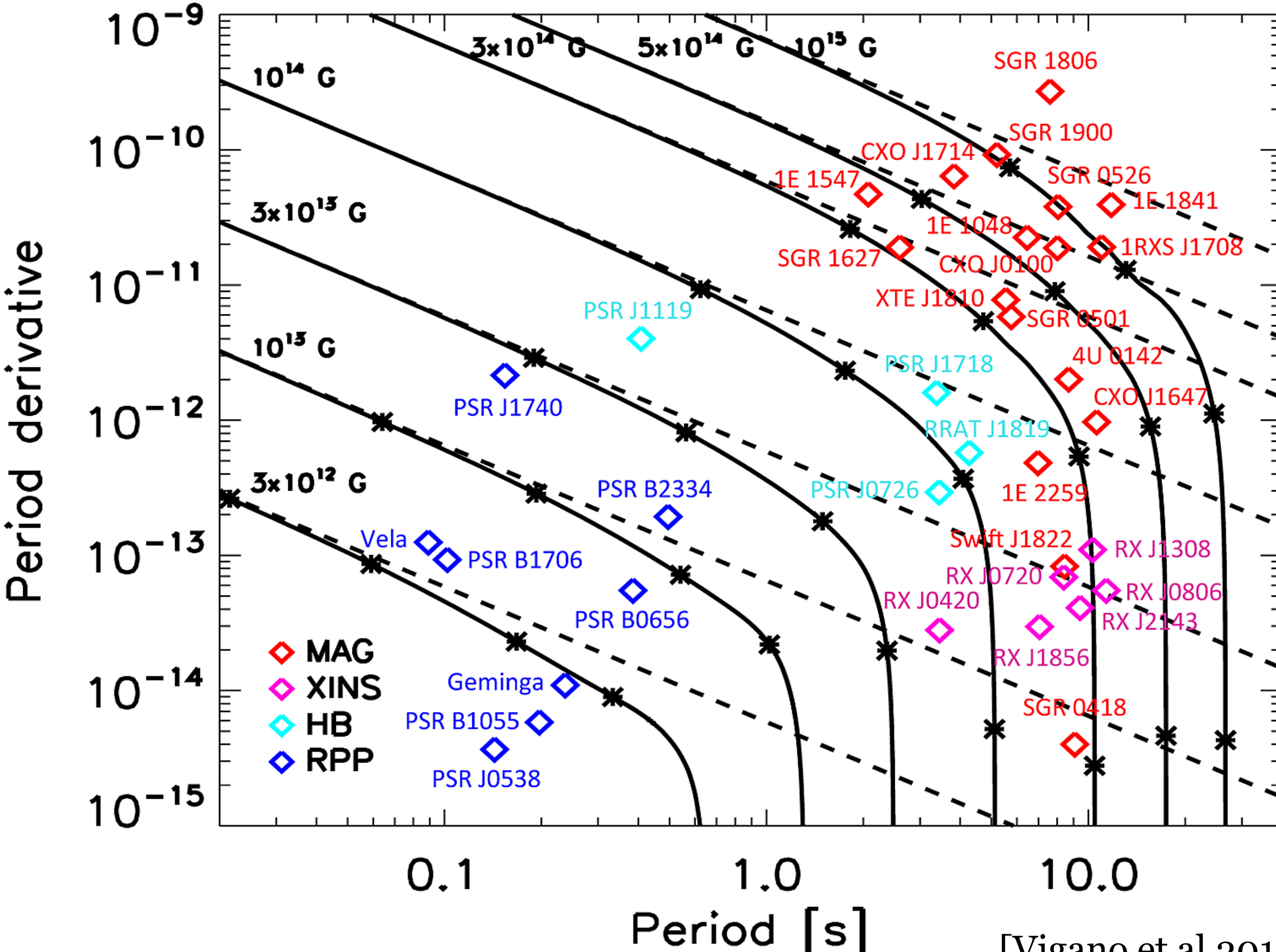
For canonical magnetic dipole radiation, the braking index should be 3. Observed systems tend to deviate significantly from this.

$n<3$ could be an indication of an increasing magnetic field (pulsars turn into magnetars as a buried field emerges?)

Expect $n=5$ for GWs ($n=7$ for r-modes).

Key point: We do not understand the origin and evolution of the magnetic field.

Intimately linked to the properties of matter at supranuclear densities (e.g. resistivity and fluxtube dynamics)



[Vigano et al 2013]

the magnetic field

The magnetosphere

- Dictates spin-evolution through different emission mechanisms.

Interior field configuration

- Interior field configuration largely "unknown". Expect a poloidal field with a (perhaps strong toroidal) component.
- Formation of the large scale field not understood (dynamos). Usually assumed to happen fast (Alfven wave crossing time).
- "Equilibrium" models appear somewhat limited (and tend to be unstable!).
- State of matter is important. If protons form type II superconductor (as expected), the magnetic field is confined to fluxtubes.

Field evolution

- Expect the field to evolve (driving observed magnetar activity) and explain spin-evolution.
- Known resistivity does not lead to expected evolution timescales…

back of the envelope

Would like to know what the **smallest** GW deformation may be.

Simple estimate (based on energetics) leads to

$$\epsilon \sim \frac{\int B^2\, dV}{GM^2/R} \sim 10^{-12} \left(\frac{B}{10^{12}\,\mathrm{G}}\right)^2$$

Poloidal field leads to oblate deformation, toroidal to prolate one.

If protons form type II superconductor (as expected), the magnetic field is confined to fluxtubes. This increases the tension by a factor of H_c/B, where $H_c \sim 10^{15}$ G, and we get

$$\epsilon \sim 10^{-9} \frac{B}{10^{12}\,\mathrm{G}}$$

But the GW emission from known pulsars would still not be detectable…

More detailed calculations (pretty much) give the same results.

The "smallest" NS mountain may simply be too small.

barotropic models

The usual starting point is to assume hydro-magnetic equilibrium (no time dependence). Can also safely take the magnetic pressure to be weak enough that we can solve the problem perturbatively. This leads to

$$\boldsymbol{\nabla}\left(\frac{\delta p}{\rho}+\delta\Phi\right)+\frac{1}{\rho^2}\left(\delta p\boldsymbol{\nabla}\rho-\delta\rho\boldsymbol{\nabla}p\right)=\frac{1}{\rho}\boldsymbol{f}_{\rm L}=\frac{1}{4\pi}\left(\boldsymbol{\nabla}\times\boldsymbol{B}\right)\times\boldsymbol{B}$$

For a barotropic equation of state $p=p(\rho)$ and it follows that we must have

$$\boldsymbol{\nabla}\times\left(\frac{\boldsymbol{f}_{\rm L}}{\rho}\right)=0$$

Moreover, since

$$\nabla\cdot\boldsymbol{B}=0$$

we can express the field in terms of stream functions

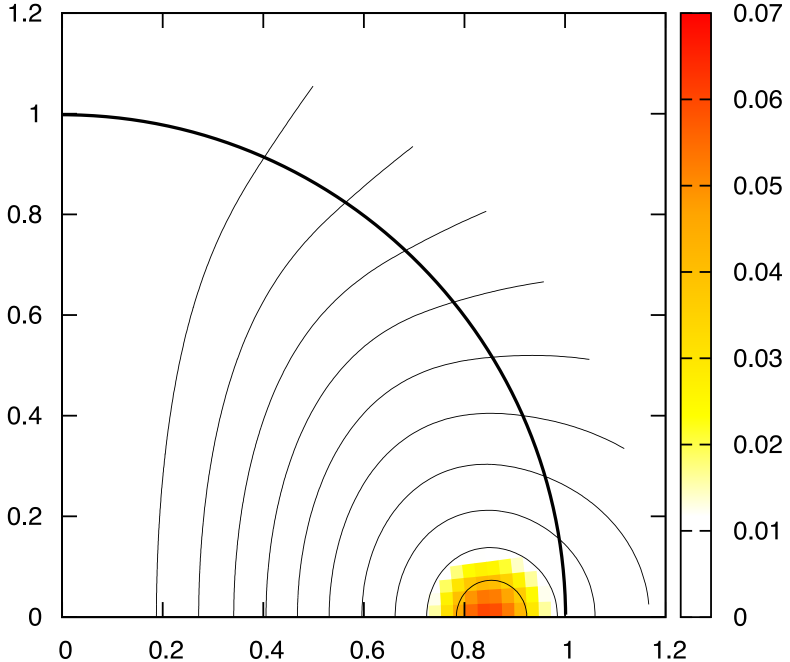
$$\boldsymbol{B}_{\rm p}=\boldsymbol{\nabla}\Psi\times\boldsymbol{\nabla}\varphi \qquad\qquad \boldsymbol{B}_{\rm t}=T\boldsymbol{\nabla}\varphi$$

If we also assume axisymmetry, we have

$$\boldsymbol{\nabla}\Psi \times \boldsymbol{\nabla}T = 0 \Rightarrow T = T(\Psi)$$

and the magnetic field follows from the **Grad-Shafranov** equation (including boundary conditions etc...)

$$\frac{\mathcal{A}}{\rho} = M(\Psi) \;\Rightarrow\; \Delta_*\Psi + TT' = 4\pi\varpi^2\rho M$$



[Lander +]

The problem now becomes one of specifying the functions Ψ and T (restricted by imagination?)

Typical configurations;

- similar maximum values of poloidal and toroidal field
- up to 5-10% of energy in toroidal component
- appear to be dynamically unstable (!)

There is a competition between the poloidal (which makes star oblate) and toroidal (which makes it prolate, and may lead to spin-flip) components.

Estimated magnetic deformations tend to be similar to rough estimates.

Equilibria appear to be poloidal dominated, but... for magnetars it is usually "assumed" that the opposite is true.

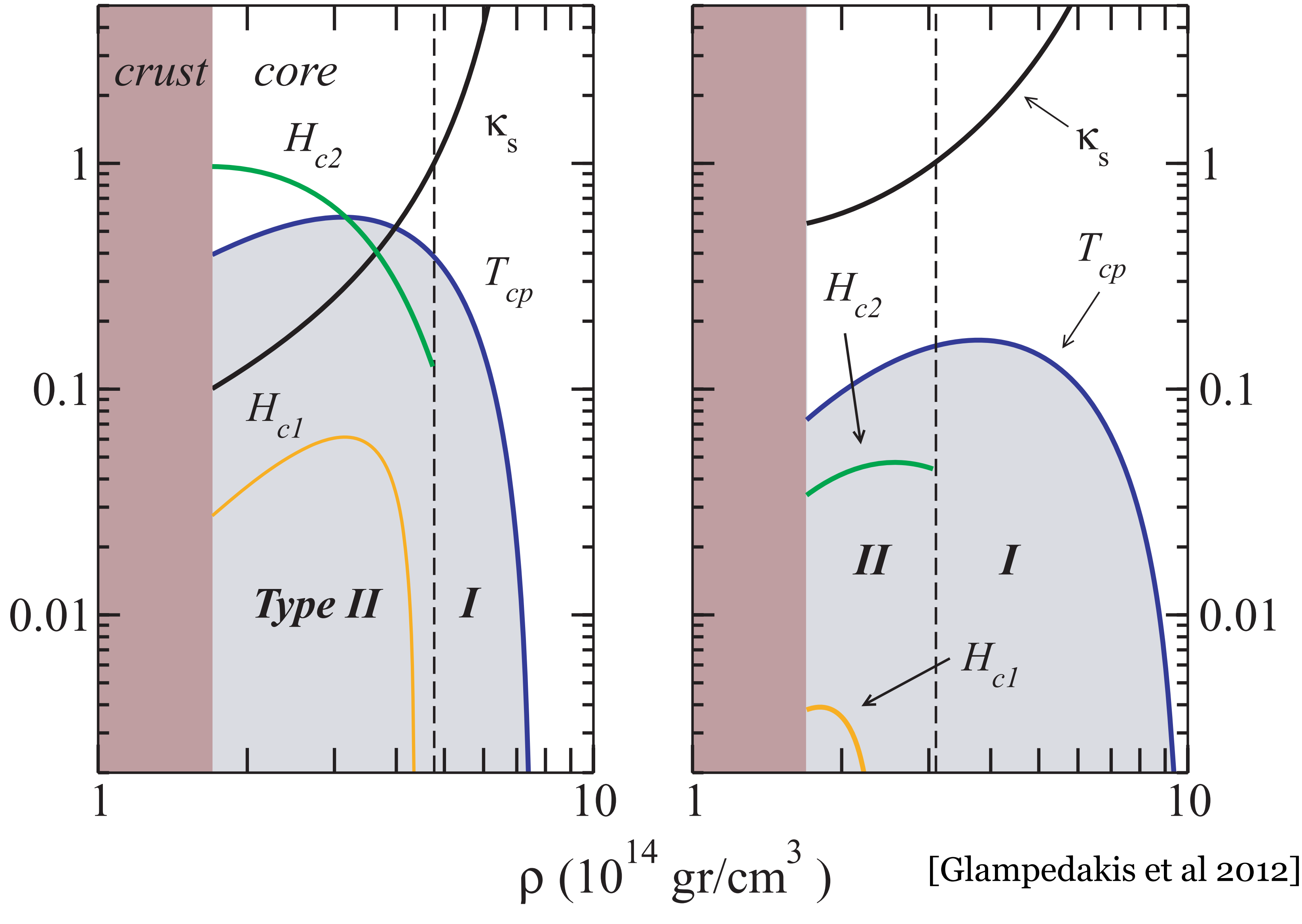
Are we missing something here?

The state of matter may be important.

In type II superconductor, the magnetic field is carried by fluxtubes. This changes the magnetic force;

$$\boldsymbol{f}_L = \frac{1}{4\pi}\left[\boldsymbol{B}\times(\nabla\times\boldsymbol{H}) + \rho_{\mathrm{p}}\nabla\left(\frac{\partial H}{\partial \rho_{\mathrm{p}}}\right)\right]$$

No longer have a Grad-Shafranov equation (and as a result, the problem has not yet been solved for mixed fields).



[Glampedakis et al 2012]

stratification

It has been argued that we are free to specify the magnetic field as we wish in stratified matter.

The argument is simple. Recall;

$$\boldsymbol{\nabla}\left(\frac{\delta p}{\rho}+\delta\Phi\right)+\frac{1}{\rho^2}\left(\delta p\boldsymbol{\nabla}\rho-\delta\rho\boldsymbol{\nabla}p\right)=\frac{1}{\rho}\boldsymbol{f}_{\mathrm{L}}$$

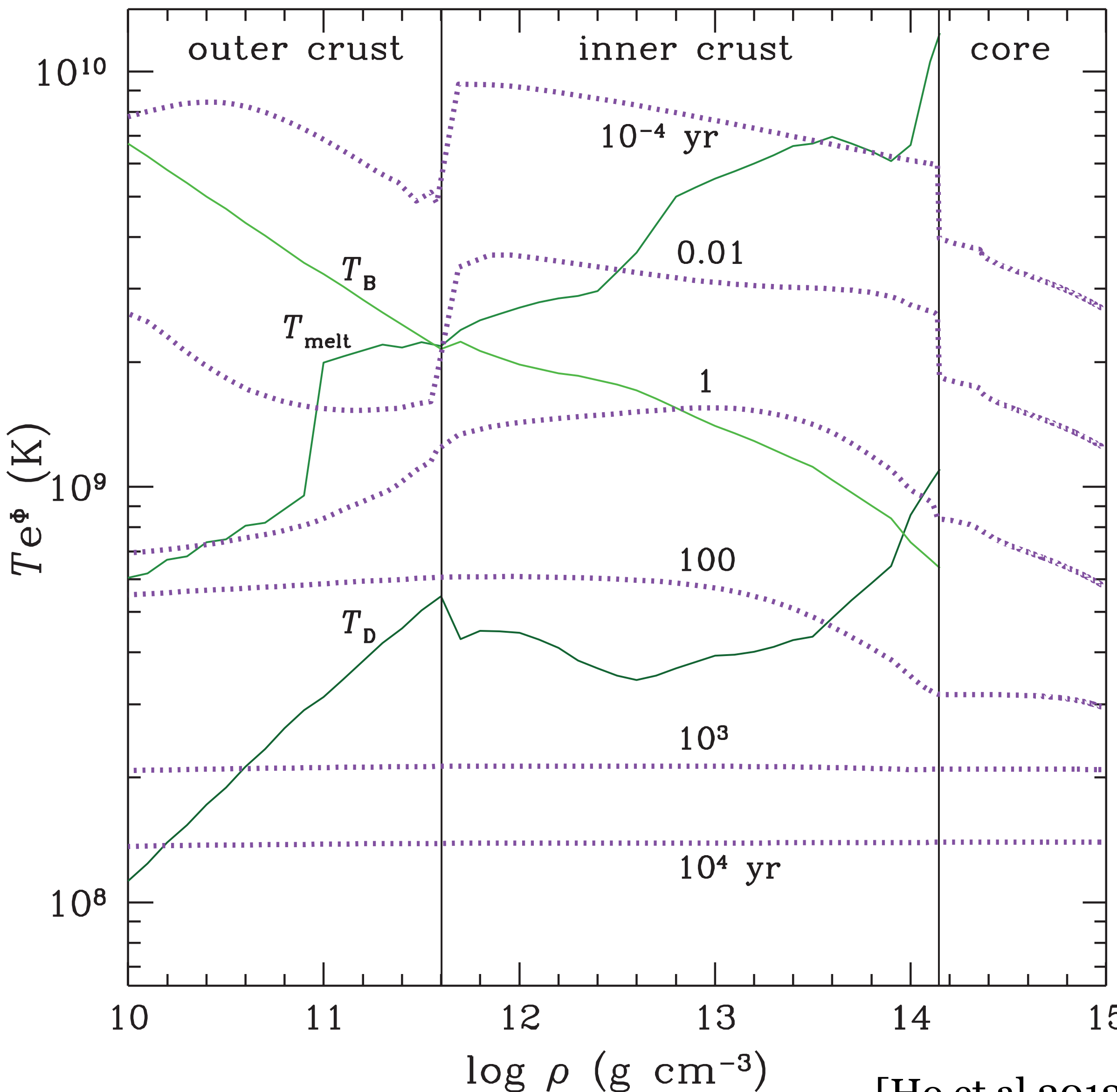
Now the second term on the left does not vanish and we do not arrive at the Grad-Shafranov equation. Specifying the magnetic force (and assuming axisymmetry) we get two equations, one for the perturbed pressure and one for the density.

Variations in the proton fraction provide the required balance;

$$\delta p=\left(\frac{\partial p}{\partial\rho}\right)_{x_{\mathrm{p}}}\delta\rho+\left(\frac{\partial p}{\partial x_{\mathrm{p}}}\right)_{\rho}\delta x_{\mathrm{p}}$$

However… the system is no longer in chemical equilibrium and one would expect the relevant reactions to reinstate equilibrium on a timescale shorter than that associated with cooling.

These configurations are unlikely to survive, but… need to understand evolution.


outer crust
inner crust
core
10^{10}
10^{9}
10^{8}
Te^{Φ} (K)
10^{-4} yr
0.01
1
100
10^3
10^4 yr
T_B
T_{melt}
T_D
10
11
12
13
14
15
log ρ (g cm^{-3})

[Ho et al 2012]

take-home message

Yes… the interior magnetic field is likely to set a lower limit on the deformations of a real neutron star,

but… we are no yet able to calculate (with confidence) the likely field configuration (and the ones we can calculate appear to be unstable),

and… when we try to find a way out we are inevitably led to evolutionary questions.

This is "unfortunate" because it means that the problem we want to solve is coupled to other (also interesting and relevant) ones, so we need to include a lot of physics.

Need **resistive**, **reactive** and **relativistic** models.

- field evolution requires "resistivity" (e.g. Ohm's law),
- reactions come into play as star evolves (ambipolar diffusion, deleptonisation…),
- general relativity required for quantitative models.

state of play

We can do this!

Build multi-fluid framework with 4 components (and could do elastic crust as well).

Example: Generalised Ohm's law

$$
\begin{aligned}
en_{\rm e}\mathcal{E}_b - \left(1-\frac{n_{\rm e}\mu_{\rm e}}{p+\varepsilon}\right)\epsilon_{bac}J^a b^c - \frac{1}{n_{\rm e}e}\left(\hat{\mathcal{R}} - \Gamma_{\rm e}s\mathcal{A}^{\rm es}\right)J_b \\
= -en_{\rm e}\epsilon_{bac}v^a_{\rm p}b^c + \mathcal{R}_{\rm en}w^{\rm np}_b + \left(\mathcal{R}_{\rm es} - \Gamma_{\rm e}s\mathcal{A}^{\rm es}\right)\left(\frac{q_b}{sT} - v^{\rm p}_b\right) \\
- n_{\rm e}\mu_{\rm e}\left[\left(v^a_{\rm p} - \frac{J^a}{en_{\rm e}}\right)\nabla_a u_b + \perp^c_b u^a\nabla_a\left(v^{\rm p}_c - \frac{J_c}{en_{\rm e}}\right) + \left(v^{\rm p}_b - \frac{J_b}{en_{\rm e}}\right)u^a\frac{1}{\mu_{\rm e}}\nabla_a\mu_{\rm e}\right] \\
+ 2n_{\rm e}u^a\nabla_{[a}s\mathcal{A}^{\rm es}w^{\rm se}_{b]} - e\Gamma_{\rm e}\left[\perp^a_b + u^a\left(v^{\rm p}_b - \frac{J_b}{en_{\rm e}}\right)\right]A_a,
\end{aligned}
$$

with e^a and b^a the electric and magnetic fields measured by the observer and the electro-chemical potential and heat flux are given by

$$
\mathcal{E}^a = e^a + \frac{1}{e}\perp^{ab}\left(\nabla_b\mu_{\rm e} - \frac{\mu_{\rm e}}{p+\varepsilon}\nabla_b p\right) \qquad\qquad q^a = sTv^a_{\rm s}
$$