Relation between scattering amplitude and Bethe-Salpeter wave function in quantum field theory

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References

TY and Kuramashi, PRD96:114511,11(2017), Namekawa and TY, arXiv:1712.10141

Multi-Hadron Systems from Lattice QCD @ INT University of Washington
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Purpose

(re)introduce a relation between scattering amplitude and BS wave function

Outline

- Wave function in finite volume method
- Bethe-Salpeter (BS) wave function outside interaction range
- BS wave function inside interaction range
- Fundamental relation in quantum mechanics
- Expansion of reduced BS wave function
- Summary

Bether-Salperter wave function

History of BS wave function in lattice comunity

1. Lüscher's finite volume method [Lüscher, NPB354:531(1991)]
2. Wave function in 2D statistical model

   [Balog *et. al*, PRD60:094508(1999); NPB618:315(2001)]

   $\delta(k)$ from wave function in $x > R$ (not explain)
3. BS wave function through LSZ reduction formula

   [Lin *et. al*, NPB619:467(2001)]
4. $I = 2$ two-pion BS wave function [CP-PACS, PRD71:094504(2005)]
5. Potential from BS wave function [Ishii *et. al*, PRL99:022001(2007)]

   Several talks in this workshop (not explain)
6. Amplitude from BS wave function [TY and Kuramashi, PRD96:114511,11(2017)]

Lüscher's finite volume method

[Lüscher, NPB354:531(1991)]

spinless two-particle elastic scattering in center of mass (CM) frame

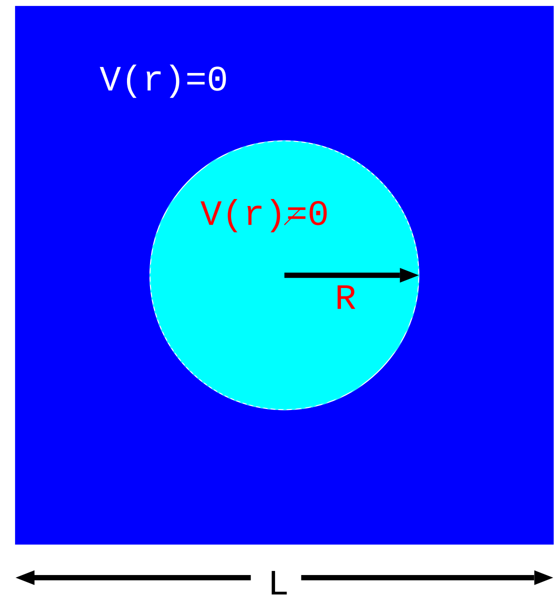
Important assumption

1. Two-particle interaction is localized.

   $\rightarrow$ Interaction range R exists.

$$V(r) \begin{cases} \neq 0 & (r \leq R) \\ = 0 \ (\sim e^{-cr}) & (r > R) \end{cases}$$

2. $V(r)$ is not affected by boundary. $\rightarrow R < L/2$



Two-particle wave function $\phi(\vec{r};k)$ satisfies Helmholtz equation.

$$\left(\Delta + k^2\right)\phi(\vec{r};k) = 0 \text{ in } r > R, \quad E = 2\sqrt{m^2 + k^2}$$

Lüscher's finite volume method

[Lüscher, NPB354:531(1991)]

Helmholtz equation on L^3

1. Solution of $(\Delta + k^2)\phi(\vec{r}; k) = 0$ in $r > R$

$$\phi(\vec{r}; k) = G(\vec{r}; k) = C \cdot \sum_{\vec{n} \in Z^3} \frac{e^{i\vec{r}\cdot\vec{n}(2\pi/L)}}{\vec{n}^2 - q^2}, \quad q^2 = (Lk/2\pi)^2 \neq \text{integer}$$

2. Expansion by spherical Bessel $j_l(kr)$ and Neumann $n_l(kr)$ functions

$$\begin{aligned} \phi(\vec{r}; k) &= \beta_0(k) n_0(kr) + \alpha_0(k) j_0(kr) + (l \geq 4) \\ &= e^{i\delta(k)} \sin(kr + \delta(k))/kx + (l \geq 4) \end{aligned}$$

3. S-wave scattering phase shift $\delta(k)$ in infinite volume

$$\frac{\beta_0(k)}{\alpha_0(k)} = \boxed{\tan\delta(k) = \frac{\pi^{3/2} q}{Z_{00}(1; q^2)}} \quad Z_{00}(s; q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\vec{n} \in Z^3} \frac{1}{(\vec{n}^2 - q^2)^s}$$

Relation between $\boxed{\delta(p) \text{ and } k}$ $\left(E = 2\sqrt{m^2 + k^2}\right)$

$\phi(\vec{r}; k)$ disappears in final formula.

Bether-Salperter wave function

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BS wave function through LSZ reduction formula

[Lin *et. al*, NPB619:467(2001)]

BS wave function of two pions in infinite volume (Only S-wave)

$$\begin{aligned}\phi(x;k) &= \langle 0|\pi_1(\vec{x}/2)\pi_2(-\vec{x}/2)|\pi_1(\vec{k})\pi_2(-\vec{k});\text{in}\rangle \\ &= e^{i\vec{k}\cdot\vec{x}} + \int \frac{d^3p}{(2\pi)^3}\frac{H(p;k)}{p^2-k^2-i\epsilon}e^{i\vec{p}\cdot\vec{x}}\end{aligned}$$

Inelastic scattering contribution and unnecessay overall factors are neglected.

Half off-shell amplitude $H(p;k)$

$$H(p;k) = \frac{E_p+E_k}{8E_pE_k}M(p;k)$$

$M(p;k)$ defined by LSZ reduction formula

$$e^{-i\mathbf{q}\cdot\mathbf{x}}\frac{-i\sqrt{Z}M(p;k)}{-\mathbf{q}^2+m^2-i\varepsilon} = \int d^4z d^4y_1 d^4y_2 K(\mathbf{p},\mathbf{z})K(-\mathbf{k_1},\mathbf{y_1})K(-\mathbf{k_2},\mathbf{y_2})G_4(\mathbf{z},\mathbf{x},\mathbf{y_1},\mathbf{y_2})$$

$$K(\mathbf{p},\mathbf{z}) = \tfrac{i}{\sqrt{Z}}e^{i\mathbf{p}\cdot\mathbf{z}}(-\mathbf{p}^2+m^2), \quad G_4(\mathbf{z},\mathbf{x},\mathbf{y_1},\mathbf{y_2}) = \langle 0|T[\pi_1(\mathbf{z})\pi_2(\mathbf{x})\pi_1(\mathbf{y_1})\pi_2(\mathbf{y_2})|0\rangle$$

$$\mathbf{p} = (E_p,\vec{p}), \quad \mathbf{k_1} = (E_k,\vec{k}), \quad \mathbf{k_2} = (E_k,-\vec{k}), \quad \underline{\mathbf{q} = (2E_k-E_p,-\vec{p})}$$

off-shell momentum

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Half off-shell amplitude $H(p;k)$

$$H(p;k) = \frac{E_p+E_k}{8E_pE_k}M(p;k)$$

$M(p;k)$ at on-shell $p=k$

$$M(k;k) = \frac{16\pi E_k}{k}e^{i\delta(k)}\sin\delta(k) \Rightarrow H(k;k) = \frac{4\pi}{k}e^{i\delta(k)}\sin\delta(k)$$

Bether-Salperter wave function

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BS wave function outside R

[CP-PACS, PRD71:094504(2005)]

Reduced BS wave function

$$h(x;k) = (\Delta + k^2)\phi(x;k) = -\int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}} H(p;k)$$

$$\phi(x;k) = e^{i\vec{k}\cdot\vec{x}} + \int \frac{d^3p}{(2\pi)^3} \frac{H(p;k)}{p^2 - k^2 - i\epsilon} e^{i\vec{p}\cdot\vec{x}}$$

Assumption: $h(x;k) = 0$ outside interaction range ($x > R$)

c.f. $(\Delta + k^2)\phi(x;k) = mV(x)\phi(x;k)$ in quantum mechanics

Using the assumption and $x > R$

$$\phi(x;k) = e^{i\delta(k)} \frac{\sin(kx + \delta(k))}{kx}$$

agrees with wave function in quantum mechanics

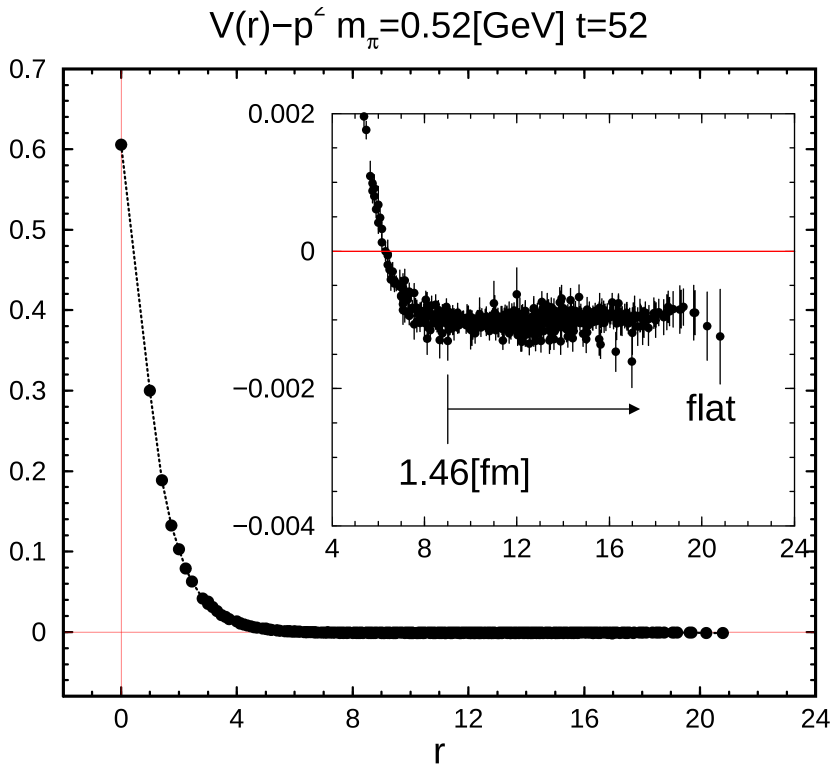
Following derivation in quantum mechanics,

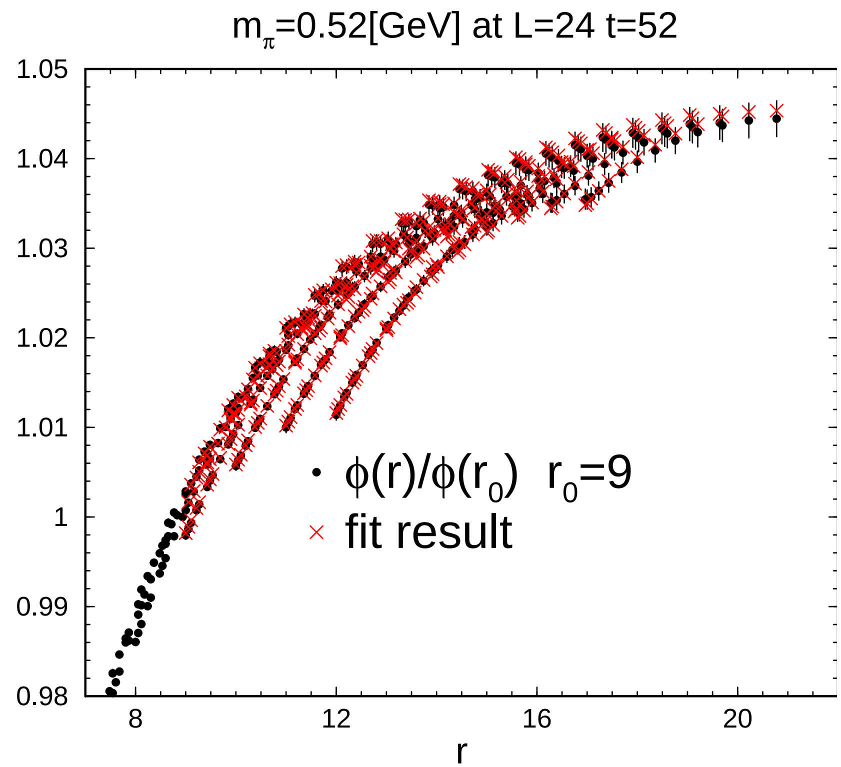
finite volume method can be derived from BS wave function.

$I = 2$ two-pion BS wave function

[CP-PACS, PRD71:094504(2005)]

$\phi(r;k)$ at $k \sim 0$ ($k = p$, $R = r_0$ in figure)





$$\frac{h(r;k)}{\phi(r;k)} - k^2$$

$h(r;k) = 0$ in $r > R$

Fit with $G(\vec{r};k)$ in $x > R$

k determined from $\phi(r;k)$ in $r > R$

$G(\vec{r};k)$: Solution of Helmholtz equation on L^3

Bether-Salperter wave function

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BS wave function inside R

[TY and Kuramashi, PRD96:114511,11(2017)]

Reduced BS wave function

$$h(x;k) = (\Delta + k^2)\phi(x;k) = -\int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}} H(p;k)$$

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$\Downarrow$ Fourier transformation

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$\Downarrow$ Fourier transformation

Fundamental relation <u>in this talk</u>

$$H(p;k) = -\int d^3x\; e^{-i\vec{p}\cdot\vec{x}} h(x;k)$$

Relation between $H(p;k)$ and $h(x;k)$ *i.e.* $\phi(x;k)$ inside R

BS wave function inside R

[TY and Kuramashi, PRD96:114511,11(2017)]

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$$h(x;k) = (\Delta + k^2)\phi(x;k) = -\int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}} H(p;k)$$

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Fundamental relation in this talk

$$H(p;k) = -\int d^3x\, e^{-i\vec{p}\cdot\vec{x}} h(x;k)$$

Relation between $H(p;k)$ and $h(x;k)$ *i.e.* $\phi(x;k)$ inside R

At on-shell $p = k$ [CP-PACS, PRD71:094504(2005), not explicitly written]

$$H(k;k) = \frac{4\pi}{k} e^{i\delta(k)} \sin\delta(k) = -\int d^3x\, e^{-i\vec{k}\cdot\vec{x}} h(x;k)$$

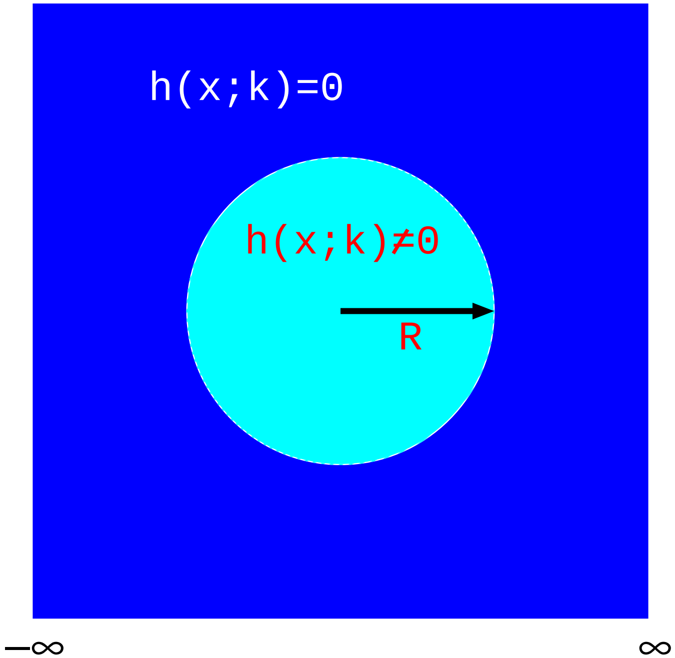
$h(x;k)$ is essential to calculate $H(p;k)$.

BS wave function inside R (cont'd)

[Namekawa and TY, arXiv:1712.10141]

Fundamental relation in finite integration range $L/2 > R$

$$H(p;k) \;=\; -\int_{-\infty}^{\infty} d^3x \; e^{-i\vec{p}\cdot\vec{x}} h(x;k)$$



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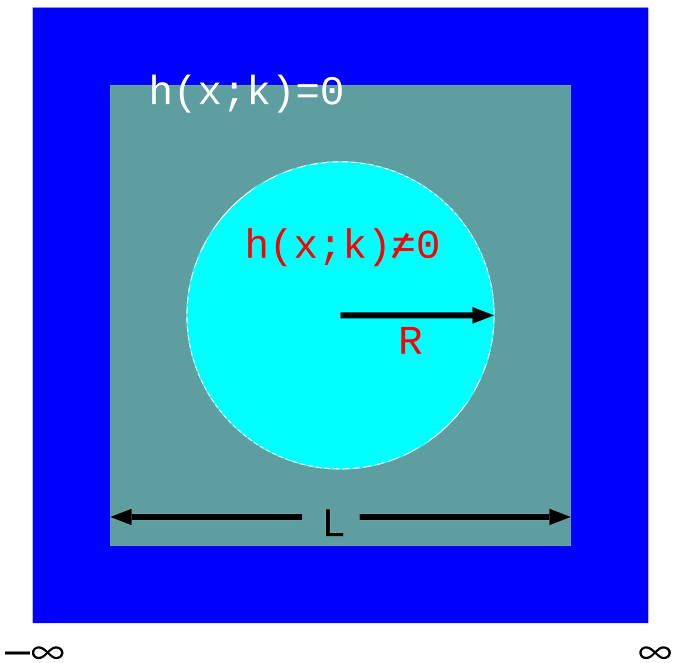
$$H(p;k) = -\int_{-\infty}^{\infty} d^3x\, e^{-i\vec{p}\cdot\vec{x}} h(x;k)$$
$$= -\int_{-L/2}^{L/2} d^3x\, e^{-i\vec{p}\cdot\vec{x}} h(x;k) - \int_{\notin L^3} d^3x\, e^{-i\vec{p}\cdot\vec{x}} h(x;k)$$



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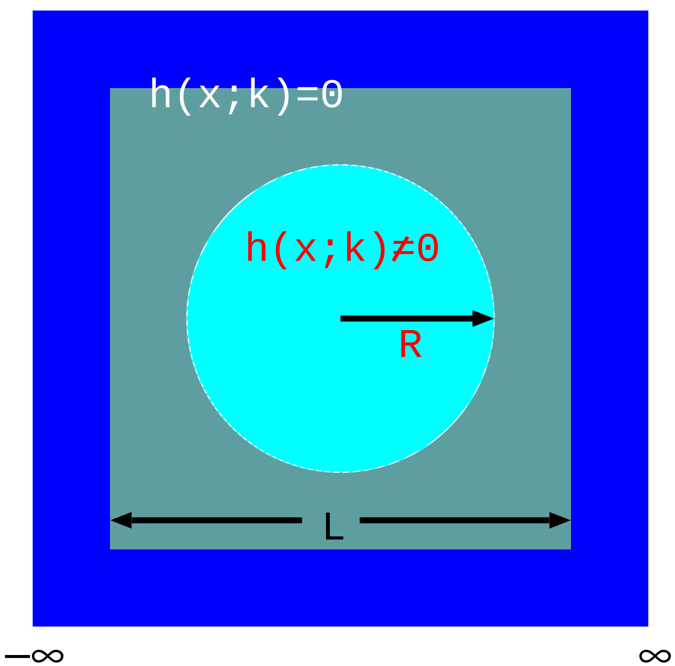
$$= -\int_{-L/2}^{L/2} d^3x\ e^{-i\vec{p}\cdot\vec{x}} h(x;k) \underline{-\int_{\notin L^3} d^3x\ e^{-i\vec{p}\cdot\vec{x}} h(x;k)}$$

$$\Rightarrow 0 \quad \because h(x;k) = 0\ (x > R)$$



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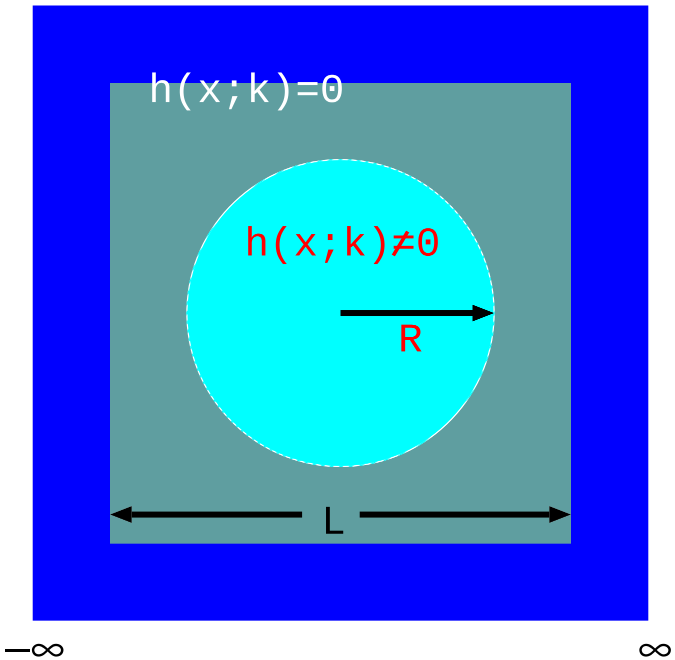

Suitable for finite volume calculation

First calculation with this relation

Next speaker Namekawa-san

BS wave function and LSZ reduction formula

[Namekawa and TY, in preparation]

LSZ reduction formula

On-shell scattering amplitude =

Fourier transformation of 4-point function × momentum factor

and on-shell limit

LSZ reduction formula in relative coordinate

$$
\begin{aligned}
H(p;k) &= -\int_{-\infty}^{\infty} d^3x \; e^{-i\vec{p}\cdot\vec{x}} (\Delta + k^2)\phi(x;k) \\
&\Downarrow \quad \text{Partial integration} \\
&= (p^2 - k^2) \int_{-\infty}^{\infty} d^3x \; e^{-i\vec{p}\cdot\vec{x}} \phi(x;k) \\
&\xrightarrow[p\to k]{} H(k;k)
\end{aligned}
$$

c.f. [Carbonell and Karmanov PLB754:270(2016)]

BS wave function and LSZ reduction formula

[Namekawa and TY, in preparation]

LSZ reduction formula in relative coordinate

$$H(p;k) = -\int_{-\infty}^{\infty} d^3x \; e^{-i\vec{p}\cdot\vec{x}} (\Delta + k^2)\phi(x;k)$$

$\Downarrow$ Partial integration

$$= (p^2 - k^2) \int_{-\infty}^{\infty} d^3x \; e^{-i\vec{p}\cdot\vec{x}} \phi(x;k)$$

c.f. [Carbonell and Karmanov PLB754:270(2016)]

LSZ reduction formula in relative coordinate with finite range $L/2 > R$

$$H(p;k) = (p^2 - k^2) \int_{-L/2}^{L/2} d^3x \; e^{-i\vec{p}\cdot\vec{x}} \phi(x;k) \underline{-\sum_j \int_{-L/2}^{L/2} d^2x \left[e^{-i\vec{p}\cdot\vec{x}} \Big(\partial_j \phi(x;k) + i p_j \phi(x;k) \Big) \right]_{-L/2}^{L/2}}$$

Surface term: integration at boundary $(j = 1, 2, 3)$

Fundamental relation in quantum mechanics

[TY and Kuramashi, PRD96:114511,11(2017)]

Interpretation of HALQCD method in this frame work

$V(x;k)$ is defined by $h(x;k)$ as

$$V(x;k) = \begin{cases} \frac{1}{m}\frac{h(x;k)}{\phi(x;k)} & (x \le R) \\ 0 & (x > R) \end{cases}$$

corresponding to LO HALQCD method

$V(x;k)$ is regarded as potential in Shrödinger equation.

$$(\Delta + p^2)\overline{\phi}(x;p) = mV(x;k)\overline{\phi}(x;p)$$

$\overline{\phi}(x;p)$ is a solution of the equation with given p.

Scattering phase shift $\overline{\delta}(p)$ from Shrödinger equation

$$\frac{e^{i\overline{\delta}(p)}\sin\overline{\delta}(p)}{p} = -\frac{m}{4\pi}\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}V(x;k)\overline{\phi}(x;p)$$

Fundamental relation in quantum mechanics

[TY and Kuramashi, PRD96:114511,11(2017)]

Scattering phase shift $\overline{\delta}(p)$ from Shrödinger equation

$$\frac{e^{i\overline{\delta}(p)}\sin\overline{\delta}(p)}{p} = -\frac{1}{4\pi}\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}\frac{h(x;k)}{\phi(x;k)}\overline{\phi}(x;p)$$

At $p = k$, $\overline{\phi}(x;k) = \phi(x;k) \quad \because (\Delta + k^2)\phi(x;k) = h(x;k)$

$$\frac{e^{i\overline{\delta}(k)}\sin\overline{\delta}(k)}{k} = -\frac{1}{4\pi}\int d^3x\, e^{-i\vec{k}\cdot\vec{x}}h(x;k) = \frac{H(k;k)}{4\pi} = \frac{e^{i\delta(k)}\sin\delta(k)}{k}$$

$$\overline{\delta}(k) = \delta(k)$$

At $p \neq k$, $\overline{\phi}(x;p) \neq \phi(x;k)$ in general

$$\frac{e^{i\overline{\delta}(p)}\sin\overline{\delta}(p)}{p} = -\frac{1}{4\pi}\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}\frac{h(x;k)}{\phi(x;k)}\overline{\phi}(x;p) \neq \frac{e^{i\delta(p)}\sin\delta(p)}{p}$$

Same $\delta(k)$ is obtained at only $p = k$, where $h(x;k)$ is defined.

Above dicussion corresponding to LO HALQCD method

Expansion of reduced BS wave function

Velocity expansion in HALQCD method

$$h(x;k) \;=\; \int d^3x' U(x;x')\phi(x';k)$$

Expansion of reduced BS wave function

Velocity expansion in HALQCD method

$$
\begin{aligned}
h(x;k) &= \int d^3x' U(x;x')\phi(x';k) \\
&= \sum_{n=0}^{\infty} V_n(x)\Delta^n \phi(x;k), \quad V_n(x) \text{ independent of } k
\end{aligned}
$$

[HALQCD, arXiv:1711.01883]

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[HALQCD, arXiv:1711.01883]

Velocity expansion expresses

$(\Delta + k^2)\phi(x;k)$ by k independent $V_n(x)$ with Δ^n.

Expansion of reduced BS wave function

[TY and Kuramashi, PRD96:114511,11(2017)]

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Convergence of expansion is unclear.

$\rightarrow$ Large number of terms would be necessary in general.

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Approximation with $V_0(x)$ and $V_1(x)$ assuming k independence

$h(x;k_1)$

$h(x;k_2)$

Expansion of reduced BS wave function

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$\rightarrow$ Large number of terms would be necessary in general.

Approximation with $V_0(x)$ and $V_1(x)$ assuming k independence

$$\begin{aligned} h(x;k_1) &= V_0(x)\phi(x;k_1) + V_1(x)\Delta\phi(x;k_1) \\ &= V_0(x)\phi(x;k_1) + V_1(x)(h(x;k_1) - k_1^2\phi(x;k_1)) \\ h(x;k_2) &= V_0(x)\phi(x;k_2) + V_1(x)(h(x;k_2) - k_2^2\phi(x;k_2)) \end{aligned}$$

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$$V_0(x) = \frac{k_1^2\phi(x;k_1)h(x;k_2) - k_2^2\phi(x;k_2)h(x;k_1)}{\phi(x;k_1)h(x;k_2) - \phi(x;k_2)h(x;k_1) + \phi(x;k_1)\phi(x;k_2)(k_1^2 - k_2^2)}$$

$$V_1(x) = \frac{\phi(x;k_1)h(x;k_2) - \phi(x;k_2)h(x;k_1)}{\phi(x;k_1)h(x;k_2) - \phi(x;k_2)h(x;k_1) + \phi(x;k_1)\phi(x;k_2)(k_1^2 - k_2^2)}$$

Expansion of reduced BS wave function

[TY and Kuramashi, PRD96:114511,11(2017)]

Velocity expansion in HALQCD method

$$h(x;k) = \sum_{n=0}^{\infty} V_n(x)\Delta^n \phi(x;k), \quad V_n(x) \text{ independent of } k$$

[HALQCD, arXiv:1711.01883]

Convergence of expansion is unclear.

$\rightarrow$ Large number of terms would be necessary in general.

Approximation with $V_0(x)$ and $V_1(x)$ assuming k independence

$$\begin{aligned} h(x;k_1) &= V_0(x)\phi(x;k_1) + V_1(x)\Delta\phi(x;k_1) \\ &= V_0(x)\phi(x;k_1) + V_1(x)(h(x;k_1) - k_1^2\phi(x;k_1)) \\ h(x;k_2) &= V_0(x)\phi(x;k_2) + V_1(x)(h(x;k_2) - k_2^2\phi(x;k_2)) \end{aligned}$$

$V_0(x) =$ … $\overline{\phi(x;k_1)h(x;k_2) - \phi(x;k_2)h(x;k_1) + \phi(x;k_1)\phi(x;k_2)(k_1^2 - k_2^2)}$

$V_0(x), V_1(x)$ depends on k_1, k_2 in general.

Inconsistent with assumption

$V_1(x) =$ … $\overline{\phi(x;k_1)h(x;k_2) - \phi(x;k_2)h(x;k_1) + \phi(x;k_1)\phi(x;k_2)(k_1^2 - k_2^2)}$

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Convergence of expansion is unclear.
$\rightarrow$ Large number of terms would be necessary in general.

Truncation of expansion could cause uncontrolled or hard to estimate systematic error.

Actual calculation of time dependent HALQCD method uses the truncated expansion.

Summary

Relation between BS wave function inside interaction range R
and half off-shell scattering amplitude $H(p;k)$

- Reduced BS wave function

$$h(x;k) = (\Delta + k^2)\phi(x;k), \quad h(x;k) = 0 \text{ in } R > x$$

- Simple relation between $H(p;k)$ and $h(x;k)$

$$H(p;k) = -\int d^3x\, e^{-i\vec{p}\cdot\vec{x}} h(x;k)$$

- LSZ reduction formula in relative coordinate

$$H(p;k) = (p^2 - k^2) \int d^3x\, e^{-i\vec{p}\cdot\vec{x}} \phi(x;k)$$

- Both formulae can be used on finite volume calculation.

- It may be possible to derive similar relations in more than two particles.

Summary

$\overline{\delta}(p)$ from Shrödinger equation with $h(x;k)/\phi(x;k)$

$$\frac{e^{i\overline{\delta}(p)}\sin\overline{\delta}(p)}{p} = -\frac{1}{4\pi}\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}\frac{h(x;k)}{\phi(x;k)}\overline{\phi}(x;p)$$

• At $p = k$, $\overline{\delta}(k) = \delta(k)$, but at $p \neq k$, $\overline{\delta}(p) \neq \delta(p)$.

Velocity expansion of $h(x;k)$

$$h(x;k) = \sum_{n=0}^{\infty} V_n(x)\Delta^n\phi(x;k), \quad V_n(x) \text{ independent of } k$$

• $V_n(x)$ depends on k truncated by finite n terms.

• Truncated expansion could cause hard to estimate systematic errors, which used in calculation of time dependent HALQCD method.

Back up

Plateau in two-nucleon channel

To understand current situation in two-nucleon calculation, variational method is necessary.

It would be useful to understand source dependence as a pilot study towards variational calculation.

$$N_f = 0\ m_\pi = 0.8\ \text{GeV},\ {}^3\text{S}_1\ \text{channel},\ L = 16, 20, 32$$

Exponential smear

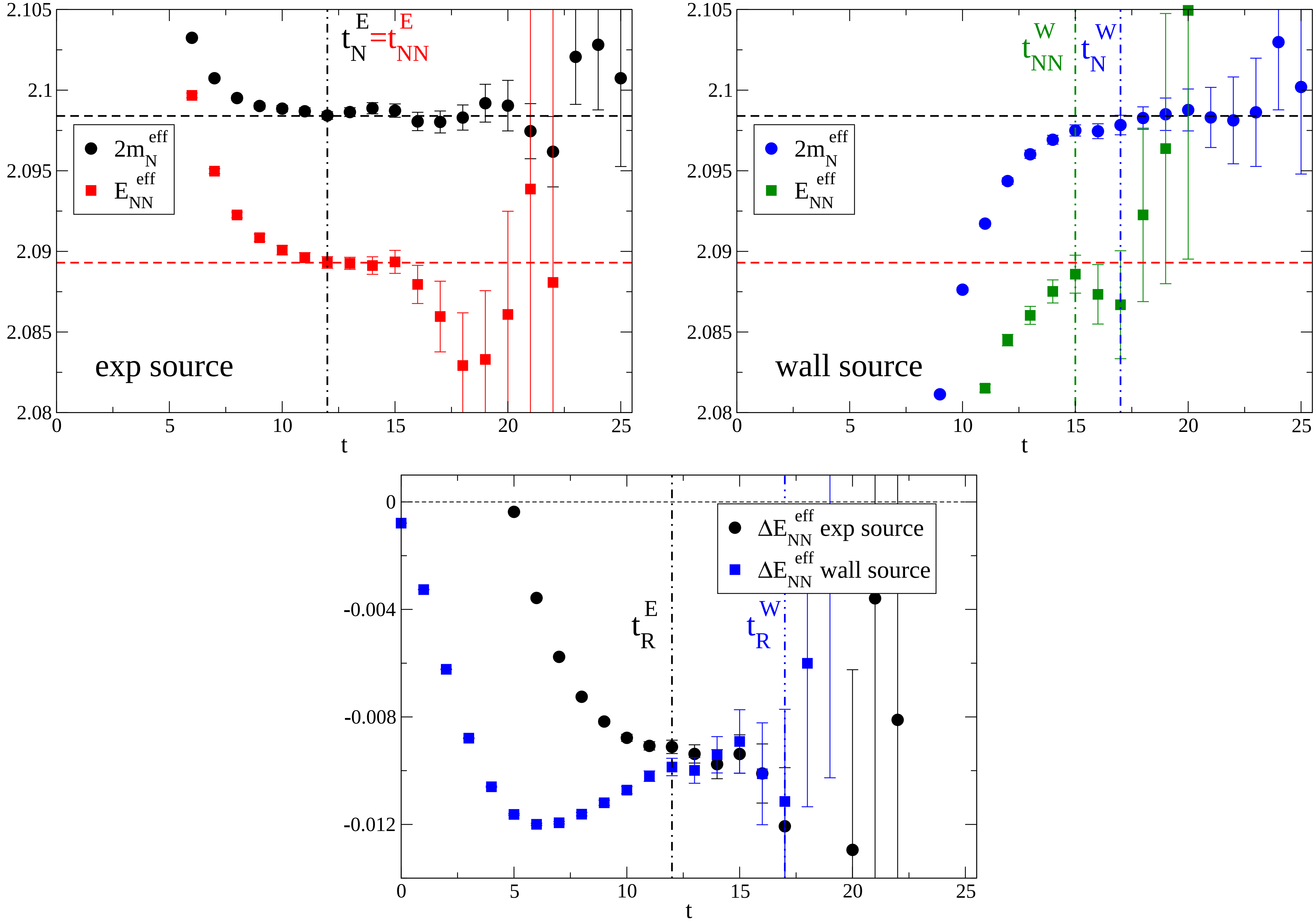
$$C^E_{NN}(t) = A^E_B\, e^{-2m_N t}\left(e^{-\Delta_B t} + B^E_{NN} e^{-\Delta_{NN} t} + \cdots\right)$$

Wall source

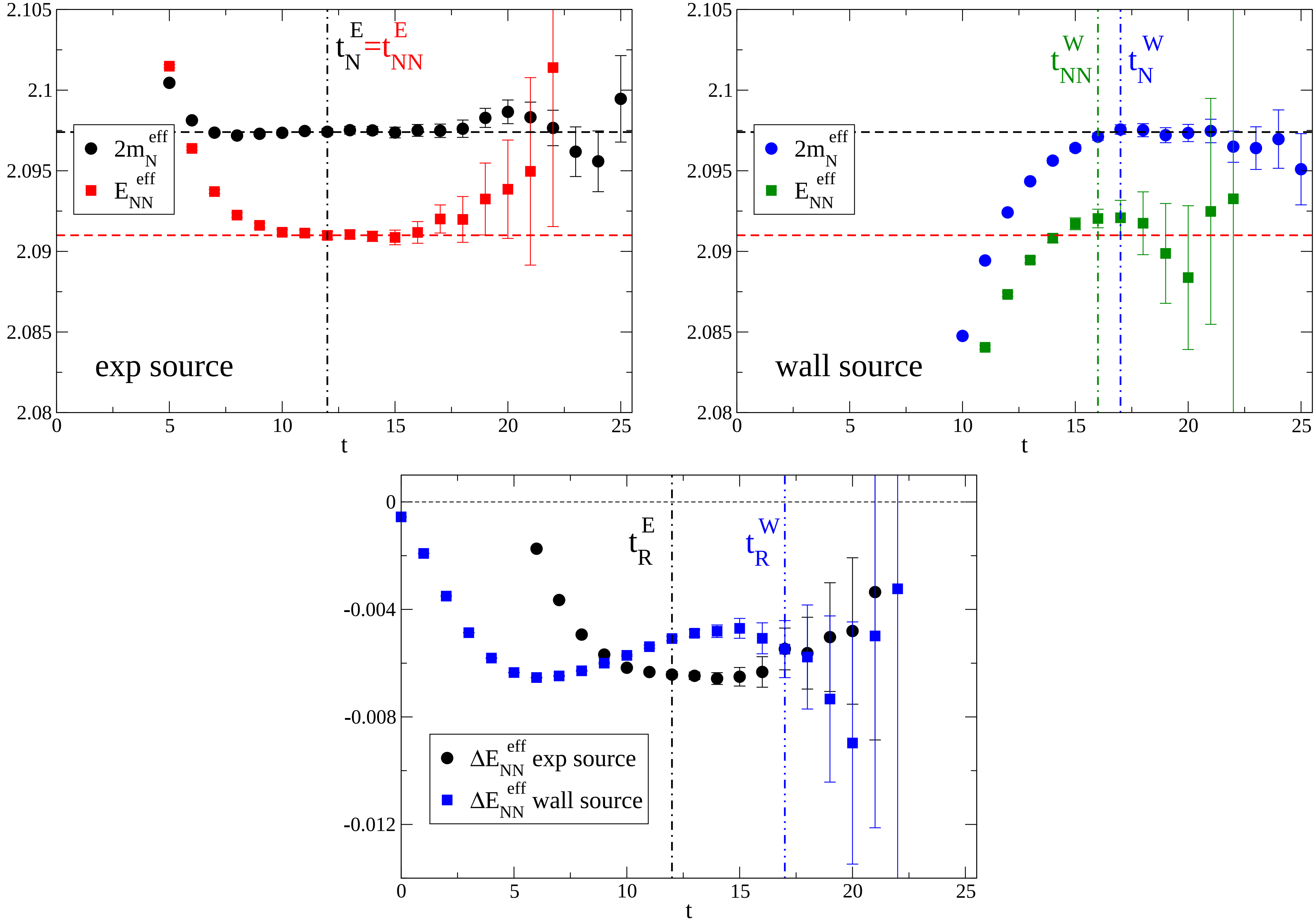
$$C^W_{NN}(t) = A^W_B\, e^{-2m_N t}\left(e^{-\Delta_B t} + B^W_{NN} L^\alpha e^{-\Delta_{NN} t} + \cdots\right)$$

$$\Delta_B < 0,\ \Delta_{NN} \propto 1/L^3 > 0,\ \alpha > 0$$

$R(t) = C_{NN}(t)/(C_N(t))^2$ in $L = 16$



$R(t) = C_{NN}(t)/(C_N(t))^2$ in $L = 20$



$R(t) = C_{NN}(t)/(C_N(t))^2$ in $L = 32$