INT Workshop INT-18-70W

Multi-Hadron Systems from Lattice QCD

February 5 - 9, 2018

Workshop goals, and introduction to Lüscher formalism for two particles

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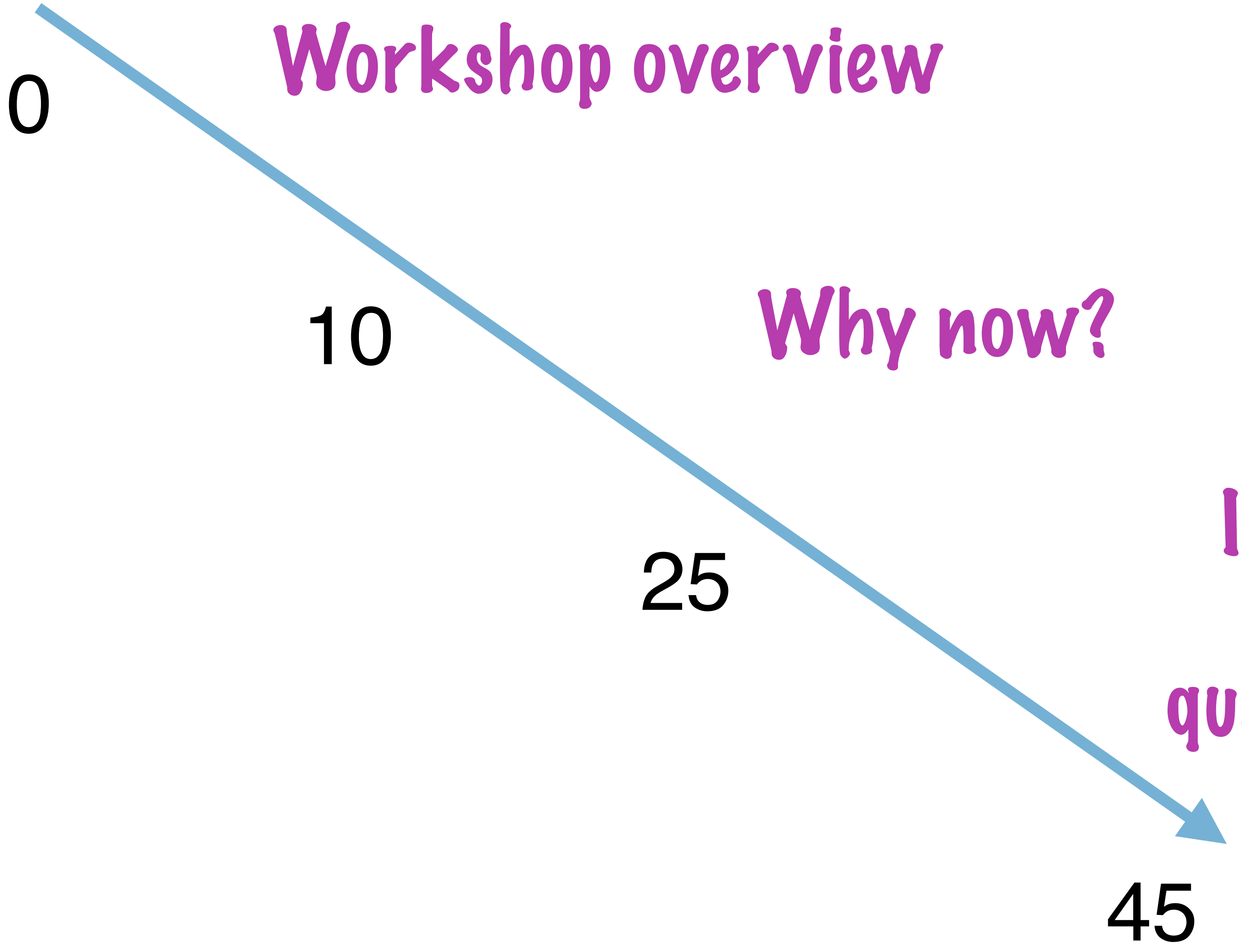
- All practical information should be in your packet
- Shared offices are C441, B470 and B474
- These rooms cannot be locked so we recommend not leaving personal belongings in the office after use
- Please send PDFs of talks to Cheryl McDaniel: chermcd@uw.edu

Outline



0 Workshop overview

10 Why now?

25 Introduction to the two-particle quantization condition

45

Overarching goals

- To clarify the landscape of methods for extracting multi-hadron observables from LQCD
- To bridge the gap between LQCD approaches and other techniques:
    - Effective field theories
    - Dispersive and amplitude analysis
    - Dyson-Schwinger equations
    - and other few-body methods….

Discussions will be key to the success of the workshop

We have moderated discussion periods at the end of most morning and afternoon sessions

Workshop outline

- Monday AM: Overview, motivation & theoretical methods for two-particle systems
- Monday PM: Lattice results for two-particle systems
- Tuesday AM: Dispersive approach to three-body physics
- Tuesday PM: Three particle quantization conditions
- Wednesday AM: Multiple baryons, part 1
- Wednesday PM: Multiple baryons, part 2
    - WORKSHOP DINNER
- Thursday AM: Multiple baryons, part 3
- Thursday PM: Alternative methods for multiple-baryon systems
- Friday A ak multihadron physics
- Friday P ide talks & Summary discussion

PUB CRAWL?!

Moderated discussions

- Monday PM: Lattice results for two-particle systems—JOHN BULAVA
- Tuesday AM: Dispersive approach to three-body physics—ADAM SZCZEPANIAK
- Tuesday PM: Three particle quantization conditions—MICHAEL DÖRING
- Wednesday AM: Multiple baryons, part 1—DEAN LEE
- Wednesday PM: Multiple baryons, part 2—MAX HANSEN
- Thursday AM: Multiple baryons, part 3—ANDRE WALKER-LOUD
- Thursday PM: Alternative methods for multiple-baryon systems—JOSE PELAEZ
- Friday AM: Electroweak multihadron physics—RAÚL BRICEÑO

If you want to show a couple of slides in a discussion session let the moderator know

5-slide talks on Friday PM

- For relevant topics that we could not fit in to the schedule
- And for ideas/comments that come up and don't make it into discussions
- Or your attempt to summarize some part of the workshop
- 5 slides means 5 slides—BEWARE!—Intro + 3 slides of results + Outlook
- 10 mins + 5 for discussion
- Let the organizers know if you want to give such a talk (so far we have 4, with room for a couple more)
- Schedule announced on Friday morning

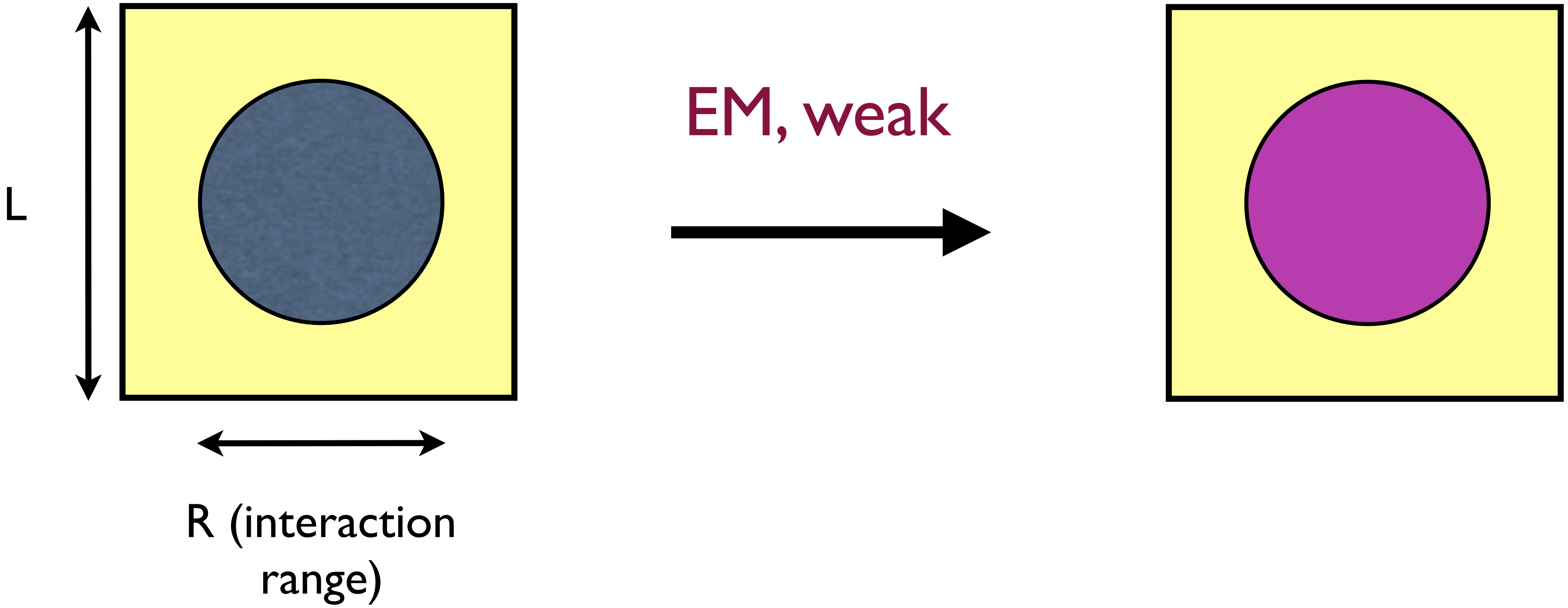
Some questions to answer

- Can LQCD calculate the finite-volume spectrum in the multihadron regime for physical quark masses?
- What can we learn about multihadron physics from results at heavier than physical quark masses?
- What is the best way (or ways) to relate the 3-particle spectrum in finite volume to physical quantities? Or should we use Bethe-Salpeter amplitudes?
- What are the best physical quantities to aim to calculate in order to connect to, or supplement, experimental results? I.E. How can we make a real impact?
- How can we combine the knowledge from EFTs, analyticity & unitarity with LQCD results in the most effective way?
- How can QED effects be included in quantization conditions?
- Can the 3-particle methods be generalized to 4+ particles, or do we need a different approach?
- …

Why this workshop is timely

Well-controlled LQCD calculations

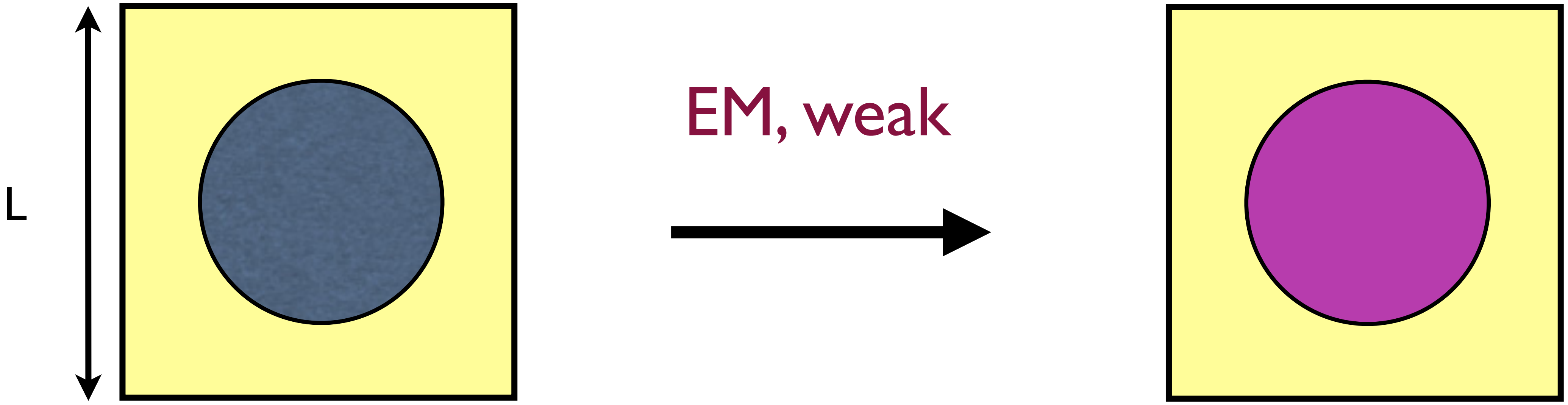
- Single particle masses and matrix elements



For large enough boxes (L>2R) dominant finite-volume effects for single-particle states fall as exp(-M_πL) [Lüscher 86,91] and can be made small

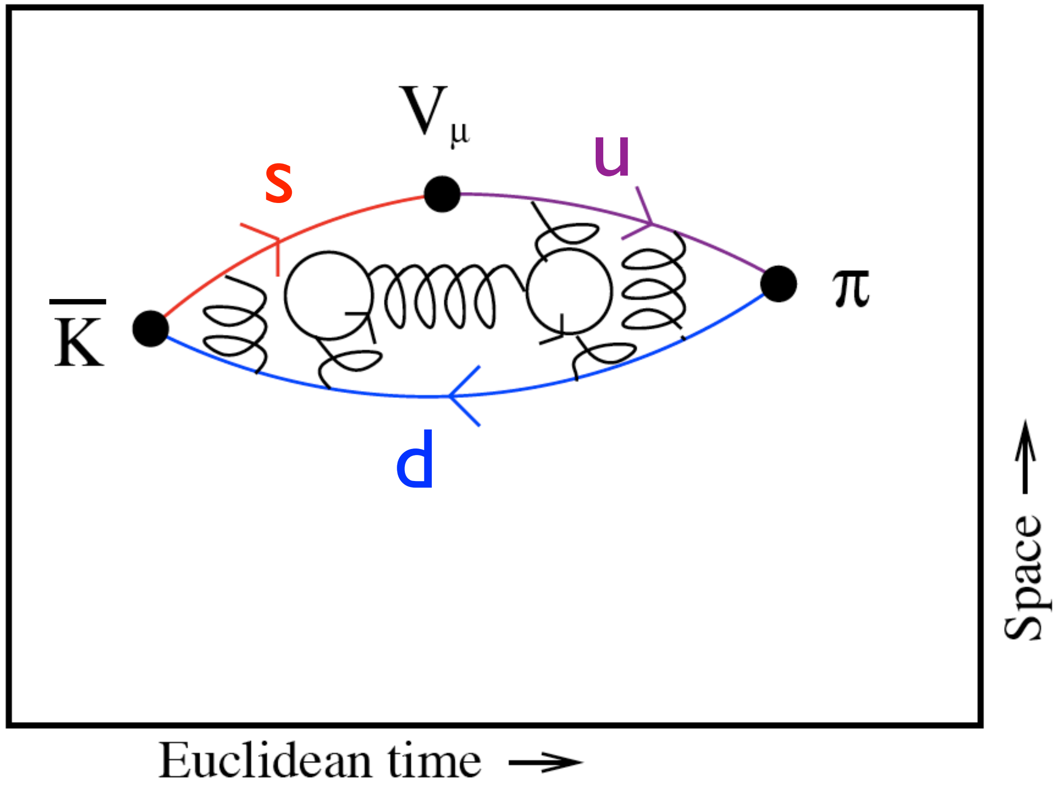
Well-controlled LQCD calculations

- Single particle masses and matrix elements

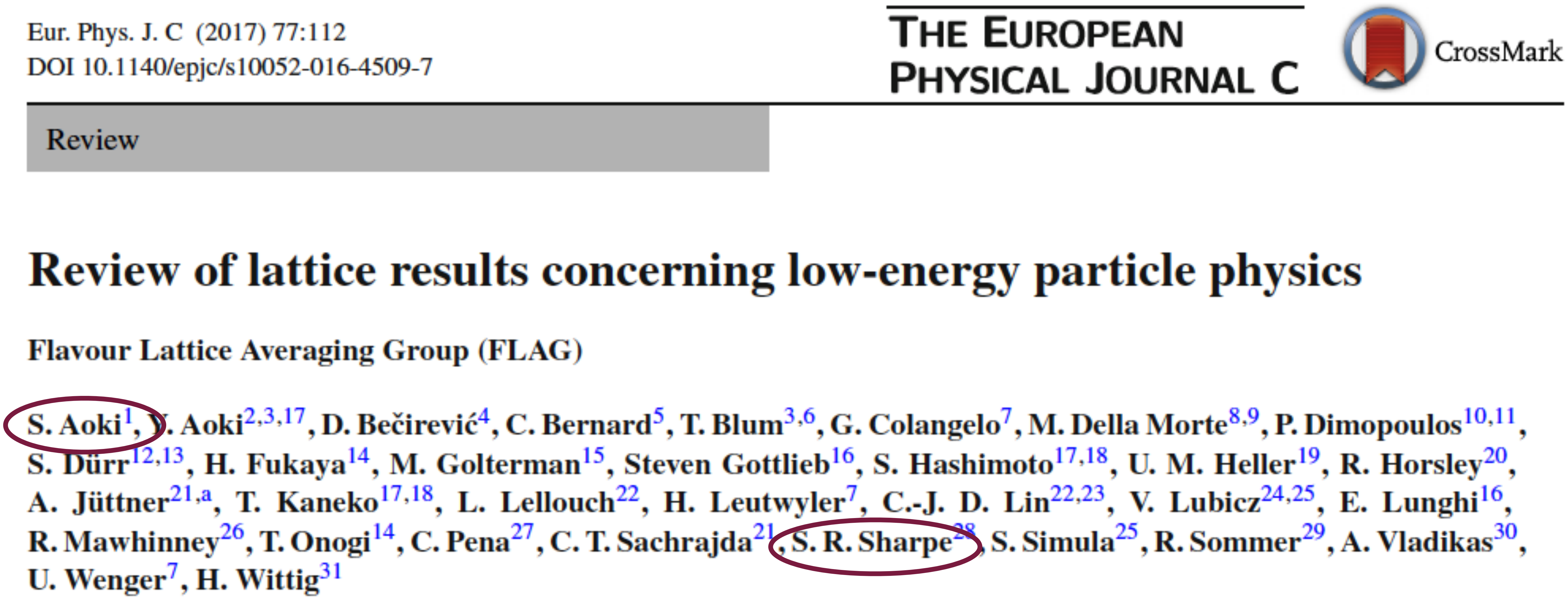


Example:
K→π form factor

$$\langle \pi(\vec{p}_2) | V_\mu(0) | \overline{K}(\vec{p}_1) \rangle$$



Flavo(u)r Lattice Averaging Group



Eur. Phys. J. C (2017) 77:112
DOI 10.1140/epjc/s10052-016-4509-7

THE EUROPEAN
PHYSICAL JOURNAL C

Review

Review of lattice results concerning low-energy particle physics

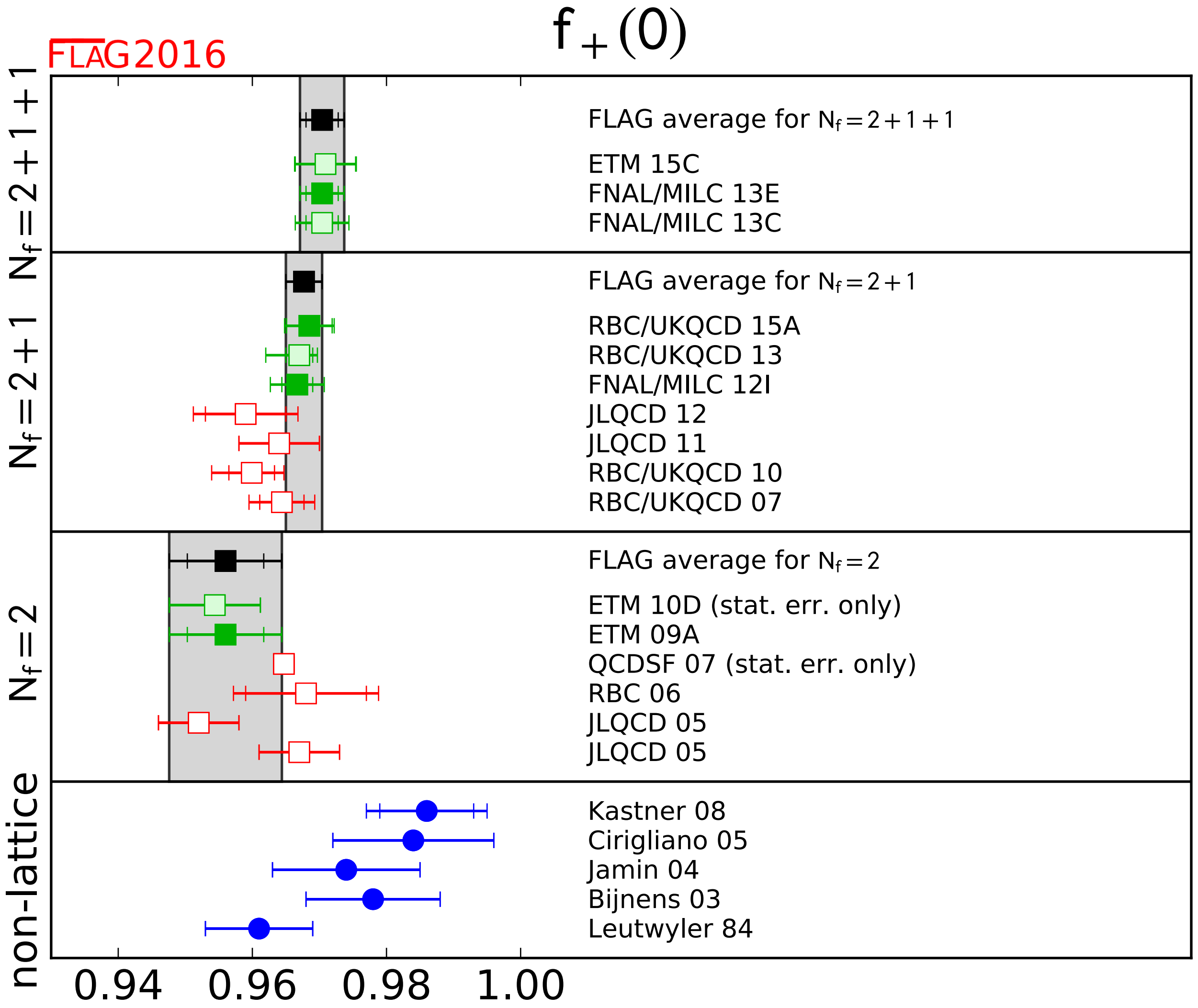
Flavour Lattice Averaging Group (FLAG)

S. Aoki[1], Y. Aoki[2,3,17], D. Bečirević[4], C. Bernard[5], T. Blum[3,6], G. Colangelo[7], M. Della Morte[8,9], P. Dimopoulos[10,11], S. Dürr[12,13], H. Fukaya[14], M. Golterman[15], Steven Gottlieb[16], S. Hashimoto[17,18], U. M. Heller[19], R. Horsley[20], A. Jüttner[21,a], T. Kaneko[17,18], L. Lellouch[22], H. Leutwyler[7], C.-J. D. Lin[22,23], V. Lubicz[24,25], E. Lunghi[16], R. Mawhinney[26], T. Onogi[14], C. Pena[27], C. T. Sachrajda[21], S. R. Sharpe[28], S. Simula[25], R. Sommer[29], A. Vladikas[30], U. Wenger[7], H. Wittig[31]

- Next FLAG review (2019) will include simple nuclear matrix elements

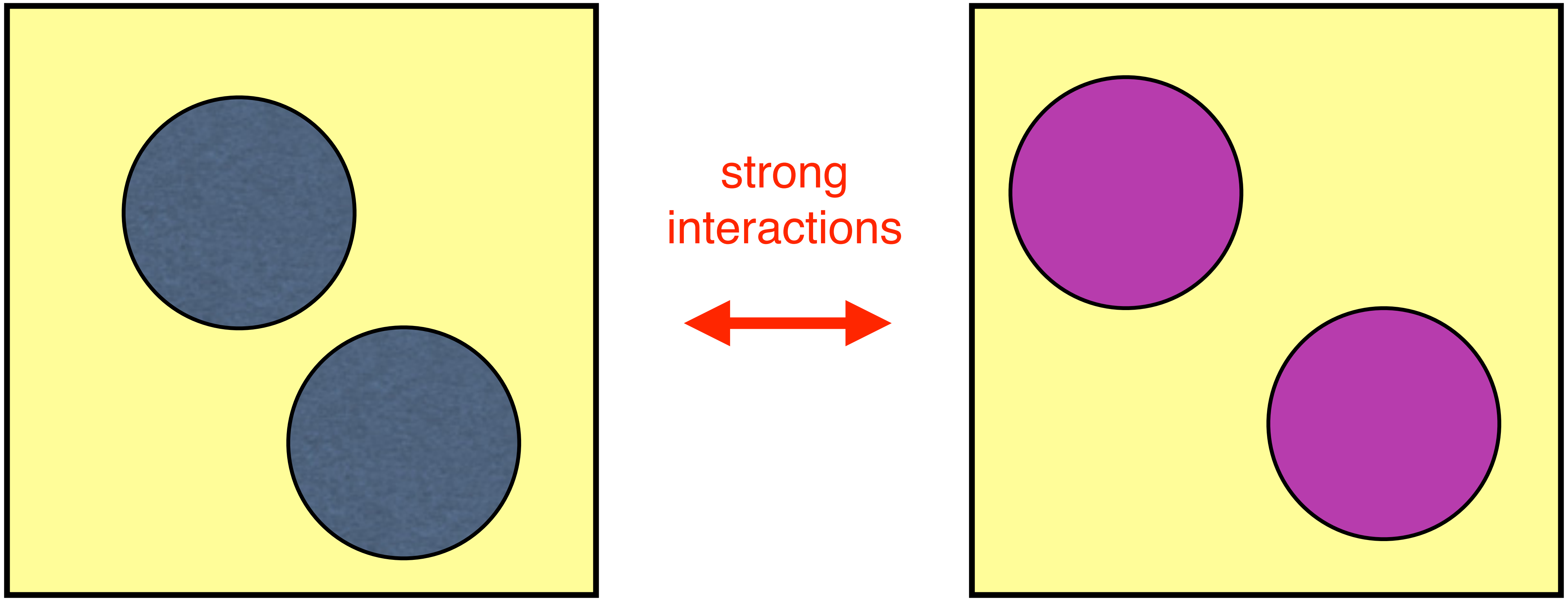
Well-controlled LQCD calculations

- Example from FLAG16: K→π form factor



S. Sharpe, "Workshop goals …" 2/5/18, INT, Seattle

Present Frontier (i)

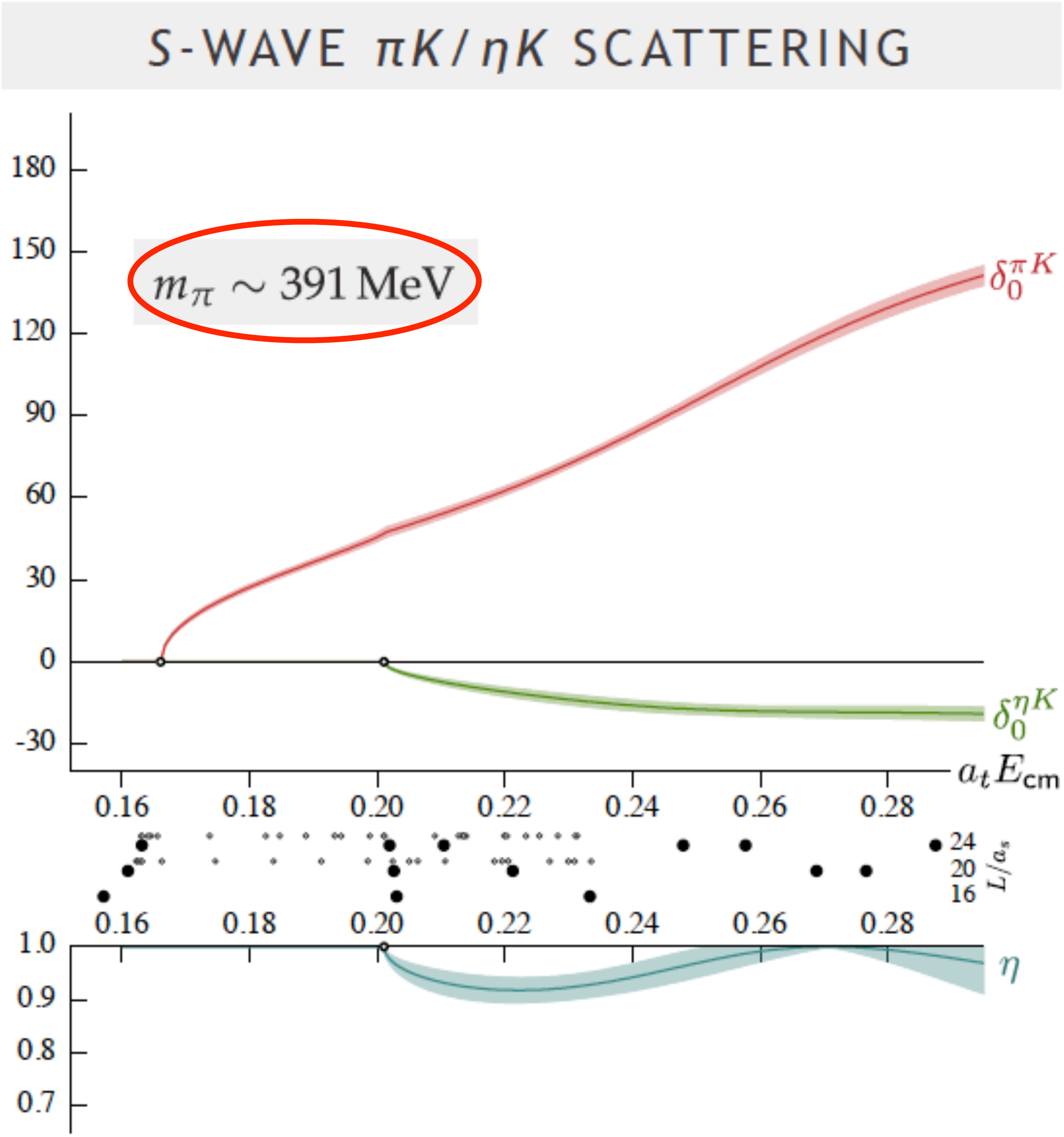


strong interactions

e.g. $\pi K \leftrightarrow \eta K$, $\pi\pi \leftrightarrow \bar{K}K$

- Issues associated with 2 particles ($1/L^n$ finite-volume effects,…) are theoretically understood [Lüscher, …]
- Can extract scattering amplitudes—infinite-volume quantities—although parametrizations are needed and must truncate in angular momentum
- Numerical implementations expanding rapidly despite computational challenges
- Easier with mesons than with baryons, although HALQCD studies baryons with near physical quark masses

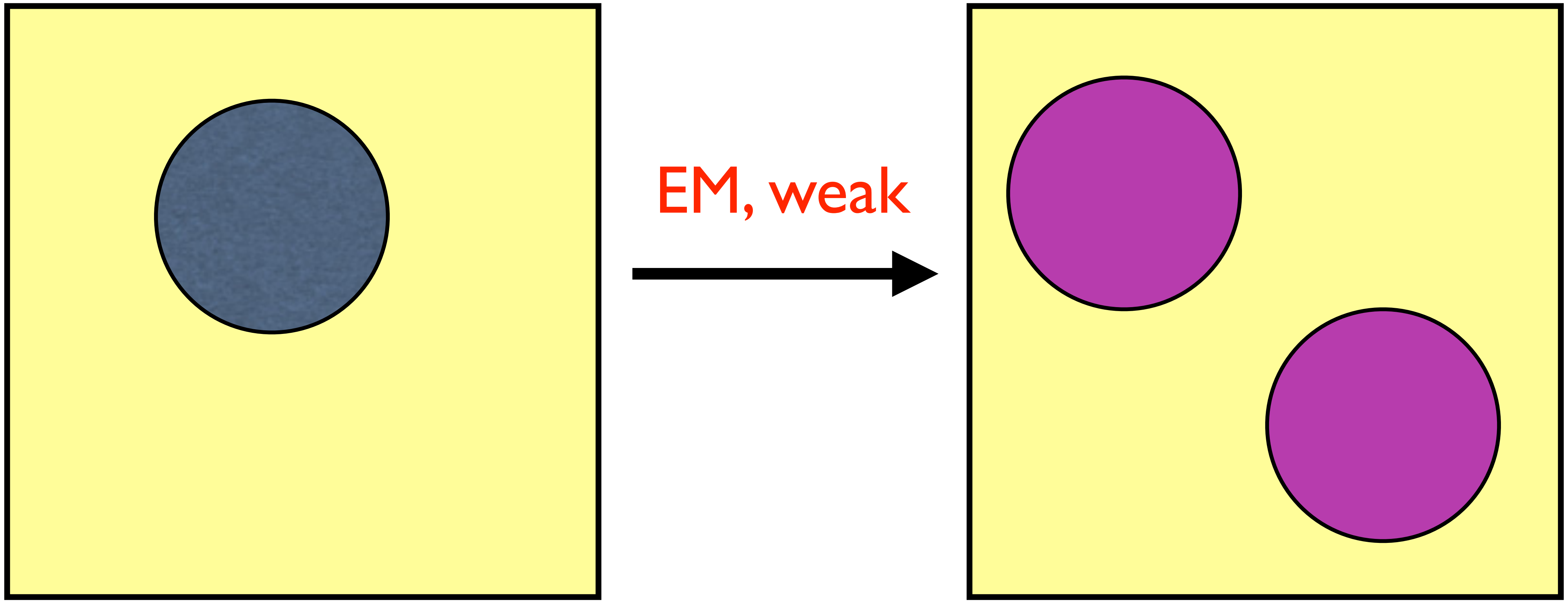
Present frontier (i)



[Dudek, Edwards, Thomas & Wilson arXiv:1406.4158]

- Theory for multiple two-particle channels [He, Feng, Liu; Bernard, …, Rusetsky; Briceño & Davoudi; Hansen & SRS]

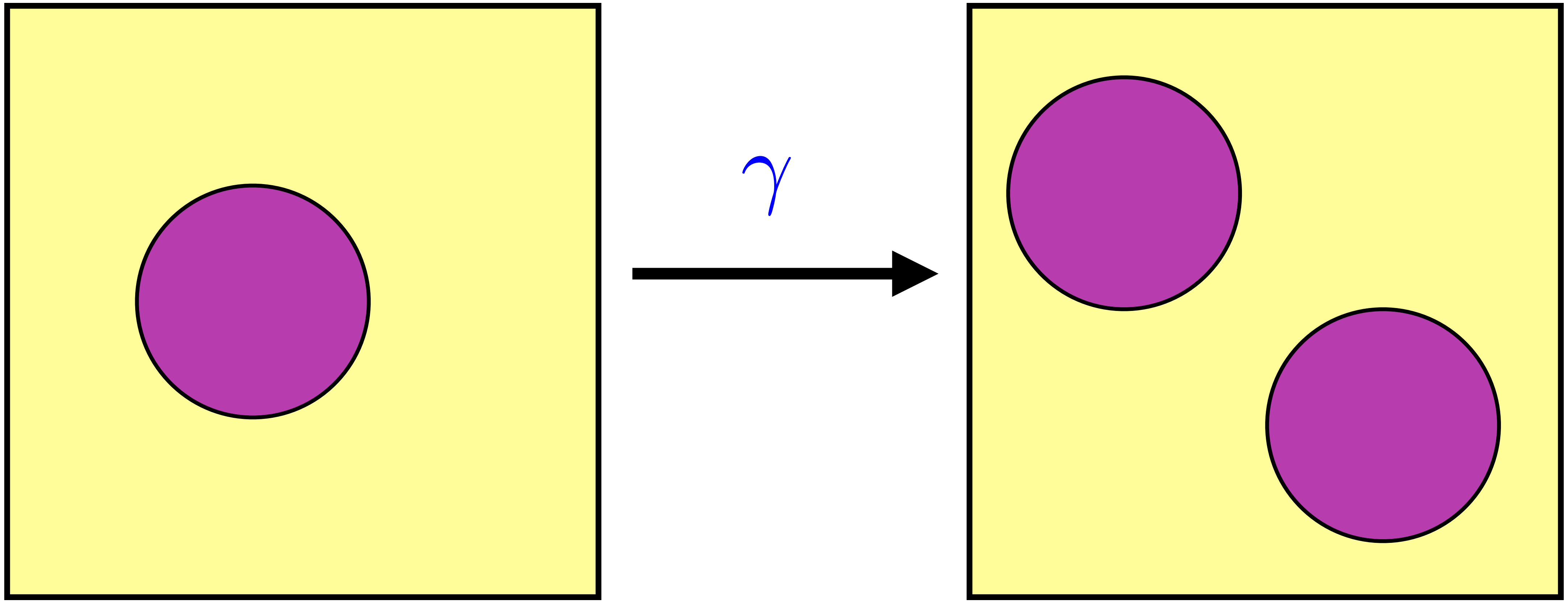
Present frontier (ii)



e.g. K→ππ decay amplitudes

- Issues associated with 2 particles ($1/L^n$ finite-volume effects,…) are theoretically understood [Lellouch & Lüscher, …]
- First lattice results obtained for decay rates (consistent with ΔI=½ rule) and preliminary results for ε′/ε [RBC/UKQCD]
- How do we include QED corrections? [Talk by Feng]

Present frontier (iii)

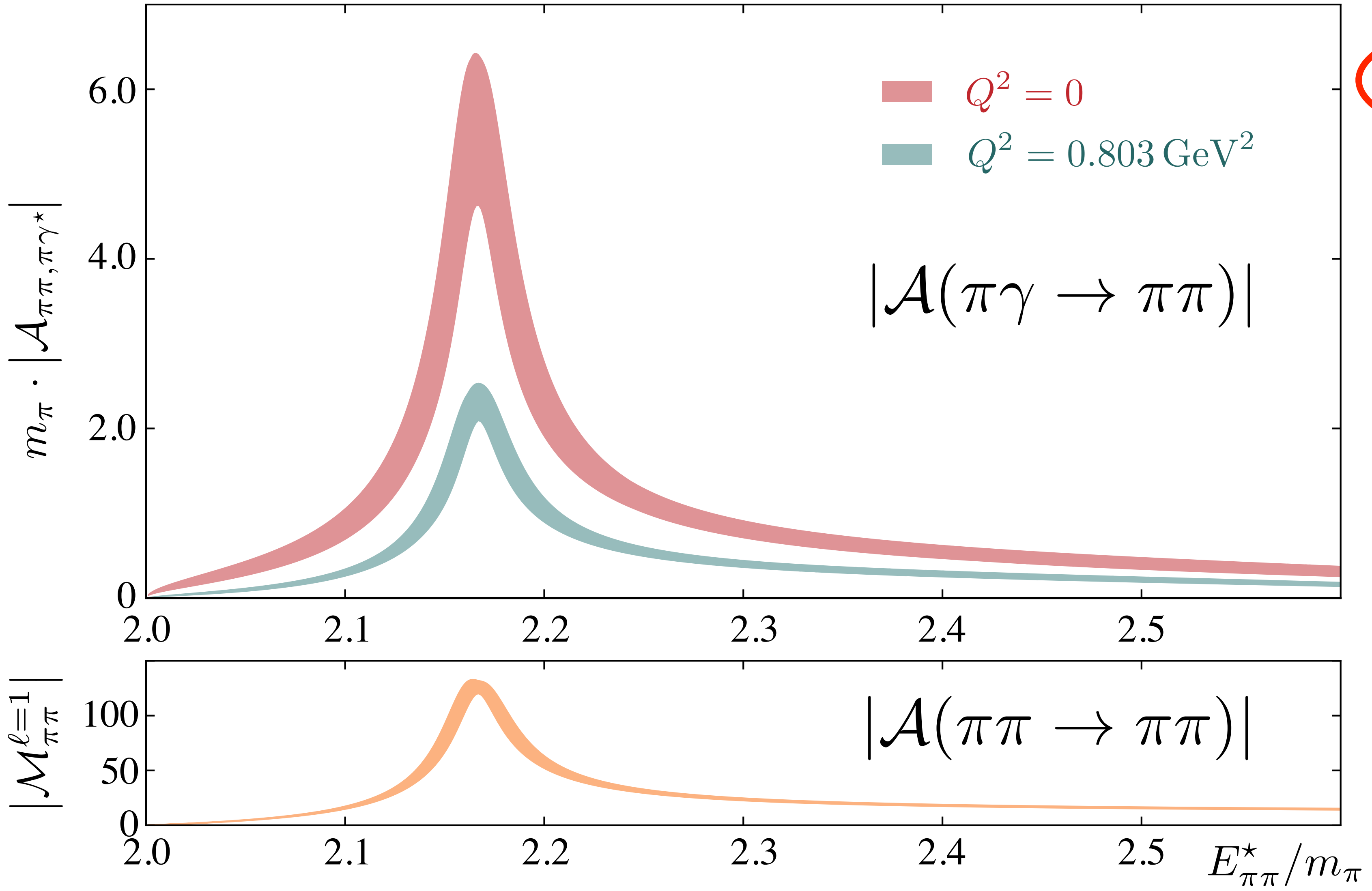


e.g. $\pi\gamma \rightarrow \rho$ amplitude

- Issues associated with 2 particles (1/L^n finite-volume effects,…) are theoretically understood [Bernard, …, Rusetsky; Briceño, Hansen & Walker-Loud, …]

Present frontier (iii)

$$\pi\gamma \to \rho$$

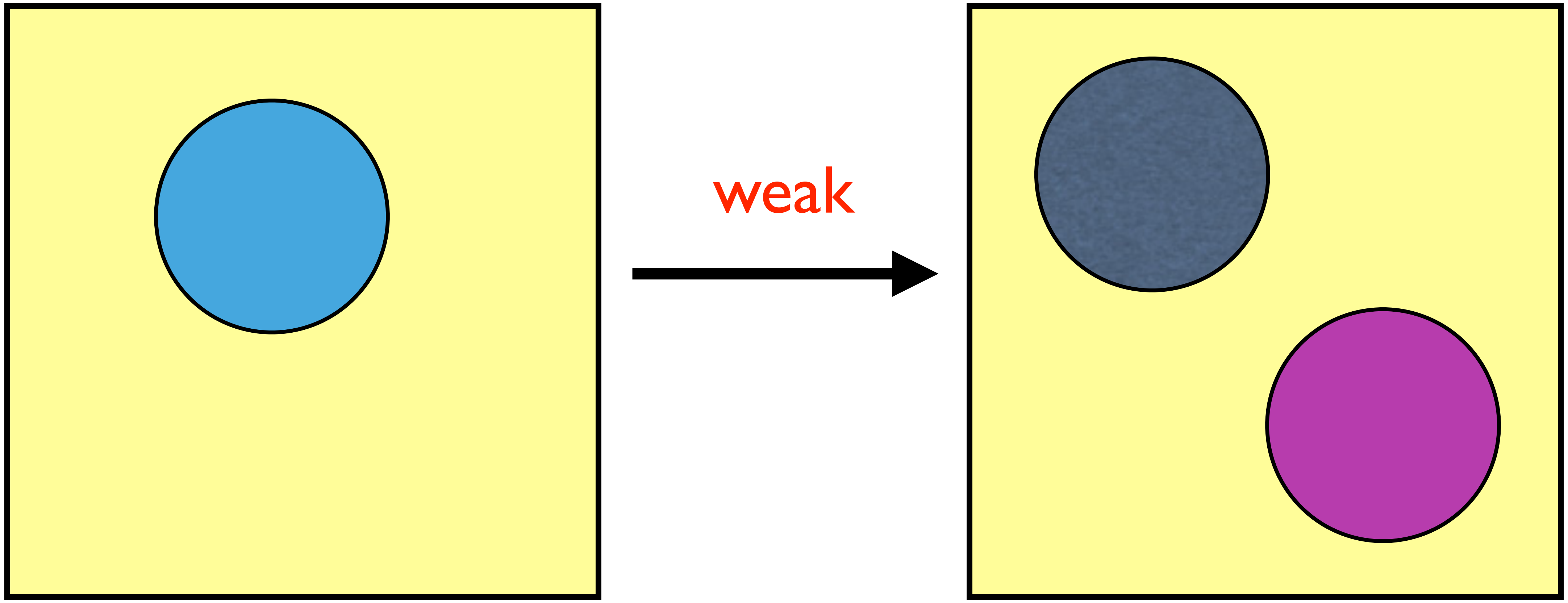


$m_\pi \approx 390$ MeV

Briceño, Dudek, Edwards, Shultz, Thomas, Wilson [HadSpec 1604.03530]

- Results also from [Leskovec, …, Meinel, …., arXiv:1611:00282]

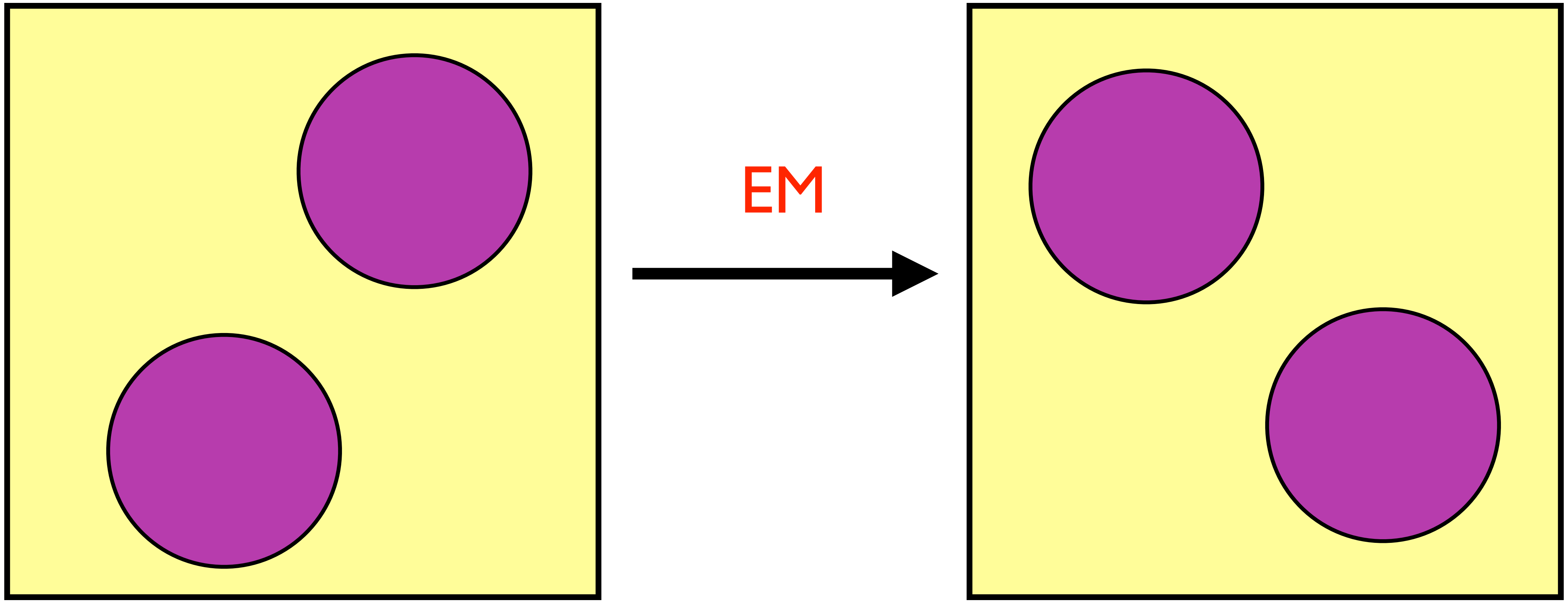
Present frontier (iv)



e.g. $B\to K^*\, l\, \nu \to K\, \pi\, l\, \nu$ decay amplitude

- Issues associated with 2 particles ($1/L^n$ finite-volume effects,…) are theoretically understood [Bernard, …, Rusetksy; Briceño, Hansen & Walker-Loud; …]
- Calculations underway [Talk by Luka Leskovec]

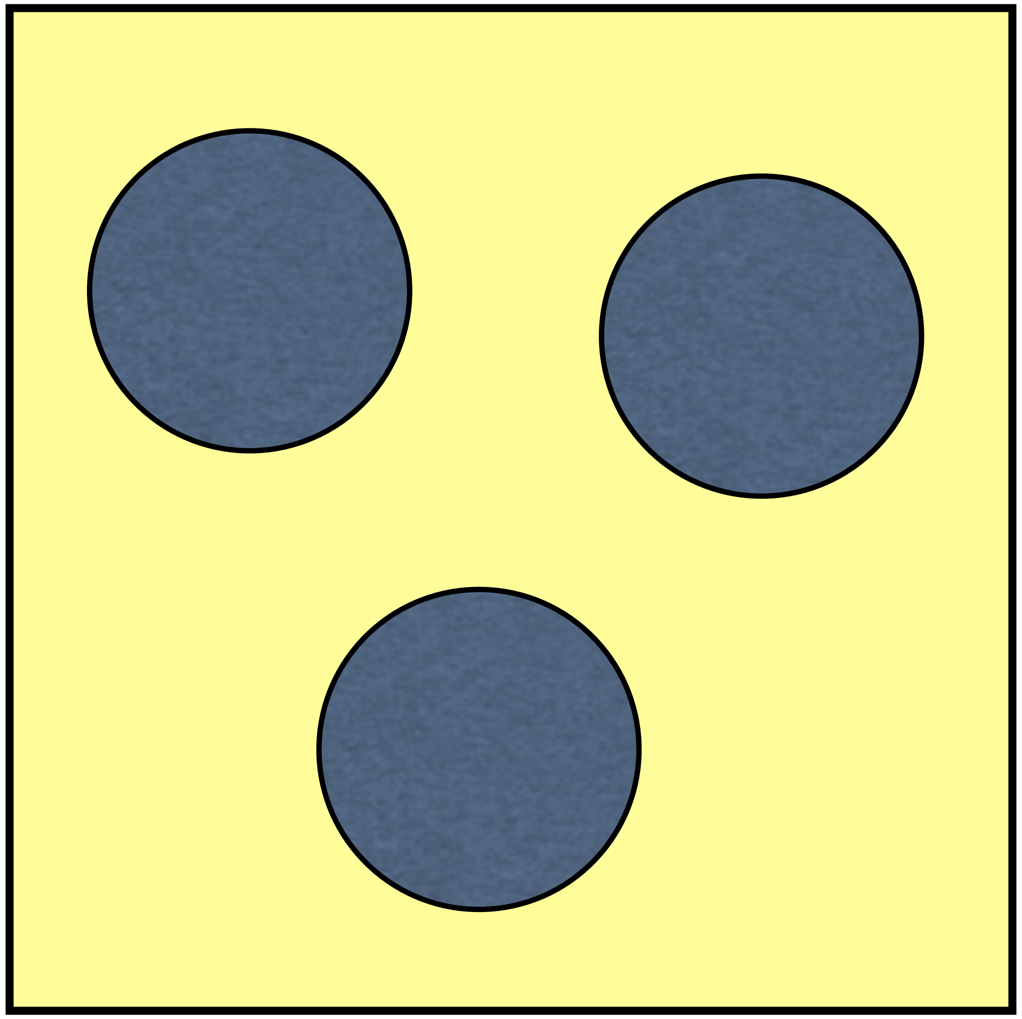
Present frontier (v)



e.g. ``ρ" form factor

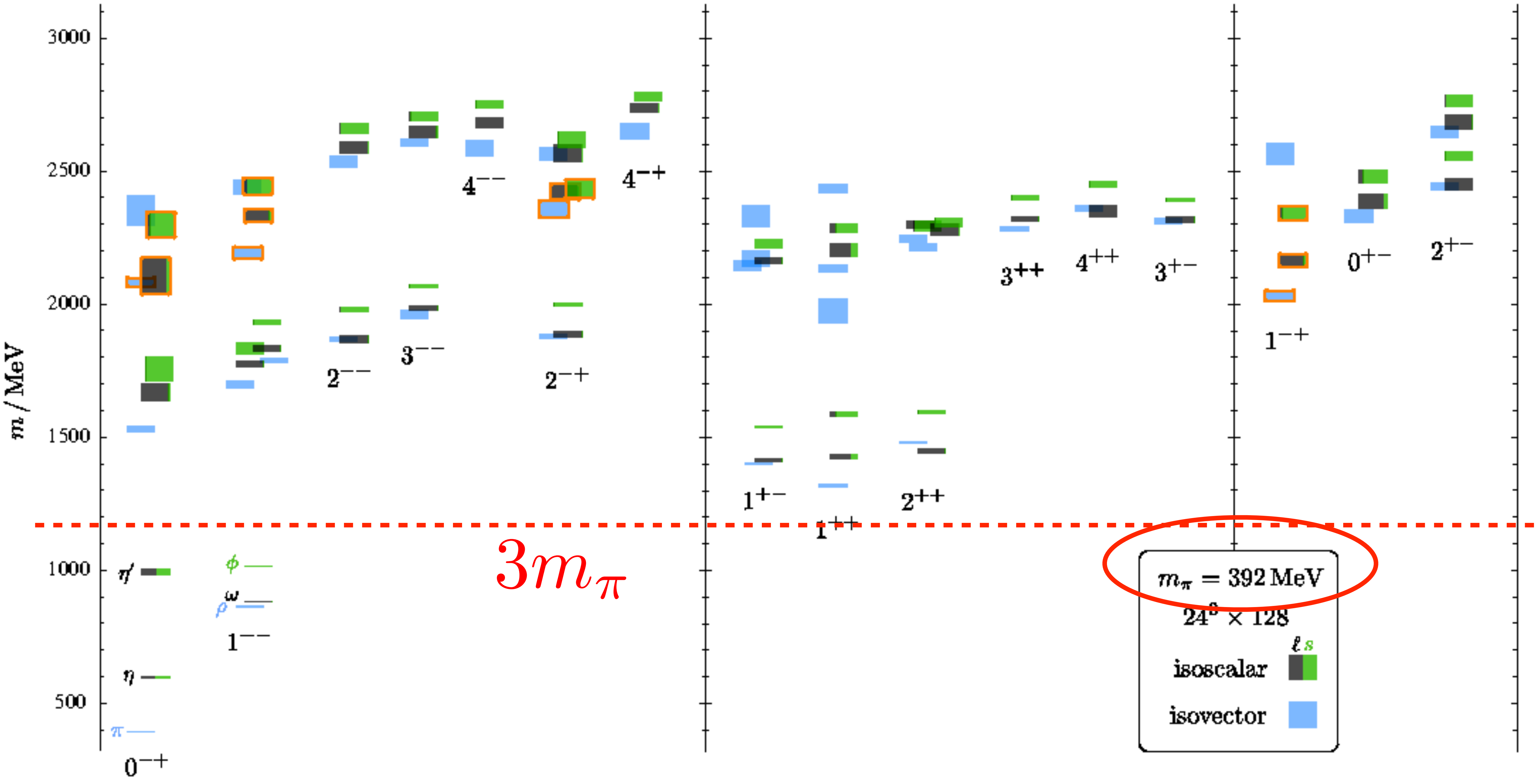
- Issues associated with 2 particles (1/L^n finite-volume effects,…) are theoretically understood [Briceño & Davoudi; Bernard, …, Rusetsky; Briceño & Hansen]
- Not yet implemented in simulations

Just beyond the frontier



- Simulations already have good results in the three-particle region of the spectrum (at least for mesons, and for unphysically heavy quark masses)
- How do we use these results? [Tuesday PM talks]

Energy levels in 3-particle regime



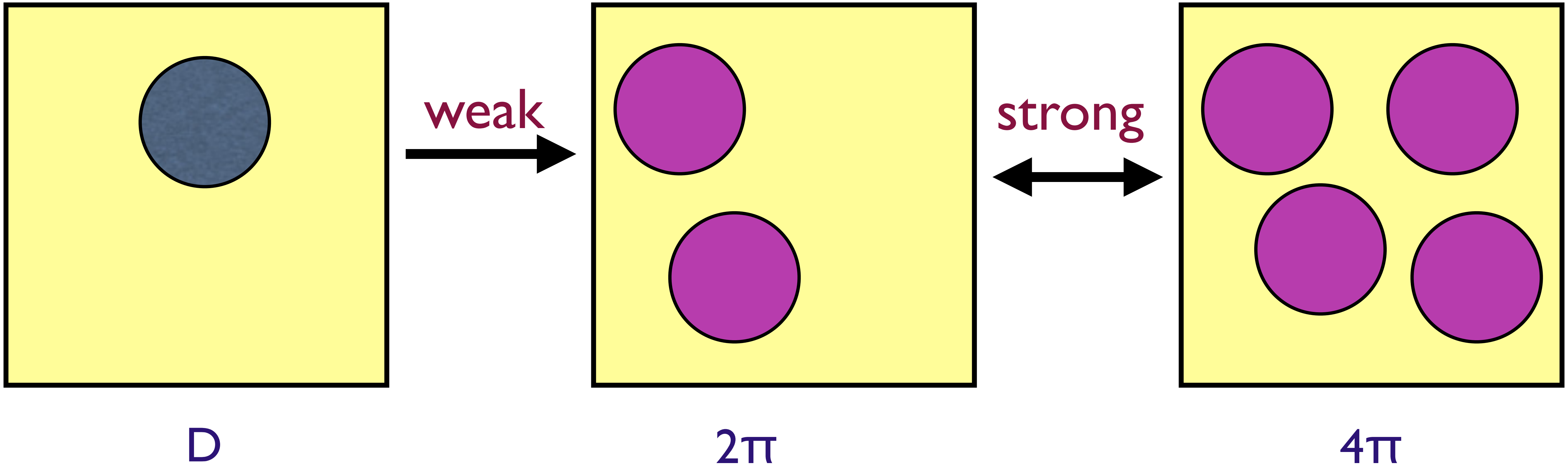
Dudek, Edwards, Guo & C.Thomas [HadSpec], arXiv:1309.2608

What can we learn from 3-particle regime?

- Understand resonances from first principles
  - e.g. $\omega(782) \to \pi\pi\pi \qquad N(1440) \to N\pi\pi$
- Electroweak decays into three particles
  - e.g. K→πππ
- NNN interaction
  - Needed for EFT treatments of larger nuclei & nuclear matter
- πππ, $\pi K\bar{K}$, … interactions
  - Needed for studying pion/kaon condensation

A more distant motivation

- Calculating CP-violation in D→ππ, $K\overline{K}$ in the Standard Model
- Finite-volume state is a mix of 2π, $K\overline{K}$, ηη, 4π, 6π, …
- Need 4 (or more) particles in the box!

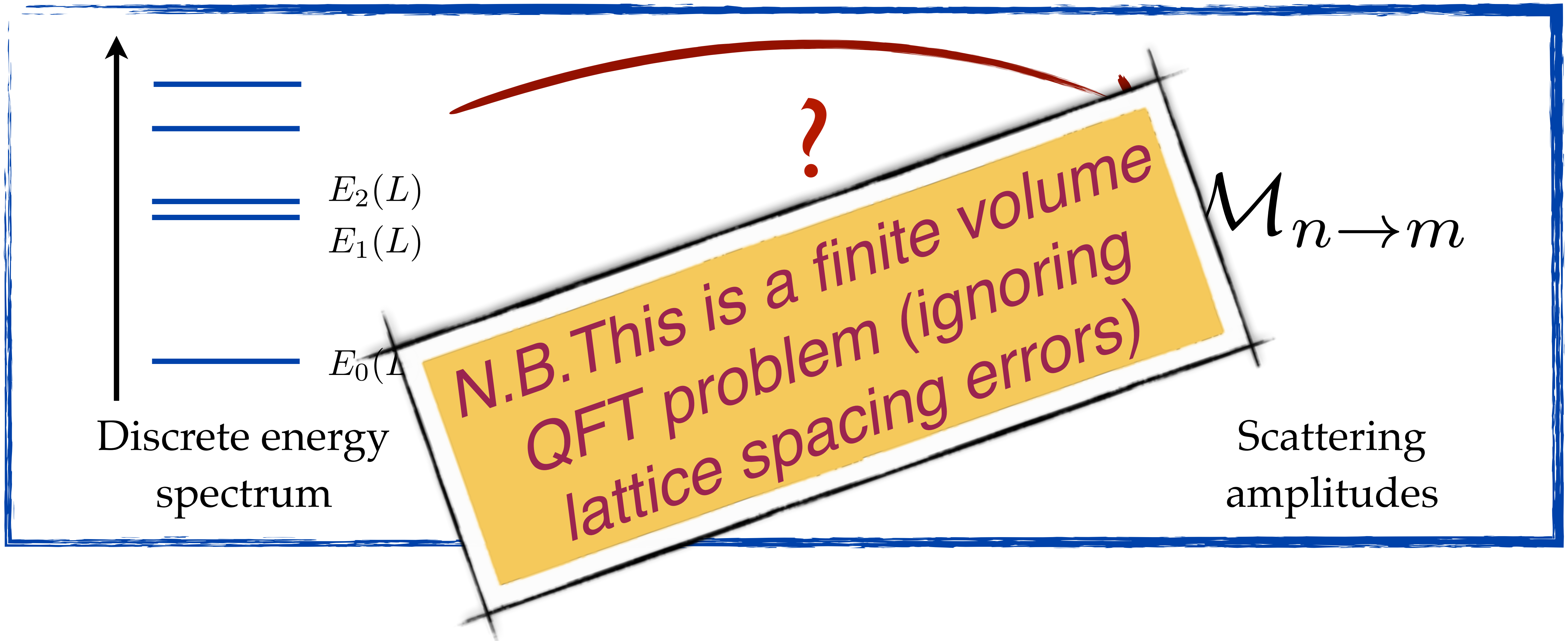


Introduction to the two-particle quantization condition

Seminal work by M. Lüscher, 1986, 1991
Many extensions and generalizations since

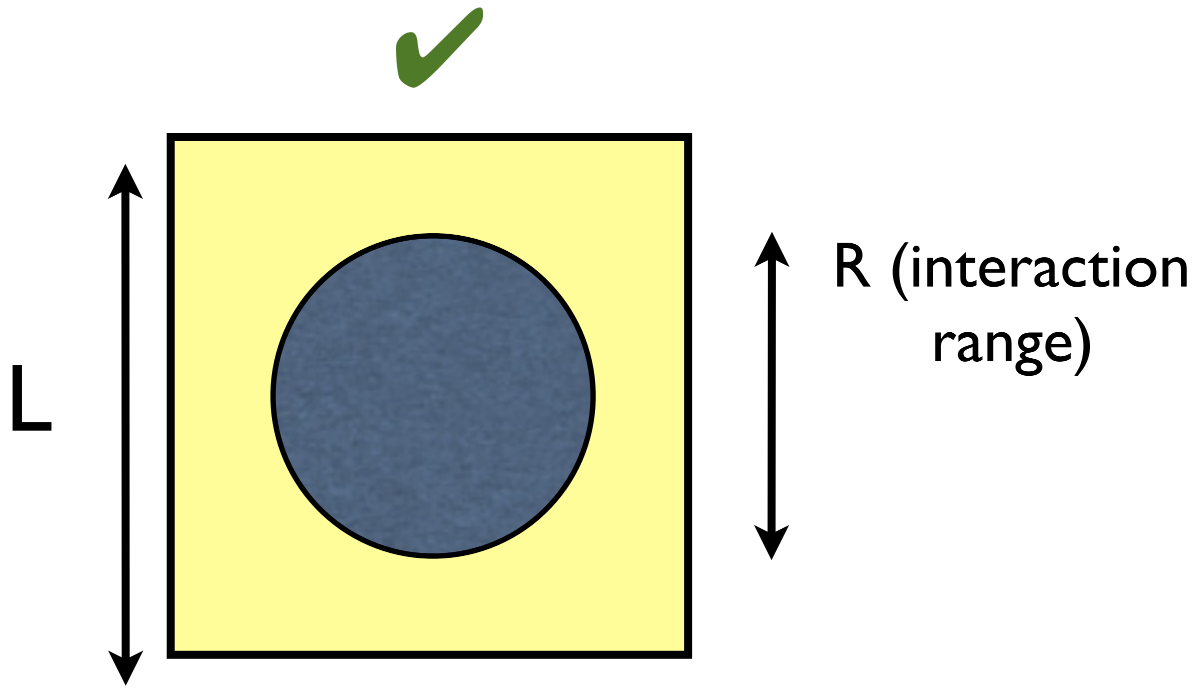
The fundamental issue

- Lattice QCD can calculate energy levels of multiparticle systems in a box
- How are these related to infinite-volume scattering amplitudes (which determine resonance properties)?

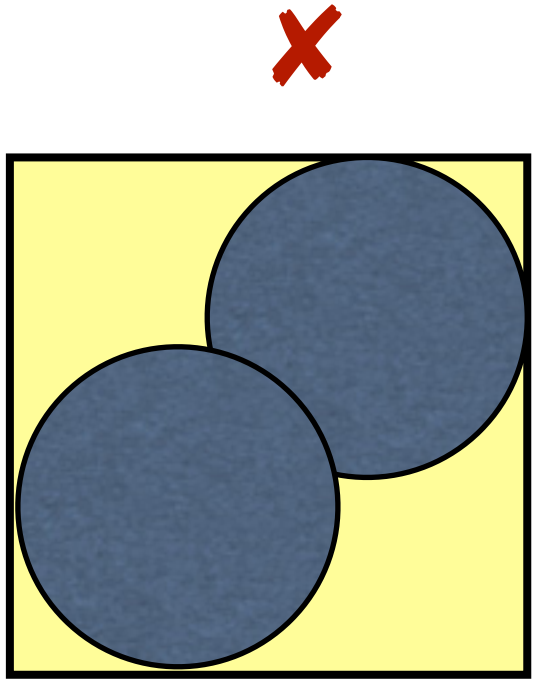


N.B. This is a finite volume QFT problem (ignoring lattice spacing errors)

When is the spectrum related to scattering amplitudes?

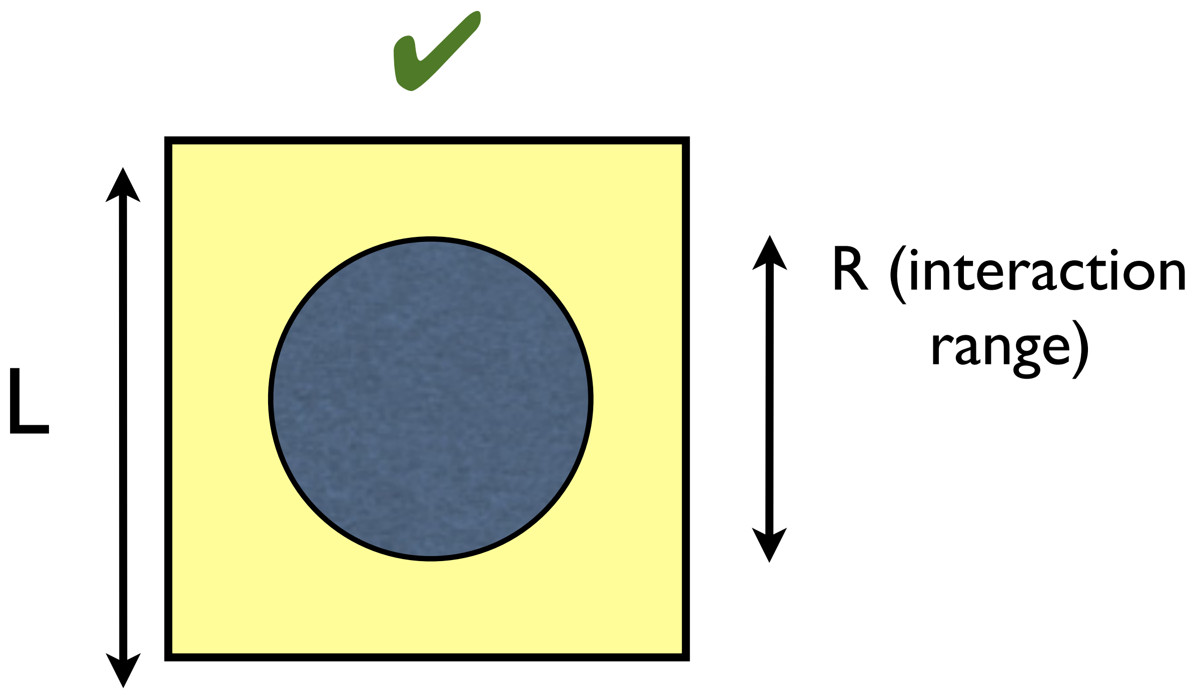


Single (stable) particle with L>R
Particle not "squeezed"
Spectrum same as in infinite volume up to corrections proportional to $e^{-M_\pi L}$
[Lüscher]



L<2R
No "outside" region.
Spectrum NOT related to scatt. amps.
Depends on finite-density properties

When is the spectrum related to scattering amplitudes?

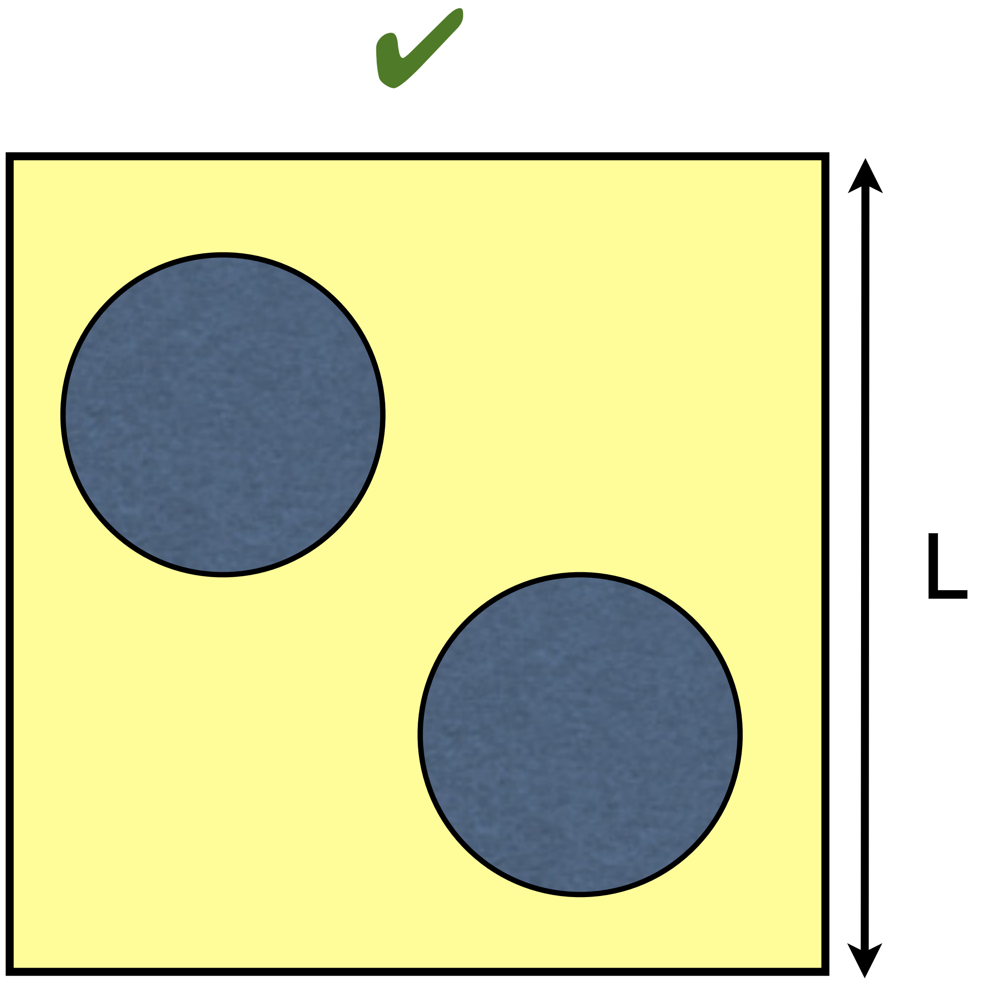


Single (stable) particle with L>R

Particle not "squeezed"

Spectrum same as in infinite volume up to corrections proportional to $e^{-M_\pi L}$

[Lüscher]



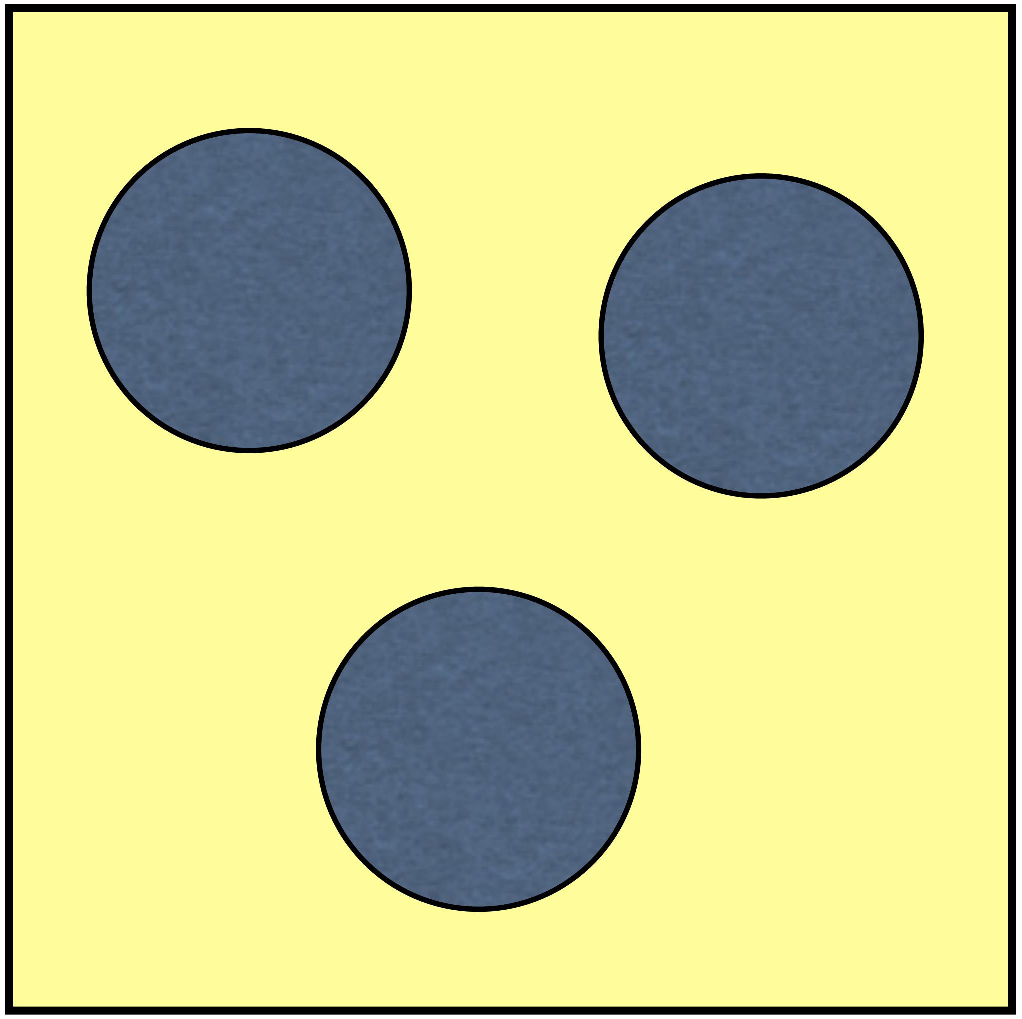
L>2R

There is an "outside" region.

Spectrum IS related to scatt. amps. up to corrections proportional to $e^{-M_\pi L}$

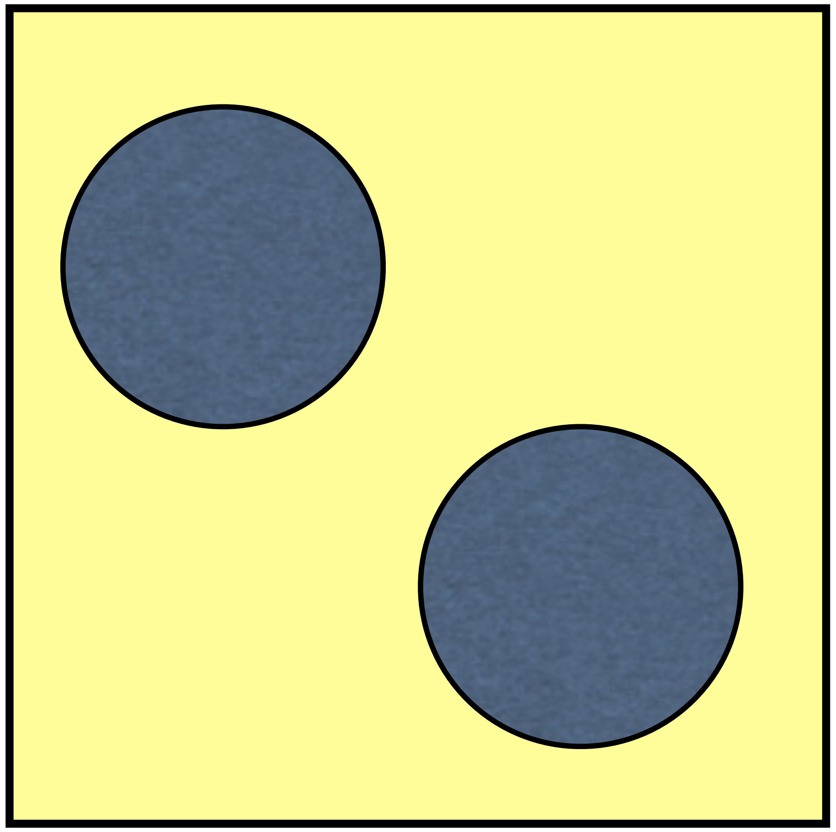
[Lüscher]

…and for 3 particles?



- Spectrum IS related to 2→2, 2→3 & 3→3 scattering amplitudes up to corrections proportional to e^{-ML} [Polejaeva & Rusetsky]
- Formalism developed in various cases under various assumptions [Talks on Tuesday & Thursday]

Lüscher's method [1991]

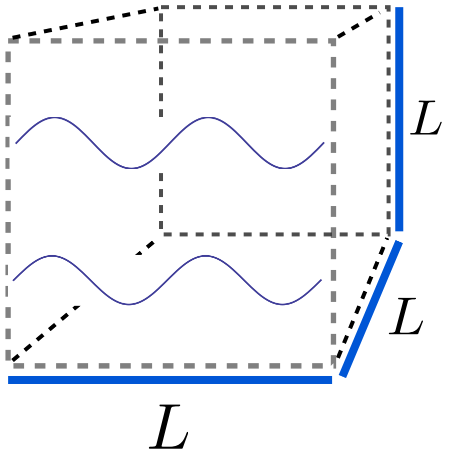


- Rewrite QFT in two-particle elastic regime as a NRQM problem with an energy-dependent potential $U_E(\mathbf{r}-\mathbf{r}')$
- Solve Schrödinger equation in periodic box using fact that there is an ``outside" region
- Leads to quantization condition (QC)
- QC depends on phase shifts, which are identical for NRQM problem and QFT

- U_E is related to the Bethe-Salpeter amplitude
- Lüscher's approach is the starting point for the HALQCD method

- Generalizing Lüscher's approach to moving frames, etc. is tricky
- Instead, here follow method of [Kim, Sachrajda & SS 05]

Set up



- Work in continuum (assume that LQCD can control discretization errors)
- Cubic box of size L with periodic BC, and infinite (Minkowski) time
  - Spatial loops are sums: $\frac{1}{L^3}\sum_{\vec{k}}$ $\vec{k} = \frac{2\pi}{L}\vec{n}$
- Consider identical scalar particles with physical mass m, interacting ***arbitrarily*** in a general relativistic effective field theory
  - Generalizations to arbitrary spin and masses ``straightforward"

S. Sharpe, "Multiparticle systems in LQCD" 8/28/2017, Santa Fe

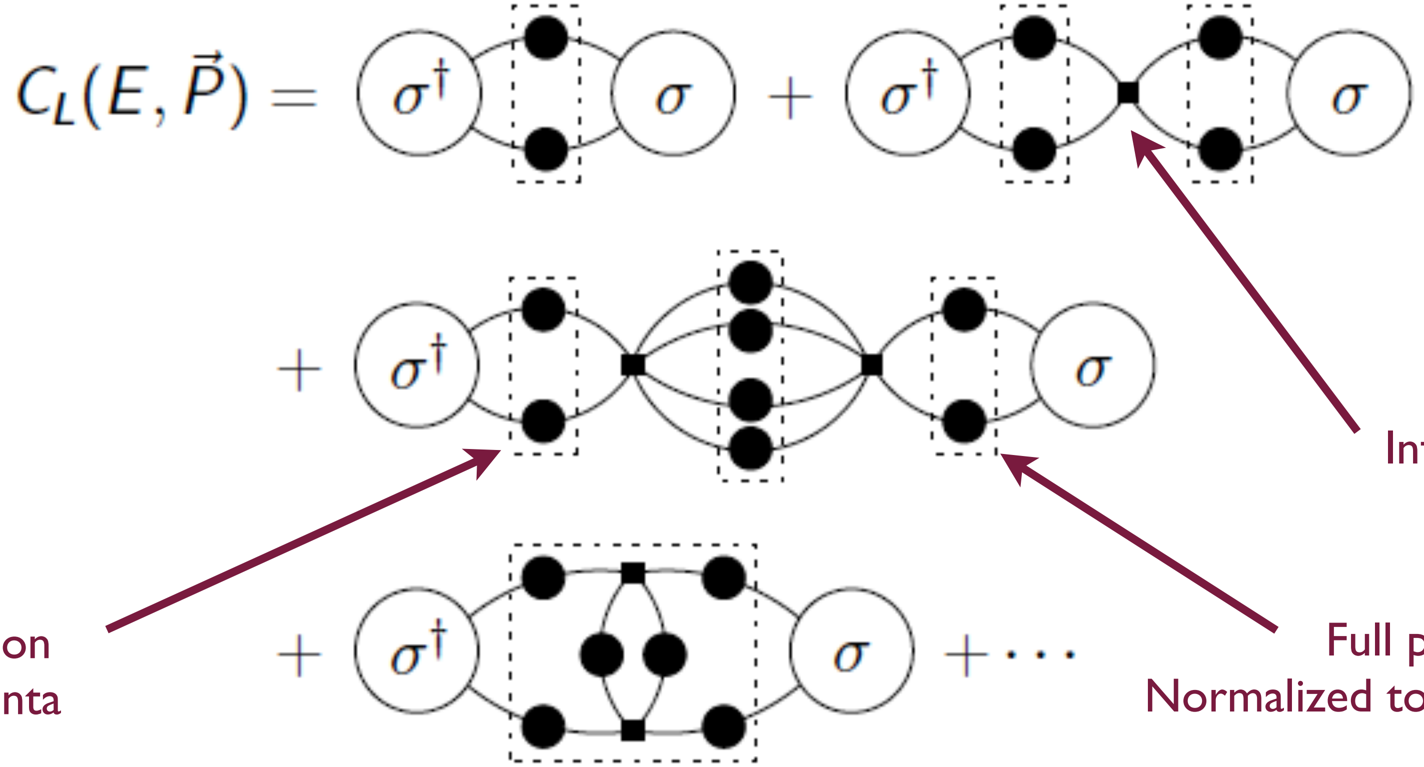
Methodology

CM energy is $E^*=\sqrt{(E^2-P^2)}$

- Calculate (for some $\mathbf{P}=2\pi\mathbf{n_P}/L$)

$$C_L(E,\vec{P}) \equiv \int_L d^4x\, e^{-i\vec{P}\cdot\vec{x}+iEt}\langle\Omega|T\sigma(x)\sigma^\dagger(0)|\Omega\rangle_L$$

- Poles in C_L occur at energies of finite-volume spectrum
- Consider here $E^* < 3m$ so 3 (or more) particles cannot go on shell
- E.g. for 2 particles (here assuming only even-legged vertices):



Infinite-volume vertices

Full propagators
Normalized to unit residue at pole

Boxes indicated summation
over finite-volume momenta

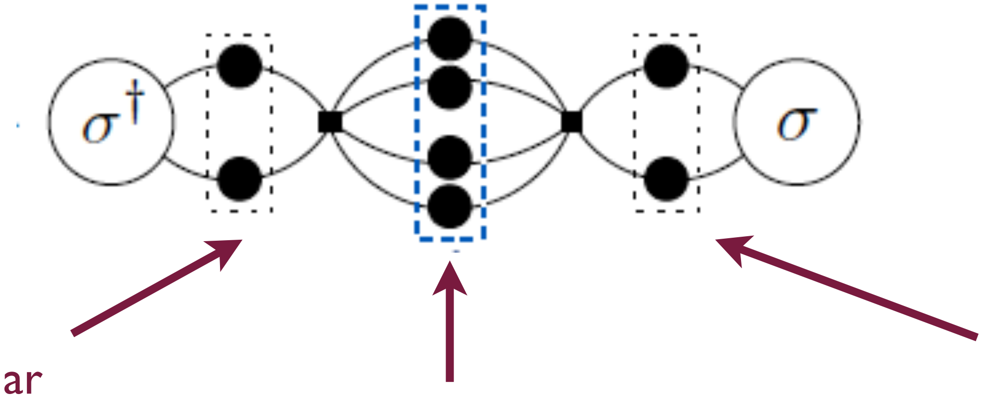
Key step 1

- Replace loop sums with integrals where possible
  - Drop exponentially suppressed terms (~e^{-ML}, $e^{-(ML)^2}$, etc.) while keeping power-law dependence

$$\frac{1}{L^3}\sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \xcancel{\sum_{\vec{l}\neq\vec{0}}\int \frac{d^3k}{(2\pi)^3} e^{iL\vec{l}\cdot\vec{k}} g(\vec{k})}$$

Exp. suppressed if g(k) is smooth and scale of derivatives of g is ~1/M

- Summand is smooth if no on-shell cuts through loop
  - For E* < 3m, this means only two-particle cuts are singular



Singular

Smooth

Singular

Key step 2

- Use "sum=integral + [sum-integral]" if integrand has pole, with

$$\left(\int\frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}}-\int\frac{d^4k}{(2\pi)^4}\right)f(k)\frac{1}{k^2-m^2+i\epsilon}\frac{1}{(P-k)^2-m^2+i\epsilon}g(k)$$

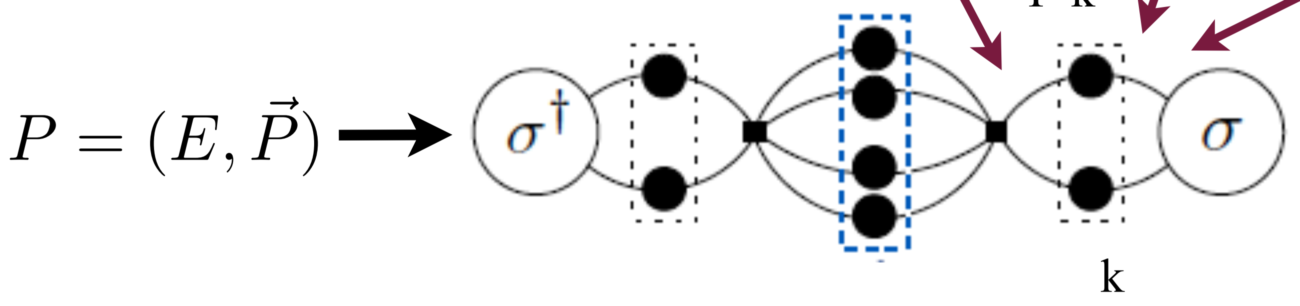
$$=\int d\Omega_{q^*}d\Omega_{q^{*\prime}}\,f^*(\hat{q}^*)\mathcal{F}\quad(q^*,q^{*'})g^*(\hat{q}^{*'})\qquad+\text{exp. suppressed}$$

q^* is relative momentum of pair on left in CM

Kinematic function

f & g evaluated for ON-SHELL momenta
Depend only on direction in CM

- Example



Focus on this loop

f is left-hand part of integrand

g is right-hand part of integrand

Key step 2

- Use "sum=integral + [sum-integral]" where integrand has pole, with [KSS]

$$\left(\int \frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^4k}{(2\pi)^4}\right) f(k)\frac{1}{k^2-m^2+i\epsilon}\,\frac{1}{(P-k)^2-m^2+i\epsilon}g(k)$$

$$= \int d\Omega_{q^*}\, d\Omega_{q^{*\prime}}\, f^*(\hat{q}^*)\mathcal{F}\ \ (q^*, q^{*\prime})g^*(\hat{q}^{*\prime})$$

- Decomposed into spherical harmonics, $\mathcal{F}$ becomes

$$F\ \ _{\ell_1,m_1;\ell_2,m_2} \equiv \quad \eta\left[\frac{\mathrm{Re}q^*}{8\pi E^*}\delta_{\ell_1\ell_2}\delta_{m_1m_2} + \frac{i}{2\pi EL}\sum_{\ell,m} x^{-\ell}\,\mathcal{Z}^P_{\ell m}[1;x^2]\int d\Omega\, Y^*_{\ell_1,m_1}Y^*_{\ell,m}Y_{\ell_2,m_2}\right]$$

$x \equiv q^*L/(2\pi)$ and $\mathcal{Z}^P_{\ell m}$ is a generalization of the zeta-function

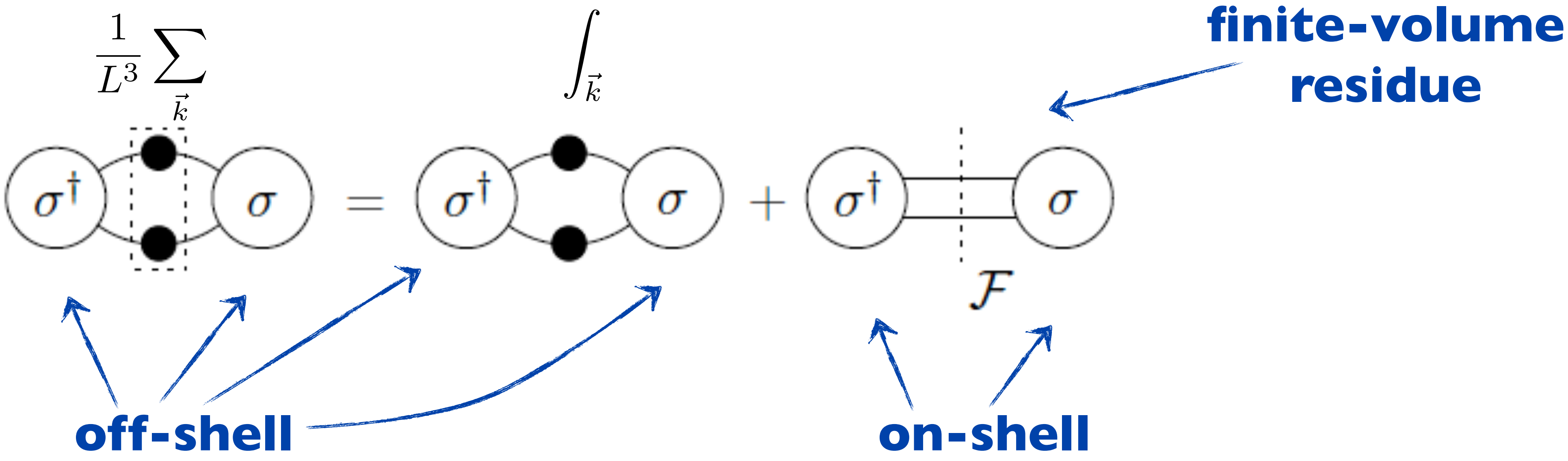
Key step 2

- Use "sum=integral + [sum-integral]" where integrand has pole, with [KSS]

$$\left(\int \frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^4k}{(2\pi)^4}\right) f(k)\frac{1}{k^2-m^2+i\epsilon}\frac{1}{(P-k)^2-m^2+i\epsilon}g(k)$$

$$= \int d\Omega_{q^*} d\Omega_{q^{*\prime}} f^*(\hat{q}^*)\mathcal{F}\ \ (q^*, q^{*'})g^*(\hat{q}^{*'})$$

- Diagrammatically



finite-volume residue

off-shell

on-shell

Variant of key step 2

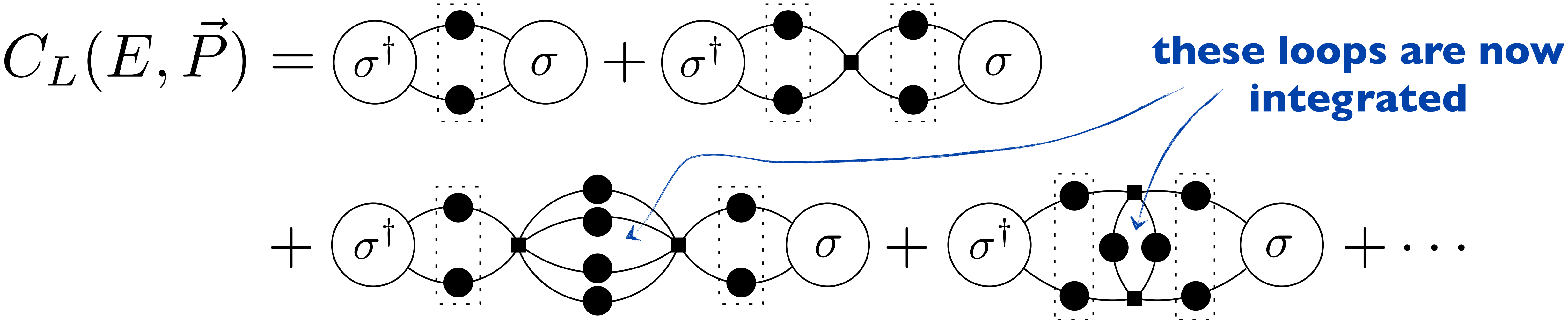
- For generalization to 3 particles use (modified) PV prescription instead of iε

$$\left(\int \frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}} - \overset{\widetilde{PV}}{\int} \frac{d^4k}{(2\pi)^4}\right) f(k)\frac{1}{k^2-m^2+\cancel{i\epsilon}}\,\frac{1}{(P-k)^2-m^2+\cancel{i\epsilon}}\,g(k)$$

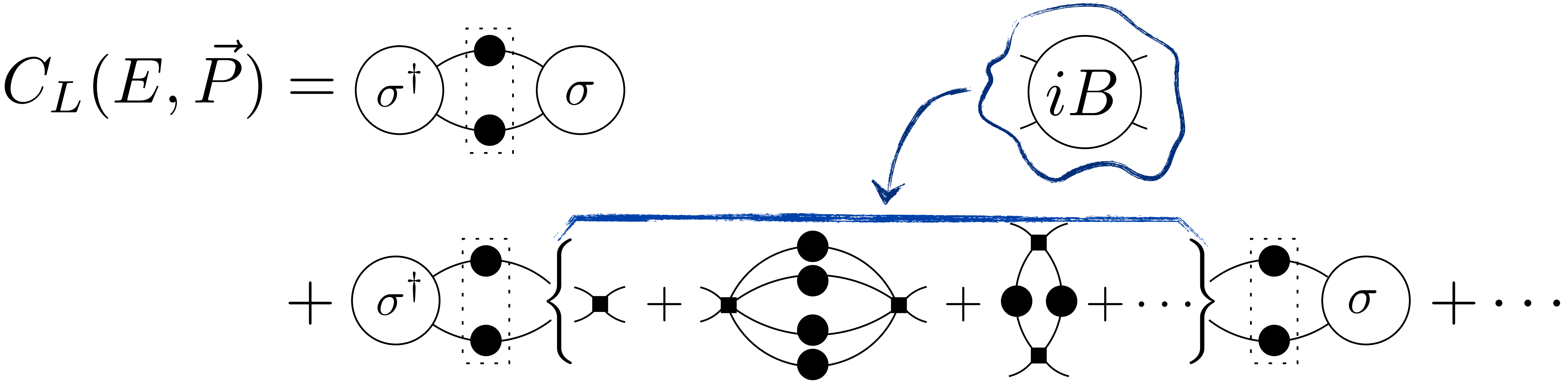
$$= \int d\Omega_{q^*}\, d\Omega_{q^{*\prime}}\, f^*(\hat{q}^*)\,\mathcal{F}_{\widetilde{PV}}(q^*, q^{*\prime})\,g^*(\hat{q}^{*\prime})$$

- Key properties of F_{PV} :(i) real; (ii) no unitary cusp at threshold

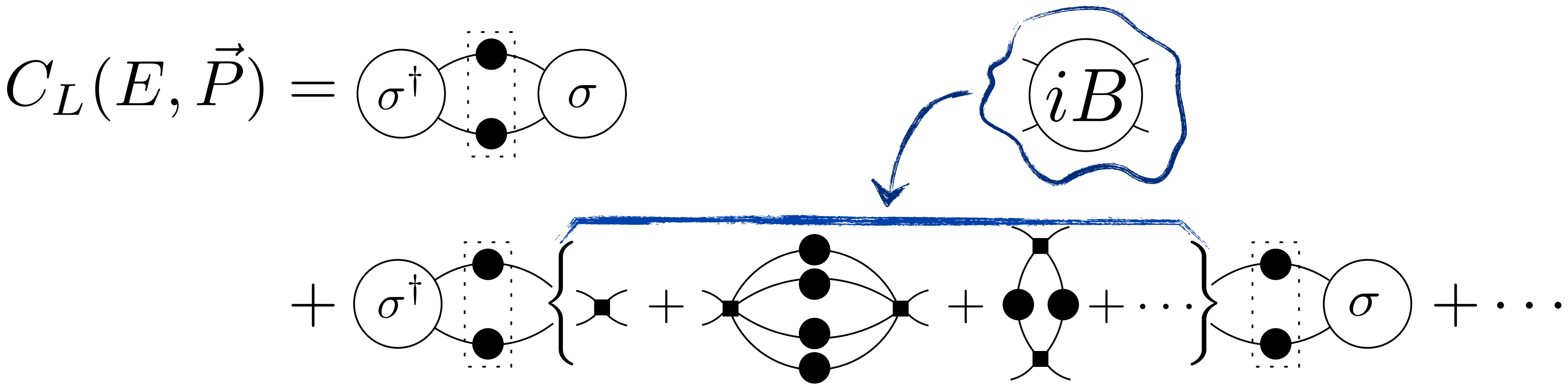
- Apply previous analysis to 2-particle correlator (0 < E* < 4M)



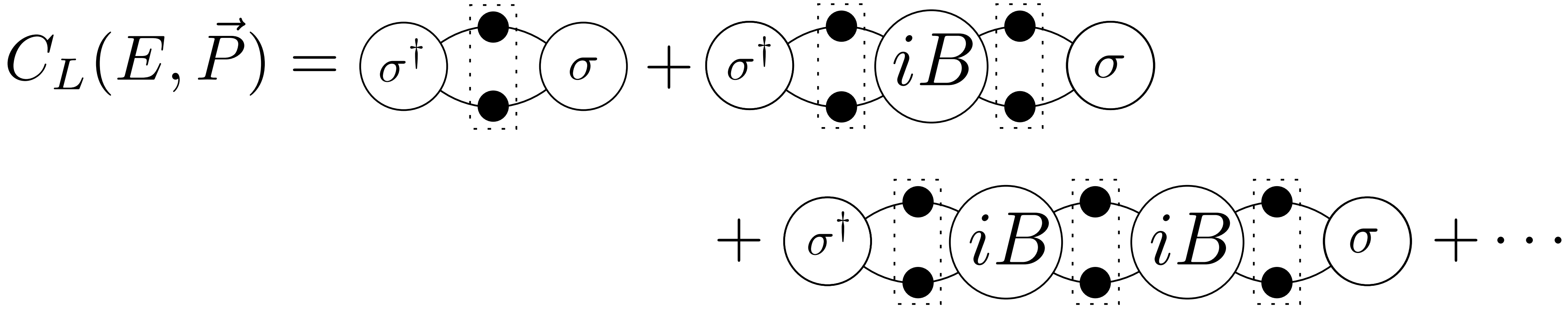
- Collect terms into infinite-volume Bethe-Salpeter kernels



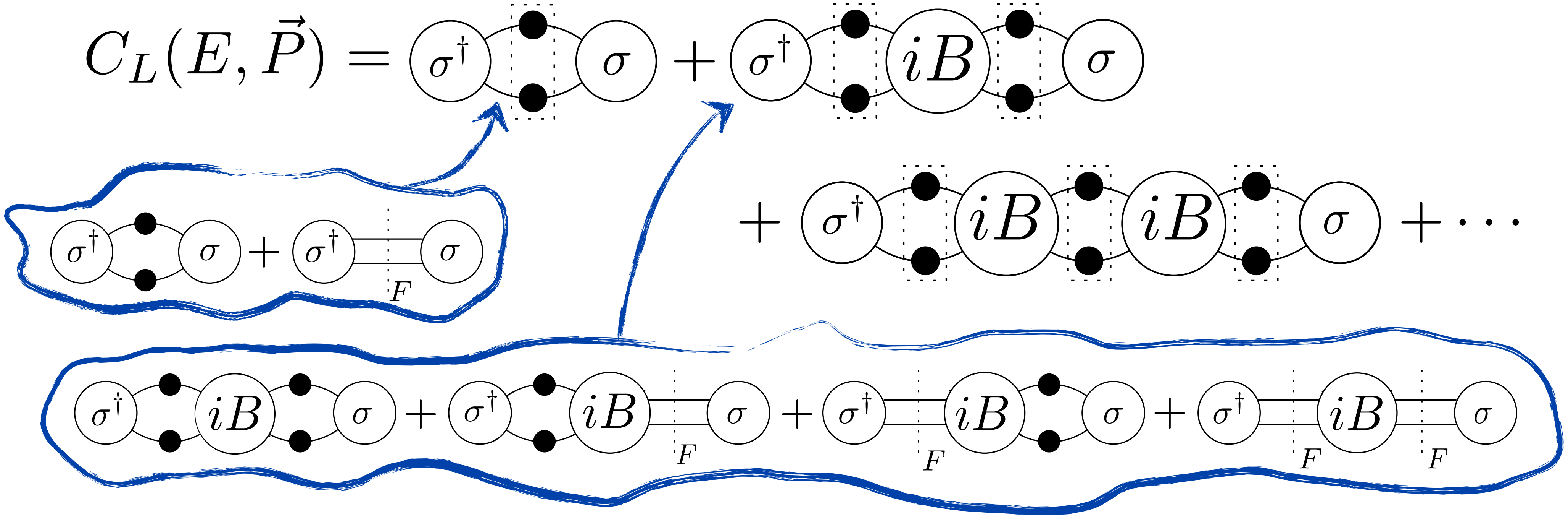
- Apply previous analysis to 2-particle correlator
- Collect terms into infinite-volume Bethe-Salpeter kernels



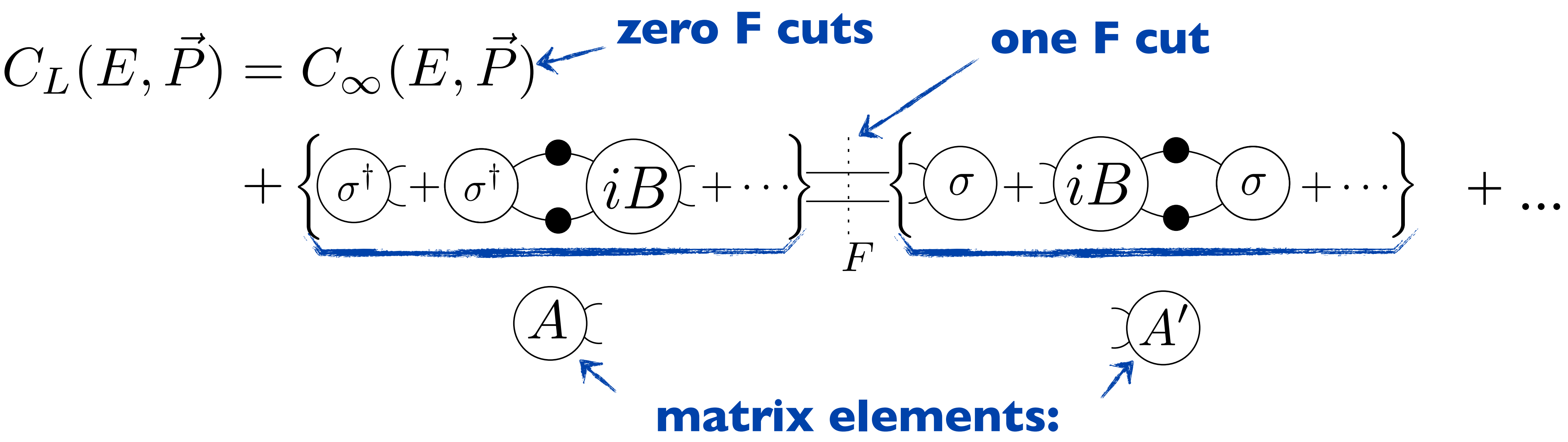
- Leading to



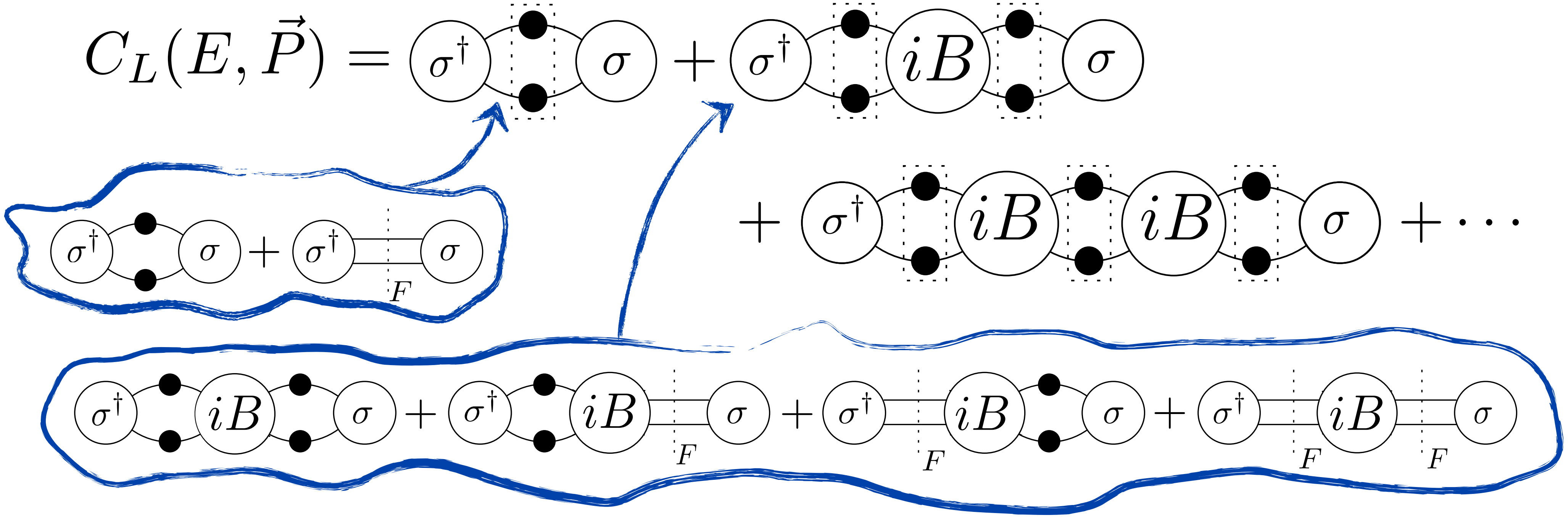
- Next use sum identity



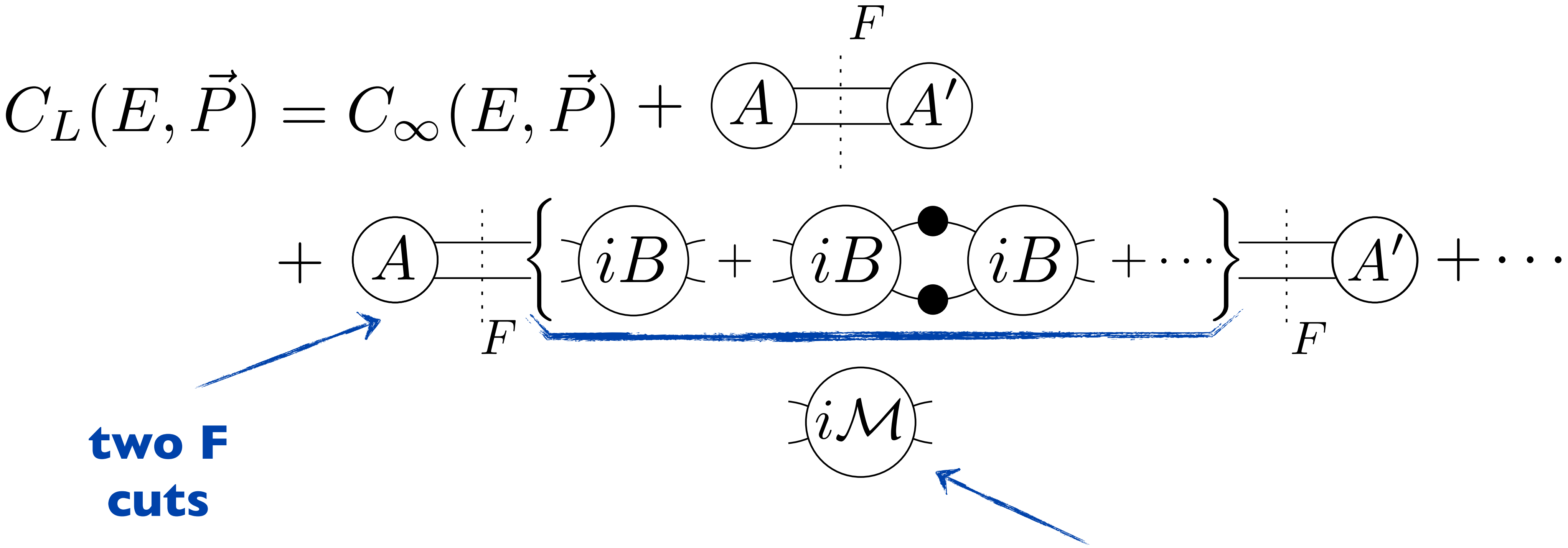
- And regroup according to number of “F cuts”



- Next use sum identity



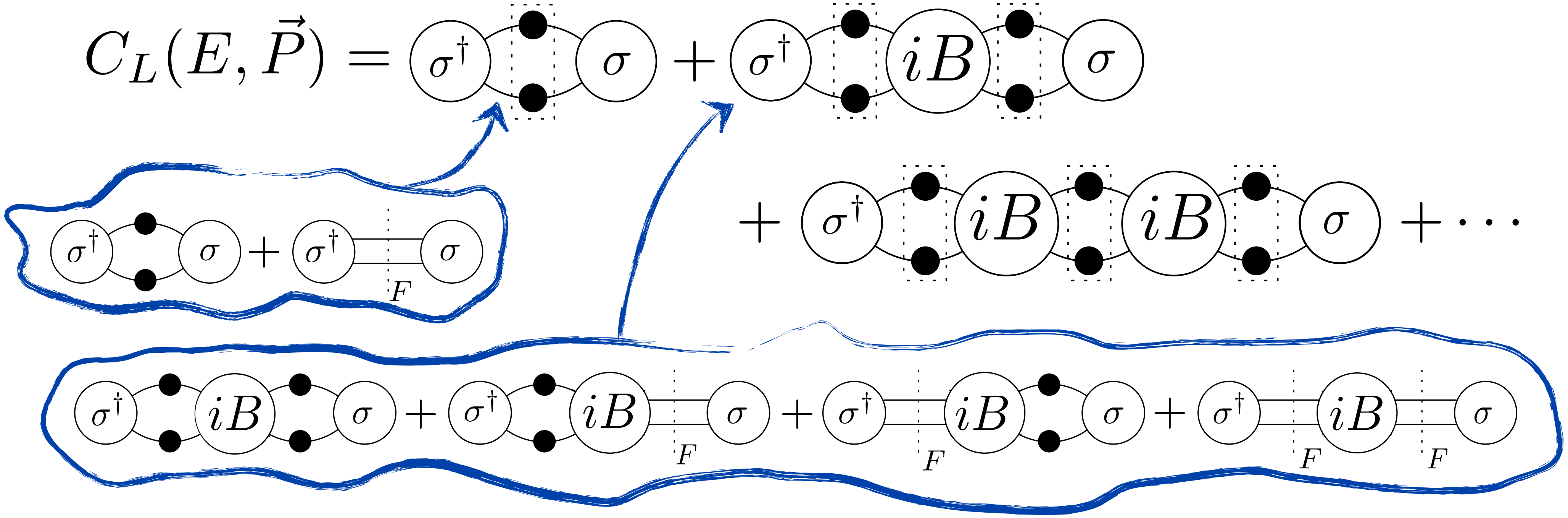
- And keep regrouping according to number of “F cuts”



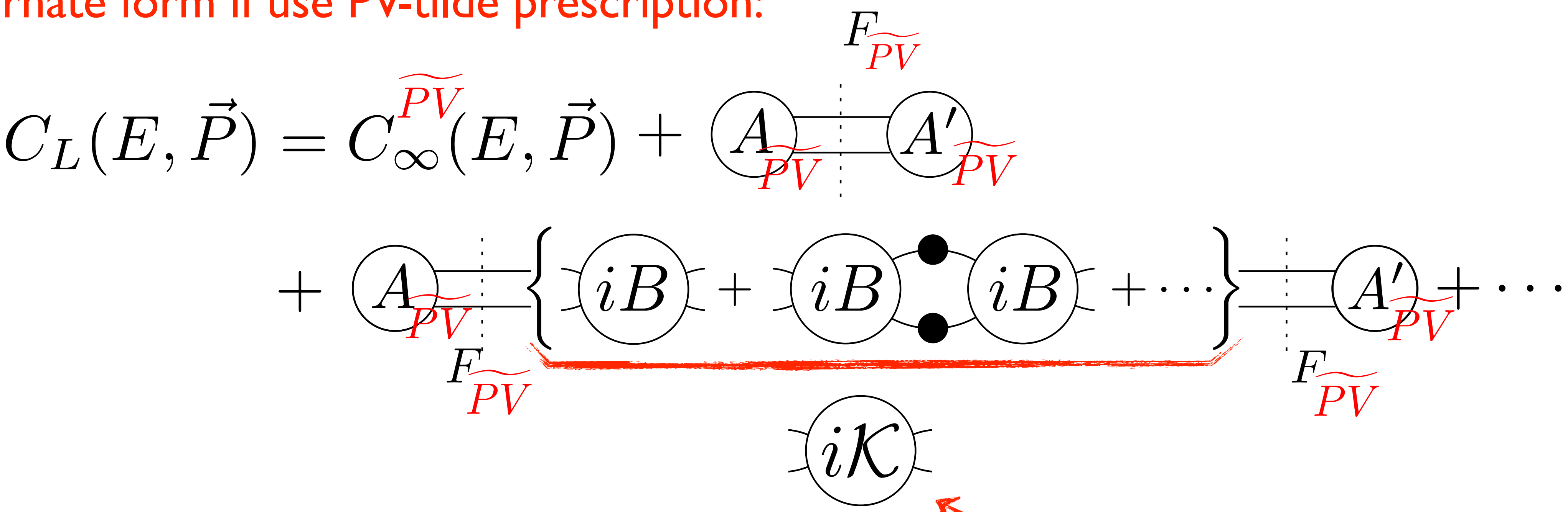
two F cuts

the infinite-volume, on-shell 2→2 scattering amplitude

- Next use sum identity



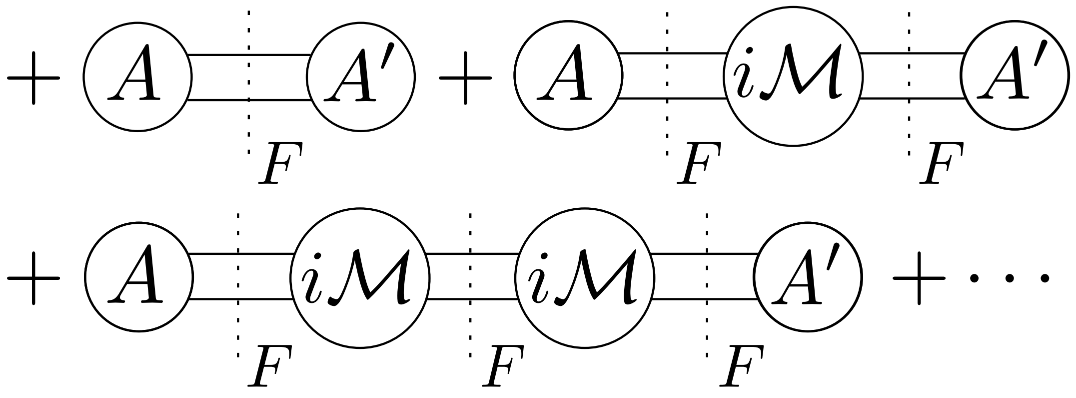
- Alternate form if use PV-tilde prescription:



the infinite-volume, on-shell 2→2 K-matrix

- Final result:

$$C_L(E,\vec{P}) = C_\infty(E,\vec{P})$$

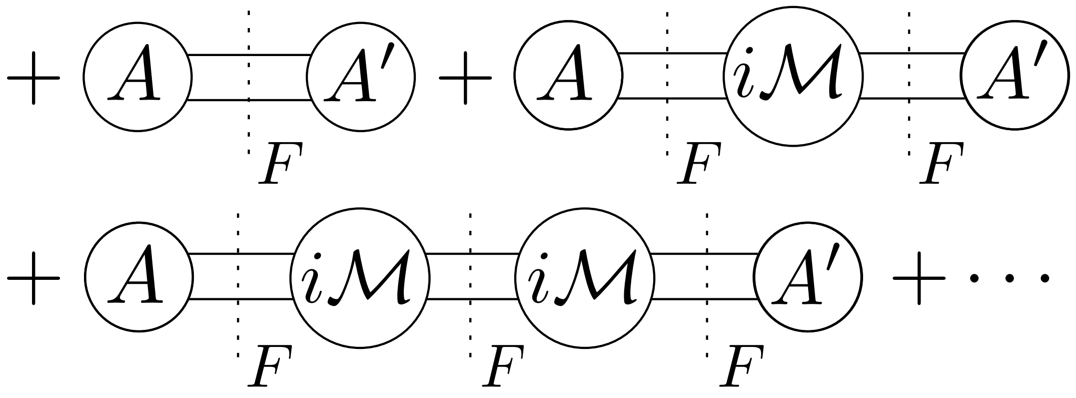


- $$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + \sum_{n=0}^{\infty} A' iF [i\mathcal{M}_{2\to2} iF]^n A$$

- Correlator is expressed in terms of infinite-volume, physical quantities and kinematic functions encoding the finite-volume effects

- Final result:

$$C_L(E,\vec P) = C_\infty(E,\vec P)$$



- $$C_L(E,\vec P) = C_\infty(E,\vec P) + \sum_{n=0}^{\infty} A' iF[i\mathcal{M}_{2\to2} iF]^n A$$

- $$C_L(E,\vec P) = C_\infty(E,\vec P) + A' iF \frac{1}{1 - i\mathcal{M}_{2\to2} iF} A$$

no poles, only cuts

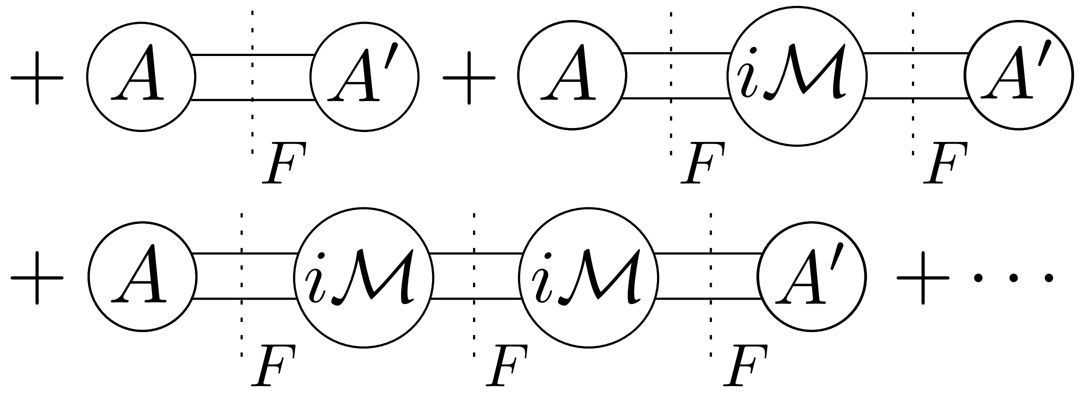
no poles, only cuts

matrices in l,m space

- $C_L(E,\vec P)$ diverges whenever $iF\dfrac{1}{1 - i\mathcal{M}_{2\to2} iF}$ diverges

- Final result:

$$C_L(E,\vec P) = C_\infty(E,\vec P)$$



- $$C_L(E,\vec P) = C_\infty(E,\vec P) + \sum_{n=0}^{\infty} A' iF [i\mathcal{M}_{2\to2} iF]^n A$$

- $$C_L(E,\vec P) = C_\infty(E,\vec P) + A' iF \frac{1}{1 - i\mathcal{M}_{2\to2} iF} A$$

no poles, only cuts

no poles, only cuts

matrices in l,m space

$$\Rightarrow \Delta_{L,\vec P}(E) = \det[(iF)^{-1} - i\mathcal{M}_{2\to2}] = 0$$

$$\Rightarrow \Delta_{L,\vec P}(E) = \det[(F_{\widetilde{PV}})^{-1} + \mathcal{K}_2] = 0$$

Alternative form

Single-channel 2-particle quantization condition

$$\det\left[(F_{\rm PV})^{-1}+\mathcal{K}_2\right]=0$$

- Infinite-dimensional determinant must be truncated to be practical; truncate by assuming that $\mathcal{K}_2$ vanishes above l_{max}

- If $l_{max}=0$, obtain one-to-one relation between energy levels and $\mathcal{K}_2 \sim \tan\delta/q$

$$E_n^* = \sqrt{E_n^2-\vec{P}^2}$$

CM energy

$$\mathcal{K}_{2,s}(E_n^*) = -\frac{1}{F_{\rm PV;00;00}(E_n,\vec{P},L)}$$

"measured"

energy-level

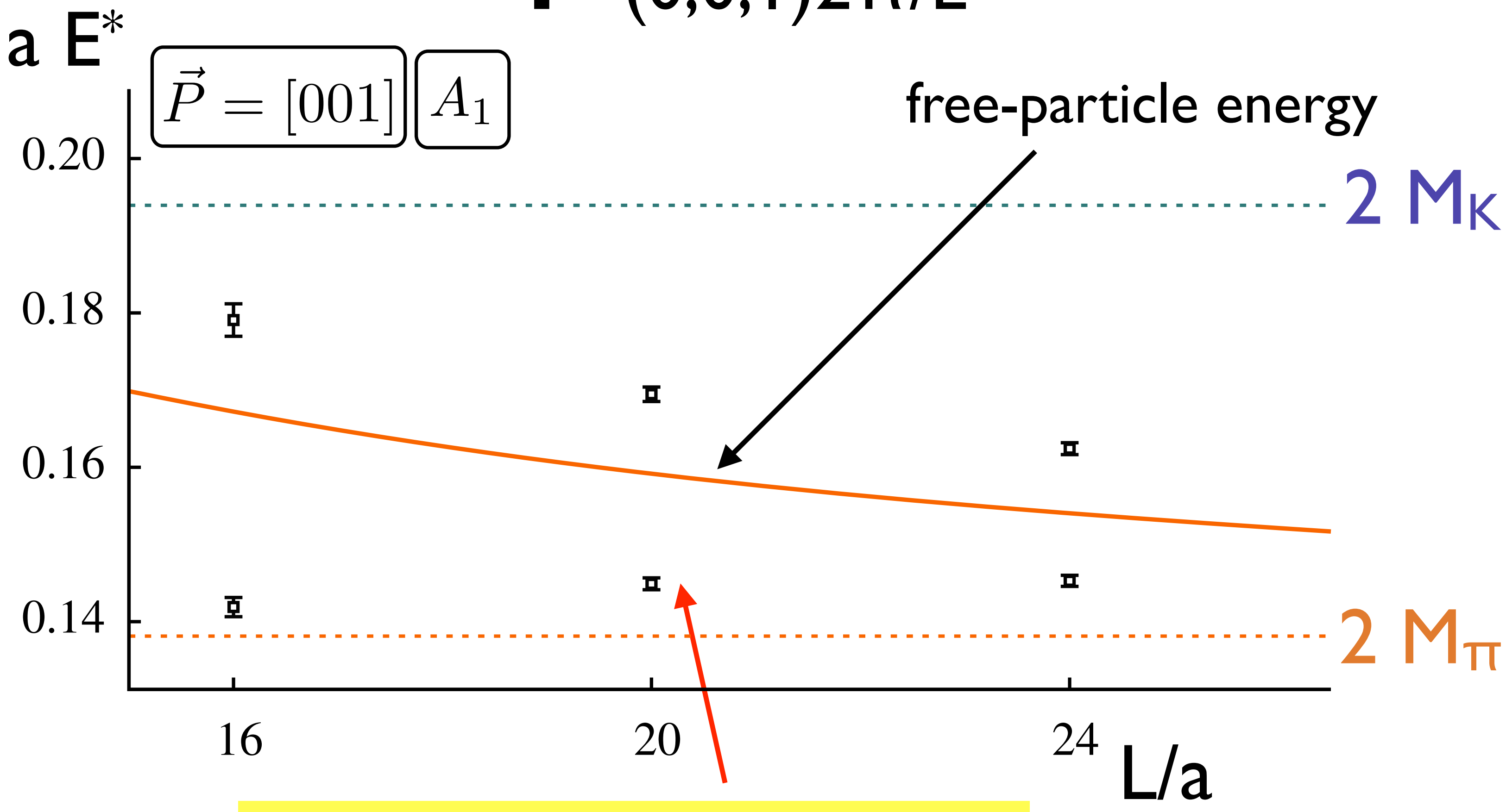
Equivalent to: $\tan[\delta(q^*)] = -\tan[\phi^P(q^*)],$

Application to ρ meson

[Dudek, Edwards & Thomas, 1212.0830]

- Proof of principle calculation with M_π ~ 400 MeV, several **P**, many spectral levels

P=(0,0,1)2π/L



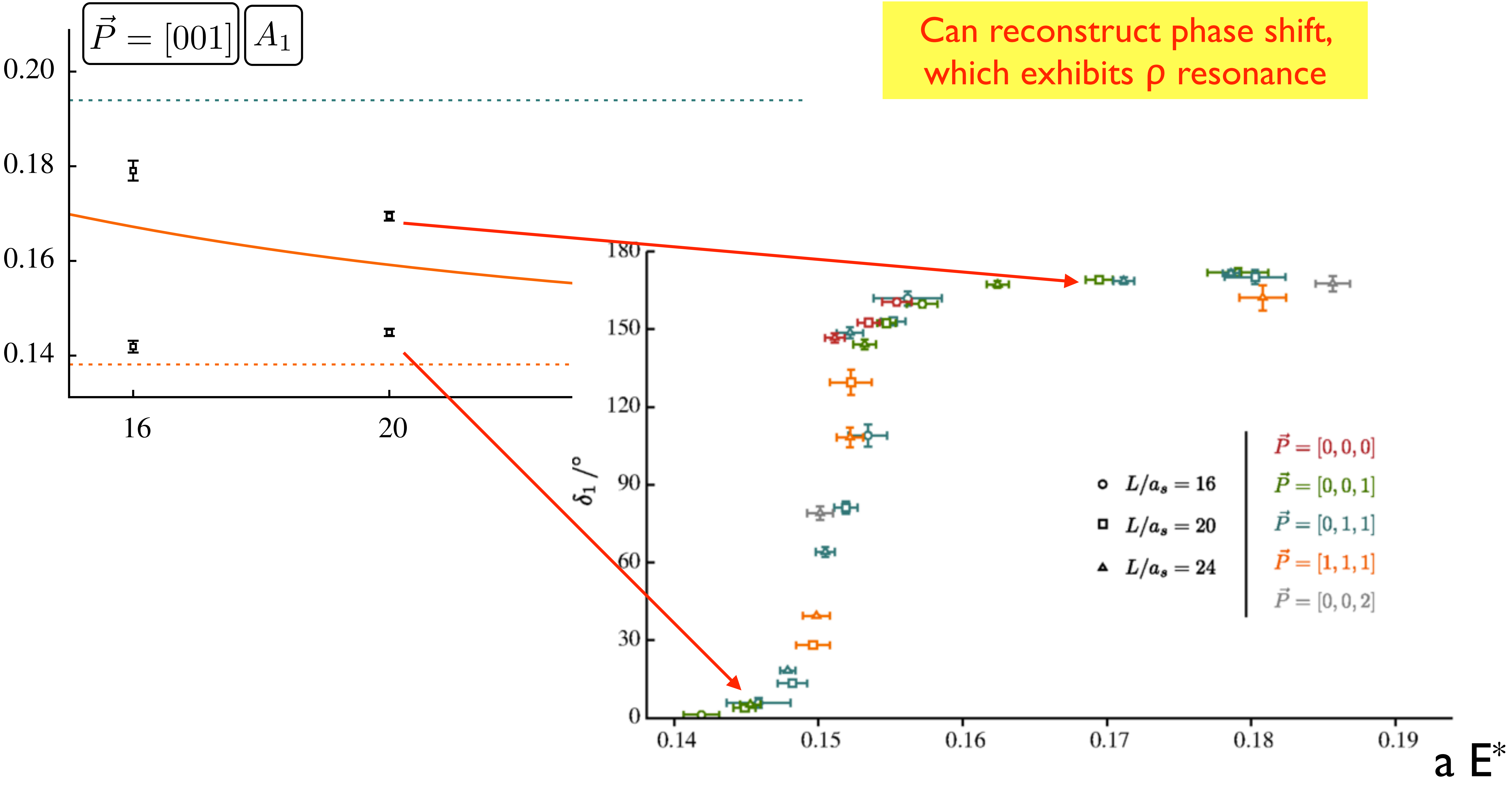
free-particle energy

Elastic region where quantization condition applies

KEY POINT: there are "extra" levels here, with no levels close to the free-particle energy

Application to ρ meson

[Dudek, Edwards & Thomas, 1212.0830]



Can reconstruct phase shift, which exhibits ρ resonance

Summary

Five years ago:

INT Workshop INT-13-53W

Nuclear Reactions from Lattice QCD

March 11-12, 2013

Organizers: Raúl Briceño, Zohreh Davoudi & Tom Luu

Progress since then?

Summary

- Enormous progress in the two-particle sector from LQCD both in formalism and simulations
    - Major opportunity to use these tools, together with EFTs & other methods, to extend the reach of first-principles calculations
- Substantial progress in the three-particle sector
    - Competing approaches, all needing extensions, e.g. to higher spins, nonidentical particles and Lellouch-Lüscher factors
    - Challenge is to develop practical methods based on these approaches
- There is much to do … but the prospects are exciting!

Questions?