Scattering length from Bethe-Salpeter wave function inside the interaction range

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Contents

1 Introduction

- Hadron interactions can be studied directly by lattice QCD using Lüscher formula and its extensions
  cf. many talks in this workshop

  ◇ Lüscher formula utilizes Bethe-Salpeter(BS) wave function outside the interaction range of two hadrons Lüscher(1986,1990),...

  ◇ A relation between on-shell scattering amplitude and BS wave function inside the interaction range was discussed in the infinite volume Lin et.al.(2001),CP-PACS(2005),Yamazaki and Kuramashi(2017)
  cf. talk by Yamazaki-san
  → We extend this approach to a finite volume

- cf. HAL QCD method (indirect method through a potential from BS wave function)
  cf. talks by Sinya-san, Iritani-san, Doi-san, Kawai-san

2 Formulation(in brief)

[Infinite volume limit $L = \infty$] Lin et.al.(2001),CP-PACS(2005),Yamazaki and Kuramashi(2017)

- Scattering amplitude $H(p;k)$ is obtained by BS wave function $\phi(\mathbf{x};k)$

  ◇ The integral range can be changed from ∞ to finite interaction range R, if $(\Delta + k^2)\phi(\mathbf{x};k) = 0$ for $x > R$
  $\therefore$ Lattice simulation for $H(p;k)$ is possible, if $R < L/2$

  ◇ NB. we consider $I = 2$ S-wave two-pion in the center of mass frame below inelastic threshold. Overall factors are omitted for simplicity.

$$\begin{aligned}
\phi(\mathbf{x};k) &= \langle 0|\pi_1(\mathbf{x}/2)\pi_2(-\mathbf{x}/2)|\pi_1(\mathbf{k})\pi_2(-\mathbf{k}); \text{in}\rangle \\
&= e^{i\mathbf{k}\cdot\mathbf{x}} + \int \frac{d^3p}{(2\pi)^3} \frac{H(p;k)}{p^2 - k^2 - i\epsilon} e^{i\mathbf{p}\cdot\mathbf{x}} + (\text{inelastic part}), \\
\therefore H(p;k) &= -\int_{-\infty}^{\infty} d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}} \left(\Delta + k^2\right) \phi(\mathbf{x};k) \\
&= -\int_{-R}^{R} d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}} \left(\Delta + k^2\right) \phi(\mathbf{x};k), \\
&\quad \text{where } \left(\Delta + k^2\right) \phi(\mathbf{x};k) = 0 \text{ for } x > R
\end{aligned}$$

[Infinite volume limit $L = \infty$(continued)]

- Once $H(p;k)$ at on-shell $p = k$ is obtained, we can extract the scattering phase shift $\delta(k)$, and the scattering length a_0.
  - ♢ Lattice simulation of $H(k;k)$ inside interaction range R gives $\delta(k)$
  - ♢ NB. $H(k;k)$ is removed in the final form of Lüscher formula
    → "Please keep $H(k;k)$. $H(k;k)$ also has scattering info."

$$\begin{aligned} H(k;k) &= \frac{4\pi}{k} e^{i\delta(k)} \sin\delta(k) \\ a_0 &= \tan\delta(k)/k + O(k^2) \end{aligned}$$

[Quick derivation on Lüscher formula(omitting some overall factors for simplicity)]

$$\begin{aligned} \phi(\mathbf{x};k) &\xrightarrow[x>R]{} v_{00} G(\mathbf{x};k), \quad G(\mathbf{x};k) : \text{solution of } (\Delta + k^2)\phi(\mathbf{x};k) = 0 \\ &= C_{00} e^{i\delta(k)} \sin(kx + \delta(k))/kx + (l \geq 4 \text{ terms}), \quad v_{00}, C_{00} : \text{constants} \end{aligned}$$

Expanding $G(\mathbf{x};k)$ by $j_l(kx)$ and $n_0(kx)$ and comparing their coefficients leads to

$$\begin{aligned} C_{00} H(k;k) &= v_{00} \\ k \cot\delta(k) C_{00} H(k;k) &= 4\pi v_{00} g_{00}(k) \end{aligned}$$

Taking a ratio of the above two equations leads to Lüscher formula,

$$k \cot\delta(k) = 4\pi g_{00}(k)$$

3 Set up of simulation

We use $I = 2$ $\pi\pi$ system in quenched lattice QCD as a test bed

- Iwasaki gauge action at $\beta = 2.334 (a^{-1} = 1.207 [\mathrm{GeV}])$ CP-PACS(2001,2005)
- Valence Clover quark action with $C_{\mathrm{SW}} = 1.398$

  ◇ Four random $Z(2)$ sources avoiding Fierz contamination
  → Six combinations of two quark propagators
  (Wall sources for comparison) + Coulomb gauge fixing

  ◇ The number of source positions is 32 i.e. every two time slices

  ◇ Periodic boundary condition in space, Dirichlet boundary condition in time

Lattice	κ_{val}	m_π [GeV]	N_{config}
$24^3 \times 64$	0.1340	0.86	200

[Observable : four-point function $\langle 0|\Phi(\mathbf{x},t)|\pi^+\pi^+, E_k\rangle$]

- $I = 2$ two-pion BS wave function $\phi(\mathbf{x};k)$ is defined by a four-point function $\langle 0|\Phi(\mathbf{x},t)|\pi^+\pi^+, E_k\rangle$ with two-pion operator $\Phi(\mathbf{x},t)$

  ◇ A_1^+ projection is performed for S-wave in center of mass frame. Overall factors are omitted for simplicity.

$$\phi(\mathbf{x};k) = \langle 0|\Phi(\mathbf{x},t)|\pi^+\pi^+, E_k\rangle e^{E_k t},$$

where

$$\Phi(\mathbf{x},t) = \sum_{\mathbf{r}} \pi^+(R_{A_1^+}[\mathbf{x}] + \mathbf{r}, t)\pi^+(\mathbf{r},t),$$

$R_{A_1^+}[\mathbf{x}]$: projector onto A_1^+ cubic group

$$E_k = 2\sqrt{m_\pi^2 + k^2}$$

$$\Delta\phi(\mathbf{x};k) = \sum_{i=1}^{3} (\phi(\mathbf{x}+\hat{i};k) + \phi(\mathbf{x}-\hat{i};k) - 2\phi(\mathbf{x};k))$$

: Laplacian on lattices

$$H(p;k) := -\int_{-\infty}^{\infty} d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}} \left(\Delta + k^2\right)\phi(\mathbf{x};k)$$

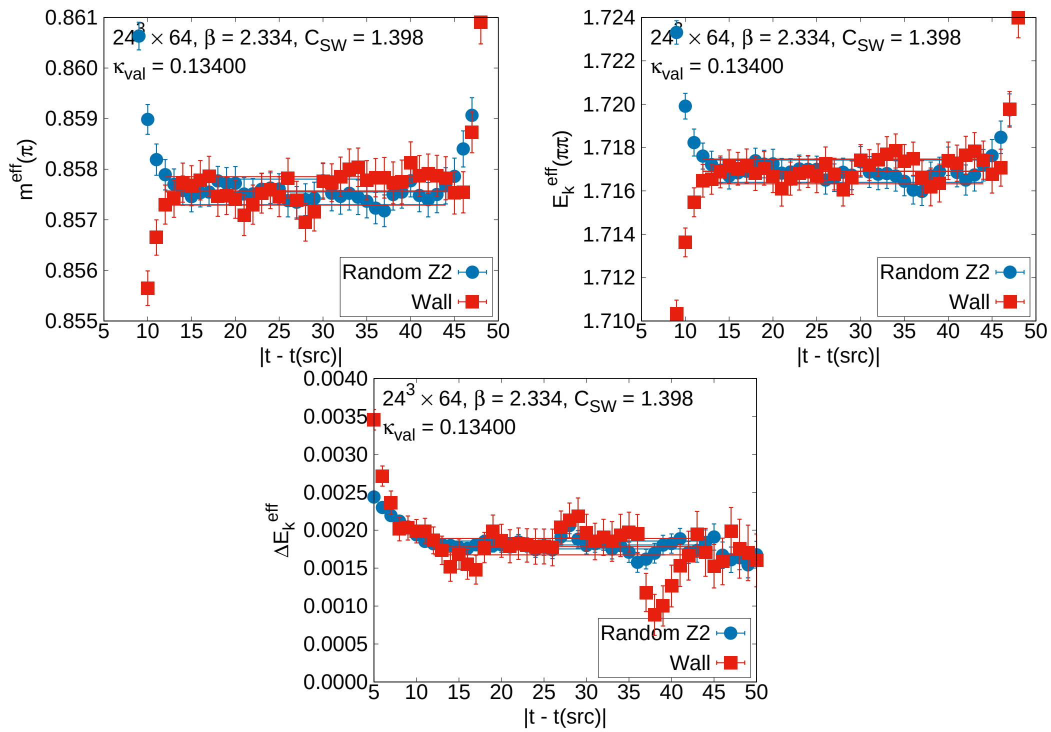
$$= -\sum_{|x|<R<L/2} e^{-i\mathbf{p}\cdot\mathbf{x}} \left(\Delta + k^2\right)\phi(\mathbf{x};k)$$

4 Result

[Check of plateau of temporal correlators]

- Effective masses of one-pion $m^{\mathrm{eff}}(\pi)$ and $I=2$ two-pion $E_k^{\mathrm{eff}}(\pi\pi)$ as well as $\Delta E_k^{\mathrm{eff}} = E_k^{\mathrm{eff}}(\pi\pi) - 2m^{\mathrm{eff}}(\pi)$ have plateau in $t=[12,44]$
$\rightarrow$ No fake plateau is observed for our case

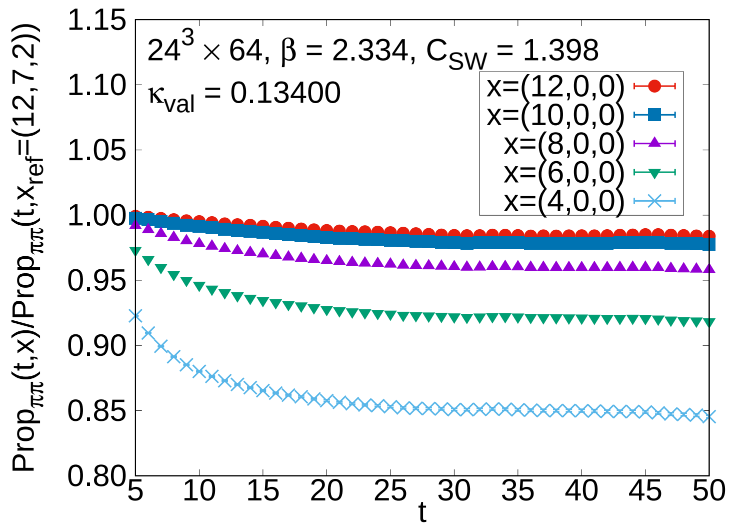
$$m^{\mathrm{eff}} = \log(prop(t)/prop(t+1))$$



[Check of plateau of a ratio of spatial wave functions $\phi(\mathbf{x};k)/\phi(\mathbf{x}_{\rm ref};k)$]

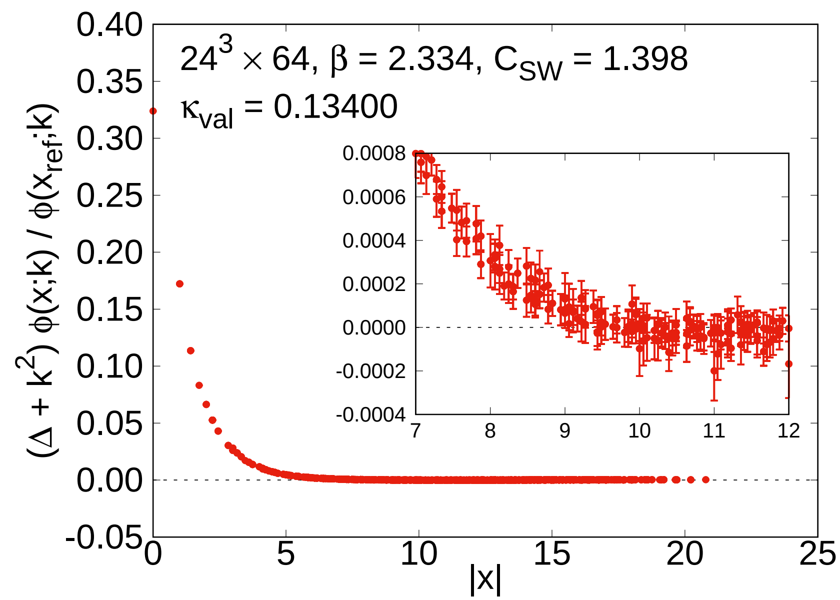
- Ratio of wave functions $\phi(\mathbf{x};k)/\phi(\mathbf{x}_{\rm ref};k)$ have plateau in $t=[32,44]$
- cf. temporal correlators have plateau in $t=[12,44]$
  - ♢ Larger t is required for wave functions, but still under control

$$\begin{aligned}\phi(\mathbf{x};k) &= \text{const} \times prop_{4\text{pt}}(\mathbf{x},t)e^{E_k t}\\ \phi(\mathbf{x};k)/\phi(\mathbf{x}_{\rm ref};k) &= prop_{4\text{pt}}(\mathbf{x},t)/prop_{4\text{pt}}(\mathbf{x}_{\rm ref},t)\end{aligned}$$



[Check of sufficient condition: $(\Delta + k^2)\phi(\mathbf{x}; k) = 0$ for $R < x < L/2$]

- We confirm $R \sim 10$, which is consistent with the result by CP-PACS(2005)
  $\rightarrow$ The sufficient condition is satisfied within our statistical errors



◇ Reference point $\mathbf{x}_{\rm ref} = (12, 7, 2)$ is chosen to minimize $l = 4$ contribution s.t.
$\phi(\mathbf{x}; k) = (l = 0 \text{ term}) + (l = 4 \text{ term}) + \ldots$
$|Y_{40}(R_{A_1^+}[\mathbf{x}_0/x_0])j_4(kx_0)/(Y_{00}(R_{A_1^+}[\mathbf{x}_0/x_0])j_0(kx_0))| < 10^{-6}$
Y_{lm} : spherical harmonics, j_l : spherical Bessel function

◇ Strictly speaking, there must be exp tail, which is below our statistical error

[Check of sufficient condition: $(\Delta + k^2)\phi(\mathbf{x}; k) = 0$ for $R < x < L/2$ (cont)]

- We also confirm $R \sim 10$ by $H_L(k; k)$ and $\phi(\mathbf{x}; k)$

$$R(\mathbf{x}) = \frac{H_L(k;k)}{\phi(\mathbf{x};k)} G(\mathbf{x};k) \xrightarrow[x>R]{} \frac{C_{00} H(k;k)}{v_{00} G(\mathbf{x};k)} G(\mathbf{x};k) = 1$$

where

$$H_L(k;k) := -\sum_{\mathbf{x}\in L^3} j_0(kx)(\Delta + k^2)\phi(\mathbf{x};k)$$
$$= C_{00} H(k;k) \quad \text{(I have omitted an overall factor } C_{00}\text{)}$$

$G(\mathbf{x}; k)$: a solution of $(\Delta + k^2)\phi(\mathbf{x}; k) = 0$ for $x > R$

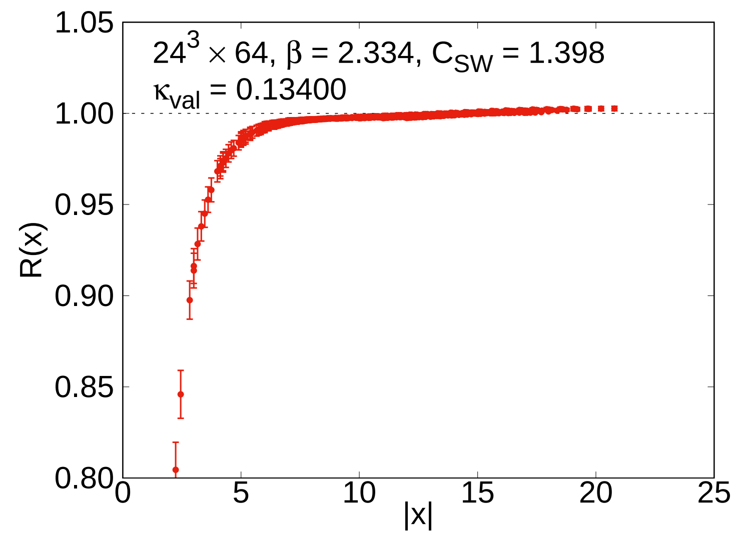
C_{00}, v_{00} : constants

[Quick derivation on Lüscher formula (again, omitting some overall factors for simplicity)]

$$\phi(\mathbf{x};k) \xrightarrow[x>R]{} v_{00} G(\mathbf{x};k)$$
$$= C_{00} e^{i\delta(k)} \sin(kx + \delta(k))/kx$$
$$+(l \geq 4 \text{ terms})$$

Expanding $G(\mathbf{x}; k)$ by $j_l(kx)$ and $n_0(kx)$ and comparing their coefficients leads to

$$C_{00} H(k;k) = v_{00}$$
$$k \cot\delta(k) C_{00} H(k;k) = 4\pi v_{00} g_{00}(k)$$

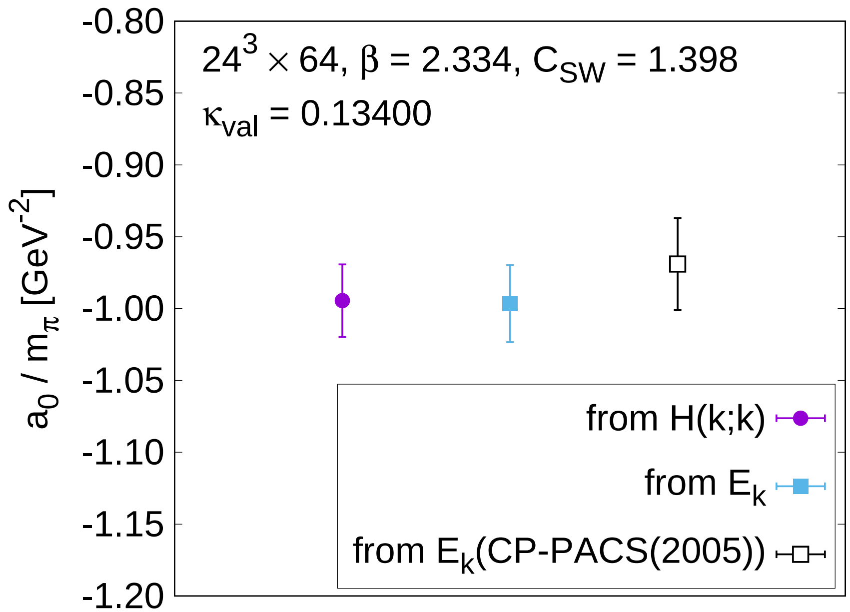


[Comparison of scattering lengths a_0]

- a_0 is evaluated by Lüscher's formula outside the interaction range, and by $H_L(k;k)$ inside the interaction range
  - ◇ Both results agree well
  - ◇ cf. CP-PACS(2005) employs Lüscher's formula with Wall sources

$$a_0/m_\pi = \tan\delta(k)/(k m_\pi) + O(k^2)$$

$$\tan\delta(k) = \frac{-\sin(k x_{\rm ref})}{4\pi x_{\rm ref}\,\phi(\mathbf{x}_{\rm ref};k)/H_L(k;k) + \cos(k x_{\rm ref})}$$

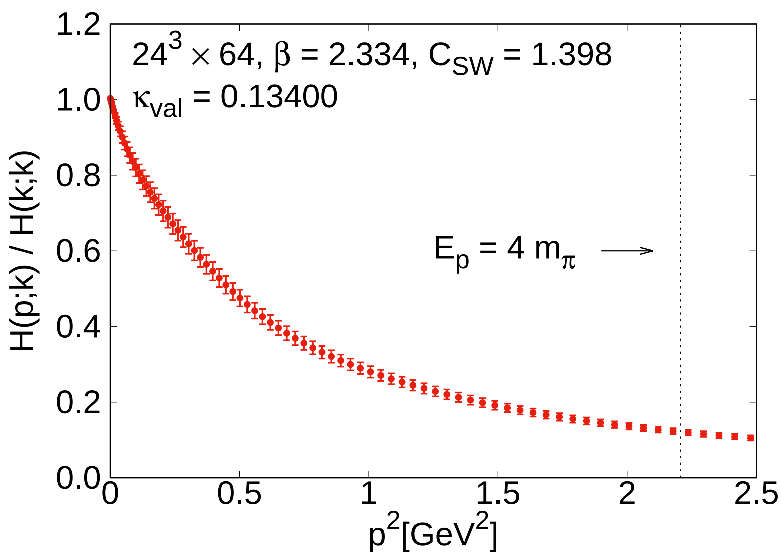


[Additional output: half off-shell scattering amplitude $H(p;k)$]

- $H(p;k)$ can be estimated by lattice QCD
  - ◇ $H(p;k)$ can be supplemental input to theoretical models of hadrons
  - ◇ NB. $H(p;k)$ / $H(k;k)$ is available below 4π threshold, although there is no true inelastic threshold in quenched QCD (quenched artificial inelastic effects may appear)

$$H_L(p;k) = -\sum_{\mathbf{x}\in L^3} j_0(px)(\Delta + k^2)\phi(\mathbf{x};k)$$

$$H(p;k)/H(k;k) = H_L(p;k)/H_L(k;k)$$



[Remark: LSZ reduction formula in momentum space]

- $H_L(p;k)$ can be calculated using LSZ reduction formula in momentum space, instead of laplacian Δ. cf. J.Carbonell and V.A.Karmanov(2016)
  - ◇ Care is needed. If we cut the integration range of $H(p;k)$ at the interaction range R, a surface term appears in general.

$$
\begin{aligned}
H(p;k) &= -\int_{-\infty}^{\infty} d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}} (\Delta + k^2)\phi(\mathbf{x};k) \\
&= -\int_{-R}^{R} d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}} (\Delta + k^2)\phi(\mathbf{x};k) \qquad \because\ (\Delta + k^2)\phi(\mathbf{x};k) = 0 \quad \text{for } x > R \\
&\Downarrow \quad \text{partial integration} \\
&= (p^2 - k^2)\int_{-R}^{R} d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{x};k) + [\text{surface term}]_{-R}^{R}
\end{aligned}
$$

At on-shell($p = k$),

$$
\begin{aligned}
H(k;k) &= [\text{surface term}]_{-R}^{R} \\
&= \frac{4\pi}{k} e^{i\delta(k)} \sin\delta(k) \qquad // \ R \text{ dependence vanishes}
\end{aligned}
$$

[Remark: LSZ reduction formula in momentum space(continued)]

- $H_L(p;k)$ using lattice laplacian Δ agrees with that using LSZ reduction formula at $p_i = (2\pi/L)n_i, n_i \in \mathbb{Z}$ with periodic boundary condition, where (surface term) = 0

$$
\begin{aligned}
H_L(p;k) &:= -\sum_{-L/2}^{L/2} e^{-i\mathbf{p}\cdot\mathbf{x}}(\Delta + k^2)\phi(\mathbf{x};k) \quad \text{supposing } R < L/2 \\
&= -\sum_{-L/2}^{L/2} e^{-i\mathbf{p}\cdot\mathbf{x}} \sum_{i=1}^{3} (\phi(\mathbf{x}+\hat{i};k) + \phi(\mathbf{x}-\hat{i};k) - 2\phi(\mathbf{x};k)) - k^2\phi(\mathbf{x};k) \\
&\Downarrow \quad \text{periodic boundary condition: } \phi(x_i = L/2+1;k) = \phi(x_i = -L/2;k) \\
&= -\sum_{-L/2}^{L/2} e^{-i\mathbf{p}\cdot\mathbf{x}} \sum_{i=1}^{3} (e^{ip_i} + e^{-ip_i} - 2)\phi(\mathbf{x};k) - k^2\phi(\mathbf{x};k) \\
&= (\tilde{p}^2 - k^2)\phi(\mathbf{p};k) \qquad \text{i.e. (surface term) } = 0
\end{aligned}
$$

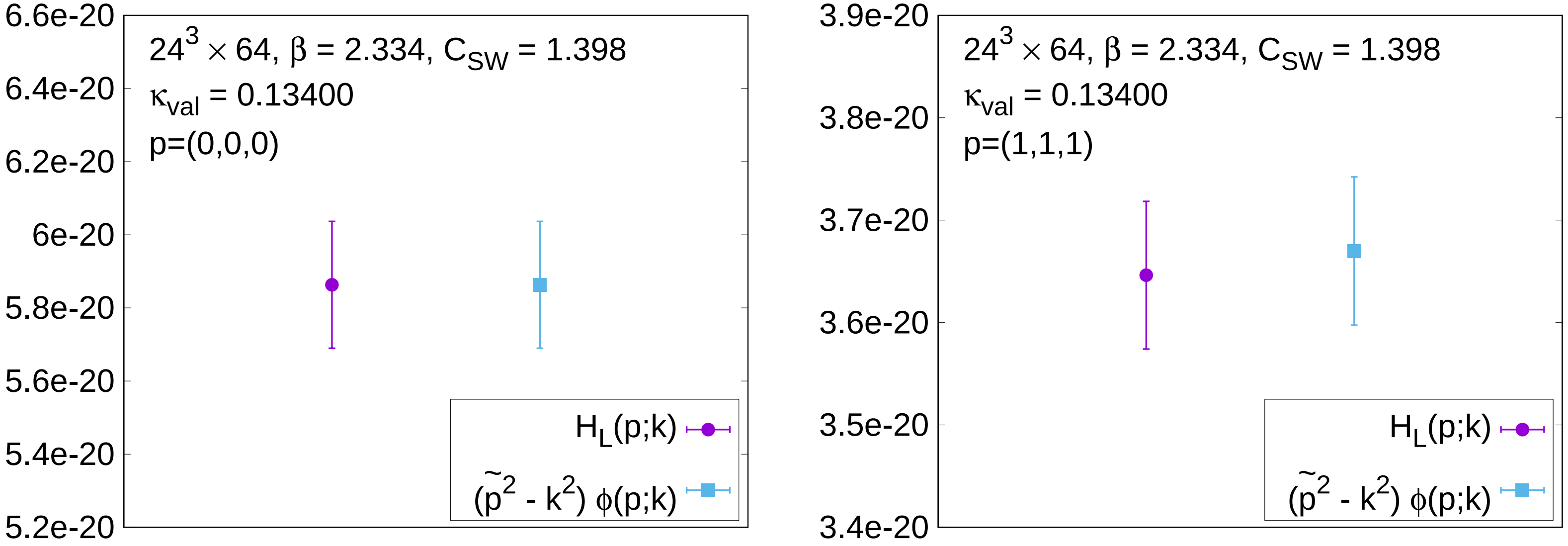
where

$$\phi(\mathbf{p};k) = \sum_{\mathbf{x}\in L^3} e^{-i\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{x};k)$$

$$\tilde{p}^2 = \sum_{i=1}^{3} \tilde{p}_i^2, \quad \tilde{p}_i := \frac{2}{a}\sin\frac{ap_i}{2}, \quad p_i = (2\pi/L)n_i, \quad n_i \in \mathbb{Z}$$

[Remark: LSZ reduction formula in momentum space(continued)]

- $H_L(p;k)$ from lattice laplacian Δ agrees with that using LSZ reduction formula at $p_i = (2\pi/L)n_i, n_i \in \mathbb{Z}$ with periodic boundary condition $\rightarrow$ Numerically checked to be correct



5 Summary

We evaluate a scattering length a_0 of $I = 2$ $\pi\pi$ system in the quenched lattice QCD using Bethe-Salpeter wave function not only outside the interaction range but also inside the interaction range

- No fake plateau is observed for our case
- Consistency is checked
  - ◇ Our result of a_0 using the scattering amplitude inside the interaction range agrees with the value of Lüscher's finite volume method using data outside the interaction range
- Additional output is obtained
  - ◇ A half off-shell scattering amplitude $H(p; k)$ can be estimated by lattice QCD, which can be supplemental input to theoretical models of hadrons

[Future work]
Apply our strategy to

- More realistic case (ex. $N_f = 2 + 1$ full QCD on the physical point)
- More complicated system (other 2-body system with not only up,down, strange but also charm quarks, and hopefully 3-body system)