DISPERSIVE APPROACH TO THREE-PARTICLE SYSTEMS

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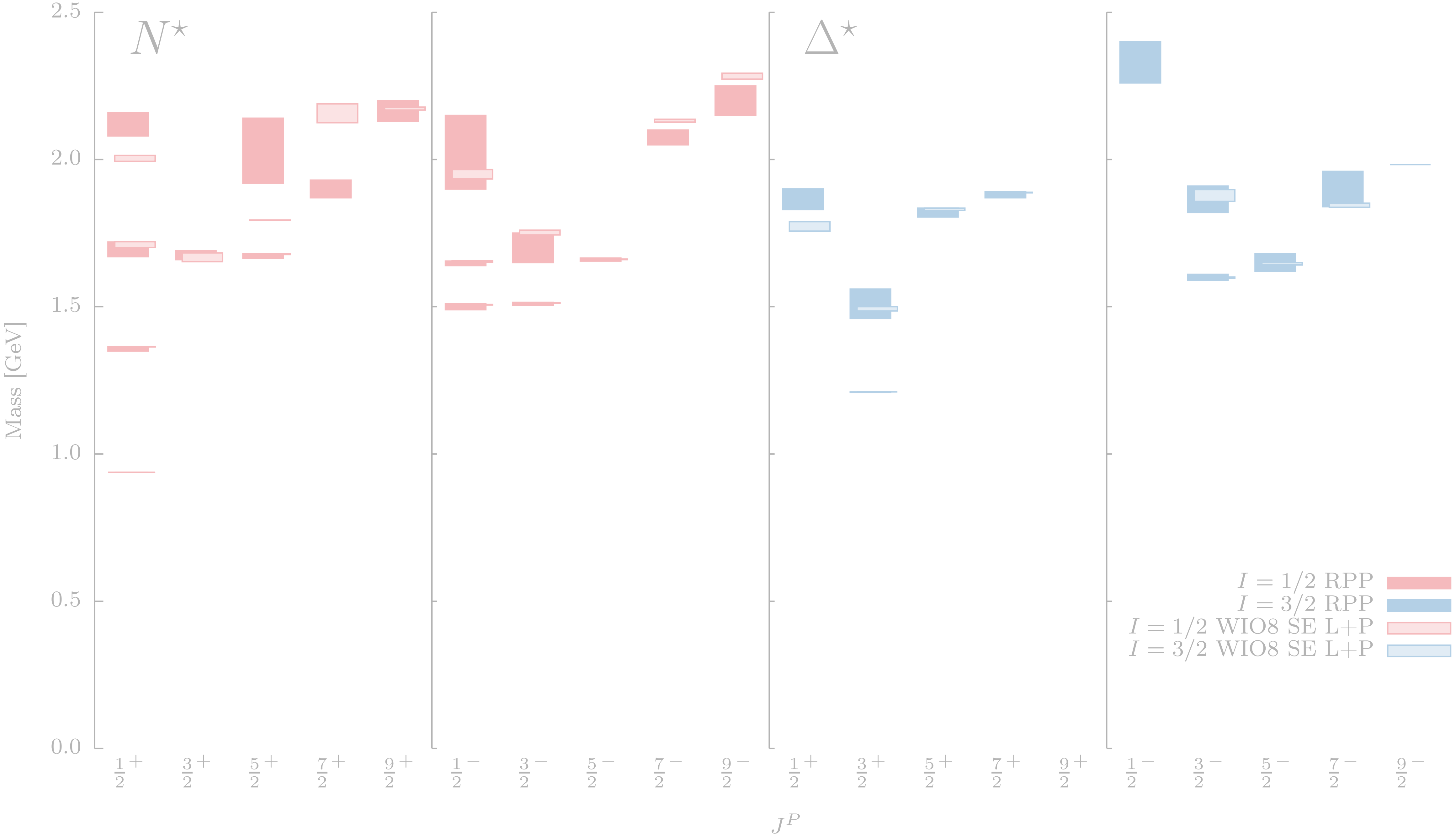
INT WORKSHOP INT-18-70W
MULTI-HADRON SYSTEMS FROM LATTICE QCD

FEBRUARY 5-9, 2018

Outline

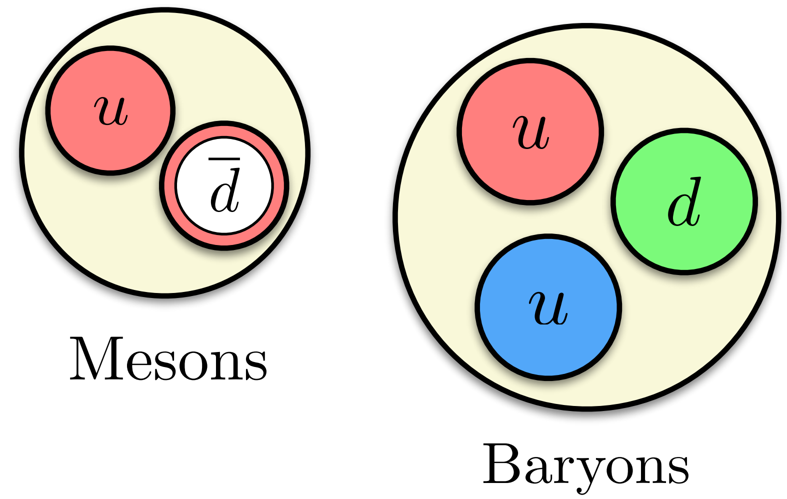
- Hadron Spectroscopy, and Phenomenology
- Review of **2→2** Reactions
- **3→3** Scattering Phenomenology
- Opportunities and Future Directions

Hadron Spectroscopy, and Phenomenology

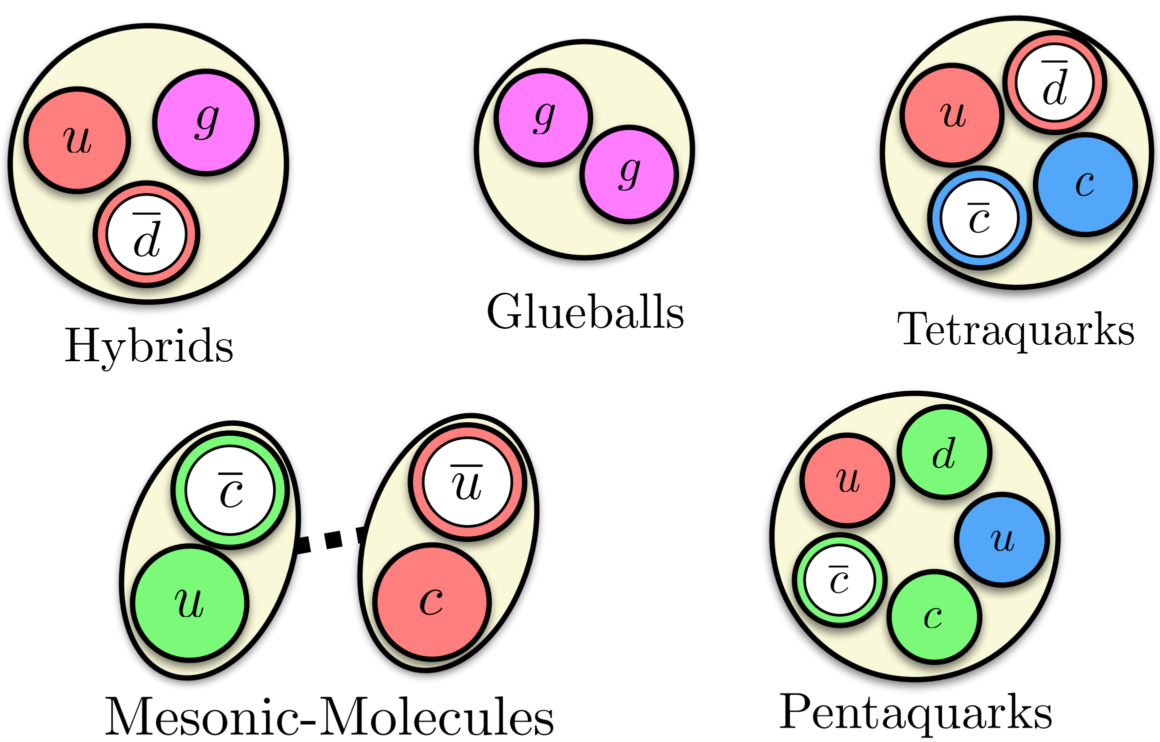


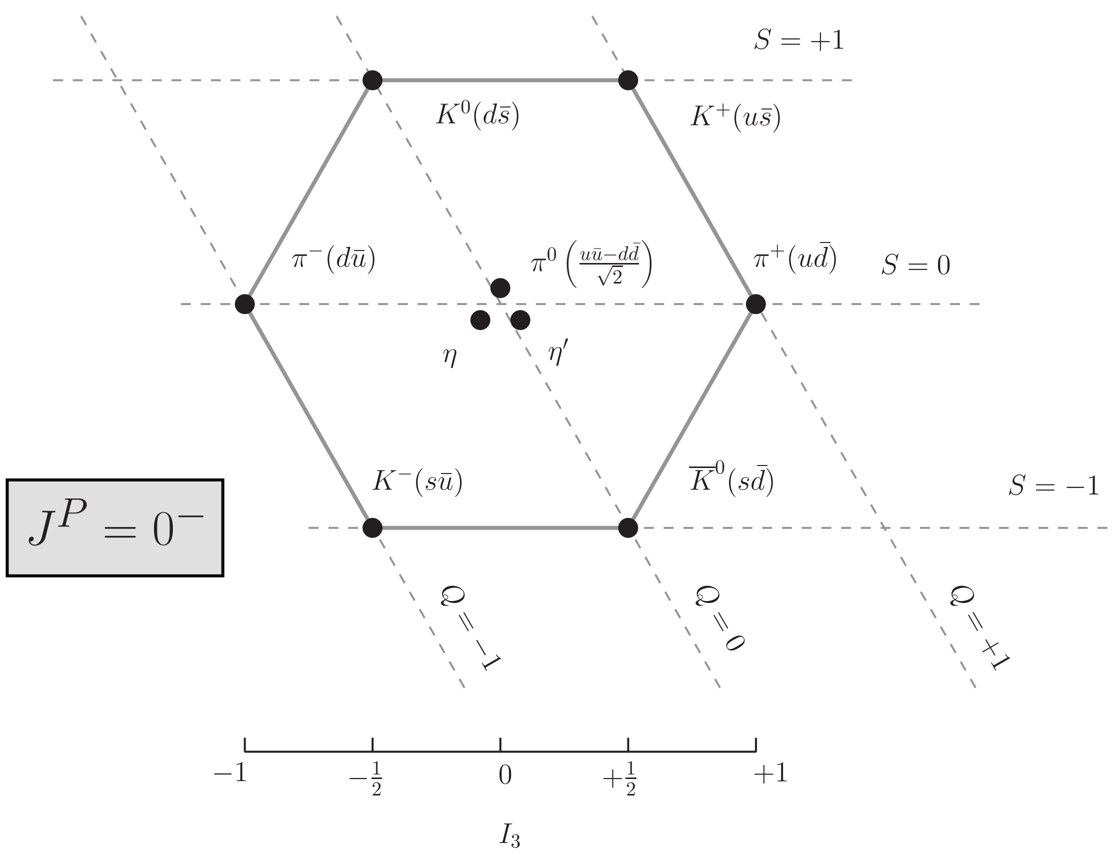
Hadron Spectroscopy

Constituent quark model has been successful in classifying the hadron spectrum, and gives guidance to the QCD substructure



Search for exotics (non-quark model) is goal of many experiments (*e.g*. GlueX), and many new states have been discovered (*XYZP*'s)





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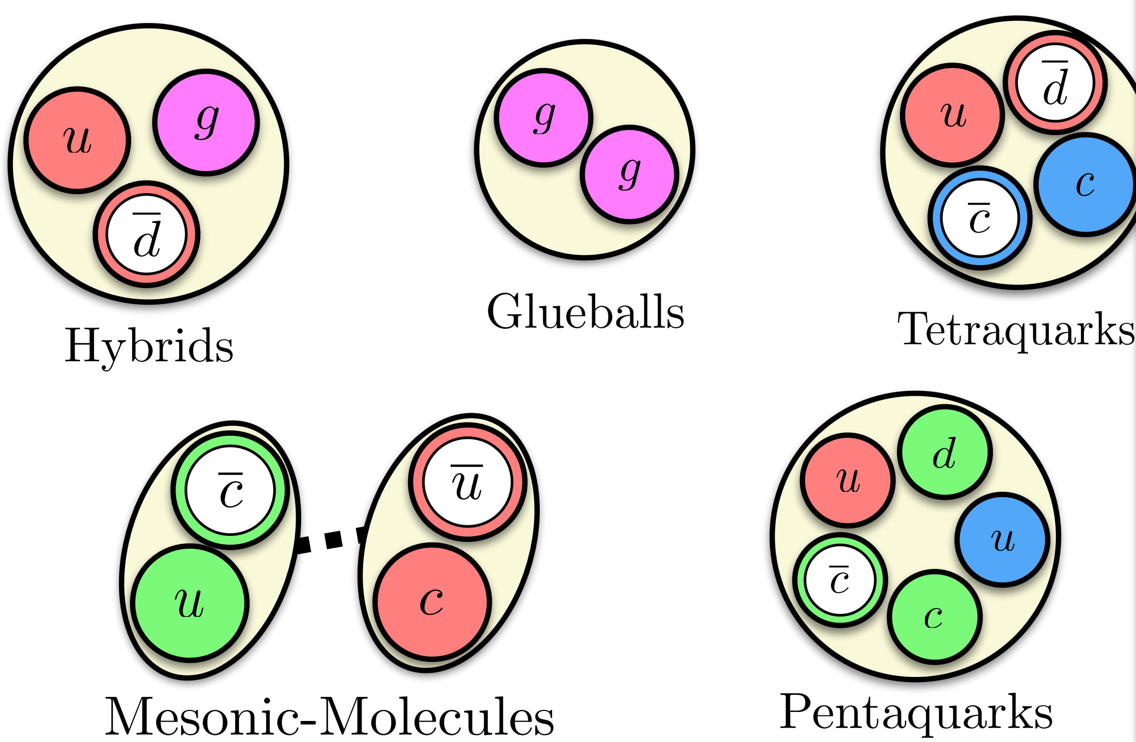
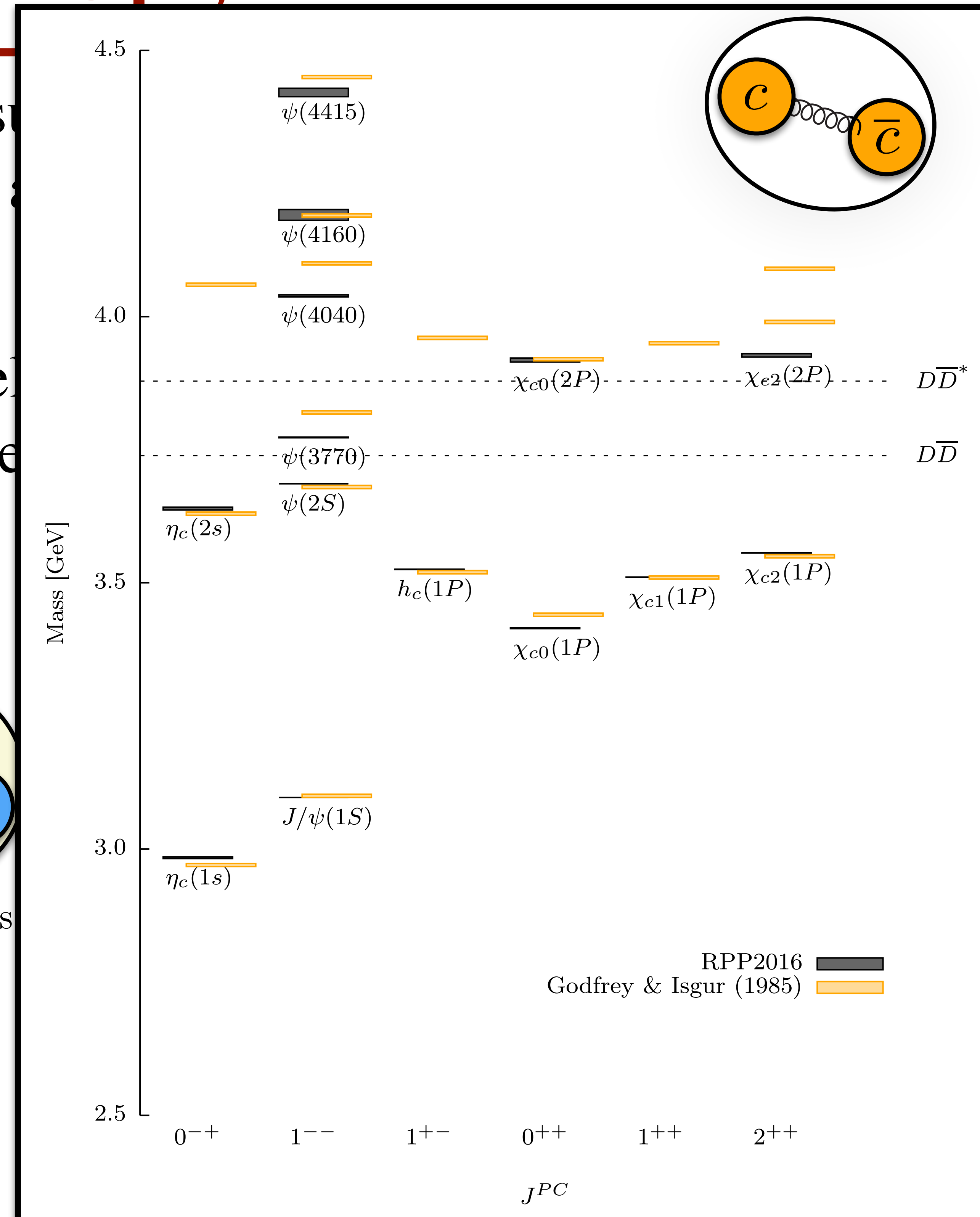




Why 3-body Physics?

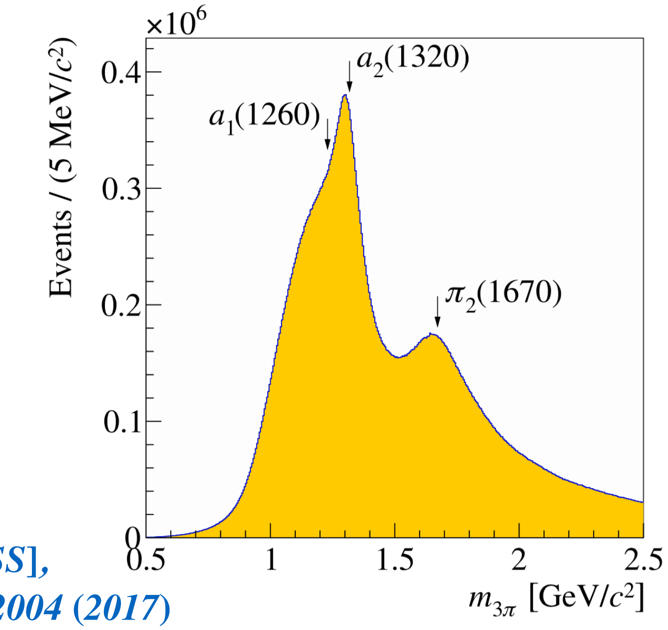
Advancements in theory and experiment require revisiting **3**-body hadron scattering

Lattice QCD has been computing scattering amplitudes - Requires **3**-body formalism for continuing amplitudes to complex energies to investigate higher mass resonances

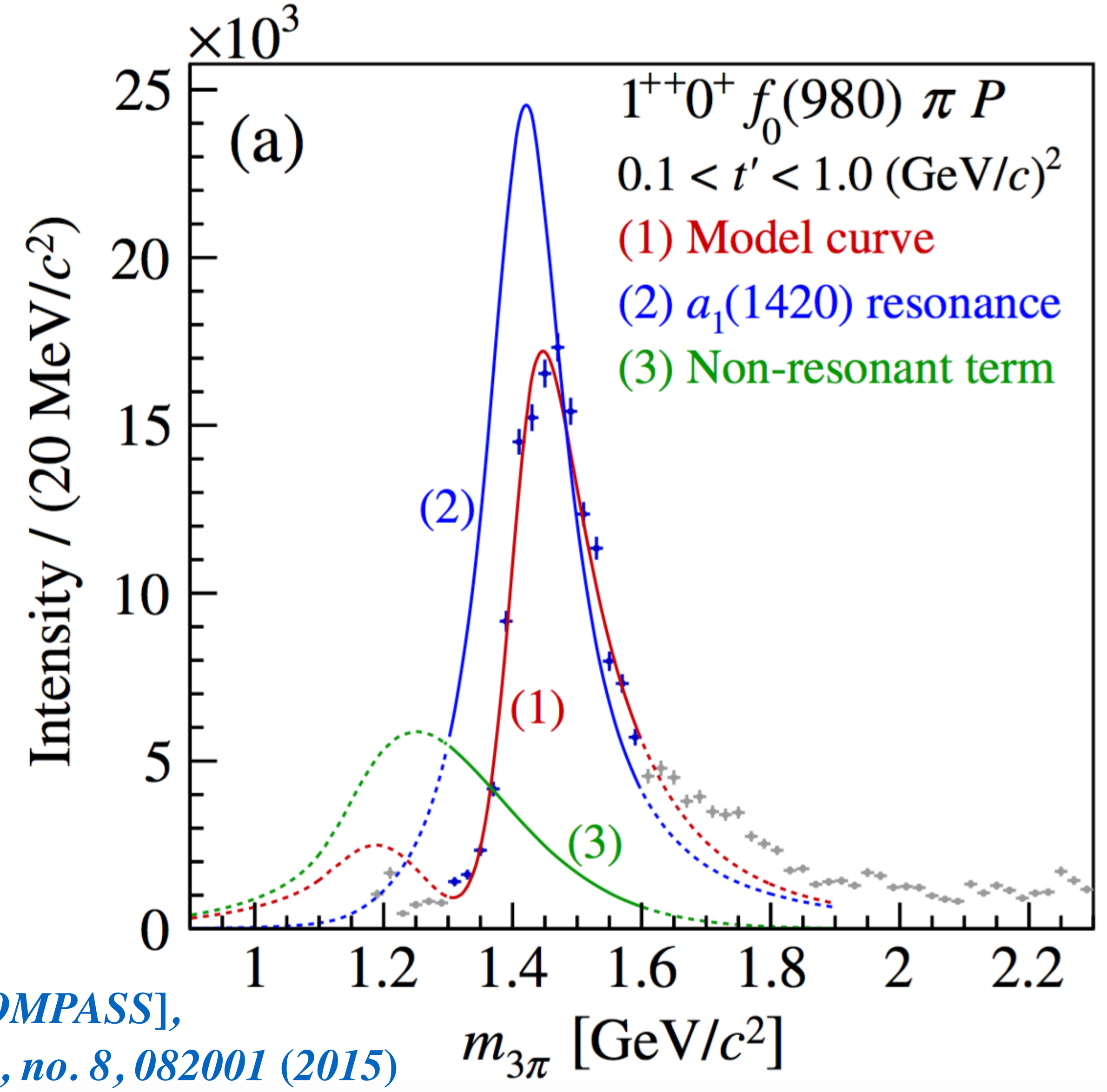
New High-precision, high-statistics data collected on many **3**-body meson systems - COMPASS, GlueX, …

New (and old) mysteries in the light-hadron sector, *e.g.*, $a_1(1420)$

$$a_1(1420) \to \pi^-\pi^-\pi^+$$



C. Adolph et al. [COMPASS], *Phys. Rev. D 95, no. 3, 032004 (2017)*



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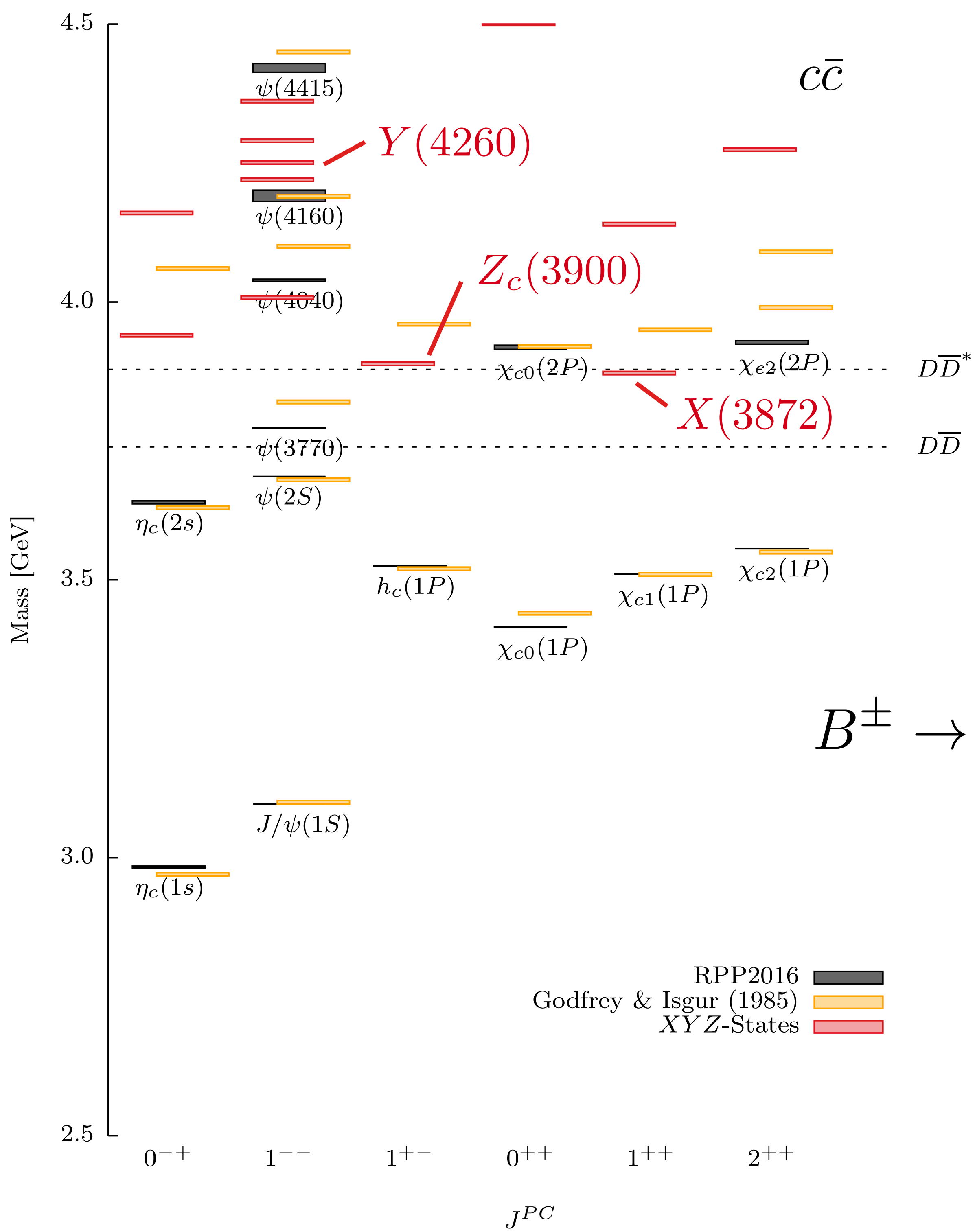


In heavy quarkonia, have discovered many non-quark model states (*XYZs*)

Many of these are found in **3**-body decays, near thresholds - could **3**-body effects contribute to the nature of these states?

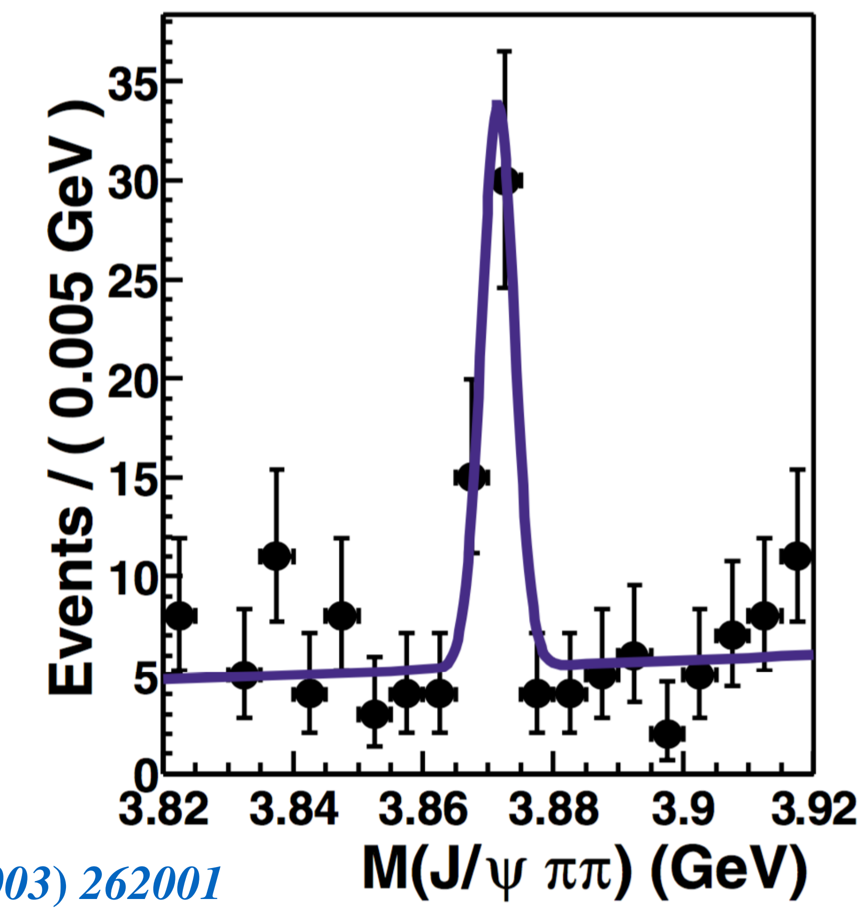
$$X(3872)/Z_c(3900) \to D\bar{D}\pi$$

$$B^{\pm} \to K^{\pm}\pi^{+}\pi^{-}J/\psi$$

$$X(3872)$$

$$\downarrow$$

$$(u\bar{u}c\bar{c})$$



S.K. Choi [Belle], Phys. Rev. Lett. 91 (2003) 262001

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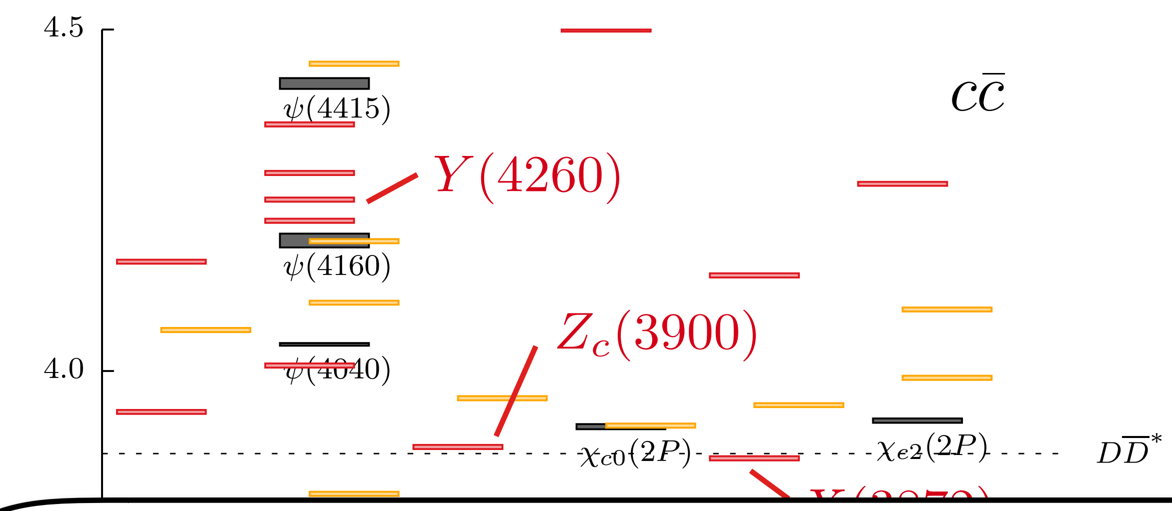

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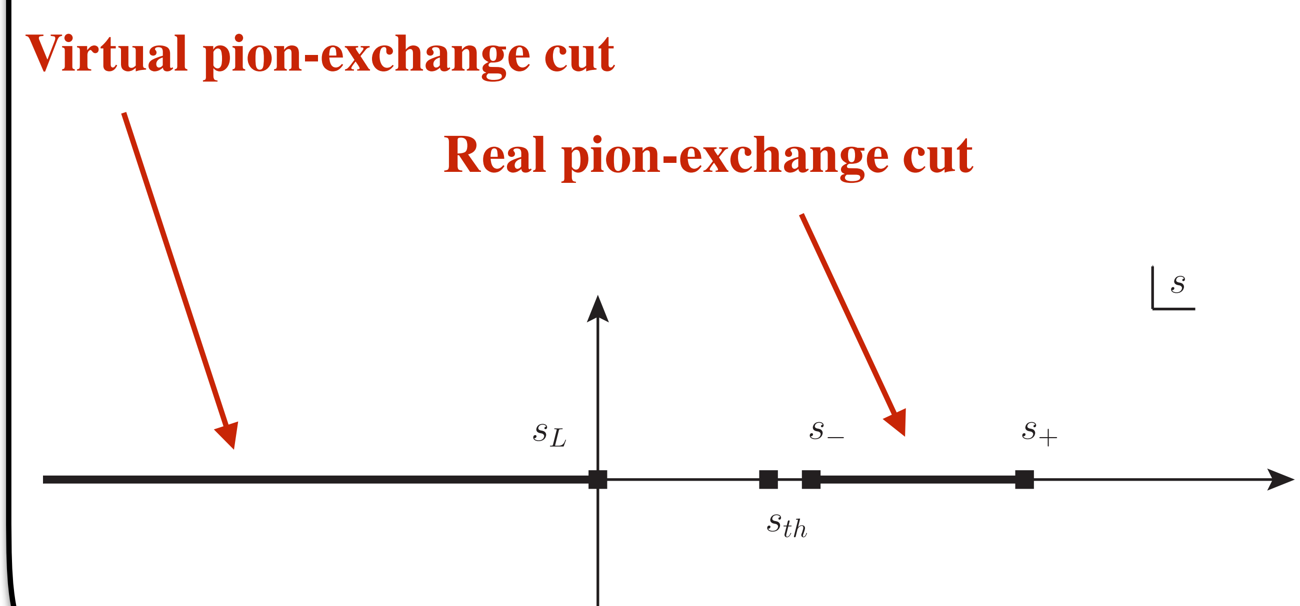
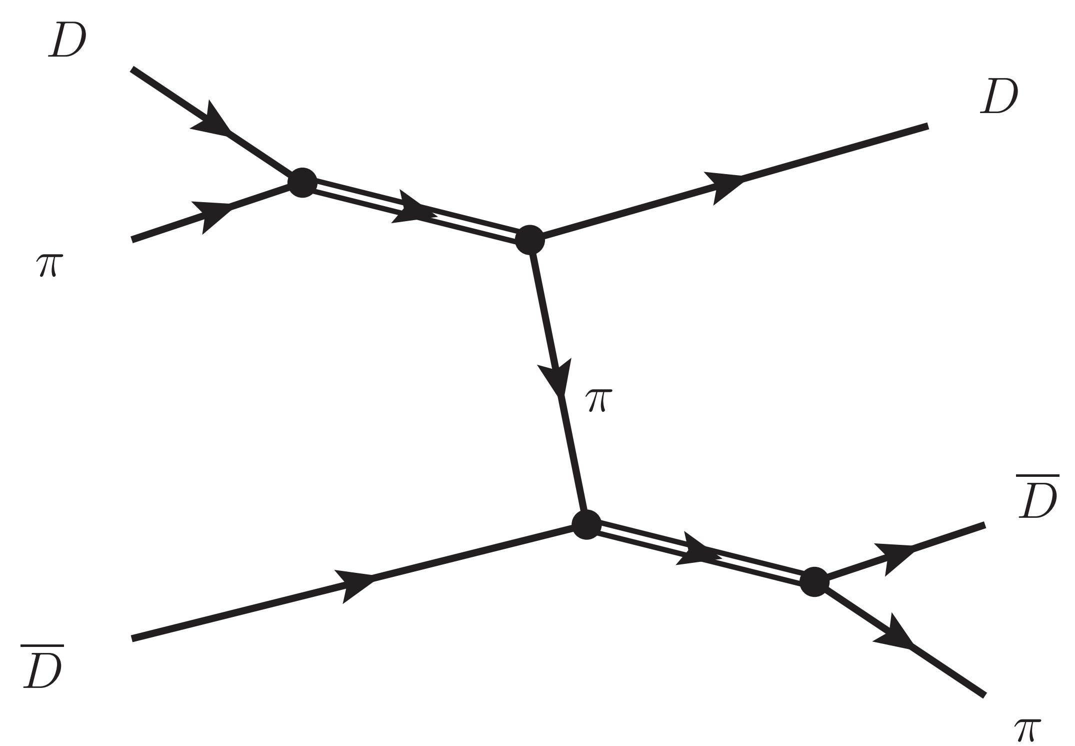
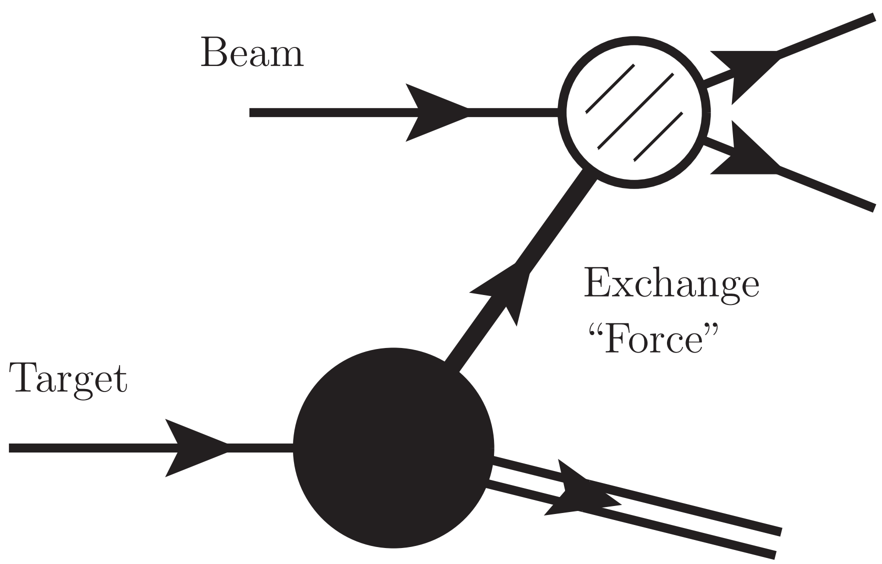


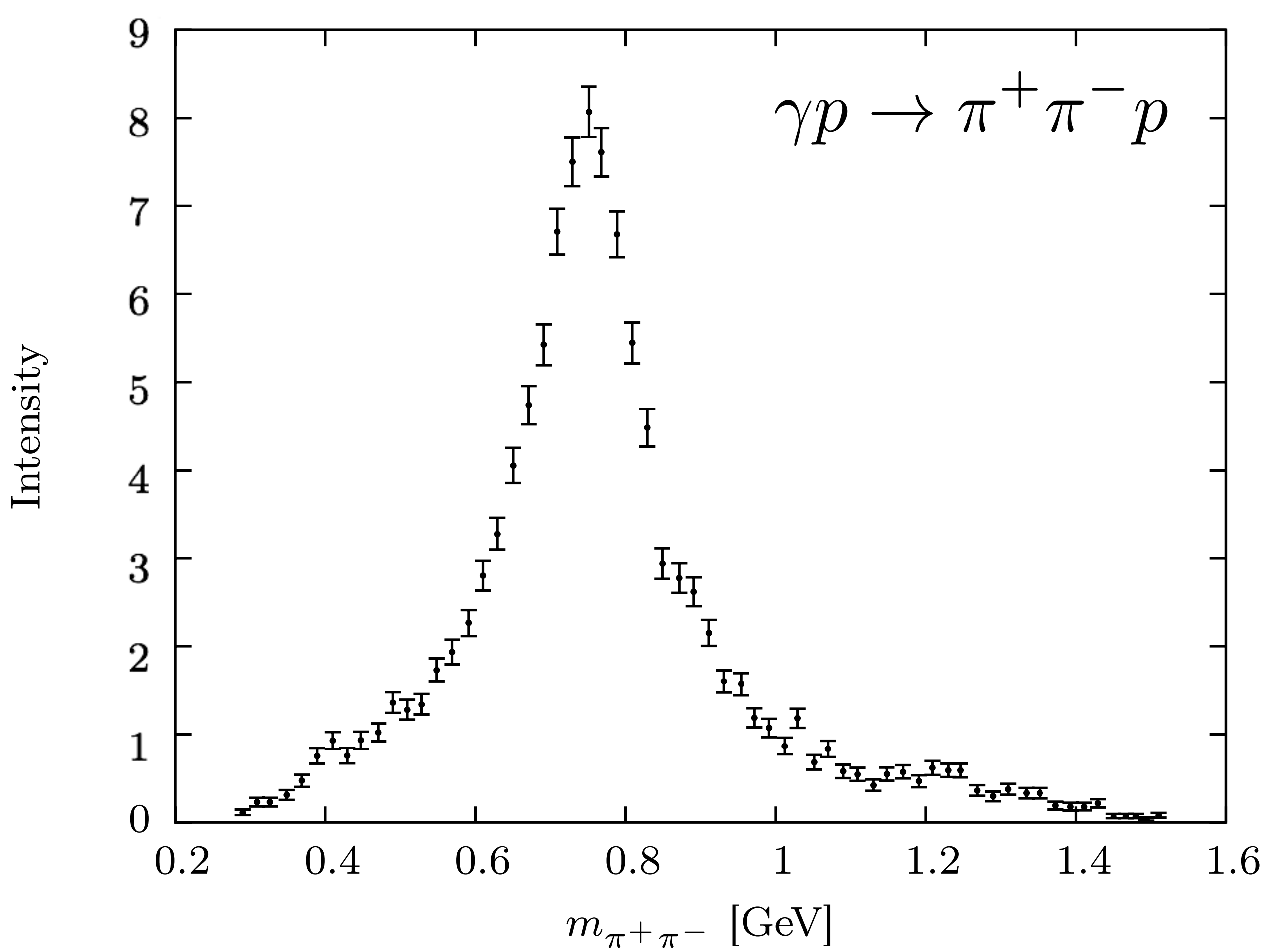


Scattering Phenomenology

Model independent methods such as S-matrix theory provide constraints for reaction amplitudes: **Unitarity**, **Analyticity**, **Crossing**, and **Poincaré Symmetry**

Data (Experimental/Lattice)





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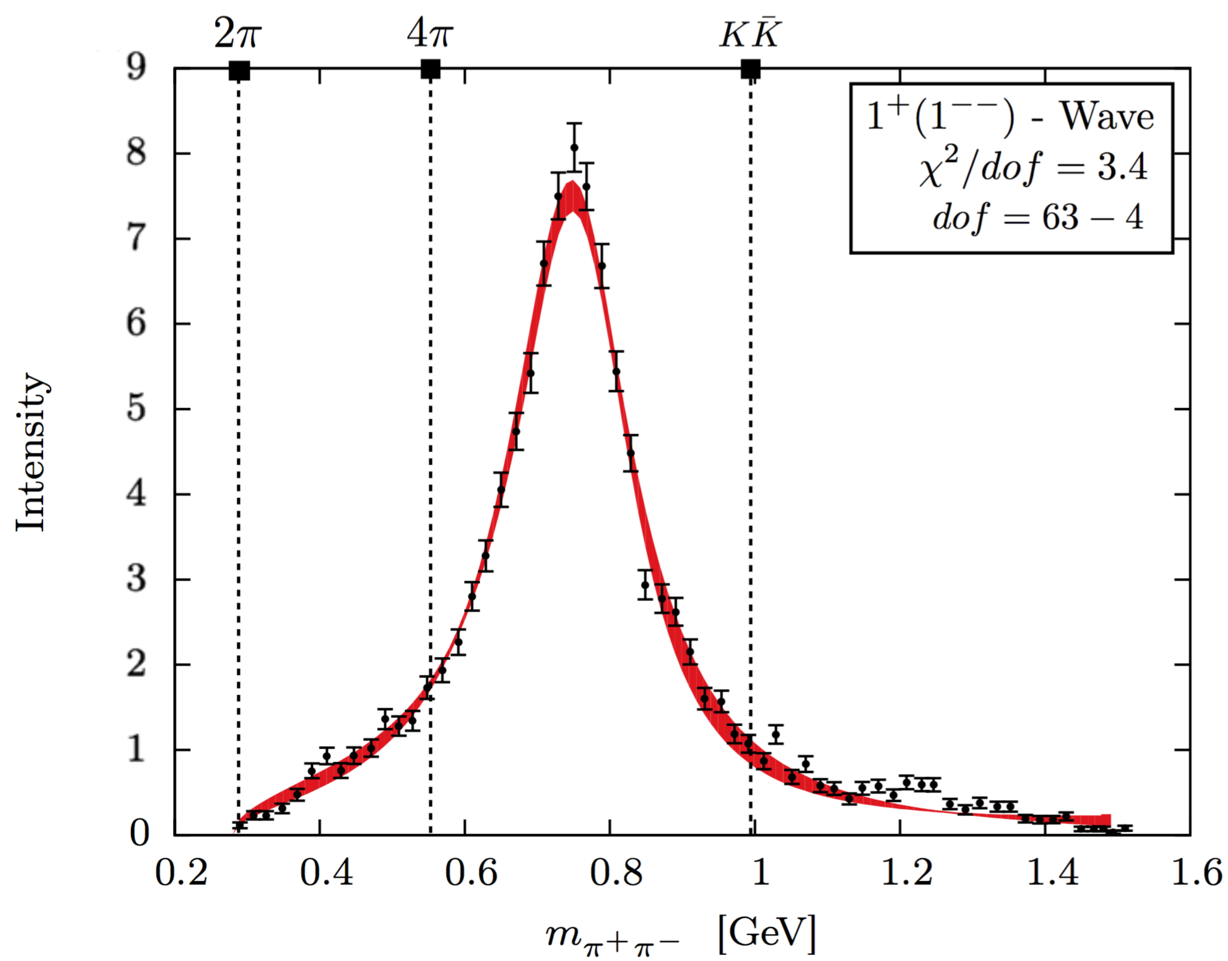
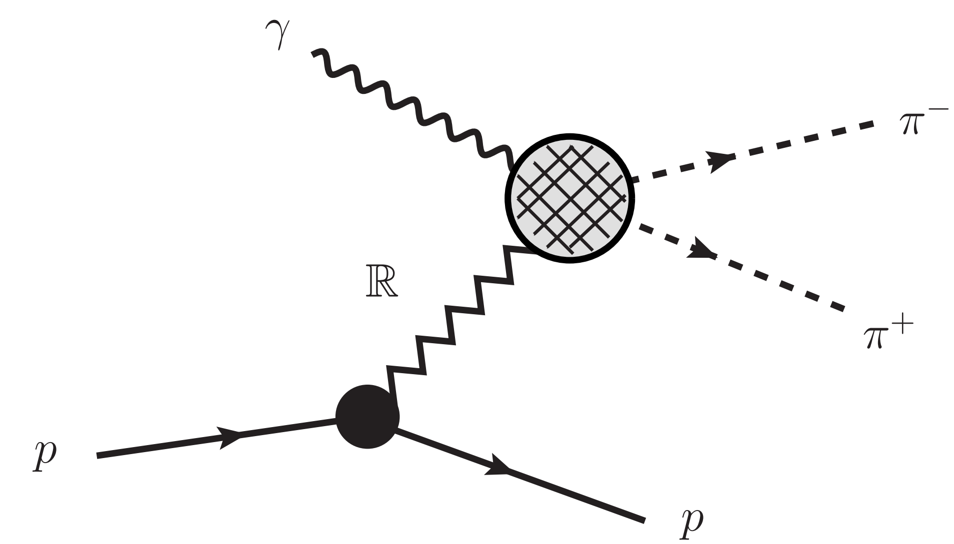
Amplitude Analysis





Amplitude Model

$$t_\ell(s) = \frac{\mathcal{N}}{c_1 + c_2 s - \frac{s}{\pi}\int_{s_{th}}^{\infty} ds' \frac{\rho(s')}{s'(s'-s)}}$$

Functional form fixed from S-matrix constrains

Scattering Phenomenology

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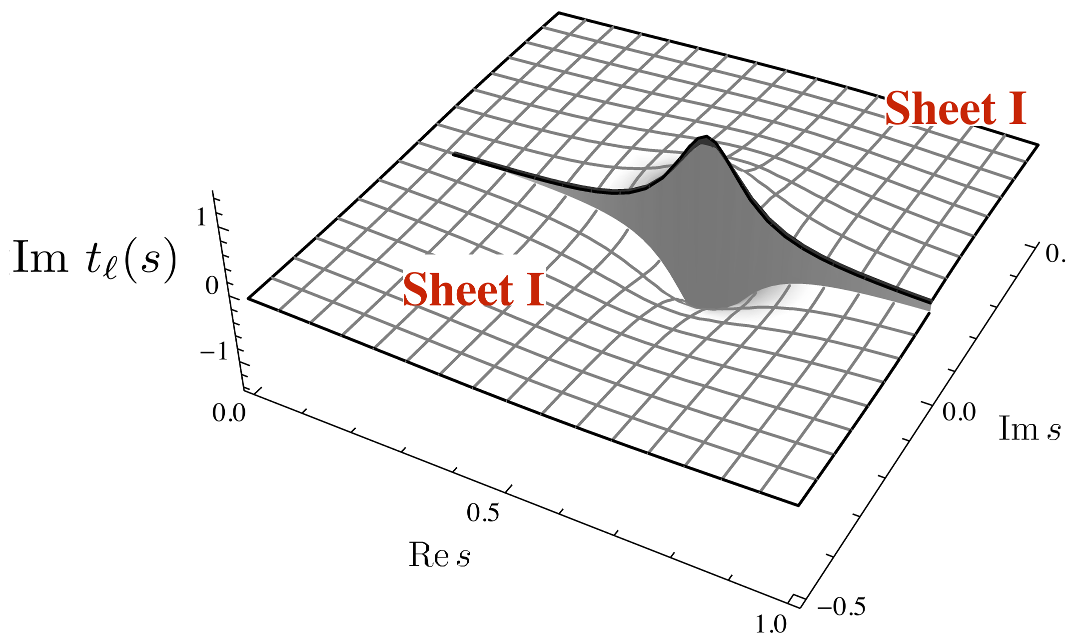
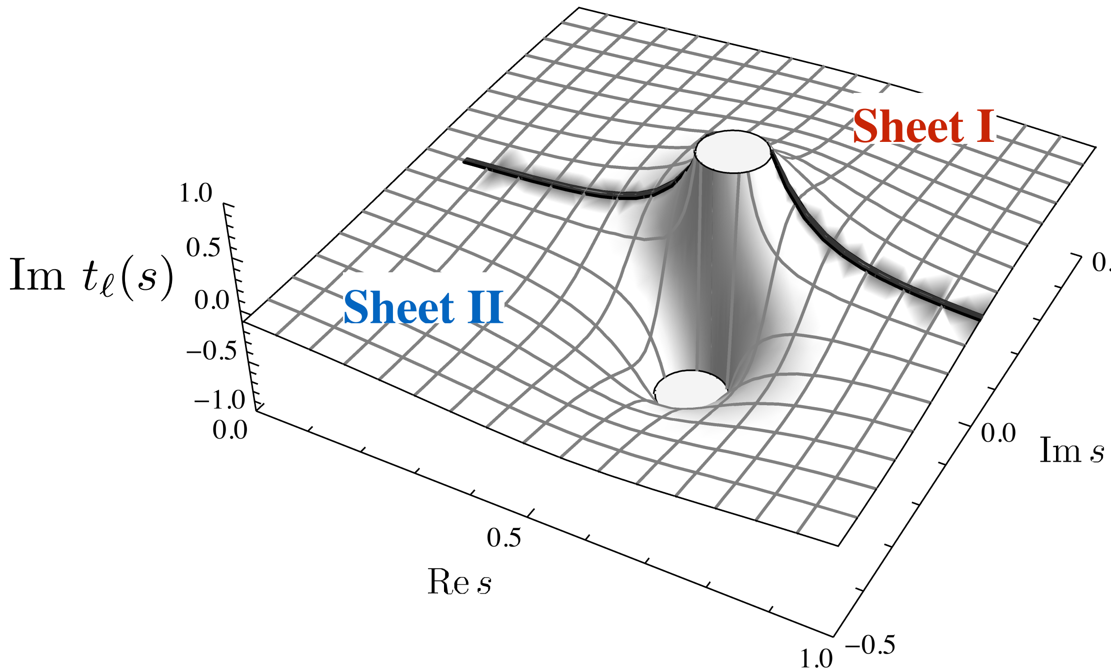
Analytic Continuation

$$t_\ell(s) \to t_\ell^{\mathrm{II}}(s) = \frac{t_\ell(s)}{1 + 2i\rho(s)t_\ell(s)}$$





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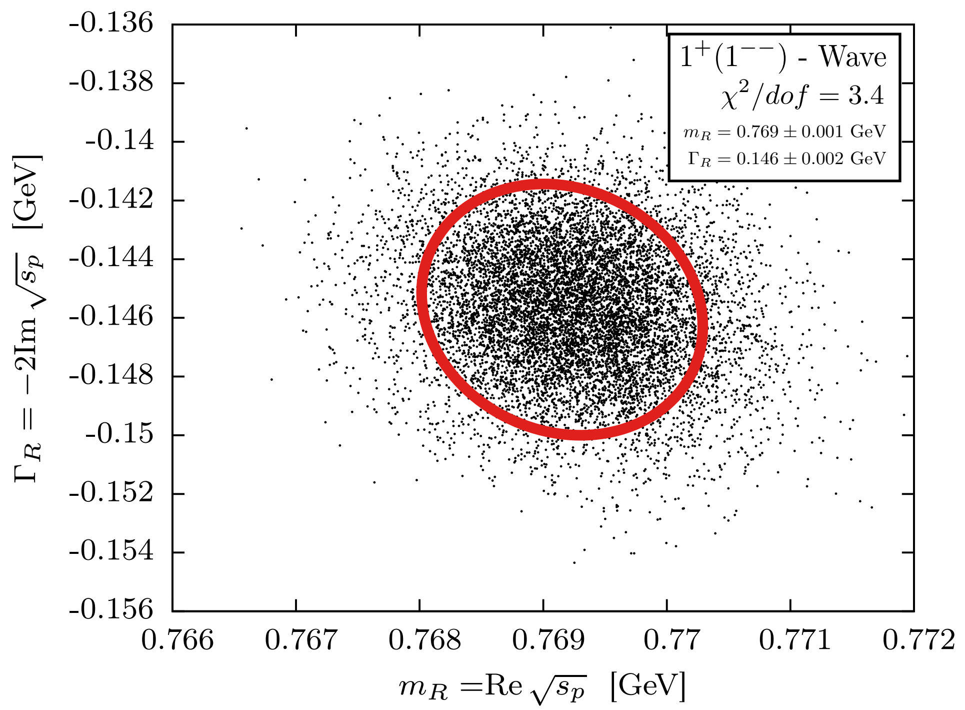
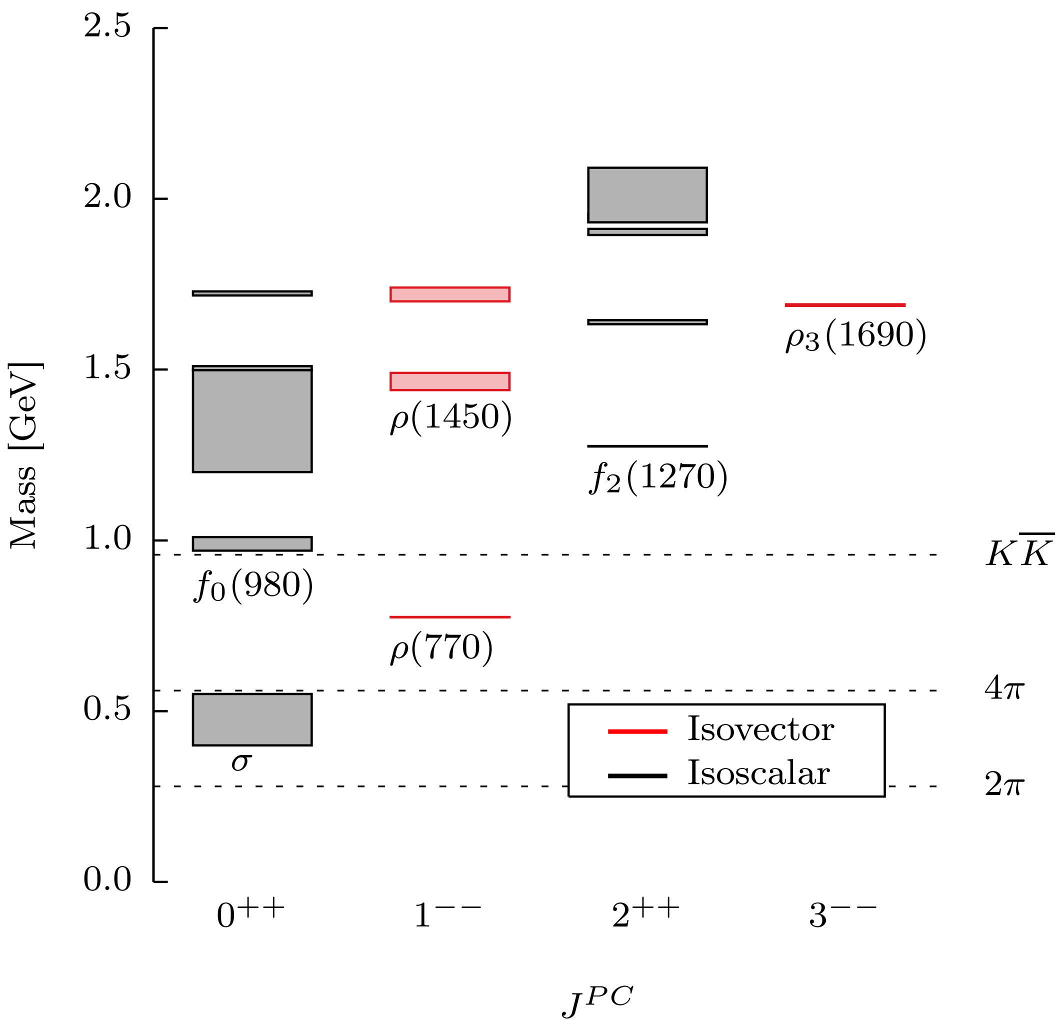
Resonance Parameters



$$m_R = 769 \pm 1\,\mathrm{MeV}$$

$$\Gamma_R = 149 \pm 2\,\mathrm{MeV}$$

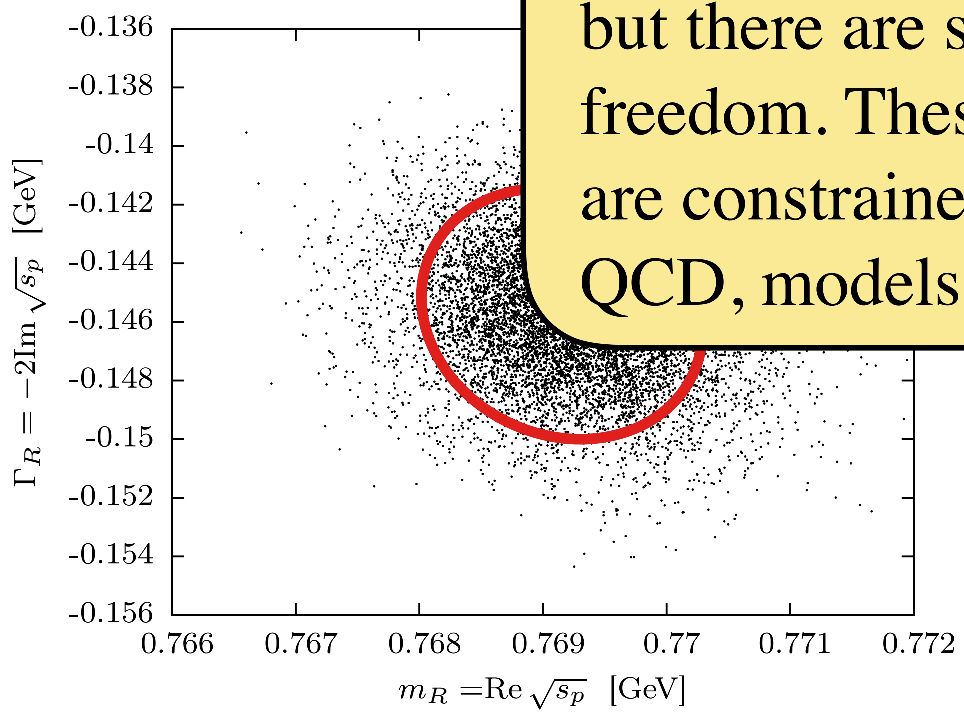


Scattering Phenomenology

Model independent methods such as S-matrix theory provide constraints for reaction amplitudes: **Unitarity**, **Analyticity**, **Crossing**, and **Poincaré Symmetry**

Resonance Parameters

Imposing these constraints gives us general forms of reaction amplitudes, but there are still some degrees of freedom. These degrees of freedom are constrained by experiment, lattice QCD, models, …



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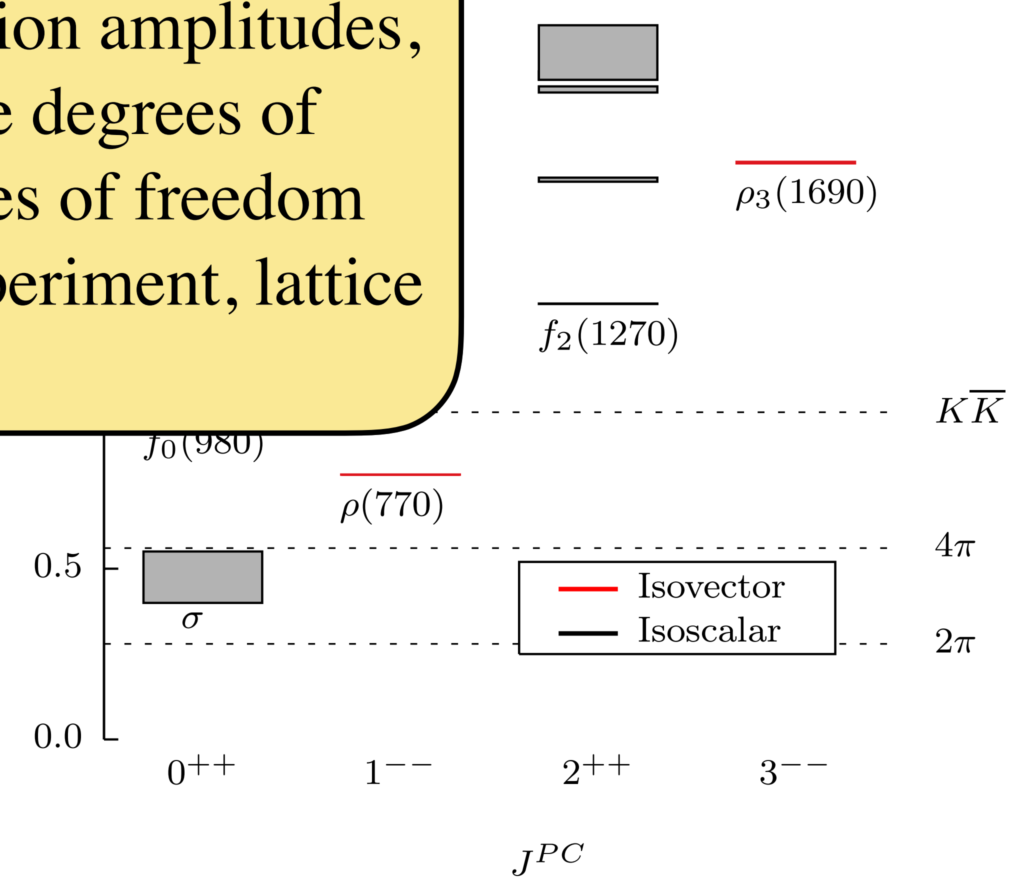
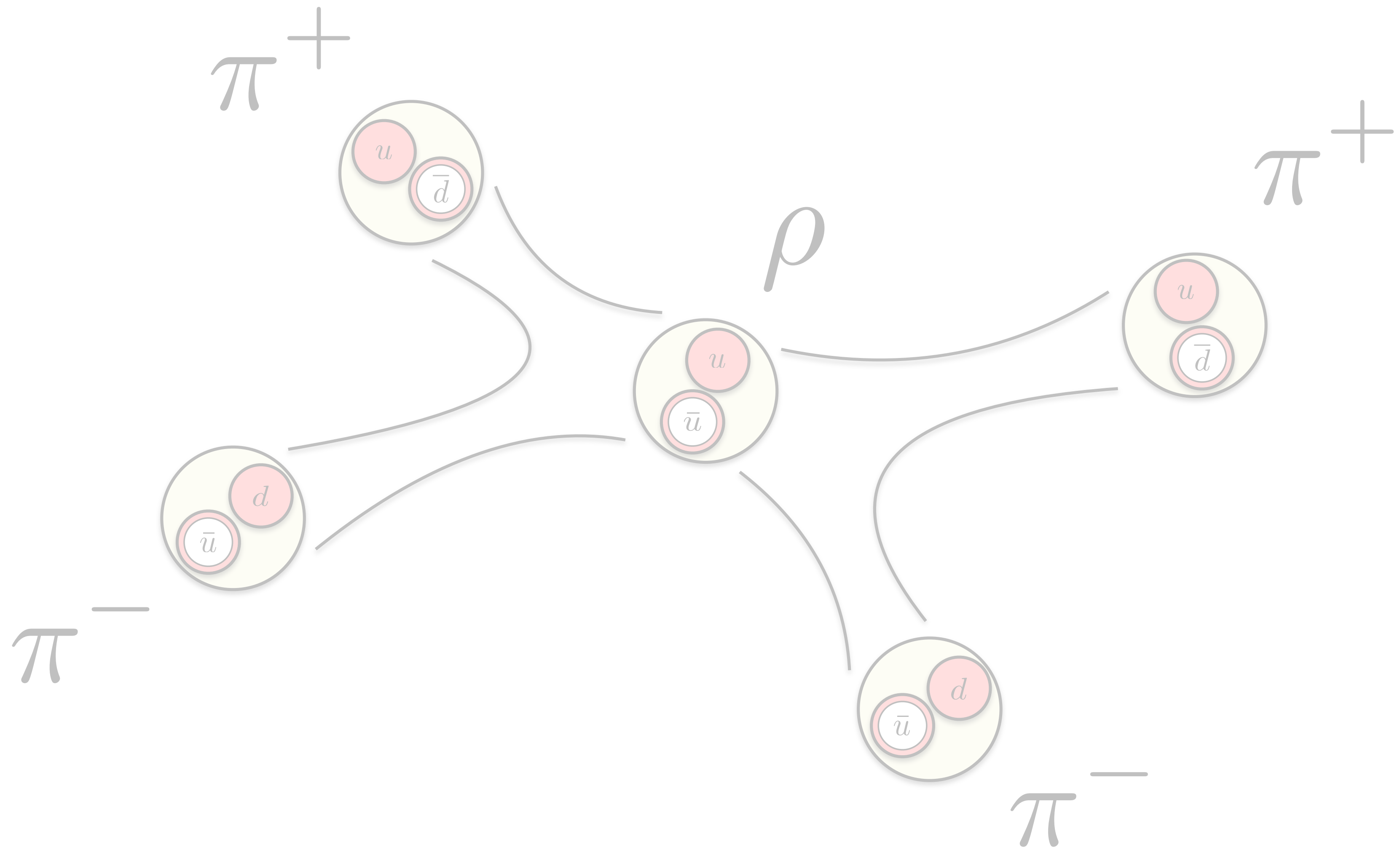


Review of 2→2 Reactions

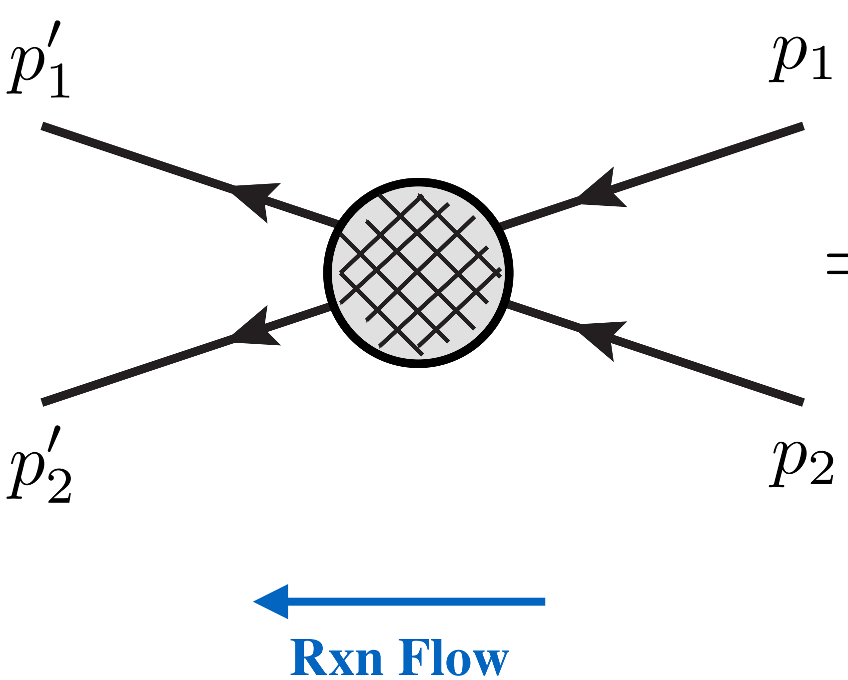


2→2 Elastic Scattering

Consider the elastic scattering of the **2→2** system $ab \to ab$, where a and b are distinguishable particles

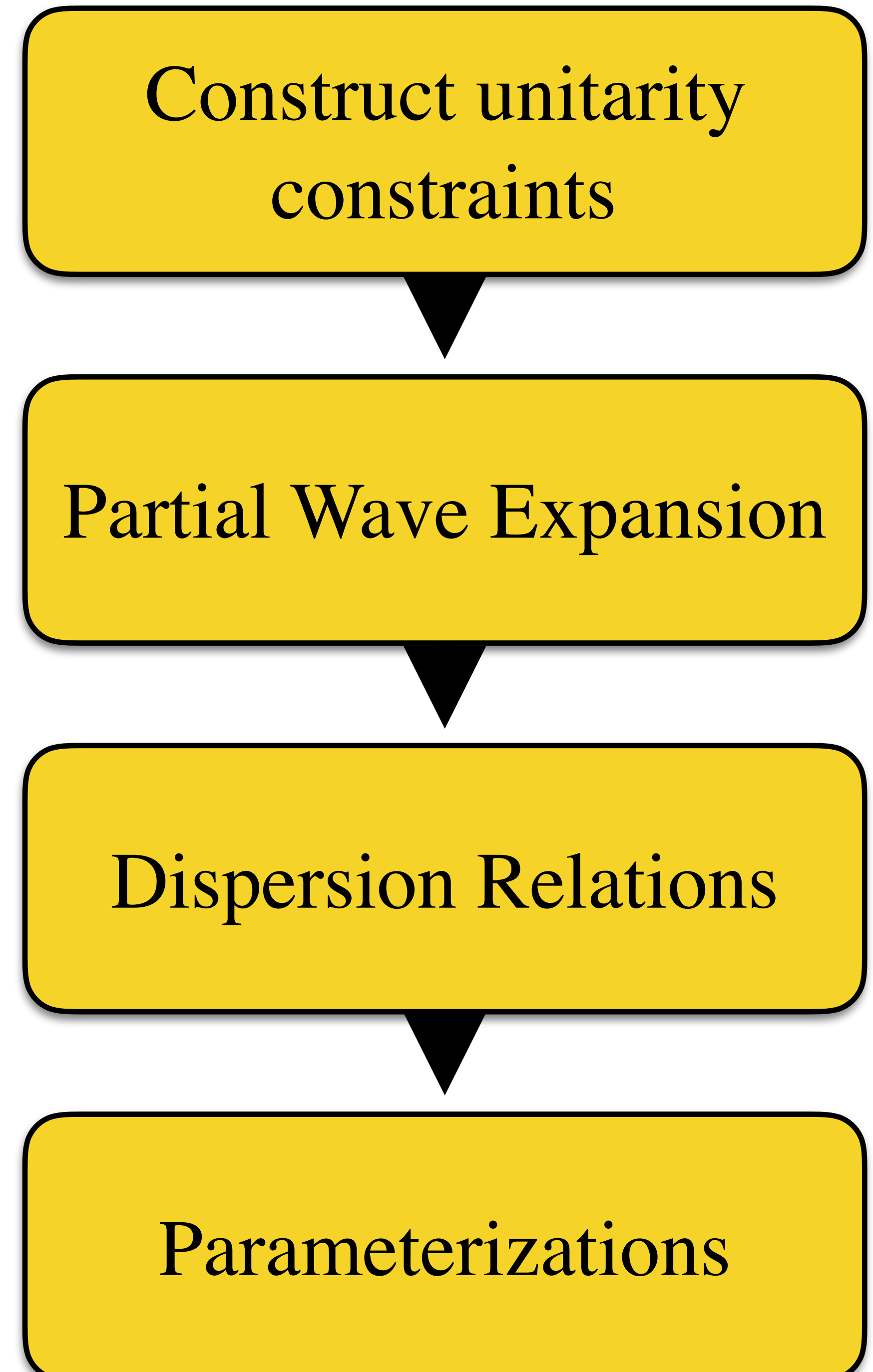
$$\langle \{\mathbf{p}'\} | \, T \, | \{\mathbf{p}\} \rangle = (2\pi)^4 \delta^{(4)}(P' - P) \mathcal{F}(\{\mathbf{p}', \mathbf{p}\})$$

Final State — **Initial State** — **2→2 Amplitude**



$$= \mathcal{F}(\{\mathbf{p}', \mathbf{p}\})$$

Relevant Kinematic Variables

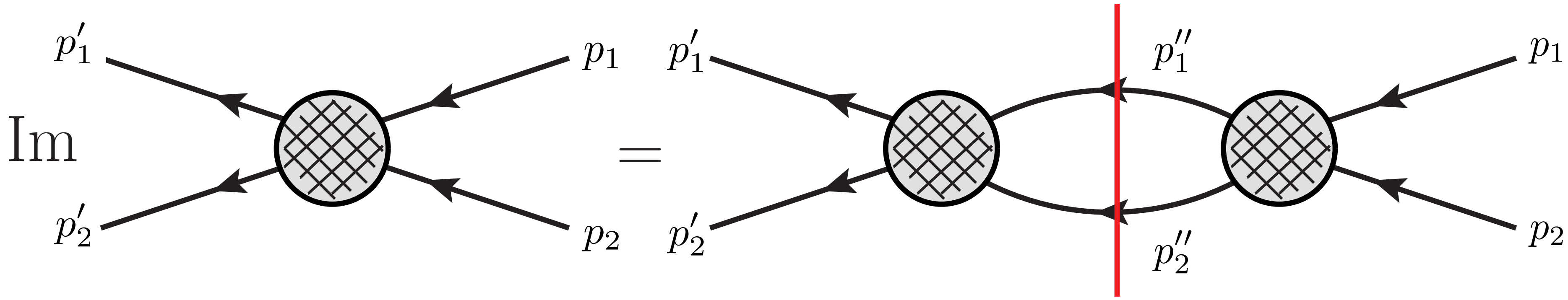


2→2 Elastic Scattering

Unitarity constrains the amplitude by fixing the imaginary part

Elastic Unitarity Relation ($s < s_{inelas}$)

$$\mathrm{Im}\ \mathcal{F}(\{\mathbf{p}',\mathbf{p}\}) = \rho_2(s)\int d\Omega_{\mathbf{p}''}\,\mathcal{F}^*(\{\mathbf{p}'',\mathbf{p}'\})\mathcal{F}(\{\mathbf{p}'',\mathbf{p}\})\Theta(s-s_{th})$$



Can reduce the unitarity relation by Partial Wave Expansion

$$\mathcal{F}(\{\mathbf{p}',\mathbf{p}\}) = \sum_{\ell=0}^{\infty}\left(\frac{2\ell+1}{4\pi}\right) f_\ell(s) P_\ell(\widehat{\mathbf{p}}'\cdot\widehat{\mathbf{p}})$$

$\widehat{\mathbf{p}}'\cdot\widehat{\mathbf{p}} = \cos\theta$

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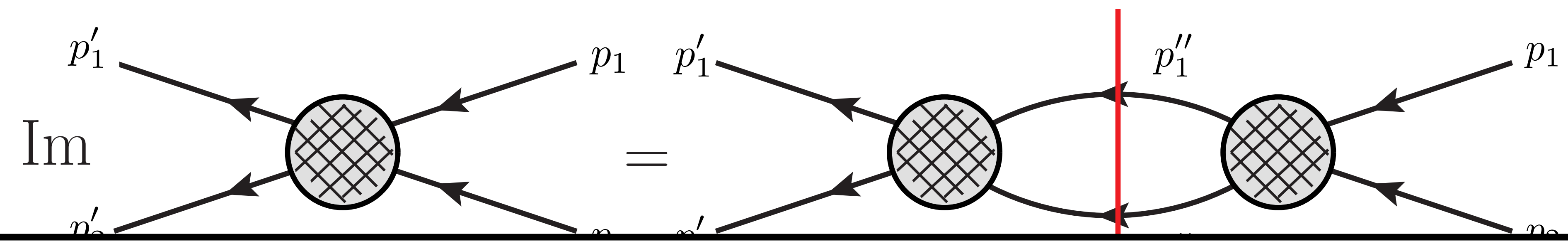


C

Partial wave unitarity relation is now algebraic

$$\text{Im}\ f_\ell(s) = \rho_2(s)|f_\ell(s)|^2\Theta(s-s_{th})$$

2→2 Elastic Scattering

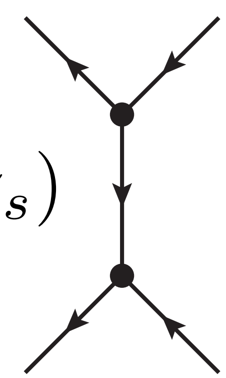
Dispersive representation for partial wave amplitudes

$$f_\ell(s) = \frac{1}{\pi}\int_{-\infty}^{s_L} ds' \, \frac{\text{Im}\ f_\ell(s')}{s'-s} + \frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \, \frac{\rho_2(s')|f_\ell(s')|^2}{s'-s}$$

Nonlinear constraint for the amplitude $f_l(s)$

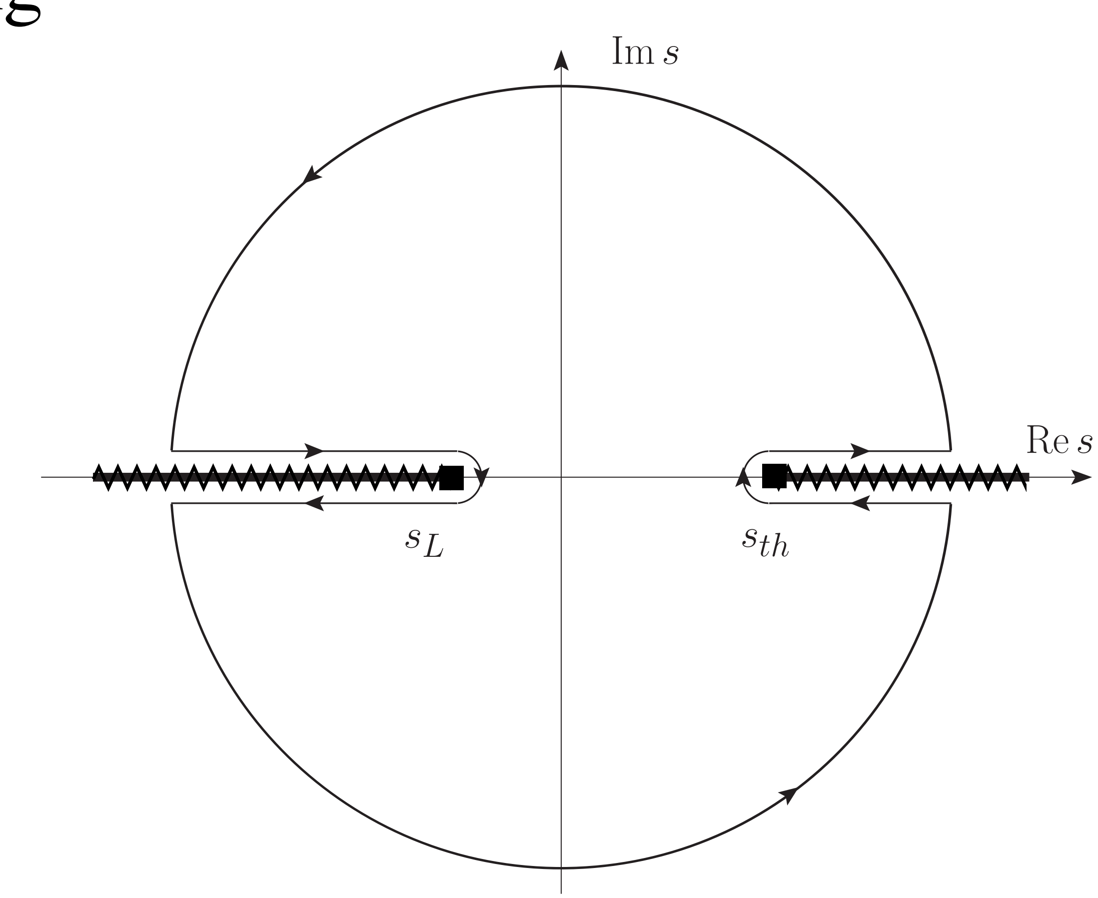
Left-hand cut physics comes from crossing

$$\int_{-1}^{1} dz_s \, P_\ell(z_s)$$



$$\sim \log\left(g(s)\right)$$

some function of kinematics



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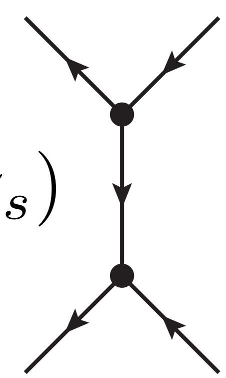

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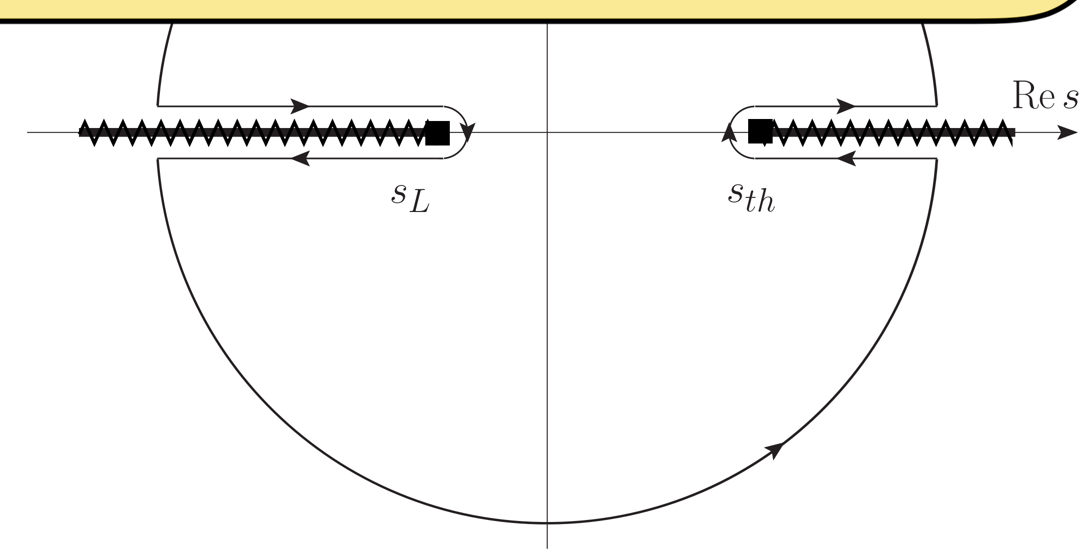
Partial wave amplitudes have more complicated analytic structures - Careful in defining dispersive representations



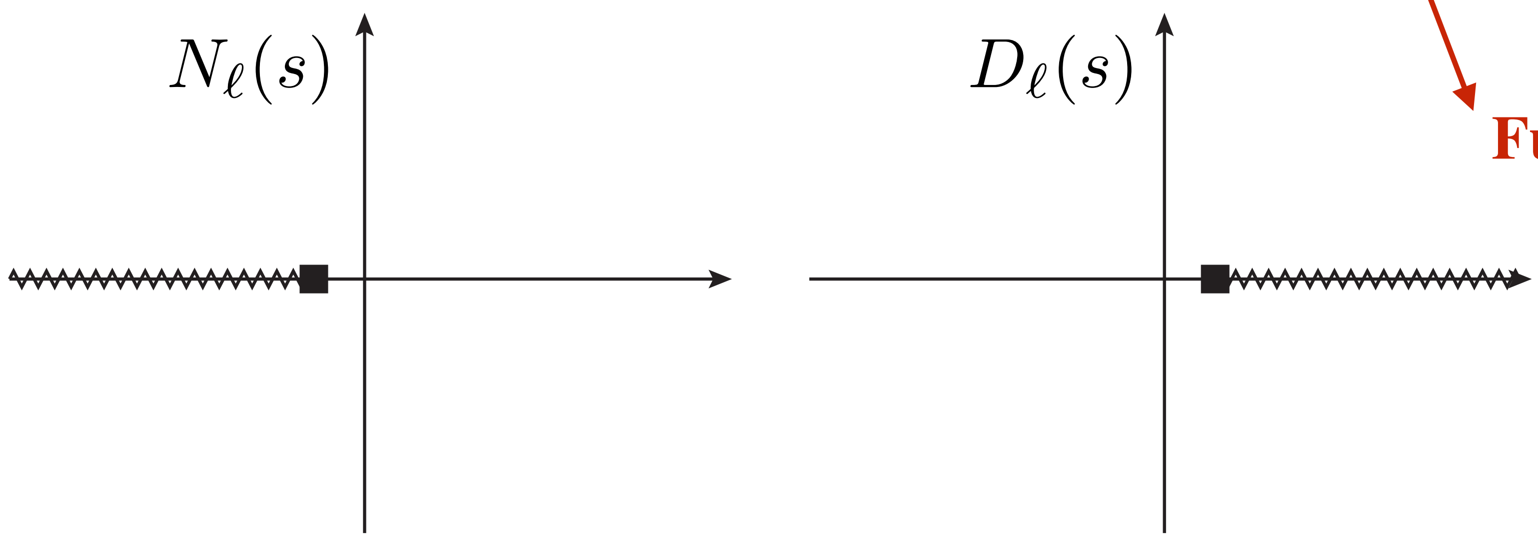
2→2 Elastic Scattering

Can linearize the system via N-over-D method

$$f_\ell(s) = \frac{N_\ell(s)}{D_\ell(s)}$$

$$N_\ell(s) = \frac{1}{\pi}\int_{-\infty}^{s_L} ds' \, \frac{D_\ell(s')\,\mathrm{Im}\ f_\ell(s')}{s'-s}$$

$$D_\ell(s) = D_\ell^{(0)}(s) - \frac{s}{\pi}\int_{s_{th}}^{\infty} ds' \, \frac{\rho_2(s')N_\ell(s')}{s'(s'-s)}$$



Function not constrained by unitarity: CDD poles, polynomials, …

Related to the K-matrix

$$f_\ell^{-1}(s) = K_\ell^{-1}(s) - \frac{s}{\pi}\int_{s_{th}}^{\infty} ds' \, \frac{\rho_2(s')}{s'(s'-s)}$$

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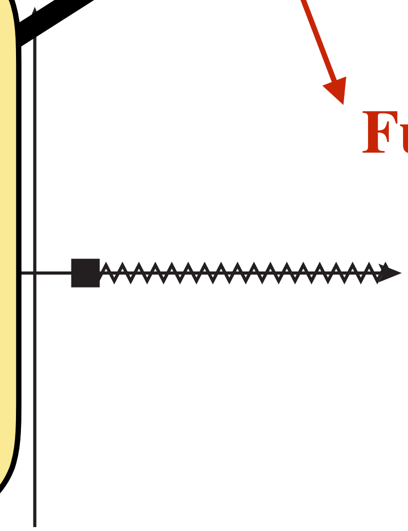
There is freedom in the function, not constrained by general principles - Must be determined by specific theory

Parameterize our Ignorance



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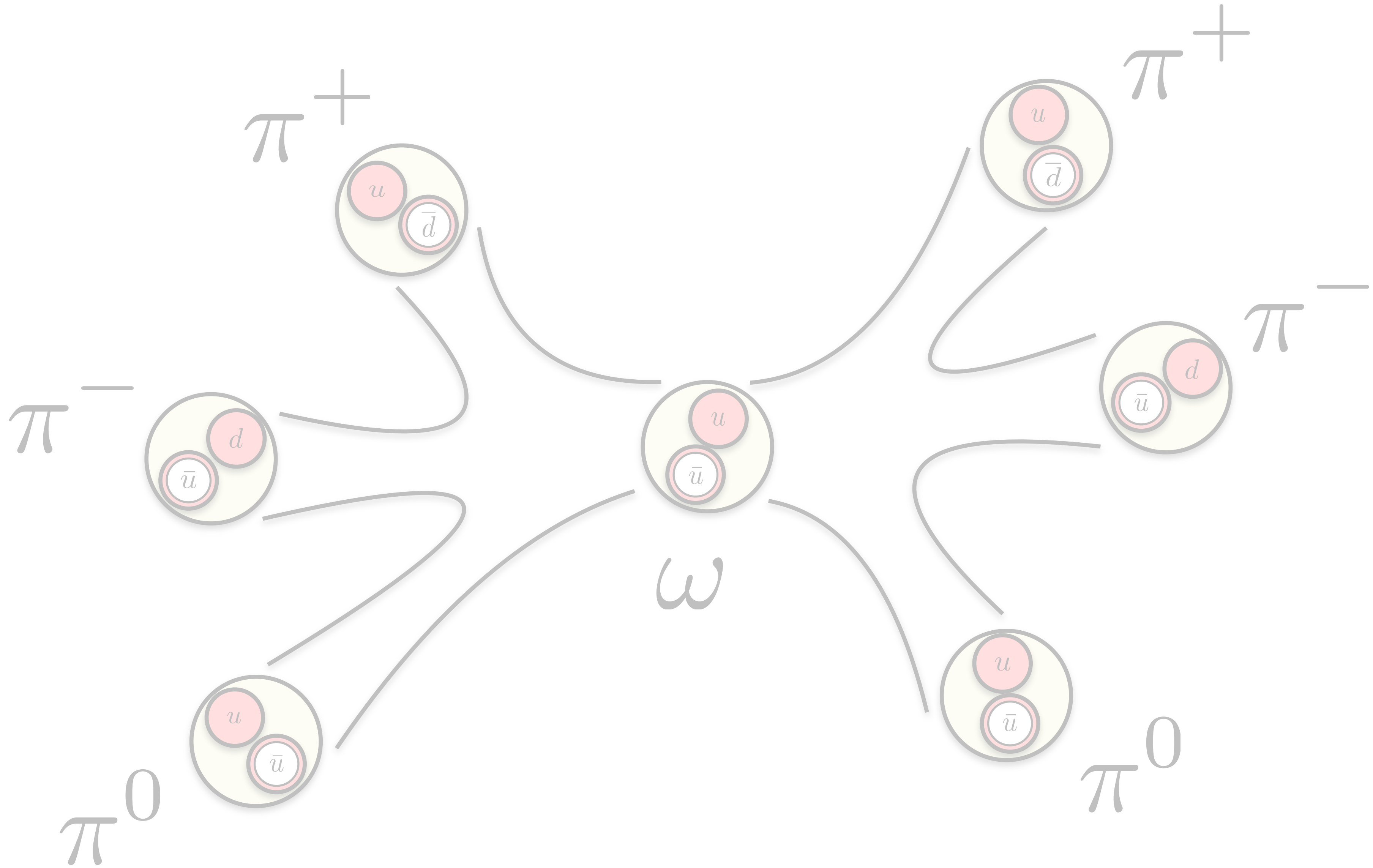
3→3 Scattering Phenomenology



3→3 Elastic Scattering

Construct unitarity constraints

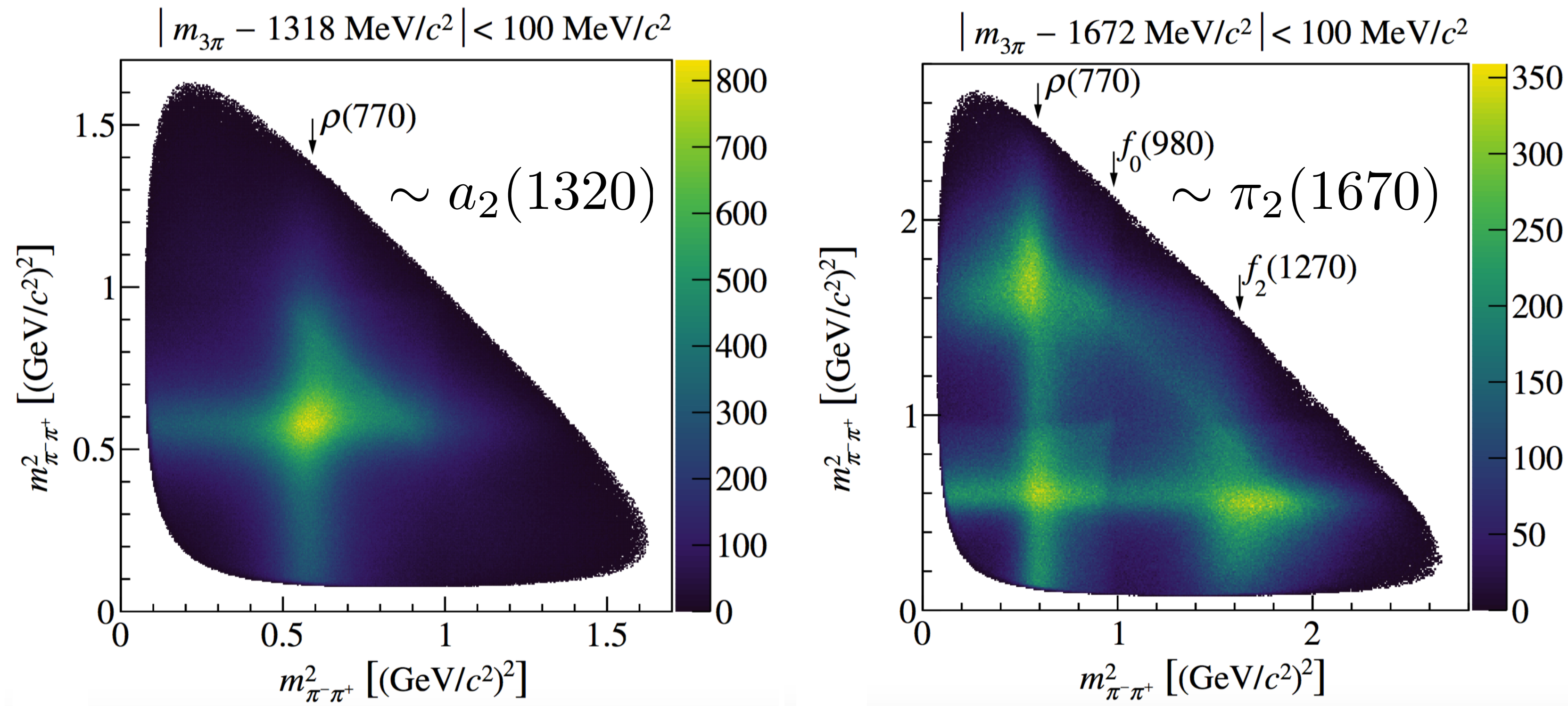
Partial Wave Expansion

Dispersion Relations

Parameterizations

Consider the elastic scattering of 3-distinguishable particles *123→123*

One approximation that is motivated by experimental analyses is the Isobar Model: Two particles resonant (called an *Isobar*) and the interact with third particle (called the *Spectator*)



C. Adolph et al. [*COMPASS*], *Phys. Rev. D 95, no. 3, 032004* (*2017*)

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Multiple variables =
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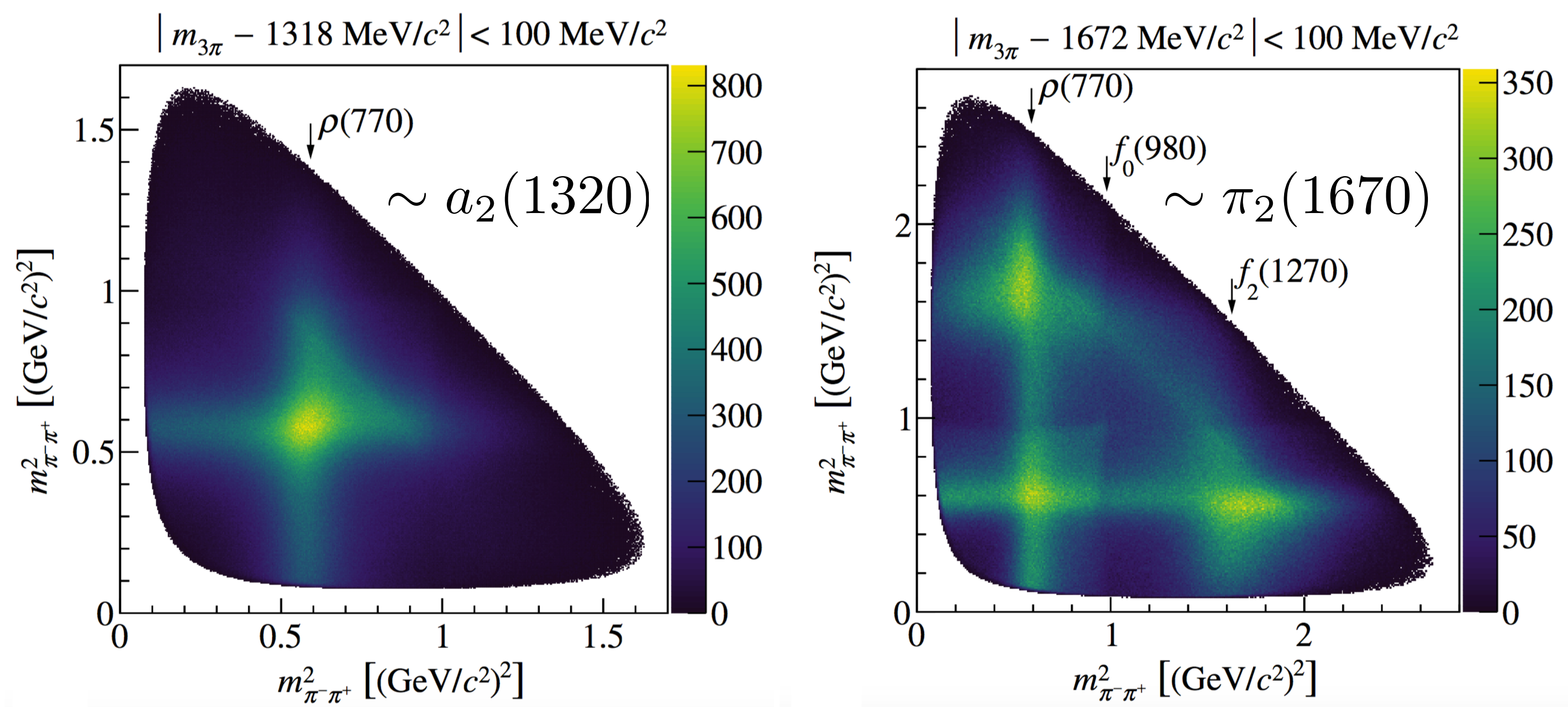

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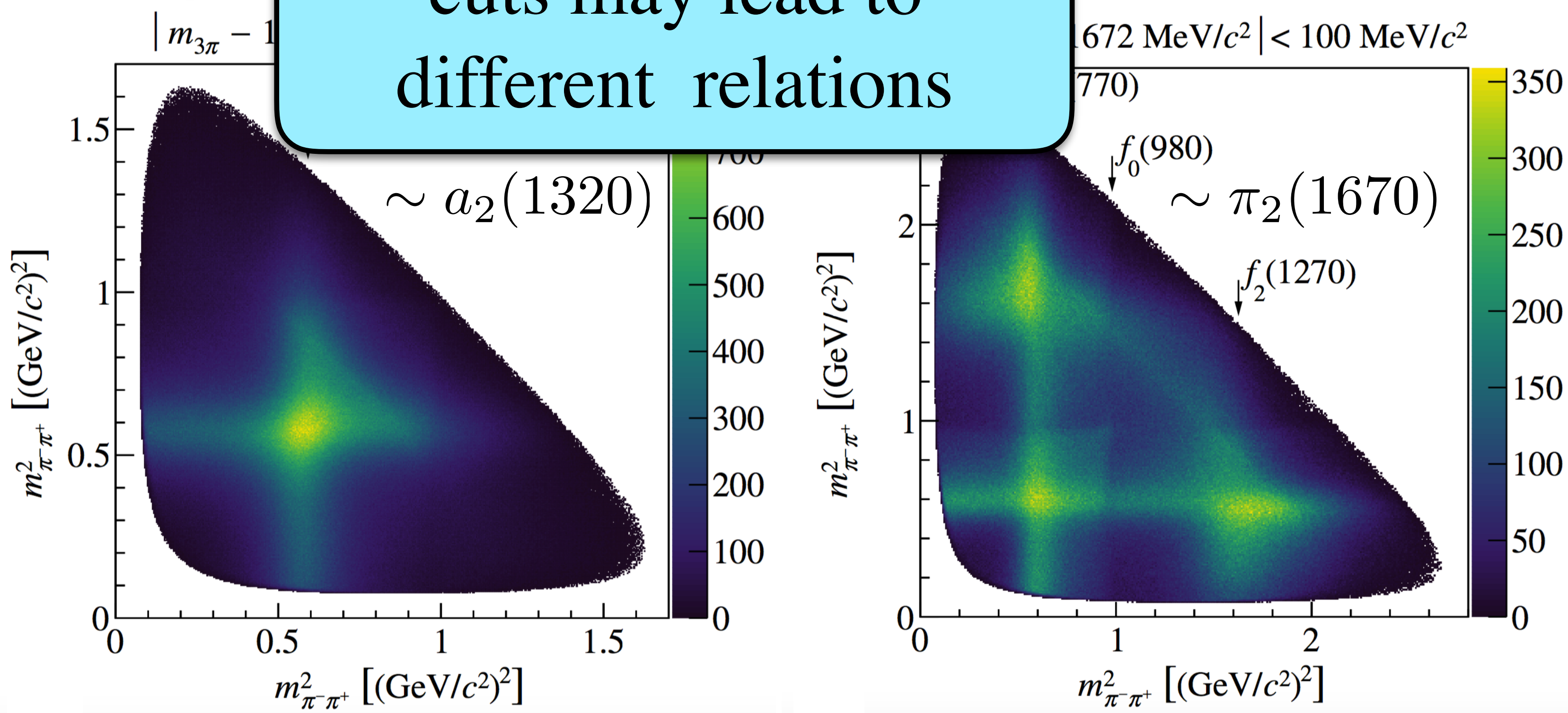
Multiple variables = Multiple discontinuities

Projecting above/below cuts may lead to different relations



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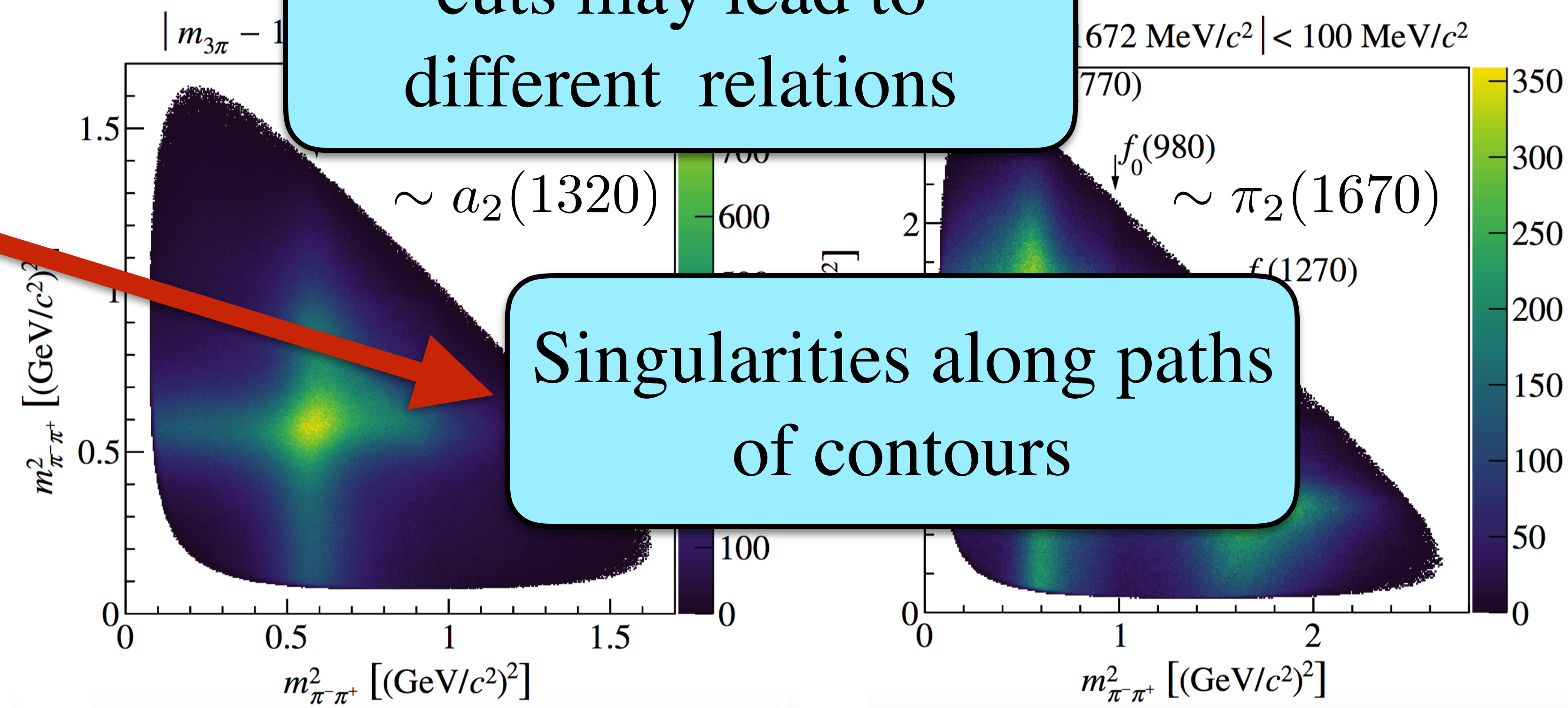
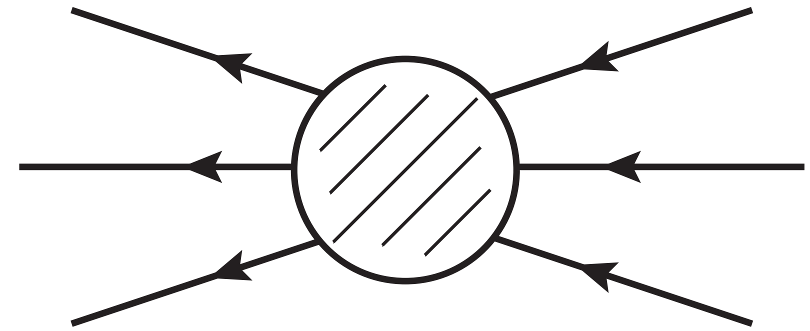


Singularities along paths of contours

C. Adolph et al. [*COMPASS*], *Phys. Rev. D 95, no. 3, 032004 (2017)*

$3\to3$ Elastic Scattering

Consider the elastic scattering of the **3→3** system *123→123*, where *1*, *2*, and *3* are distinguishable particles



The S-matrix is decomposed as

$$\langle\{\mathbf{p}'\}|\, S\,|\{\mathbf{p}\}\rangle = \langle\{\mathbf{p}'\}|\{\mathbf{p}\}\rangle$$

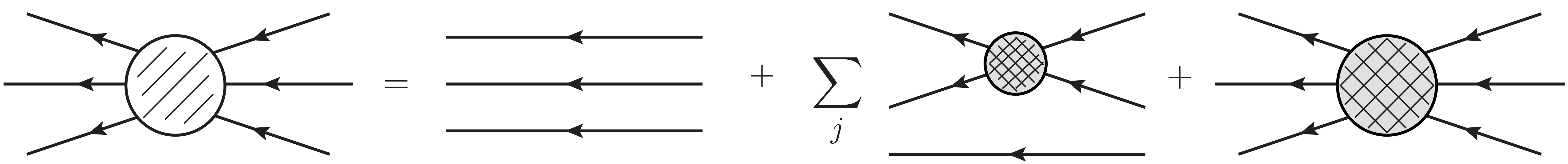
Completely Disconnected

$$+\, i\sum_j \widetilde{\delta}(p'_j - p_j)(2\pi)^4\delta^{(4)}(Q'_j - Q_j)\mathcal{F}_j(\{\mathbf{p}',\mathbf{p}\}_j)$$

Disconnected

$$+\, i(2\pi)^4\delta^{(4)}(P'-P)\mathcal{A}(\{\mathbf{p}',\mathbf{p}\})$$

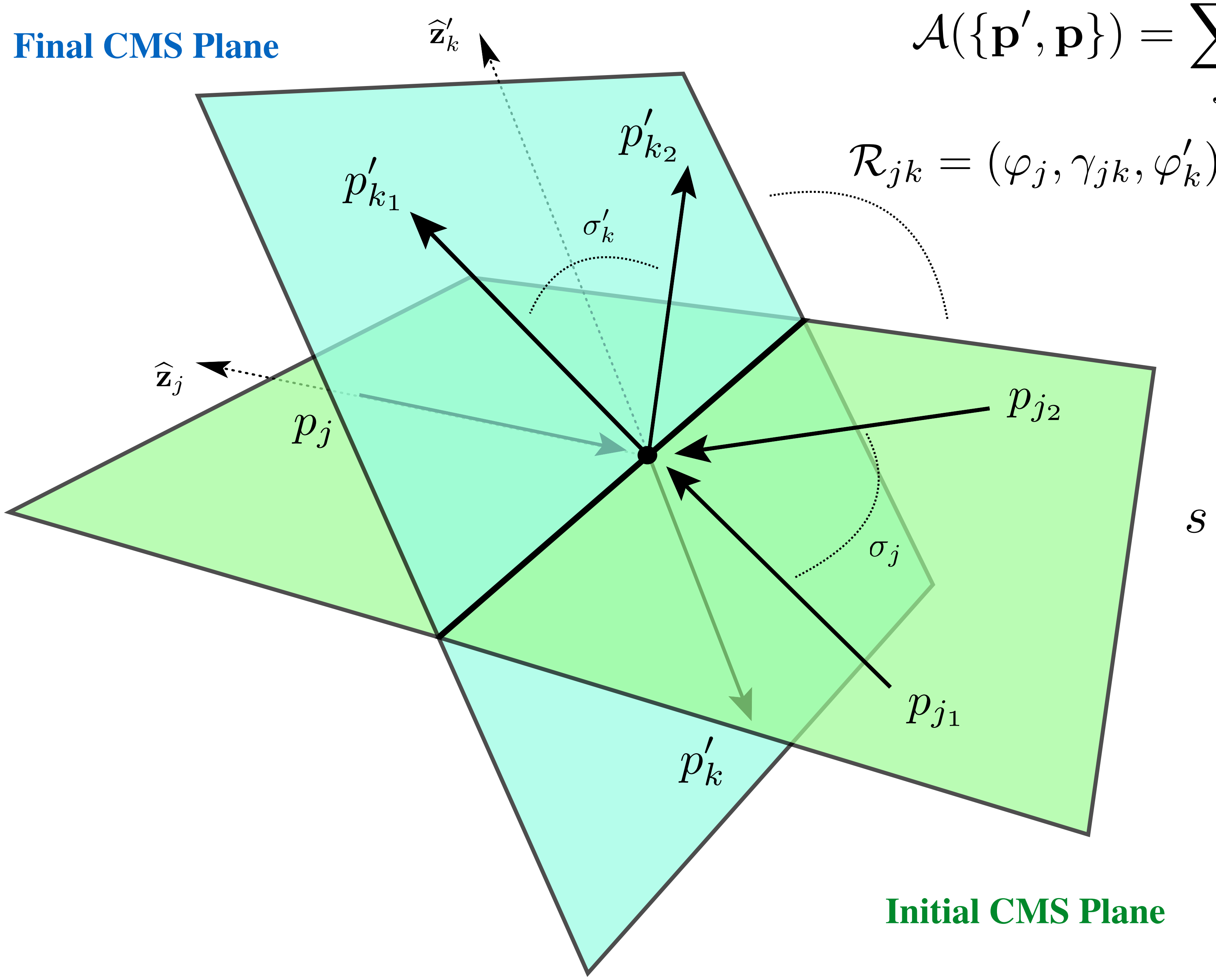
Connected



$\mathcal{F}_j(\{\mathbf{p}',\mathbf{p}\}_j)$ $\mathcal{A}(\{\mathbf{p}',\mathbf{p}\})$

3→3 Elastic Scattering

3→3 amplitudes depend on 8 independent variables. One representation is



$$\mathcal{A}(\{\mathbf{p}', \mathbf{p}\}) = \sum_J \sum_{\lambda,\lambda'} \left(\frac{2J+1}{8\pi^2}\right) \mathcal{A}^J_{\lambda'\lambda}(\{E\}) \mathcal{D}^{(J)}_{\lambda'\lambda}(\mathcal{R})$$

Invariant energies **Euler angles**

$$\{E\} = \{\sigma'_1, \sigma'_2, s, \sigma_1, \sigma_2\}$$

$$s + t_{jk} + u_{jk} = \sigma_j + \sigma'_k + m_j^2 + m_k^2$$

$$\sum_{j=1}^{3} \sigma_j = s + \sum_{j=1}^{3} m_j^2$$

$$\sum_{k=1}^{3} \sigma'_k = s + \sum_{k=1}^{3} m_k^2$$

Unitarity Relations

Disconnected 2→2 Unitarity Relation

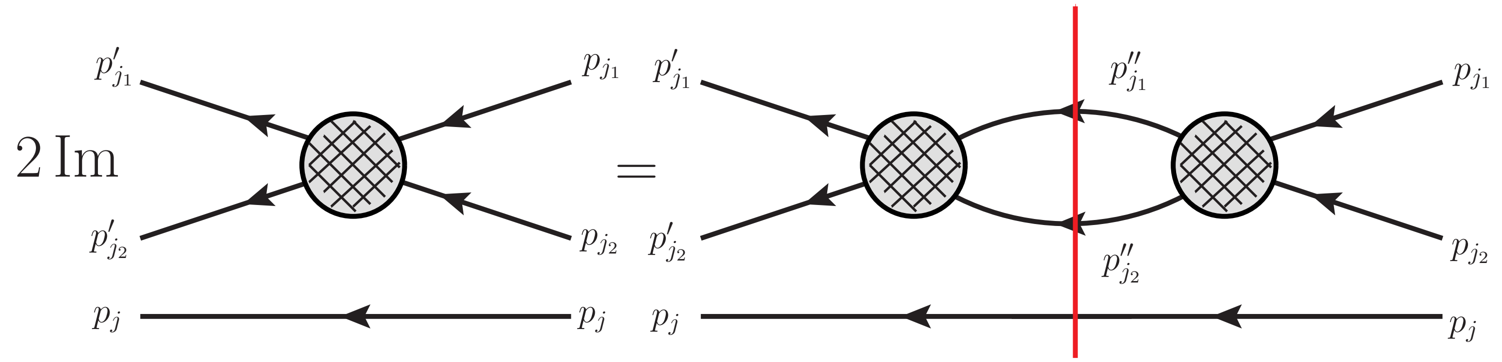
$$2\,\mathrm{Im}\;\mathcal{F}_j(\{\mathbf{p}',\mathbf{p}\}_j) = \rho_2(\sigma_j)\int d\Omega''_j\,\mathcal{F}^*_j(\{\mathbf{p}'',\mathbf{p}'\}_j)\mathcal{F}_j(\{\mathbf{p}'',\mathbf{p}\}_j)$$

Connected 3→3 Unitarity Relation

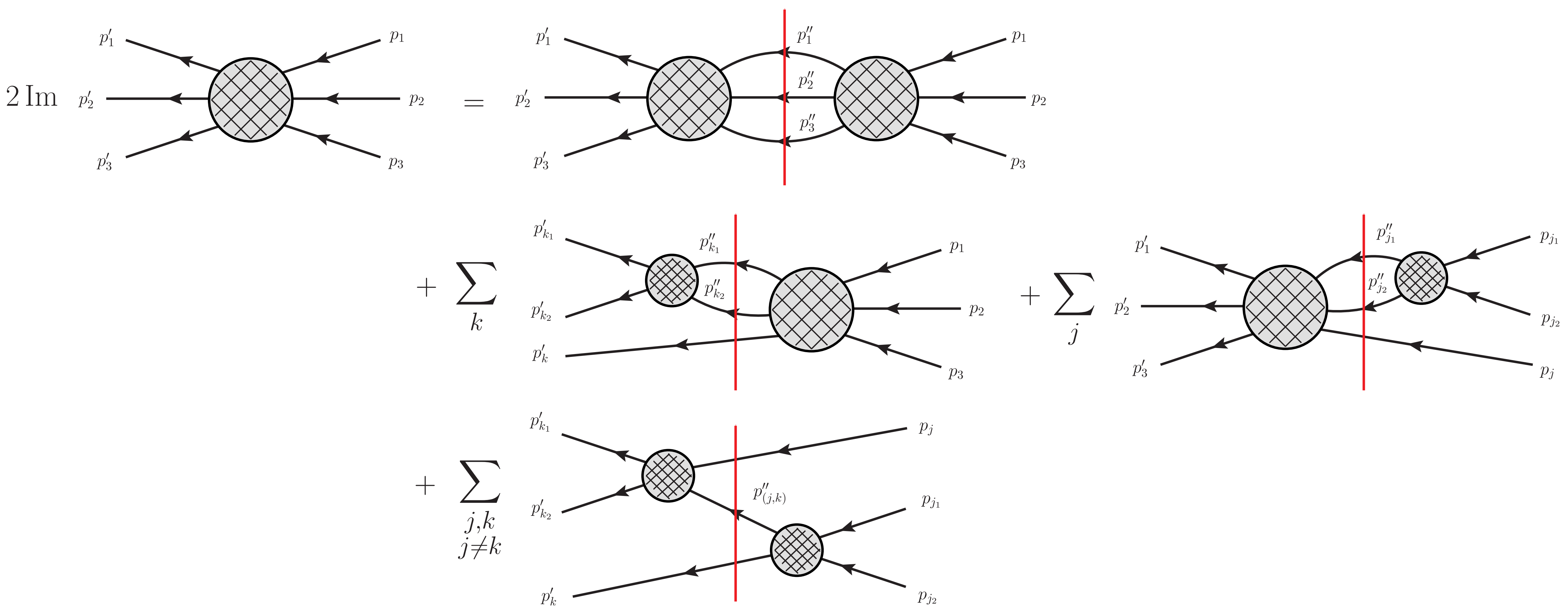
$$\begin{aligned}2\,\mathrm{Im}\;\mathcal{A}(\{\mathbf{p}',\mathbf{p}\}) = &\int \widetilde{d}p''_1\widetilde{d}p''_2\widetilde{d}p''_3\,(2\pi)^4\delta^{(4)}(P''-P)\,\mathcal{A}^*(\{\mathbf{p}'',\mathbf{p}'\})\mathcal{A}(\{\mathbf{p}'',\mathbf{p}\})\\ &+\sum_k \rho_2(\sigma'_k)\int d\Omega''_k\,\mathcal{F}^*_k(\{\mathbf{p}'',\mathbf{p}'\}_k)\mathcal{A}(\{\mathbf{p}'',\mathbf{p}\})\Theta(\sigma'_k-\sigma^{(k)}_{th})\\ &+\sum_j \rho_2(\sigma_j)\int d\Omega''_j\,\mathcal{A}^*(\{\mathbf{p}'',\mathbf{p}'\})\mathcal{F}(\{\mathbf{p}'',\mathbf{p}\}_j)\Theta(\sigma_j-\sigma^{(j)}_{th})\\ &+\sum_{\substack{j,k\\ j\neq k}} 2\pi\,\delta(u_{jk}-m^2_{(jk)})\,\mathcal{F}^*_k(\{\mathbf{p}'',\mathbf{p}'\}_k)\mathcal{F}_j(\{\mathbf{p}'',\mathbf{p}\}_j)\end{aligned}$$

Unitarity Relations

Disconnected 2→2 Unitarity Relation



Connected 3→3 Unitarity Relation

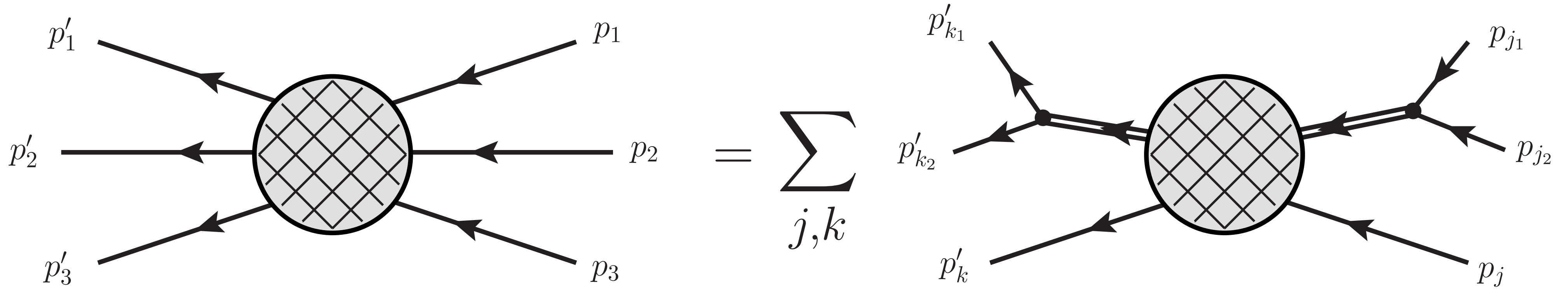


The Isobar Model

Assume that the amplitude can be expanded into *Isobar Amplitudes*

$$\mathcal{A}(\{\mathbf{p}',\mathbf{p}\}) = \sum_{j,k} \mathcal{A}_{kj}(\{\mathbf{p}',\mathbf{p}\}_{kj})$$

Two particles interact before interacting with spectator



Sum over all allowed isobars

The Isobar Model

Assume that the amplitude can be expanded into *Isobar Amplitudes*

$$\mathcal{A}(\{\mathbf{p}',\mathbf{p}\}) = \sum_{j,k} \mathcal{A}_{kj}(\{\mathbf{p}',\mathbf{p}\}_{kj})$$

Two particles interact before interacting with spectator

$$\mathcal{A}_{kj} \to \sum_{s_j,s'_k}\sum_{\lambda_j,\lambda'_k} \mathcal{A}_{kj}(\sigma'_k, s, t_{jk}, \sigma_j) Y^*_{s_k}(\Omega_k) Y_{s_j}(\Omega_j)$$

Model involves only finite number of isobars

Sum over all allowed isobars

Isobar Model Unitarity Relations

Factorizes the sub-energy rescattering

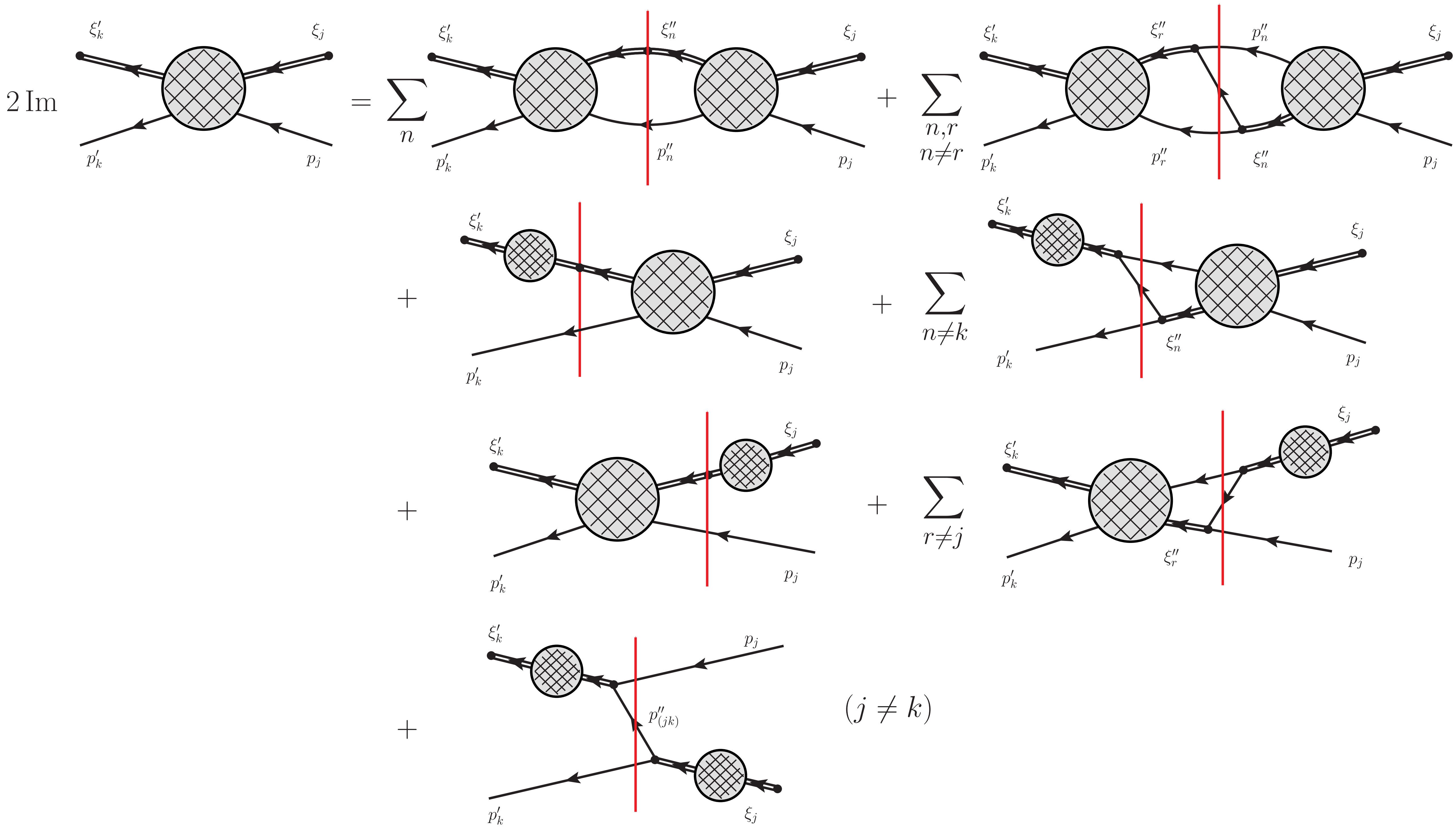
$$\mathcal{A}_{kj}(\sigma'_k, s, t_{jk}, \sigma_j) = \frac{1}{D_k(\sigma'_k)} \widehat{\mathcal{A}}_{kj}(\sigma'_k, s, t_{jk}, \sigma_j) \frac{1}{D_j(\sigma_j)}$$

Still sub-energy dependence

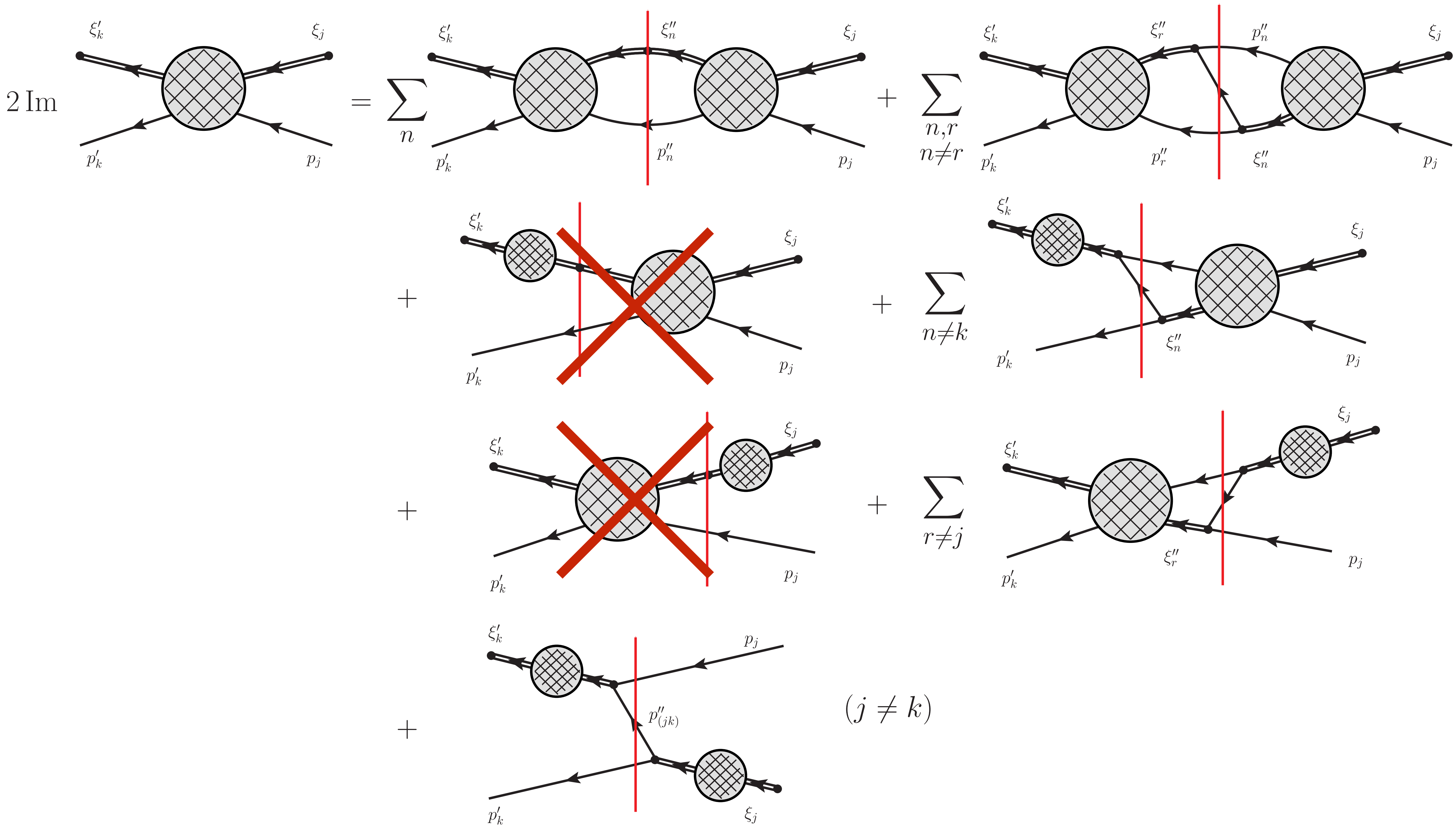
2→2 Rescattering

$$f_j(\sigma_j) = N_j(\sigma_j)/D_j(\sigma_j)$$

Isobar Model Unitarity Relations

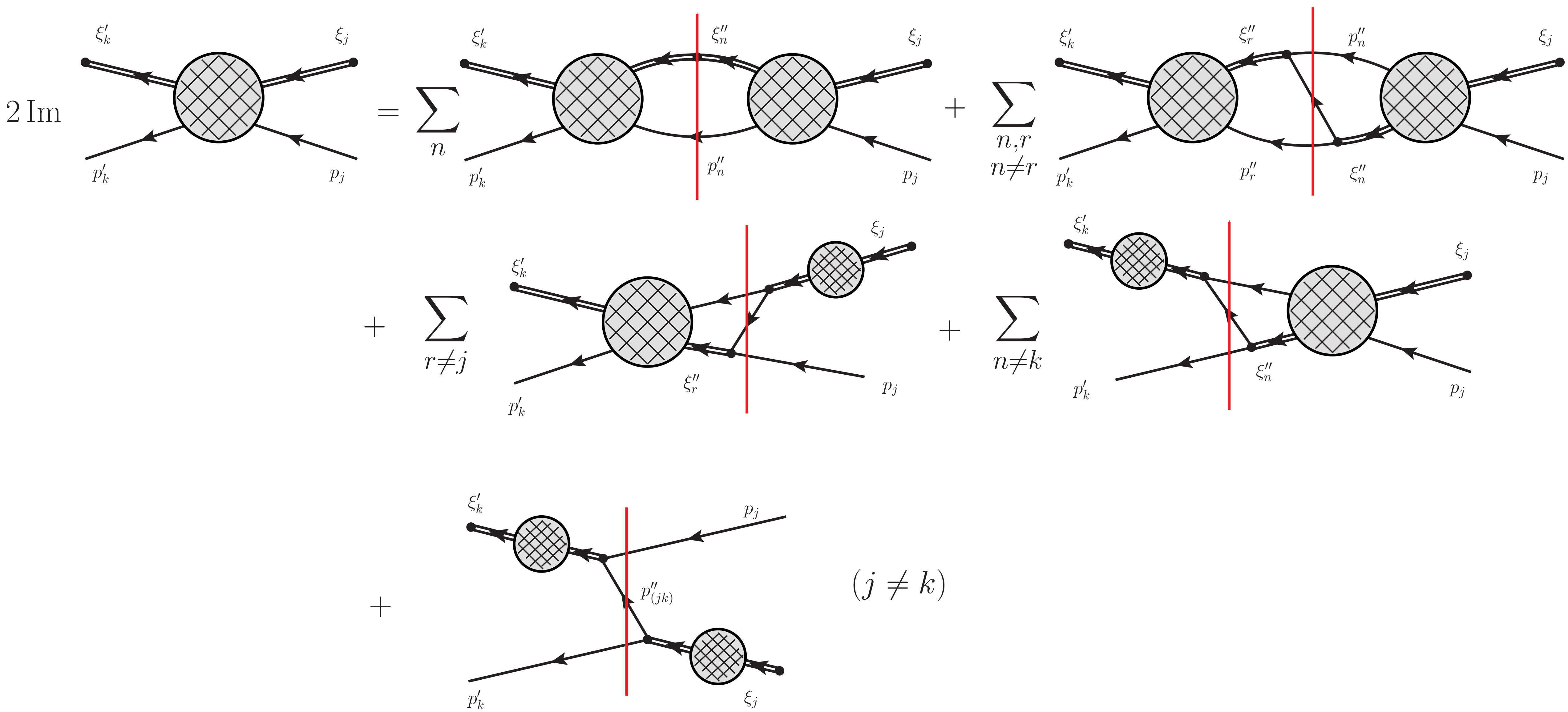


Isobar Model Unitarity Relations



A. Jackura, Indiana University (ajackura@indiana.edu)

Isobar Model Unitarity Relations



Analytic Structure

We can split the imaginary part into discontinuities across all variables

Need to be careful on which direction we approach the real axis from the complex planes

$$
\begin{aligned}
2i \ \mathrm{Im} \ \widehat{\mathcal{A}}_{kj}(\sigma_k', s, t_{jk}, u_{jk}, \sigma_j) = & \ \Delta_{\sigma_k'} \widehat{\mathcal{A}}_{kj}(s_+, t_{jk_+}, u_{jk_+}, \sigma_{j_+}) \\
& + \Delta_s \widehat{\mathcal{A}}_{kj}(\sigma_{k_-}', t_{jk_+}, u_{jk_+}, \sigma_{j_+}) \\
& + \Delta_{t_{jk}} \widehat{\mathcal{A}}_{kj}(\sigma_{k_-}', s_-, u_{jk_+}, \sigma_{j_+}) \\
& + \Delta_{u_{jk}} \widehat{\mathcal{A}}_{kj}(\sigma_{k_-}', s_-, t_{jk_-}, \sigma_{j_+}) \\
& + \Delta_{\sigma_j} \widehat{\mathcal{A}}_{kj}(\sigma_{k_-}', s_-, t_{jk_-}, u_{jk_-})
\end{aligned}
$$

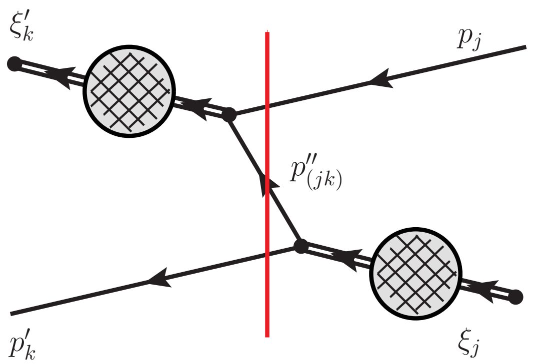
$$x_\pm = x \pm i\epsilon$$

Analytic Structure

We can split the imaginary part into discontinuities across all variables

Need to be careful on which direction we approach the real axis from the complex planes

For $j \neq k$, have to worry about singularities in u_{jk} from One Particle Exchange (OPE)



$$(j \neq k) \quad \sim \delta(u_{jk} - m_{(jk)})$$

$$jk_+, \sigma_{j+})$$

$$, u_{jk+}, \sigma_{j+})$$

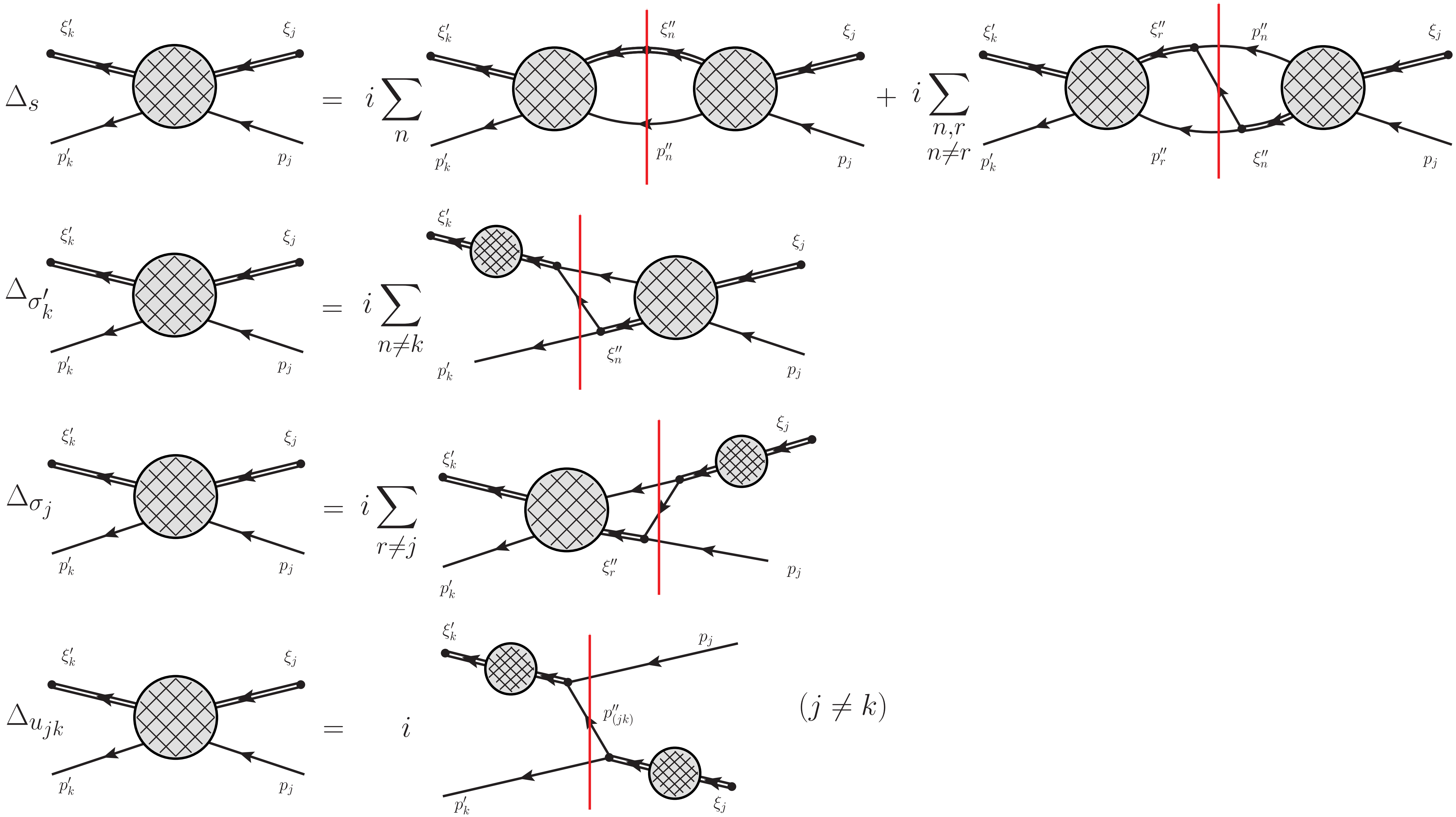
$$, u_{jk+}, \sigma_{j+})$$

$$, t_{jk-}, \sigma_{j+})$$

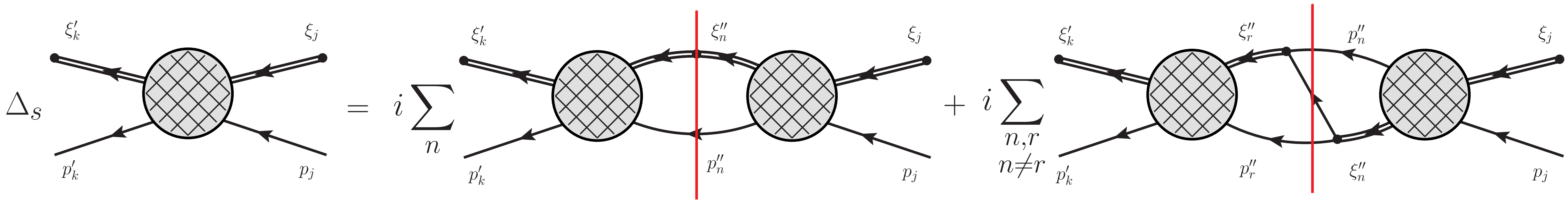
$$+ \Delta_{\sigma_j} \mathcal{M}_{kj}(\sigma_{k-}, s_-, t_{jk-}, u_{jk-})$$

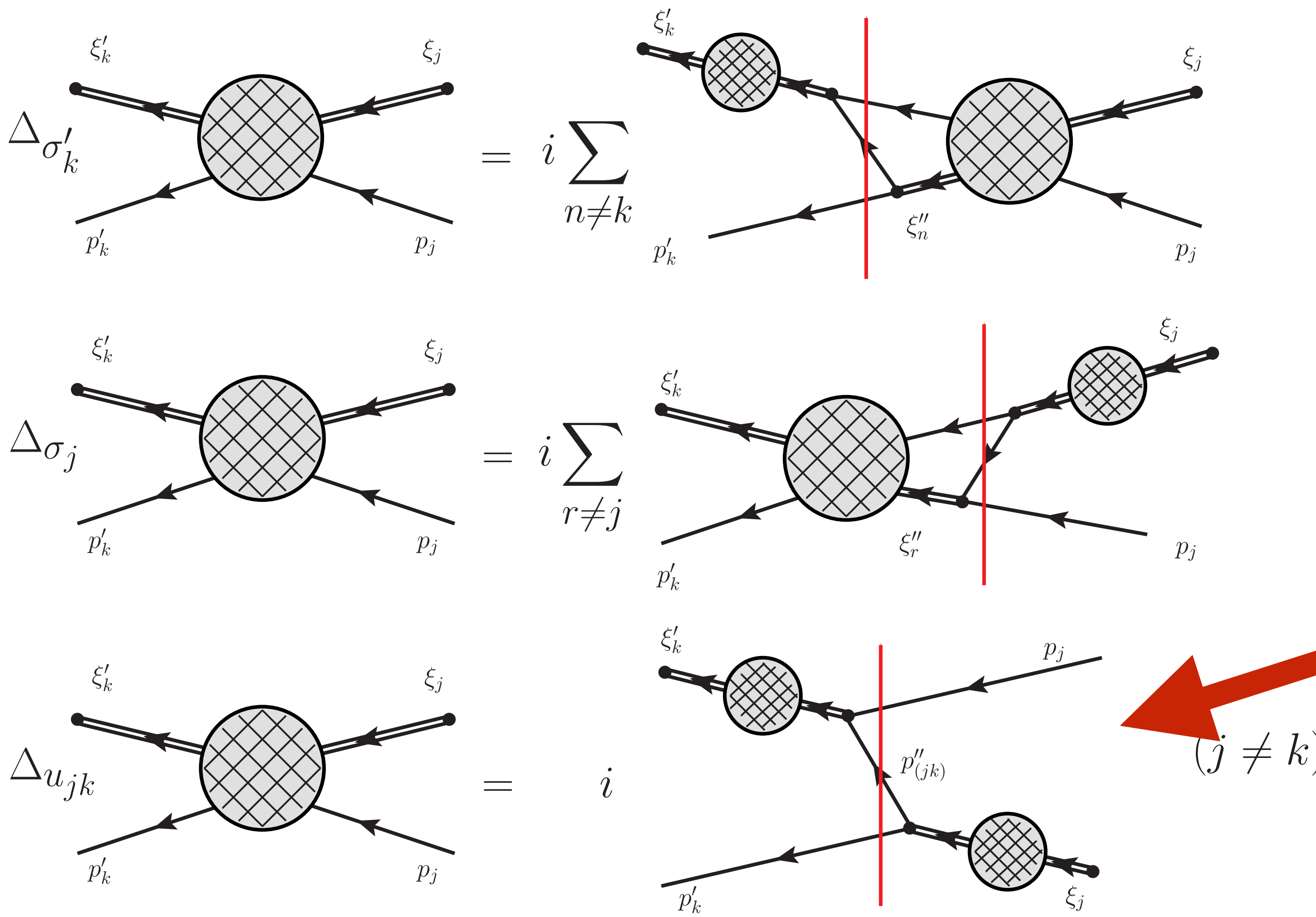
$$x_\pm = x \pm i\epsilon$$

Analytic Structure



Analytic Structure





Will turn this into an s-channel cut via Partial Wave Projection

$$u_{jk} = u_{jk}(\sigma'_k, s, z_{jk}, \sigma_j)$$

Partial Wave Amplitudes

We now want to consider partial wave projections of the amplitude

To simplify the expressions, let's consider the case for $J = 0$, and spin-0 isobars

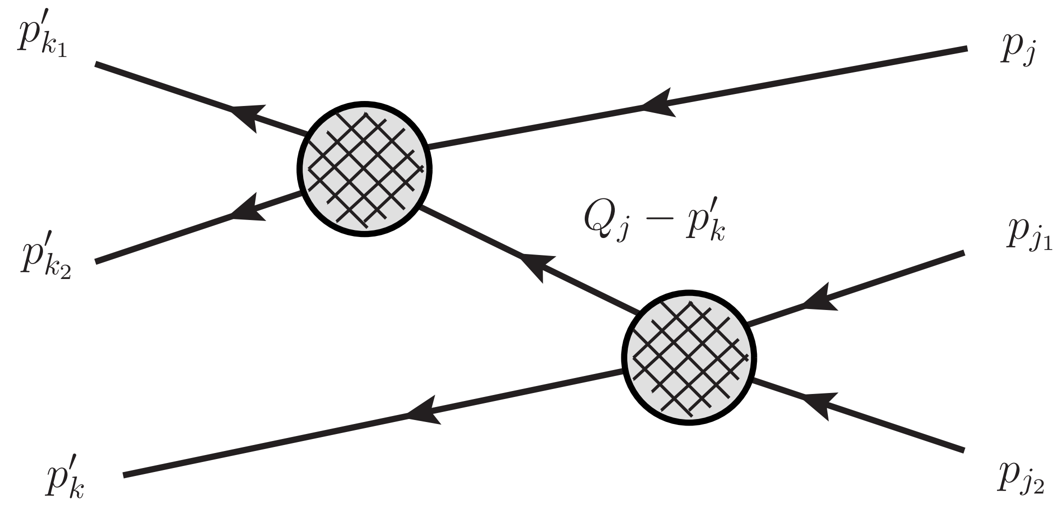
$$\mathcal{C}_{kj}(\sigma_k', s, \sigma_j) = \int_{-1}^{+1} dz_{jk}\, \widehat{\mathcal{A}}_{kj}(\sigma_k', s, t_{jk}(s, z_{jk}), \sigma_j)$$

We can proceed to project out the discontinuities

Note : The off-diagonal ($j \neq k$) amplitudes have a subtlety because of the OPE amplitude

One-Particle-Exchange

Partial wave projection of the OPE term gives an extra cut in the complex s-plane

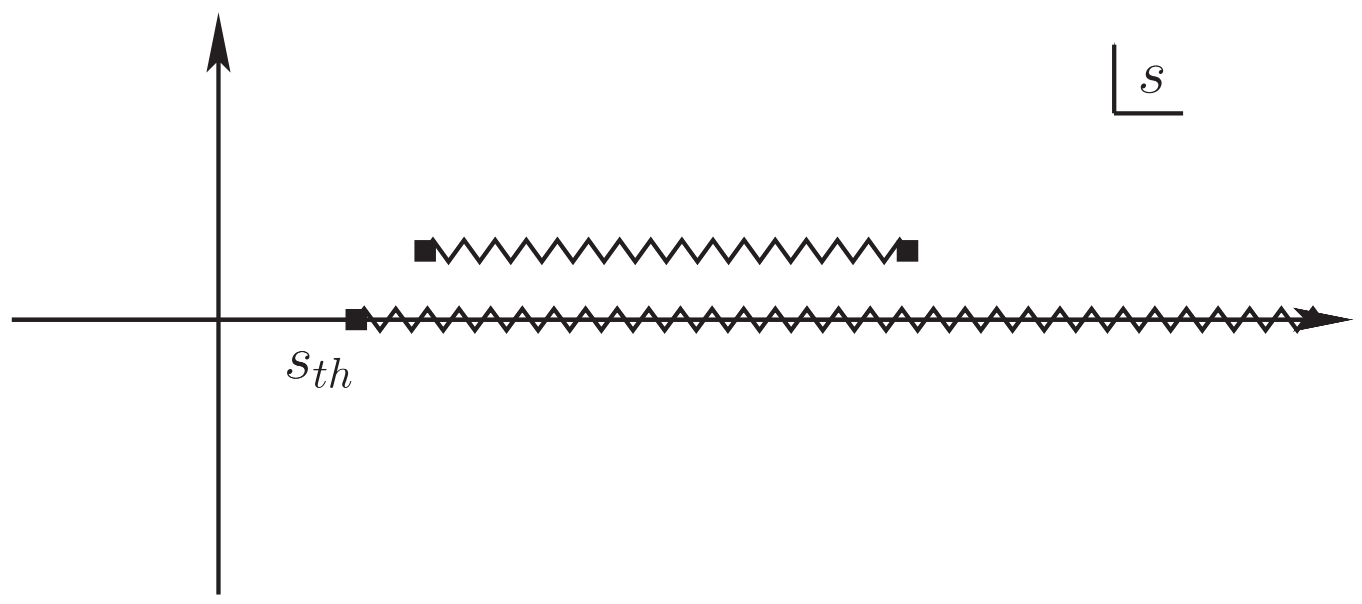


Exchange Mass

$$\int_{-1}^{+1} dz_{jk}\, \delta(u_{jk}(s, z_{jk}) - m^2_{(jk)})$$

$$\sim \frac{2s}{\lambda^{1/2}(s, \sigma_j, m_j^2)\lambda^{1/2}(s, \sigma'_k, m_k^2)} \Theta(s - s^{(+)})\Theta(s^{(-)} - s)$$

Non-zero in Dalitz region



One-Particle-Exchange

Want partial wave projection of

$$\Delta_s \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_+, u_{jk_+}, \sigma_{j_+}) + \Delta_{u_{jk}} \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_-, u_{jk_+}, \sigma_{j_+})$$

$$u_{jk_+} = u_{jk} + i\epsilon$$

$$u_{jk_-} = u_{jk} - i\epsilon$$

$$s + t_{jk} + u_{jk} = \sigma_j + \sigma_k' + m_j^2 + m_k^2$$

$$s \pm i\epsilon \implies u_{jk} \mp i\epsilon$$

$$\Delta_s \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_+, u_{jk_+}, \sigma_{j_+}) = \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_+, u_{jk_+}, \sigma_j) - \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_-, u_{jk_+}, \sigma_j)$$

$$\Delta_{u_{jk}} \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_-, u_{jk_+}, \sigma_{j_+}) = \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_-, u_{jk_+}, \sigma_j) - \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_-, u_{jk_-}, \sigma_j)$$

One-Particle-Exchange

Want partial wave projection of

$$\Delta_s \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_+, u_{jk_+}, \sigma_{j_+}) + \Delta_{u_{jk}} \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_-, u_{jk_+}, \sigma_{j_+})$$

$$u_{jk_+} = u_{jk} + i\epsilon$$

$$u_{jk_-} = u_{jk} - i\epsilon$$

$$s + t_{jk} + u_{jk} = \sigma_j + \sigma_k' + m_j^2 + m_k^2$$

$$s \pm$$

Above Unitarity
Below OPE

Below Unitarity
Below OPE

$$\Delta_s \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_+, u_{jk_+}, \sigma_{j_+}) = \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_+, u_{jk_+}, \sigma_j) - \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_-, u_{jk_+}, \sigma_j)$$

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Below Unitarity
Below OPE

Below Unitarity
Above OPE

One-Particle-Exchange

Want partial wave projection of

$$\Delta_s \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_+, u_{jk_+}, \sigma_{j_+}) + \Delta_{u_{jk}} \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_-, u_{jk_+}, \sigma_{j_+})$$

$$u_{jk_+} = u_{jk} + i\epsilon$$

$$u_{jk_-} = u_{jk} - i\epsilon$$

$$s + t_{jk} + u_{jk} = \sigma_j + \sigma_k' + m_j^2 + m_k^2$$

$s \pm$

Above Unitarity
Below OPE

$B(s_+)$

Below Unitarity
Below OPE

$B(s_-)$

$$\Delta_s \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_+, u_{jk_+}, \sigma_{j_+}) = \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_+, u_{jk_+}, \sigma_j) - \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_-, u_{jk_+}, \sigma_j)$$

$$\Delta_{u_{jk}} \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_-, u_{jk_+}, \sigma_{j_+}) = \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_-, u_{jk_+}, \sigma_j) - \widehat{\mathcal{A}}_{kj}(\sigma_k{}'_-, s_-, u_{jk_-}, \sigma_j)$$

$B(s_-)$

Below Unitarity
Below OPE

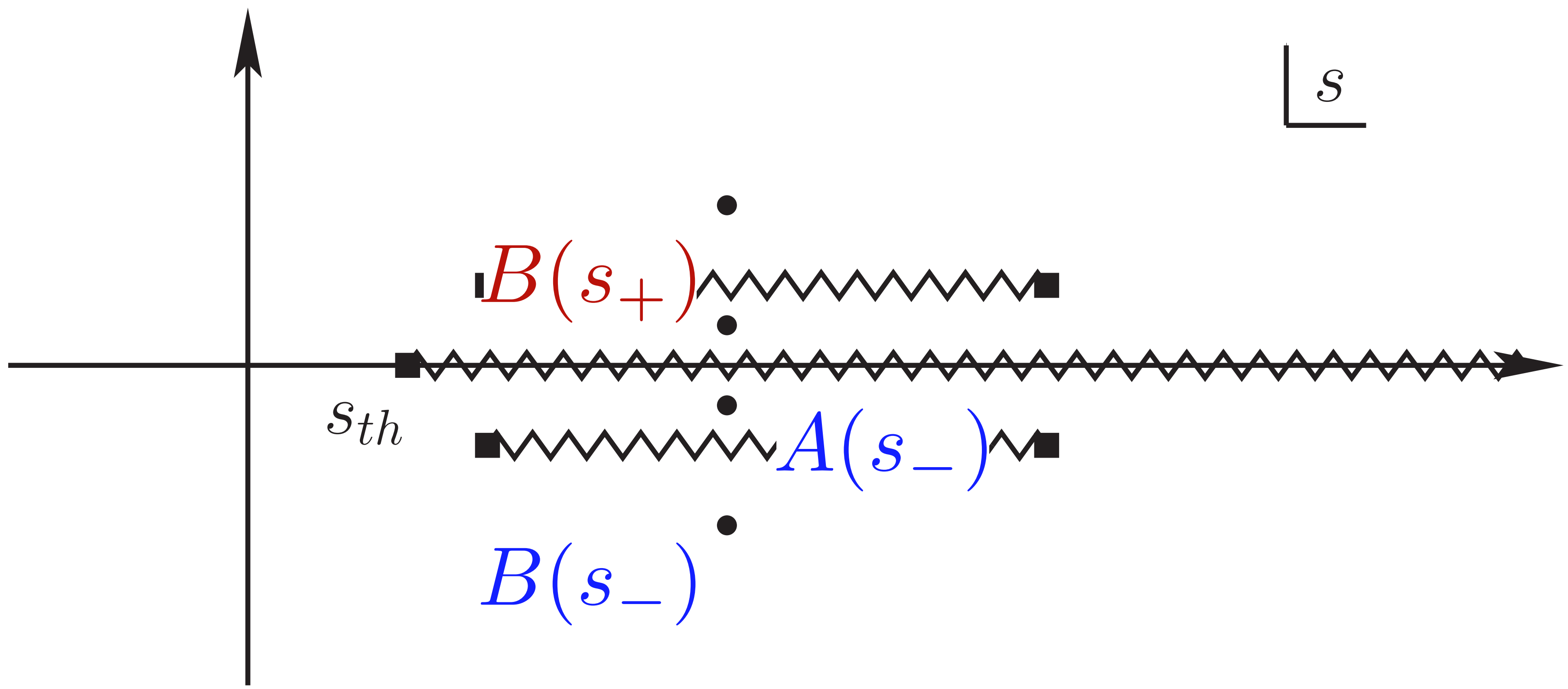
Below Unitarity
Above OPE

$A(s_-)$

One-Particle-Exchange

$$\int_{-1}^{1} dz_{jk} \left[\Delta_s \widehat{\mathcal{A}}_{kj}(\sigma_{k}{}'_{-}, s_+, u_{jk_+}, \sigma_{j_+}) + \Delta_{u_{jk}} \widehat{\mathcal{A}}_{kj}(\sigma_{k}{}'_{-}, s_-, u_{jk_+}, \sigma_{j_+}) \right]$$

$$= B(s_+) - B(s_-) - (A(s_-) - B(s_-))$$



One-Particle-Exchange

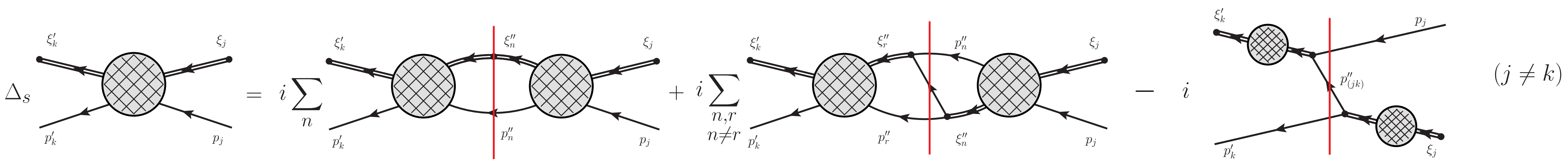
$$\int_{-1}^{1} dz_{jk} \left[\Delta_s \widehat{\mathcal{A}}_{kj}(\sigma_k{'}_{-}, s_+, u_{jk_+}, \sigma_{j_+}) + \Delta_{u_{jk}} \widehat{\mathcal{A}}_{kj}(\sigma_k{'}_{-}, s_-, u_{jk_+}, \sigma_{j_+}) \right]$$

$$= B(s_+) - B(s_-) - (A(s_-) - B(s_-))$$

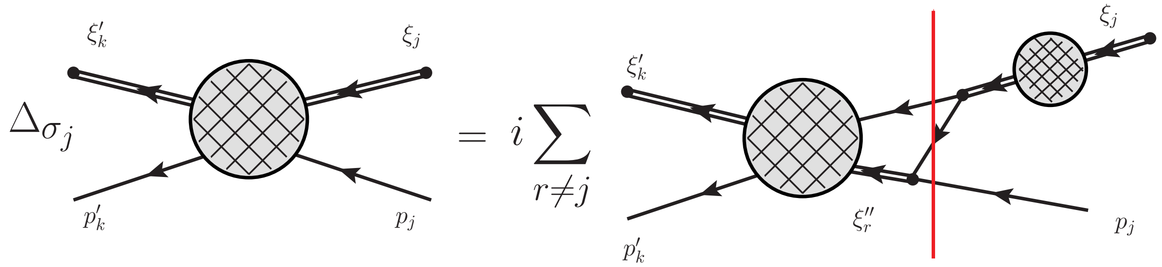
s

leads to discontinuity across *s*

$$\Delta_s\, \mathcal{C}_{kj}(\sigma'_{k\,-}, s_+, \sigma_{j\,+}) = \Delta_s[\text{Boxes}] - \Delta[\text{OPE}]$$



Triangle Diagrams

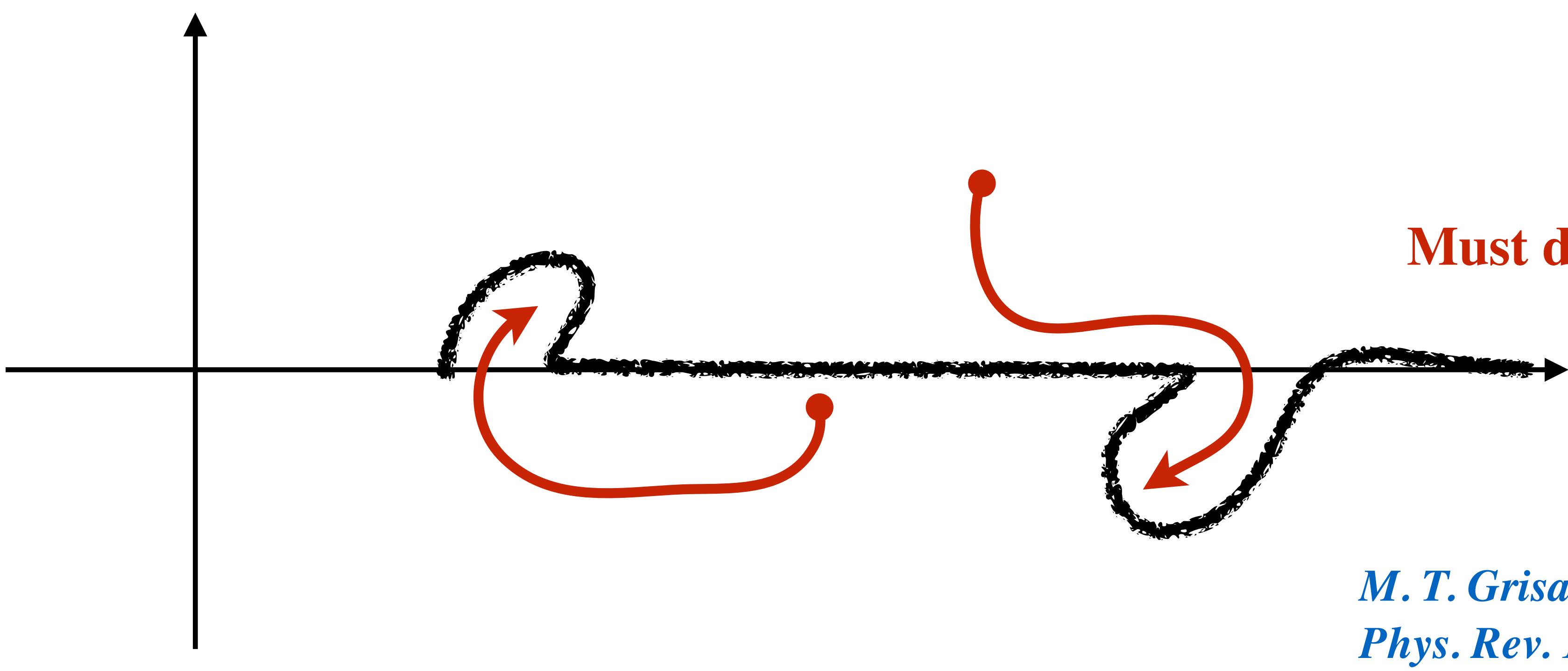


Kinematics may require deformation of dispersive contours

$$\Delta_{\sigma_1}\mathcal{C}_{31}(\sigma_3{}'_-, s_-, \sigma_{1+}) = i\rho_2(\sigma_{1+})N_1(\sigma_{1+})\int d\sigma_3''\, D_3^{-1}(\sigma_3'')\mathcal{C}_{33}(\sigma_{3-}, s_-, \sigma_3{}'_-)$$

Fix $s, \sigma_3{}'$, investigate contour in σ_1

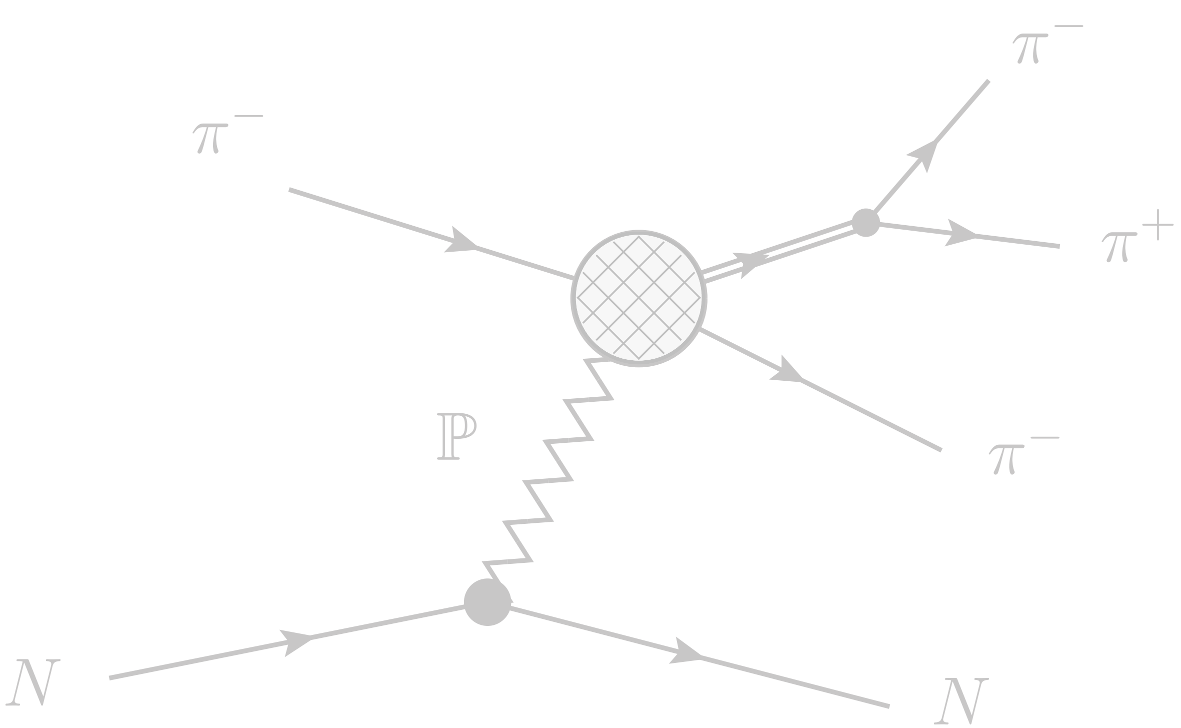
$$\mathcal{C}_{31}(\sigma_3{}'_-, s_-, \sigma_{1+}) = \frac{1}{\pi}\int_{\sigma_{th}^{(1)}}^{(\sqrt{s_-}-m_1)^2} d\widehat{\sigma}\,\frac{1}{\widehat{\sigma}-\sigma_{1+}}\rho_2(\widehat{\sigma})N_1(\widehat{\sigma})b(\widehat{\sigma}, s_-, \sigma_3{}'_-)$$

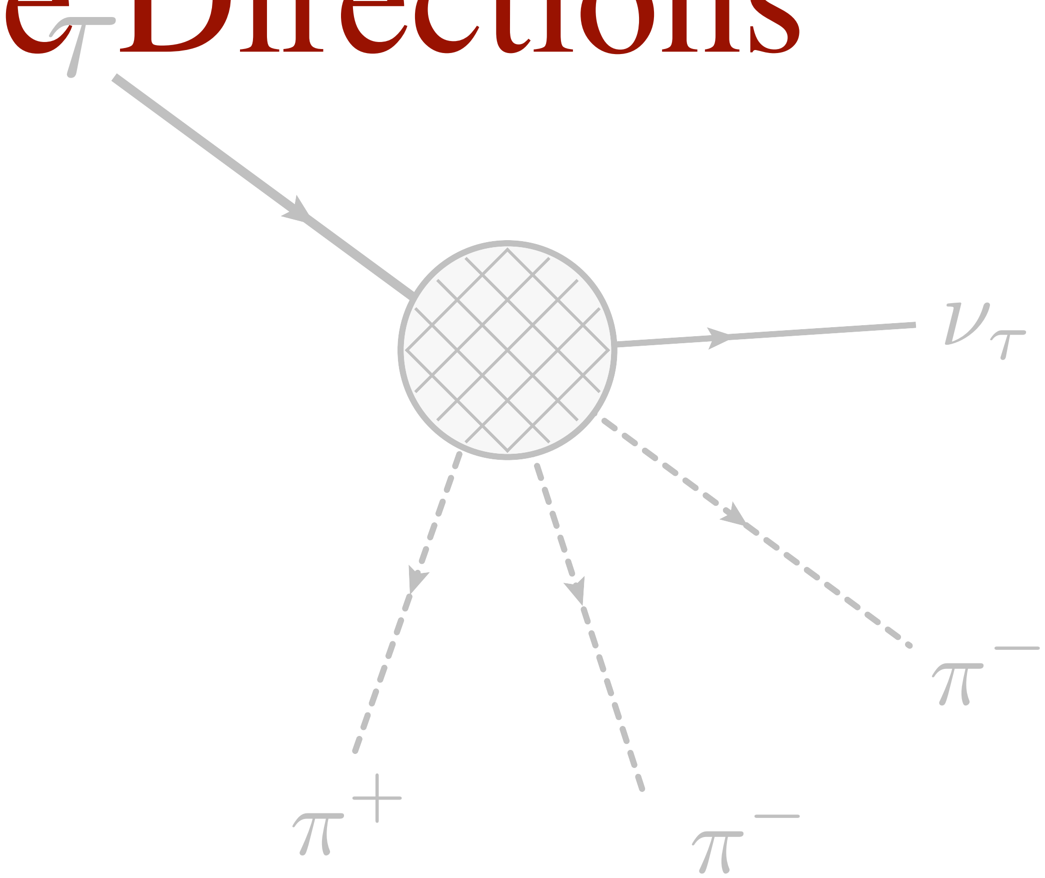


Must deform contour

M. T. Grisaru
Phys. Rev. 146, 1098 (1966)

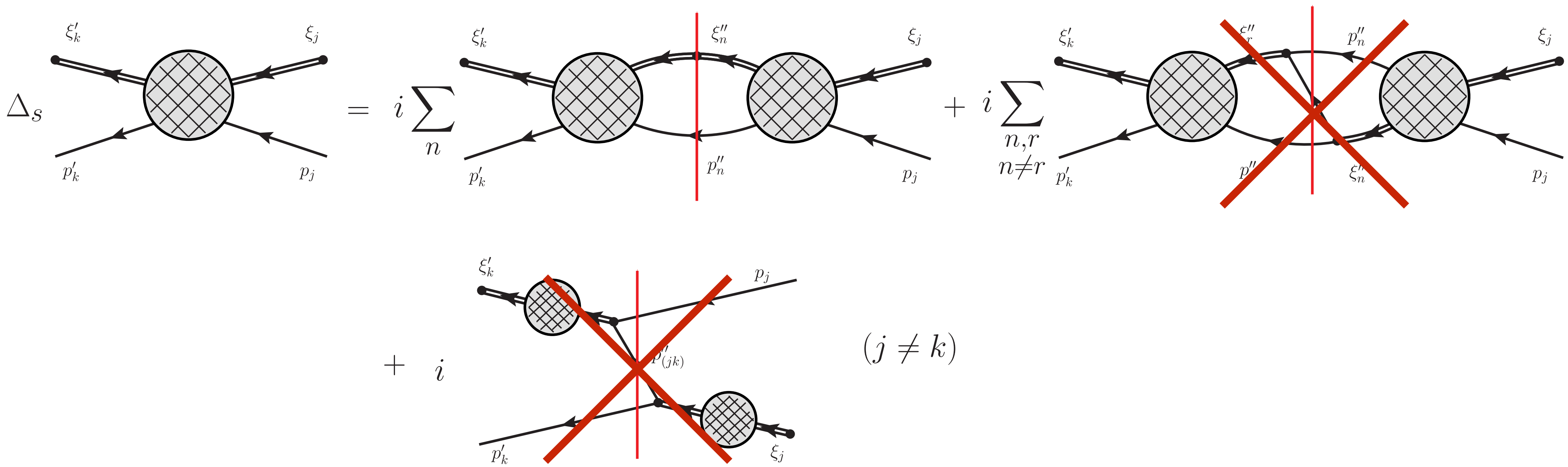
Opportunities and Future Directions





Quasi-2-Body Approximation

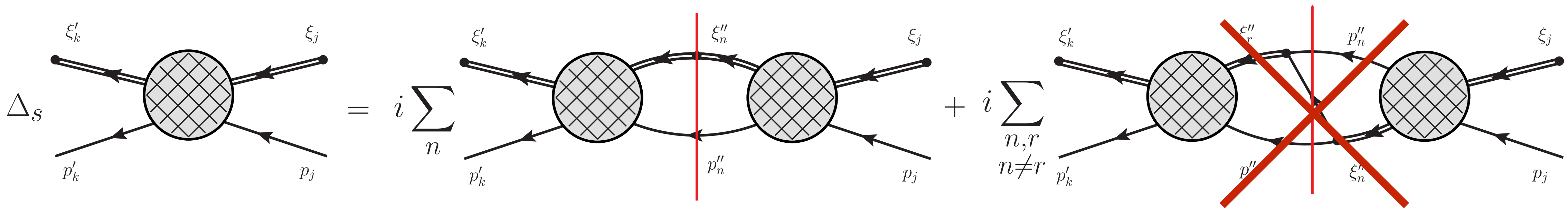
As a first approximation, we consider that the isobars are “quasi-stable” ⇒ Effective **2→2** system, with isobar decay correction in intermediate state

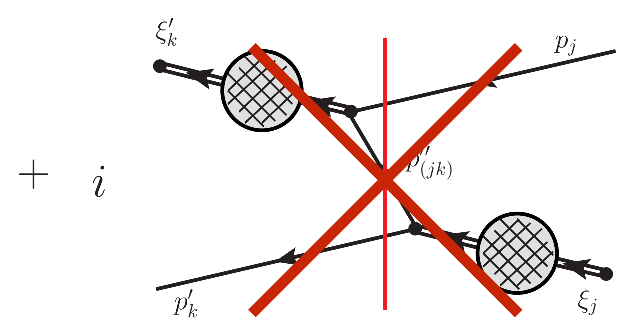


Effects of other terms can be estimated for cases where resonance is far from isobar-spectator threshold

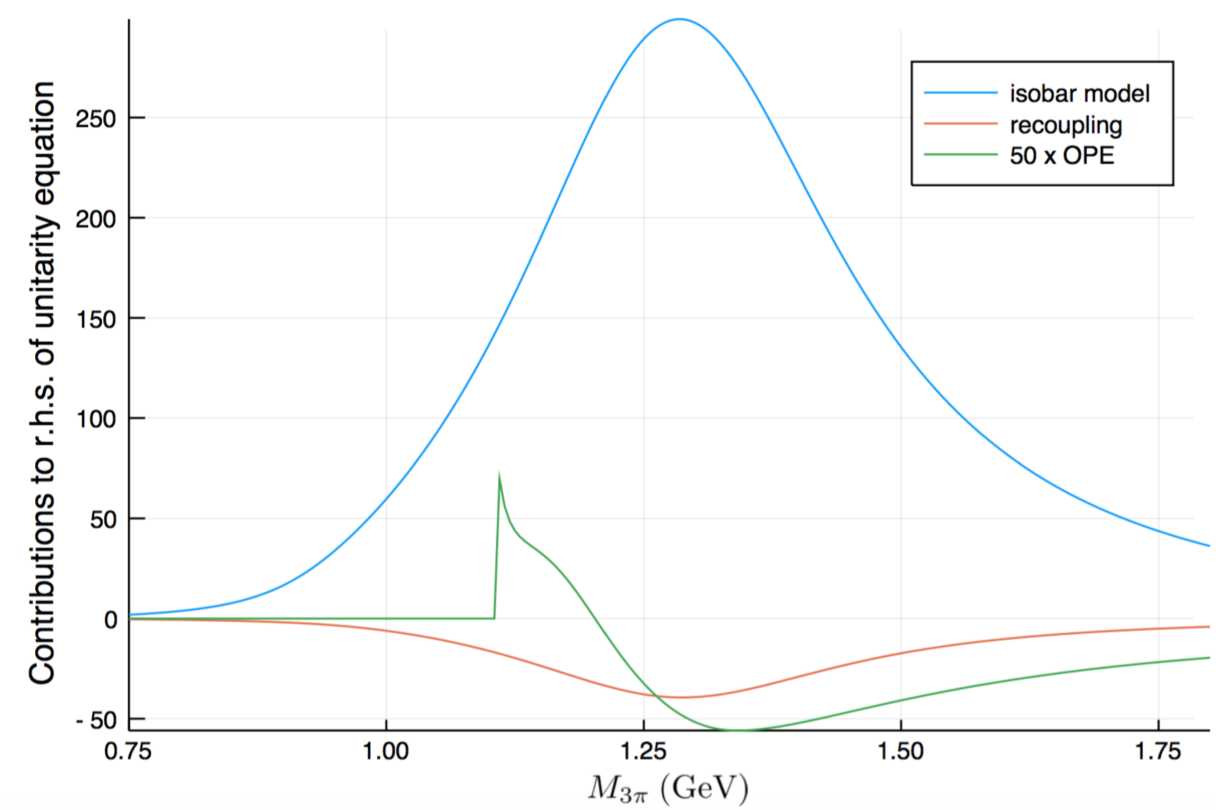
Quasi-2-Body Approximation

As a first approximation, we consider that the isobars are “quasi-stable” ⇒ Effective **2→2** system, with isobar decay correction in intermediate state





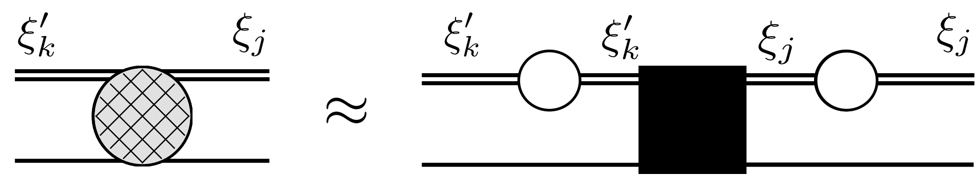
$$\pi^-\pi^-\pi^+ \to \pi^-\pi^-\pi^+$$



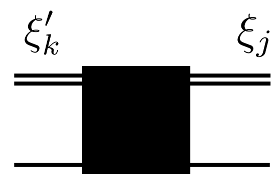
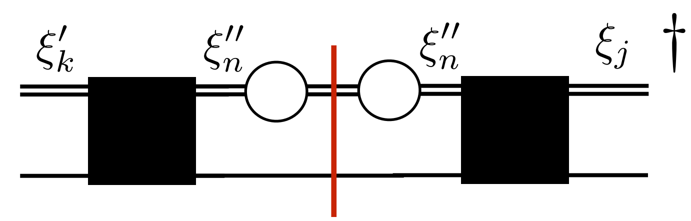
Effects of other terms can be estimated for cases where resonance is far from isobar-spectator threshold

Quasi-2-Body Approximation

Assume that entire sub-energy dependence is purely isobar amplitude



$$\widehat{\mathcal{A}}^J_{kj}(\sigma'_k, s, \sigma_j) \approx \widehat{\mathcal{A}}^J_{kj}(s)$$

$\frac{1}{2i}\,\Delta_s$  $= \frac{1}{2}$ 

$$\mathrm{Im}\ \widehat{\mathcal{A}}^J_{kj}(s) = \sum_n \int_{\sigma^{(n)}_{th}}^{(\sqrt{s}-m_n)^2} d\sigma''_n\, \rho_2(s,\sigma''_n, m_n^2)\,\mathrm{Im}\ D_n^{-1}(\sigma''_n)\widehat{\mathcal{A}}^{J\,*}_{kn}(s)\widehat{\mathcal{A}}^J_{nj}(s)$$

$$\equiv \sum_n \widetilde{\rho}_n(s)\widehat{\mathcal{A}}^{J\,*}_{kn}(s)\widehat{\mathcal{A}}^J_{nj}(s)$$

Quasi - 2→2 Unitarity

Quasi-2-Body Approximation

Have turned **3**-body system into quasi-**2**→**2** coupled-channel system

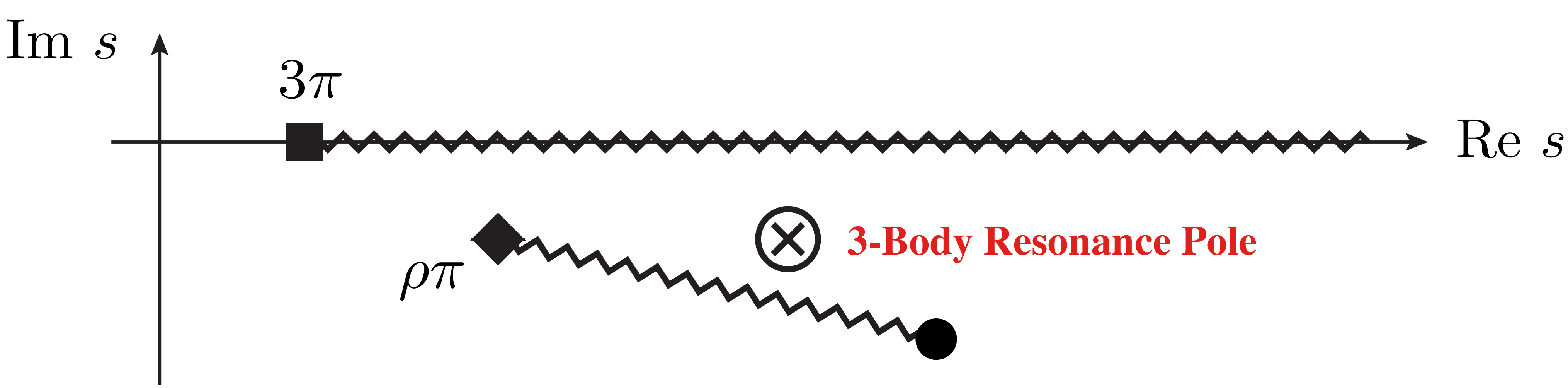
$$\text{Im}\ \widehat{\mathcal{A}}^{J}_{kj}(s) = \sum_{n} \widetilde{\rho}_n(s) \widehat{A}^{J\,*}_{kn}(s) \widehat{A}^{J}_{nj}(s)$$

Can parameterize with *N*/*D* method

Isobar decay effects encoded into quasi-**2**-body phase space

$$\widetilde{\rho}_n(s) = \int_{\sigma_{th}^{(n)}}^{(\sqrt{s}-m_n)^2} d\sigma_n''\, \rho_2(s, \sigma_n'', m_n^2)\, \text{Im}\ D_n^{-1}(\sigma_n'')$$

Introduces additional cuts in *s*-plane (Woolly cuts)



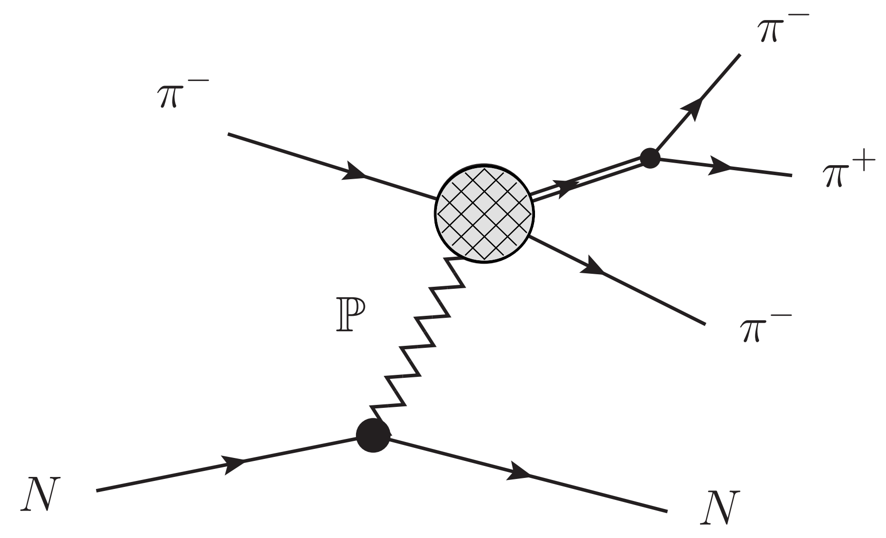
3π at COMPASS

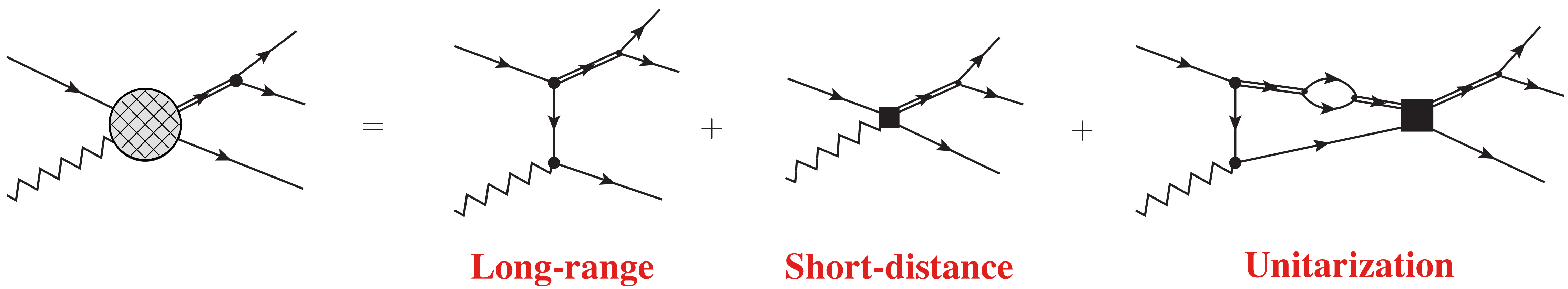
COMPASS has largest dataset for 3π resonance production

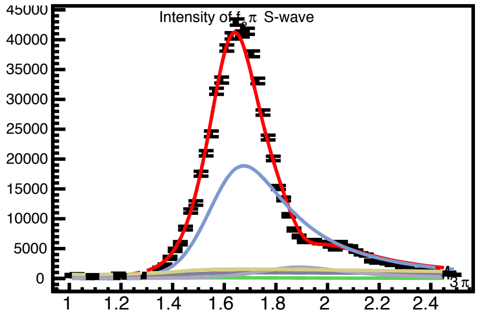
JPAC in collaboration with COMPASS, developing analytic model to extract resonance poles for partial wave intensities

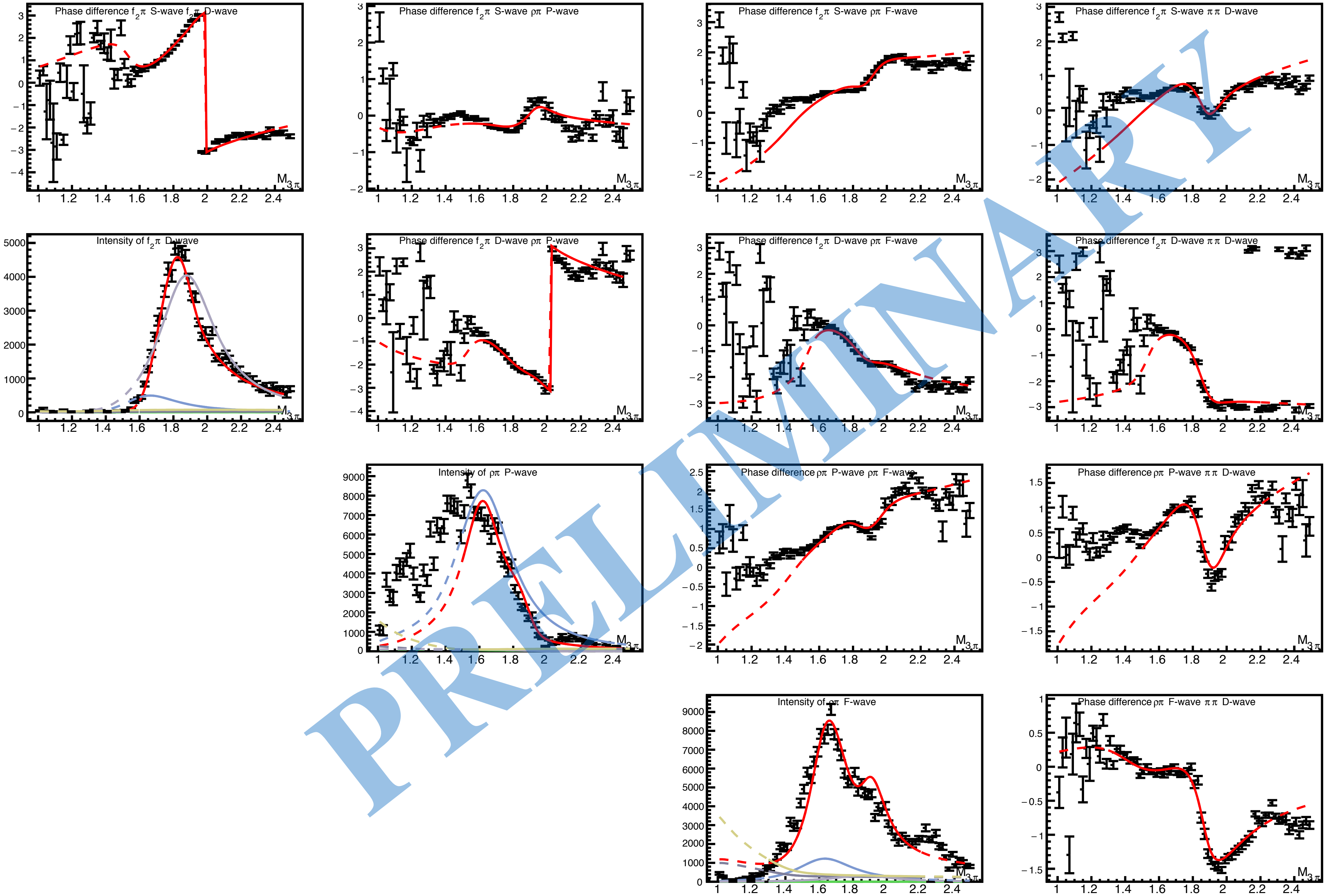
Interested in $J^{PC} = 2^{-+}, 1^{++}$ to investigate π_2- and a_1-systems, and non-resonant production mechanisms (*e.g.* Deck)

Implement quasi-**2**-body unitarity - High-energy process (190 GeV π^--beam), can assume factorization of nuclear recoil









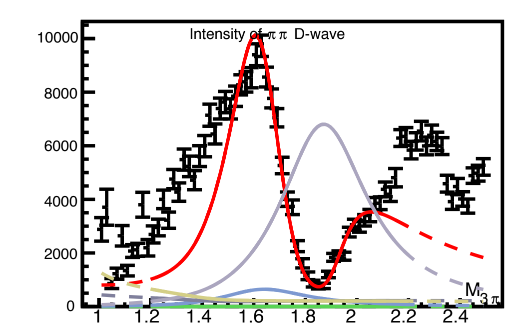
$f_2\pi\ S$

$(\pi\pi)_S\pi\ D$

$\rho\pi\ F$

$\rho\pi\ P$

$f_2\pi\ D$



M. Mikhasenko, AJ [JPAC], In Preparation

Simultaneous fit for 11 t'-bins

Tau-decay

The resonance $a_1(1260)$ pole position can be tested with the quasi-two-body model

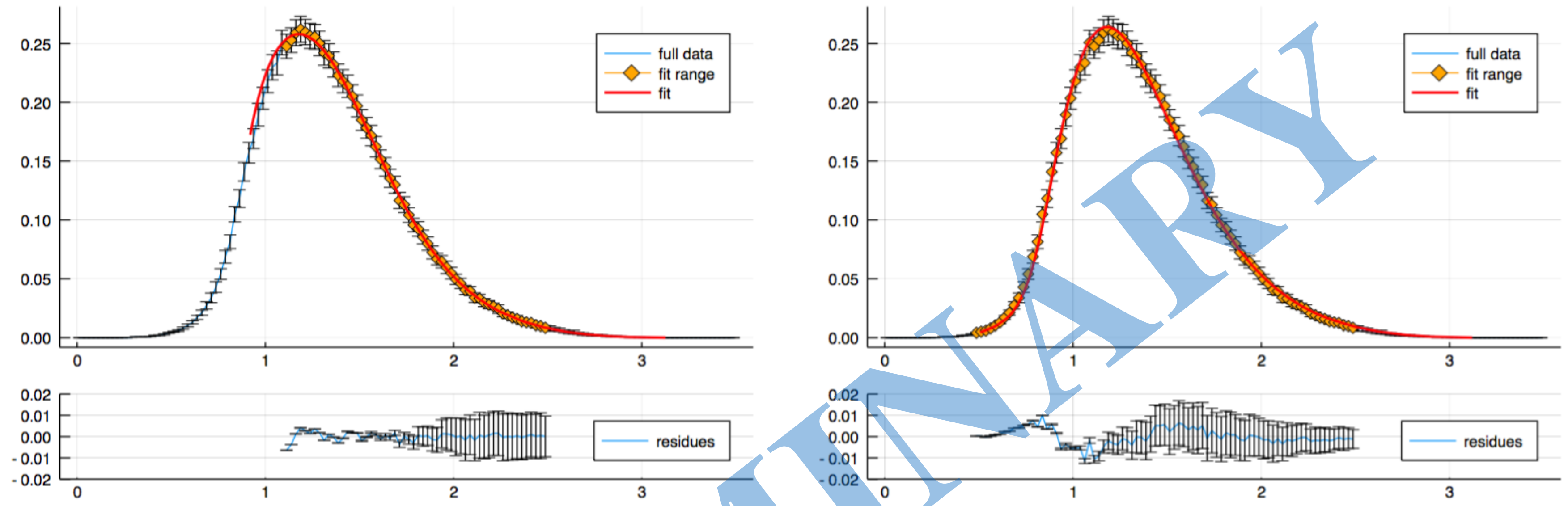
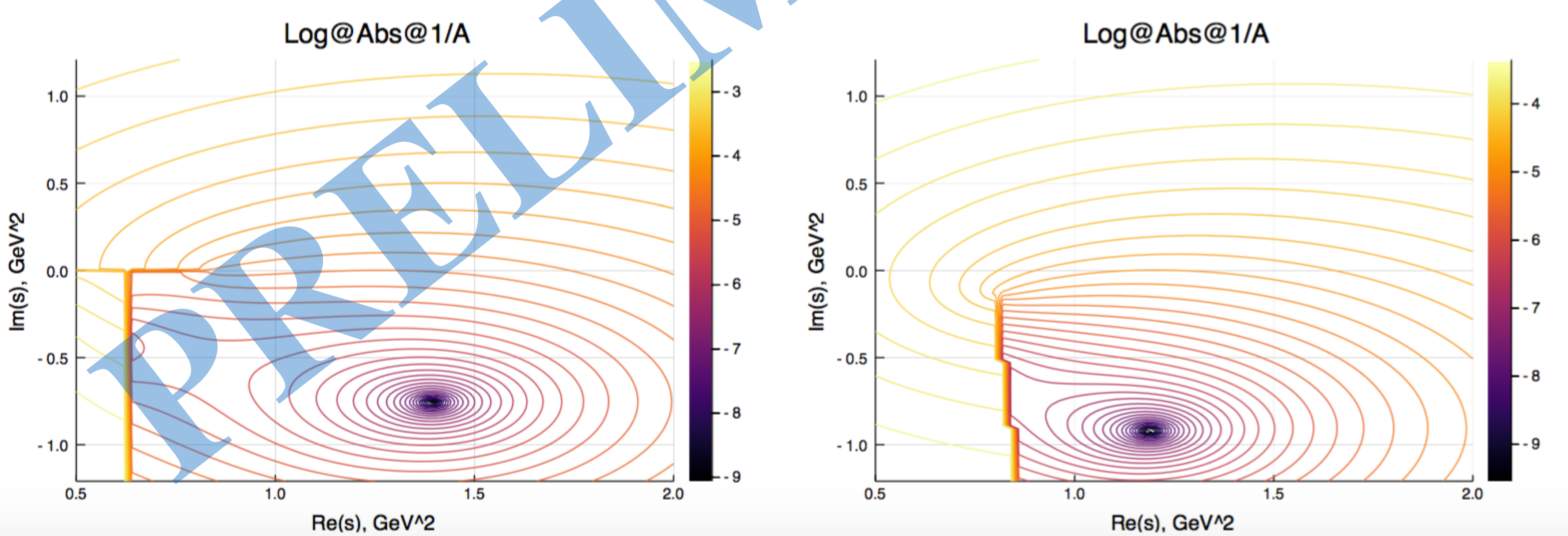
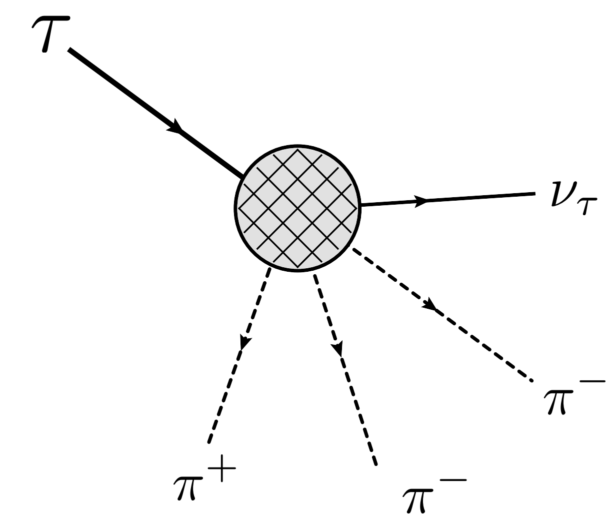


FIG. 2: Intensity of the $a_1(1260)$ for the partial waves.





M. Mikhasenko, AJ [JPAC], In Preparation

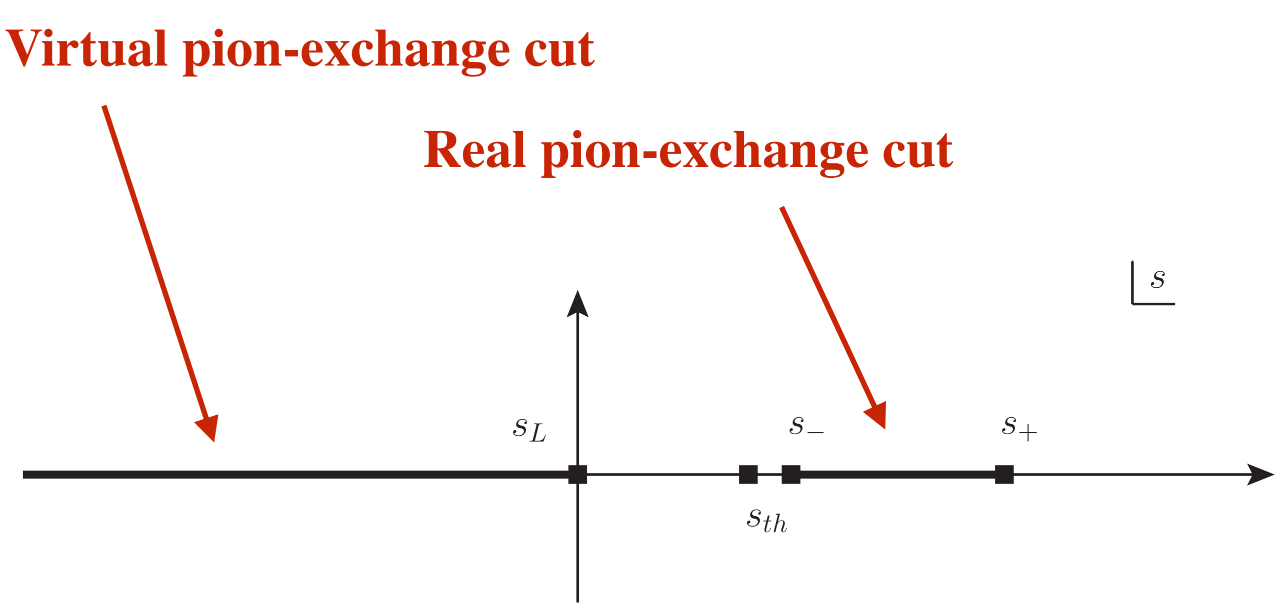
$X(3872)$

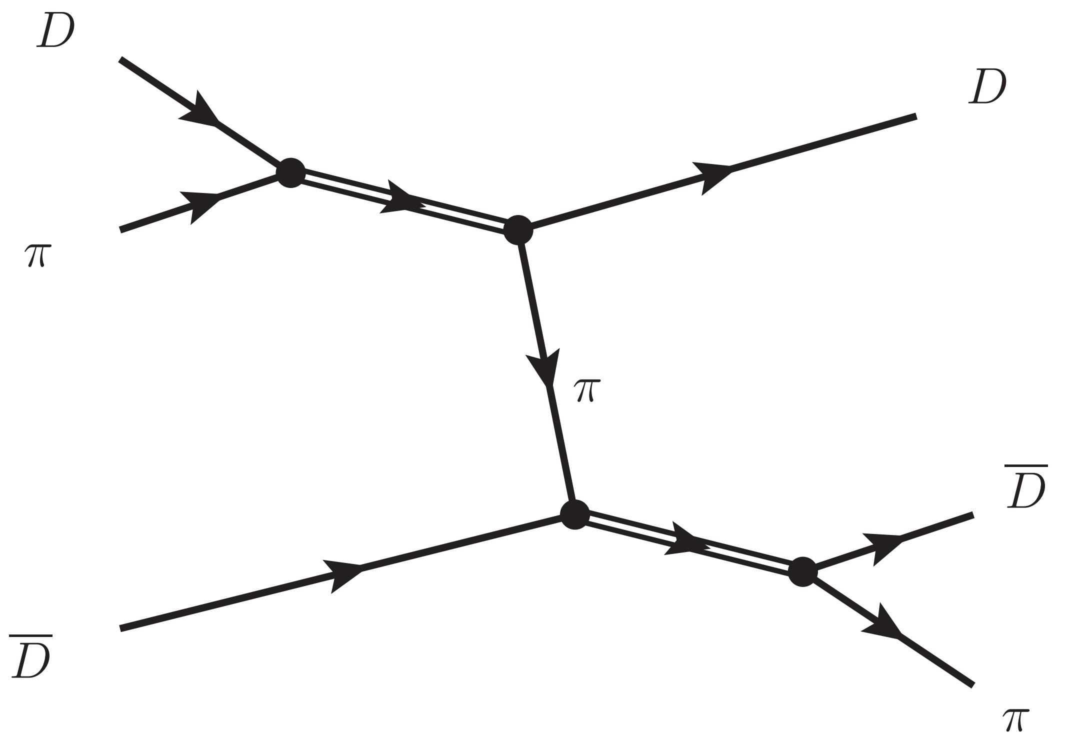
$X(3872)$ is the most well-known XYZ state - Still controversial on the nature of the state (mesonic molecule, tetraquark, …)

Primary decay mode: $X(3872) \longrightarrow \bar{D}D\pi$

Investigate effects of single pion exchange - Unitarization may result in pole

AJ et al. [JPAC], In Preparation





Outlook and Future Directions

Unitarity and Analyticity give consistent constraints on reaction amplitudes

3→3 relations involve functions taken at different points in the complex planes - difficult to find an ‘easy’ parameterization

Work on-going to investigate the analytic structure, and derive a set of relations one could use for various parameterizations

Certain approximations (narrow-width, etc.) may potentially lead to some parameterizations that can be used in analyses