$\gamma^*\pi \to \pi\pi$: analyticity, unitarity, and quark mass dependence

Martin Hoferichter

Institute for Nuclear Theory

University of Washington

INT Workshop on

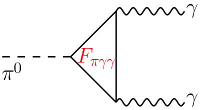
Multi-Hadron Systems from Lattice QCD

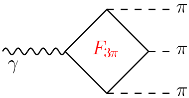
Seattle, February 9, 2018

PRD 86 (2012) 116009, PRD 96 (2017) 114016 with B. Kubis, D. Sakkas, and M. Zanke

work in progress with B. Kubis and M. Niehus

Chiral anomaly and resonance couplings



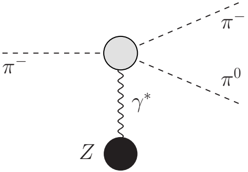


- **Anomalous pion–photon processes** related by low-energy theorem

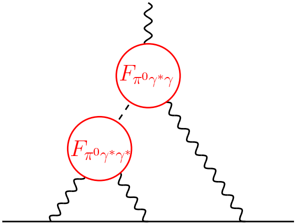
$$F_{\pi\gamma\gamma} = eF_\pi^2 F_{3\pi} = \frac{e^2 N_c}{12\pi^2 F_\pi}$$

- Experimental error on $F_{\pi\gamma\gamma}$ down to **1.5%** PrimEx 2011
- **Low-energy theorem** for $F_{3\pi}$ only tested at the **10% level**

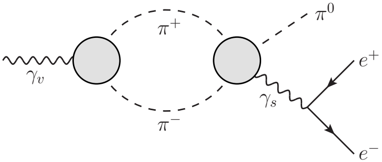
  $\hookrightarrow$ extract from $\gamma\pi \to \pi\pi$ cross section
- Radiative **resonance couplings**: $\rho, \rho_3 \to \pi\gamma$
- **Primakoff program** at COMPASS

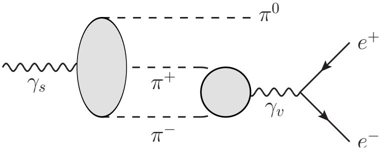


Anomalous magnetic moment of the muon: pion pole



- **Pion transition form factor $F_{\pi^0\gamma^*\gamma^*}$**
  $\hookrightarrow$ pion pole in **light-by-light scattering**
- In $\pi^0 \to \gamma^*\gamma^*$, one photon each isovector/isoscalar
- Discontinuities in isovector channel dominated by $\pi\pi$, e.g. for $e^+e^- \to \pi^0\gamma$:





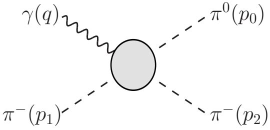
- Key input: **$\gamma^*\pi \to \pi\pi$** and **pion form factor**

Outline

1. Dispersion relations for $\gamma\pi \to \pi\pi$
2. Resonance couplings
3. Quark mass dependence
4. From $\gamma\pi \to \pi\pi$ to $\gamma^*\pi \to \pi\pi$

Kinematics and partial-wave expansion

- **Amplitude decomposition**



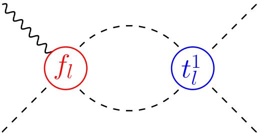
$$\mathcal{M}(s,t,u) = i\epsilon_{\mu\nu\alpha\beta}\epsilon^\mu p_1^\nu p_2^\alpha p_0^\beta \mathcal{F}(s,t,u)$$

$$\mathcal{F}(0,0,0) = F_{3\pi}$$

- **Partial-wave expansion**

$$\mathcal{F}(s,t,u) = \sum_{\text{odd } \ell} f_\ell(s) P'_\ell(z)$$

- **Unitarity**



$$\text{Im}\, f_\ell(s) = \sigma_\pi(s) f_\ell(s) \big(t_\ell^1(s)\big)^* \theta\big(s - 4M_\pi^2\big)$$

$$\sigma_\pi(s) = \sqrt{1 - \frac{4M_\pi^2}{s}} \qquad t_\ell^1(s) = \frac{e^{2i\delta_\ell^1(s)} - 1}{2i\sigma_\pi(s)}$$

- Partial waves $\ell \geq 3$ and higher intermediate states $(4\pi, K\bar{K}, \ldots)$

  negligible below 1 GeV

Dispersive representation

- **Analyticity**: **dispersion relation** for $\mathcal{F}(s,t,u)$

$$\mathcal{F}(s,t,u) = C_2 + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^2}\bigg\{\frac{s^2}{s'-s}+\frac{t^2}{s'-t}+\frac{u^2}{s'-u}\bigg\}\text{Im}\, f_1(s')$$

Dispersive representation

- **Analyticity**: **dispersion relation** for $\mathcal{F}(s,t,u)$

$$\mathcal{F}(s,t,u) = C_2 + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \frac{ds'}{s'^2}\left\{\frac{s^2}{s'-s} + \frac{t^2}{s'-t} + \frac{u^2}{s'-u}\right\}\text{Im}\, f_1(s')$$

- Decomposition into **single-variable functions**

$$\mathcal{F}(s,t,u) = \mathcal{F}(s) + \mathcal{F}(t) + \mathcal{F}(u)$$

$$\mathcal{F}(s) = \frac{1}{3}\big(C_2^{(1)} + C_2^{(2)} s\big) + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \frac{ds'}{s'^2}\frac{s^2}{s'-s}\text{Im}\,\mathcal{F}(s') \qquad C_2^{(1)} + C_2^{(2)} M_\pi^2 = C_2$$

Dispersive representation

- **Analyticity**: **dispersion relation** for $\mathcal{F}(s,t,u)$

$$\mathcal{F}(s,t,u) = C_2 + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{d}s'}{s'^2}\bigg\{\frac{s^2}{s'-s}+\frac{t^2}{s'-t}+\frac{u^2}{s'-u}\bigg\}\mathrm{Im}\, f_1(s')$$

- Decomposition into **single-variable functions**

$$\mathcal{F}(s,t,u) = \mathcal{F}(s)+\mathcal{F}(t)+\mathcal{F}(u)$$

$$\mathcal{F}(s) = \frac{1}{3}\big(C_2^{(1)}+C_2^{(2)}s\big)+\frac{1}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{d}s'}{s'^2}\frac{s^2}{s'-s}\mathrm{Im}\,\mathcal{F}(s') \qquad C_2^{(1)}+C_2^{(2)}M_\pi^2 = C_2$$

- **Partial-wave projection**

$$f_1(s) = \mathcal{F}(s)+\hat{\mathcal{F}}(s) \qquad \hat{\mathcal{F}}(s) = 3\big\langle(1-z^2)\mathcal{F}\big\rangle \qquad \langle z^n\mathcal{F}\rangle = \frac{1}{2}\int_{-1}^{1}\mathrm{d}z z^n\mathcal{F}(t)$$

- **Unitarity**

$$\mathrm{Im}\, f_1(s) = \mathrm{Im}\,\mathcal{F}(s) = \big(\mathcal{F}(s)+\hat{\mathcal{F}}(s)\big)\theta\big(s-4M_\pi^2\big)\sin\delta_1^1(s)e^{-i\delta_1^1(s)}$$

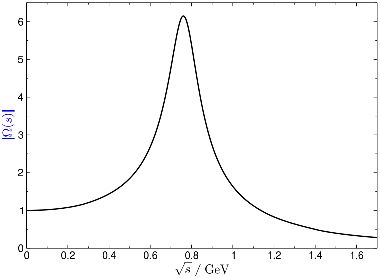
Iterative solution: Omnès functions

Omnès solution for $\mathcal{F}(s)$

$$\mathcal{F}(s) = \Omega(s)\left\{\frac{C_2^{(1)}}{3}\big(1-\dot{\Omega}(0)s\big) + \frac{C_2^{(2)}}{3}s + \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\hat{\mathcal{F}}(s')\sin\delta_1^1(s')}{s'^2(s'-s)|\Omega(s')|}\right\}$$

- Solution in terms of **Omnès function**

$$\Omega(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s'-s)}\right\}$$



- Solve for $\mathcal{F}(s)$ by iteration
- $\hat{\mathcal{F}}(s)$ corresponds to crossed-channel **$\pi\pi$ rescattering**

Iterative solution: basis functions

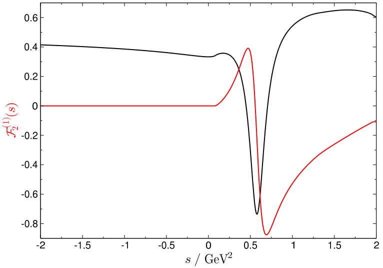
Omnès solution for $\mathcal{F}(s)$

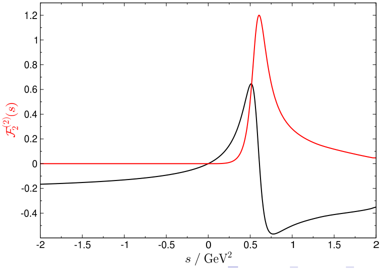
$$\mathcal{F}(s) = \Omega(s)\left\{\frac{C_2^{(1)}}{3}\big(1-\dot{\Omega}(0)s\big) + \frac{C_2^{(2)}}{3}s + \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\hat{\mathcal{F}}(s')\sin\delta_1^1(s')}{s'^2(s'-s)|\Omega(s')|}\right\}$$

- Important observation: $\mathcal{F}(s)$ **linear** in $C_2^{(i)}$

$$\mathcal{F}(s) = C_2^{(1)}\mathcal{F}_2^{(1)}(s) + C_2^{(2)}\mathcal{F}_2^{(2)}(s)$$

$\hookrightarrow$ **basis functions** $\mathcal{F}_2^{(i)}(s)$ can be calculated independently of the $C_2^{(i)}$!



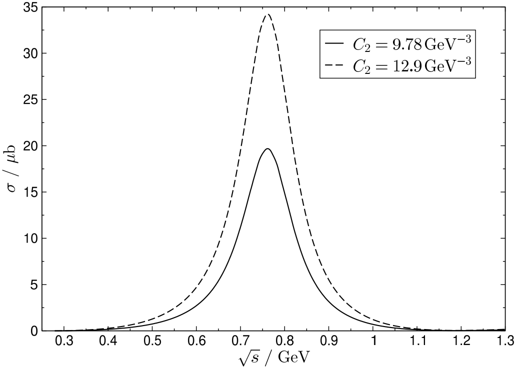


Fitting the subtraction constants

- Representation of the cross section in terms of **two parameters**

  $\hookrightarrow$ fit $C_2^{(i)}$ to data (COMPASS)

- $C_2 = C_2^{(1)} + C_2^{(2)} M_\pi^2 \simeq F_{3\pi}$ strongly sensitive to the ρ peak



Radiative coupling of the $\rho(770)$

- Unitarity relates Riemann sheets:

$$f_{1,\mathrm{I}}(s) - f_{1,\mathrm{II}}(s) = -2\sigma^\pi(s) f_{1,\mathrm{I}}(s) t^1_{1,\mathrm{II}}(s) \qquad \sigma^\pi(s) = \sqrt{\frac{4M_\pi^2}{s} - 1}$$

- Near the pole

$$t^1_{1,\mathrm{II}}(s) = \frac{g^2_{\rho\pi\pi}(s-4M_\pi^2)}{48\pi(s_\rho - s)} \qquad f_{1,\mathrm{II}}(s) = \frac{2e g_{\rho\pi\gamma} g_{\rho\pi\pi}}{s_\rho - s} \qquad \Rightarrow \qquad \frac{e g_{\rho\pi\gamma}}{g_{\rho\pi\pi}} = i\frac{s_\rho \sigma^3_\pi(s_\rho)}{48\pi} f_{1,\mathrm{I}}(s_\rho)$$

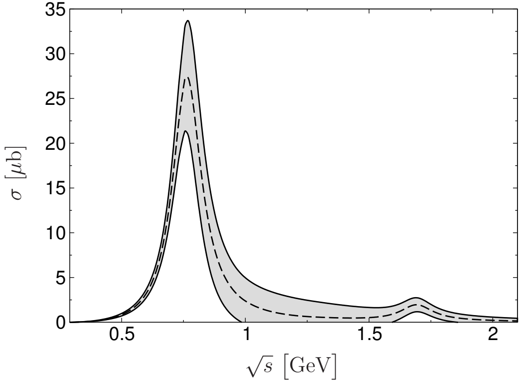
- $g_{\rho\pi\pi}$ and s_ρ known from $\pi\pi$ Roy equations
- Once $C_2^{(i)}$ known, calculate $f_{1,\mathrm{I}}(s_\rho)$ from dispersion relation, numerically

$$f_{1,\mathrm{I}}(s_\rho) = C_2^{(1)}\big(0.588(5) + 0.193(7)i\big) - C_2^{(2)}\big(0.071(7) + 0.570(5)i\big)\ \mathrm{GeV}^2$$

$\hookrightarrow$ determine $g_{\rho\pi\gamma}$ aka "$\Gamma_{\rho\to\pi\gamma}$"

- Can extract the residue $g_{\rho\pi\gamma}$ in a model-independent way from the cross section!

Prediction for the cross section



- Current knowledge of $F_{3\pi}$ + $\omega \to \pi^0\gamma$ + $SU(3)$ fixes $C_2^{(i)}$

$$C_2^{(1)} = 9.9(1.0)\,\text{GeV}^{-3} \qquad C_2^{(2)} = 24.1(2.5)\,\text{GeV}^{-5}$$

$\hookrightarrow$ prediction for COMPASS (including VMD for $\rho_3(1690)$)

- In the future: extract $C_2^{(i)}$ from data $\Rightarrow$ improve $F_{3\pi}$, $g_{\rho\pi\gamma}$

$\pi\pi$ phase shift from unitarized ChPT

- Want $\delta_1^1(s)$ as function of M_π

  $\hookrightarrow$ unitarized ChPT, **inverse amplitude method (IAM)**

- For $SU(2)$ **dispersive justification** based on dispersion relation for $1/t_1^1(s)$ and ChPT for the left-hand cut Gómez Nicola, Peláez, Ríos 2008

- Amounts to

$$t_1^1(s) = t_2(s) + t_4(s) + \mathcal{O}(p^6) \qquad \Rightarrow \qquad t_1^1(s) = \frac{1}{\text{Re}\,\frac{1}{t_1^1(s)} - i\sigma_\pi(s)} = \frac{\big(t_2(s)\big)^2}{t_2(s) - t_4(s)}$$

- Free parameters up to $\mathcal{O}(p^4)$: $\bar{l}_2 - \bar{l}_1$, F_0 (pion decay constant in the chiral limit)

  $\hookrightarrow$ with $F_\pi/F_0 = 1.064(7)$ FLAG there is **only one free parameter**!

- Note: using F_π instead only differs at $\mathcal{O}(p^6)$

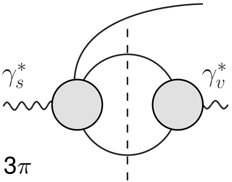
  $\hookrightarrow$ weak sensitivity to l_4 (but lots of other p^6 terms too!)

$\pi\pi$ phase shift from unitarized ChPT: fits

source/fit	$\bar{l}_2 - \bar{l}_1$	$\bar{l}_4$	χ^2/dof
Roy equations + 2-loop ChPT Colangelo, Gasser, Leutwyler 2001	4.7(6)		
fit of IAM to $\delta_1^1(s)$ from Roy equations (without $\bar{l}_4$)	5.8		
fit of IAM to $\delta_1^1(s)$ from Roy equations (including $\bar{l}_4$, our variant)	6.1	11.3	
FLAG 2+1		4.10(45)	
fit of IAM to HadSpec 2015 (including $\bar{l}_4$, Bolton, Briceño, Wilson 2016)	7.0(2)	$-1.0\left(^{+1.1}_{-2.0}\right)$	1.3
fit of IAM to HadSpec 2015 (including $\bar{l}_4$, our variant)	7.0	10.3	1.3
fit of IAM to HadSpec 2015 (without $\bar{l}_4$)	5.9		3.2
fit of IAM to HadSpec 2015 (generic M_π^4 term at $\mathcal{O}(p^6)$)	7.0		1.3

- Pion mass dependence: $\bar{l}_4 = 16\pi^2 l_4^r(\mu) - \log\frac{M_\pi^2}{\mu^2}$
- Fully consistent fit to lattice data appears to require **2-loop amplitudes**
- With $\delta_1^1(s; M_\pi)$ fixed, $\mathcal{F}_2^{(i)}(s; M_\pi)$ follows as before

From $\gamma\pi \to \pi\pi$ to $\gamma^*\pi \to \pi\pi$



- For $q^2 < 9M_\pi^2$ generalization of dispersion relations easy, but: how to fix the **normalization** $a(q^2)$?

  $\hookrightarrow$ $a(0) = F_{3\pi}$ for $\gamma\pi \to \pi\pi$, related to widths for $\omega, \phi \to 3\pi$

- Fit to $e^+e^- \to 3\pi$ MH, Kubis, Leupold, Niecknig, Schneider 2014

$$a(q^2) = \alpha + \beta q^2 + \frac{q^4}{\pi}\int_{s_{\text{thr}}}^{\infty} ds' \frac{\text{Im}\,\mathcal{A}(s')}{s'^2(s'-q^2)}$$

$$\mathcal{A}(q^2) = \frac{c_\omega}{M_\omega^2 - q^2 - i\sqrt{q^2}\,\Gamma_\omega(q^2)} + \frac{c_\phi}{M_\phi^2 - q^2 - i\sqrt{q^2}\,\Gamma_\phi(q^2)}$$

- α fixed by $F_{3\pi}$, $\Gamma_{\omega/\phi}(q^2)$ include 3π, $K\bar{K}$, $\pi^0\gamma$ channels

- Good analytic properties, free parameters: β, c_ω, c_ϕ

- Quark mass dependence of vector meson masses Bijnens, Gosdzinsky 1996, polynomial for (small) space-like virtualities?

Conclusions

- Parameterization of amplitudes for $\gamma^{(*)}\pi \to \pi\pi$ below 1 GeV consistent with **analyticity**, **unitarity**, and **crossing symmetry**
- Consistent analytic continuation to ρ pole
- COMPASS: **chiral anomaly** and **radiative decay of ρ(770)** [and $\rho_3(1690)$]
- Quark mass dependence with $SU(2)$ **inverse amplitude method**
- Outlook:
  - basis functions for unphysical pion masses
  - generalization to non-zero virtuality
  - extrapolate lattice results Briceño et al. 2015, 2016 for $\gamma^{(*)}\pi \to \pi\pi$ to physical point

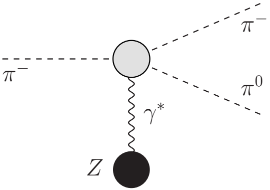
Chiral anomaly: experimental tests

Low-energy theorem for $\gamma\pi \to \pi\pi$

$$F_{3\pi} = \frac{eN_c}{12\pi^2 F_\pi^3} = 9.76(3)\,\text{GeV}^{-3}$$

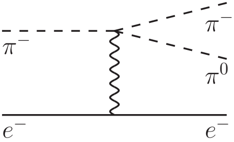
- **Primakoff measurement** of $\gamma\pi^- \to \pi^0\pi^-$
  Ametller et al. 2001

  $$F_{3\pi} = 10.7(1.2)\,\text{GeV}^{-3}$$



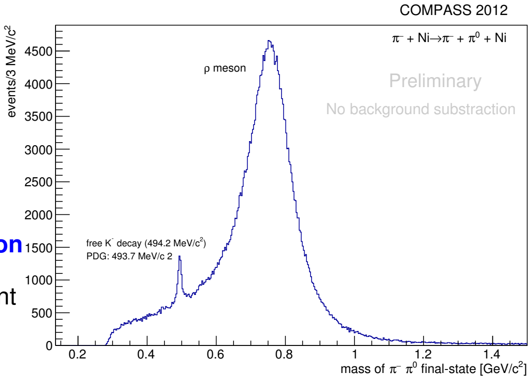
- $\pi^- e^- \to \pi^- e^- \pi^0$ Giller et al. 2005

  $$F_{3\pi} = 9.6(1.1)\,\text{GeV}^{-3}$$



$\Rightarrow$ $F_{3\pi}$ tested at the **10% level**

Chiral anomaly: Primakoff measurement



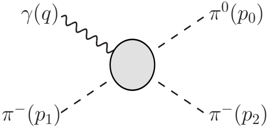
J. Seyfried, MSc thesis 2017

- Previous extractions rely on **ChPT analysis** of Serpukhov 1987
  $\Rightarrow$ restricted to **threshold region**
- COMPASS: Primakoff measurement of full spectrum
- Use **dispersion relations** to exploit **all data up to 1 GeV** for the anomaly extraction
- Basic idea: include the $\rho(770)$ model-independently in terms of the ***P*-wave $\pi\pi$ phase shift**

Crossing symmetry of $\mathcal{F}(s,t,u)$

- **Amplitude decomposition**

$$\mathcal{M}(s,t,u) = i\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu}p_1^{\nu}p_2^{\alpha}p_0^{\beta}\mathcal{F}(s,t,u)$$



$$s = (q+p_1)^2 = (p_0+p_2)^2 \qquad t = (p_1-p_2)^2 = (q-p_0)^2 \qquad u = (q-p_2)^2 = (p_1-p_0)^2$$

- Crossing symmetry in $s \leftrightarrow u$ and $t \leftrightarrow u \Rightarrow$ **fully crossing symmetric**
- **Isospin symmetry**

$$\langle\gamma\pi^-|\mathcal{M}|\pi^0\pi^-\rangle = \underbrace{\frac{1}{\sqrt{2}}\mathcal{M}^1}_{I_\gamma=0} + \underbrace{\frac{1}{2}(\mathcal{M}^1+\mathcal{M}^2)}_{I_\gamma=1} \qquad \langle\gamma\pi^0|\mathcal{M}|\pi^+\pi^-\rangle = \underbrace{\frac{1}{\sqrt{2}}\mathcal{M}^1}_{I_\gamma=0} + \underbrace{\frac{1}{3}(\mathcal{M}^2-\mathcal{M}^0)}_{I_\gamma=1}$$

$\Rightarrow$ photon isospin $I_\gamma = 0$, total isospin $I = 1$

$\Rightarrow$ only odd partial waves

Extracting the chiral anomaly: towards the chiral limit

- **Quark mass renormalization**

$$C_2 = F_{3\pi}\left(1 + 3M_\pi^2 \bar{C}\right)$$

$\hookrightarrow$ **not calculable** in dispersion theory $\Rightarrow$ need ChPT here

- At 1-loop

$$\bar{C} = -\frac{64\pi^2}{3e} C_2^{\mathrm{r}}(\mu) - \frac{1}{96\pi^2 F_\pi^2}\left(1 + \log\frac{M_\pi^2}{\mu^2}\right)$$

- **Resonance saturation**

$$C_2^{\mathrm{r}}(\mu) = -\frac{3e}{128\pi^2 M_\rho^2} \qquad \Rightarrow \qquad 3M_\pi^2 \bar{C} = 4.9\% + 1.8\%$$

- Theoretical uncertainty due to the low-energy constant $C_2^{\mathrm{r}}(\mu)$ around **1%**

Sum rule for $f_{\pi^0\gamma}$

- Unsubtracted dispersion relation implies

$$f_{\pi^0\gamma}(0) = \frac{F_{\pi\gamma\gamma}}{2} = \frac{1}{12\pi^2}\int_{4M_\pi^2}^{\infty} ds' \frac{q_\pi^3(s')}{s'^{3/2}} \big(F_\pi^V(s')\big)^* f_1(s')$$

- Indeed fulfilled by HLS amplitude + NWA + KSFR relation $2F_\pi^2 g_{\rho\pi\pi}^2 = M_\rho^2$

$$\mathcal{F}(s,t,u) = F_{3\pi}\bigg\{1 + \frac{1}{2}\big(D_\rho(s) + D_\rho(t) + D_\rho(u) - 3\big)\bigg\} \qquad D_\rho(s) = \frac{M_\rho^2}{M_\rho^2 - s}$$

- More realistic input ($C_2 = 1.066 F_{3\pi}$, cutoff $\Lambda = 1.2\,\text{GeV}$, phenomenological pion form factor) $\Rightarrow$ Saturation of **87% . . . 90%**
- Remainder due to higher energies and higher intermediate states

Off-shell terms and dispersion relations

- Dispersion relations only valid **on-shell**, otherwise extra term

$$\big(s+t+u-3M_\pi^2\big)\mathcal{G}(s,t,u)$$

- Matching with ChPT

$$C_2+\big(s+t+u-3M_\pi^2\big)\mathcal{G}(s,t,u)=F_{3\pi}\big(1+\bar{C}(s+t+u)\big)$$

$\Rightarrow \mathcal{G}(s,t,u)$ constant up to higher orders

$\Rightarrow C_2-3M_\pi^2\mathcal{G}=F_{3\pi}$ and $\mathcal{G}=F_{3\pi}\bar{C}\Rightarrow C_2=F_{3\pi}\big(1+3M_\pi^2\bar{C}\big)$

- **Quark mass renormalization not accessible in dispersion theory**