Update on the relativistic three-particle finite-volume formalism

Maxwell T. Hansen
CERN

February 6th, 2018

A good time for three particles in a box!

- At this workshop, three competing groups are presenting finite-volume three-particle formalism

- Not just that!…
  All three groups are presenting examples of numerical implementation

- In a nutshell…

Raul, Steve and I

Relativistic, EFT-independent, follows the approach of Lüscher (ala Kim, Sachrajda and Sharpe) to the extent possible

Akaki, (H.W. Hammer, J.-Y. Pang)

Non-relativistic, EFT based, focuses on extracting LECs, simpler derivation and formulae

Maxim and Michael

Relativistic, built on unitary constraints + replacing integrals with sums over shells

- Aims for today:
  - Better understand, compare and contrast the methods
  - Understand when (if) each method is useful or even best
  - Discuss if ideas can be combined to reach an optimal approach

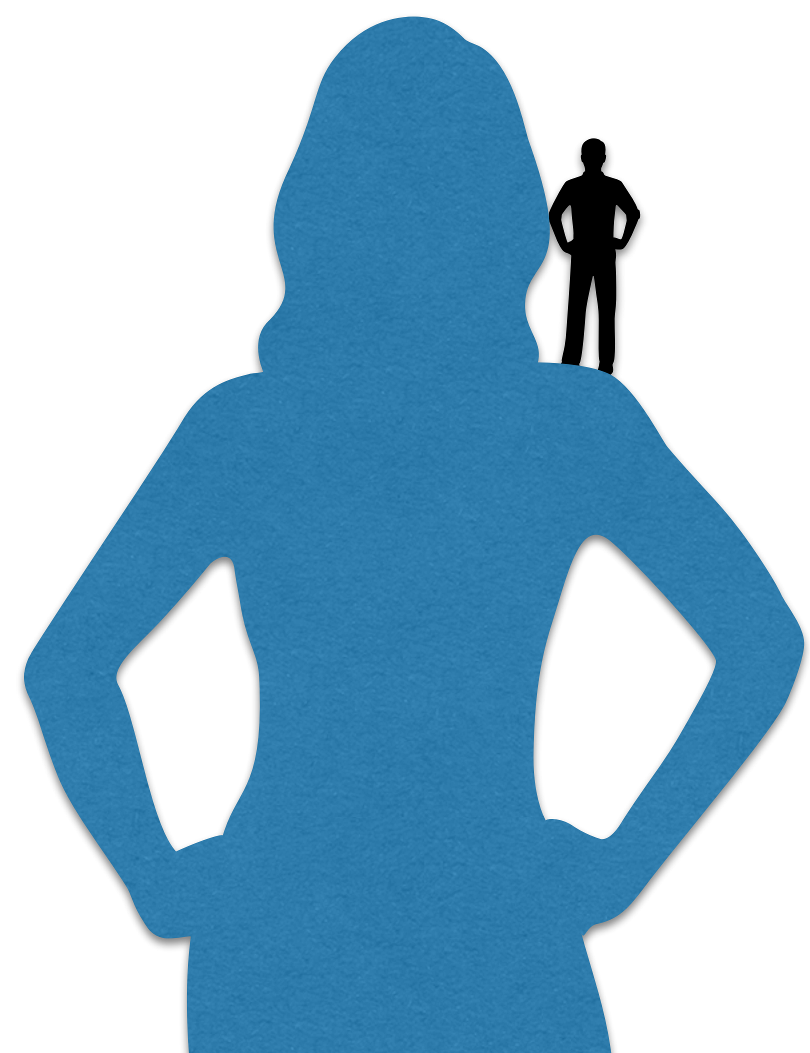
If I have seen further…

- These ideas build on a great deal of earlier work
  Agadjanov, Beane, Bernard, Briceño, Christ, Davoudi, Detmold, Döring, Fu, Guo, Huang, Kim, Lellouch, Leskovec, Lüscher, Luu, Mai, Meißner, Meyer, Oset, Polejaeva, Prelovsek, Rios, Rusetsky, Savage, Sachrajda, Sharpe, Tan, Walker-Loud, Yamazaki, Yang

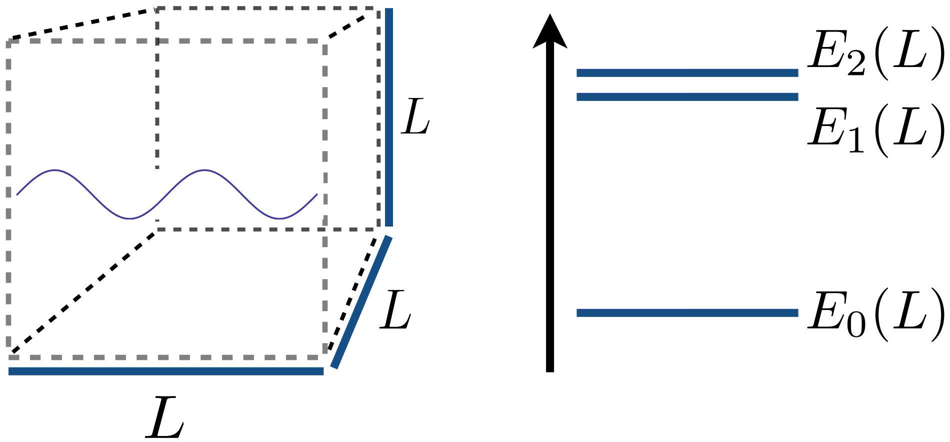
- To go forward we will undoubtedly need input from non-finite-volume experts here and elsewhere

- Most importantly, this would all be quite useless without the remarkable numerical progress in this field

Let's jump in!



Basic set-up



cubic, spatial volume (extent L)

periodic boundary conditions

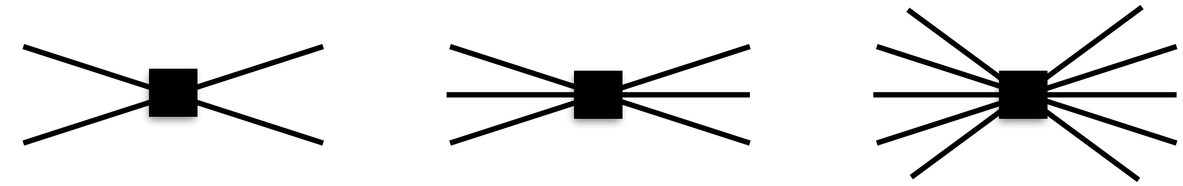
$$\vec{p} \in (2\pi/L)\mathbb{Z}^3$$

time direction **infinite**

L large enough to ignore e^{-mL}

Generic relativistic QFT

1. Include all interactions



2. no power-counting scheme

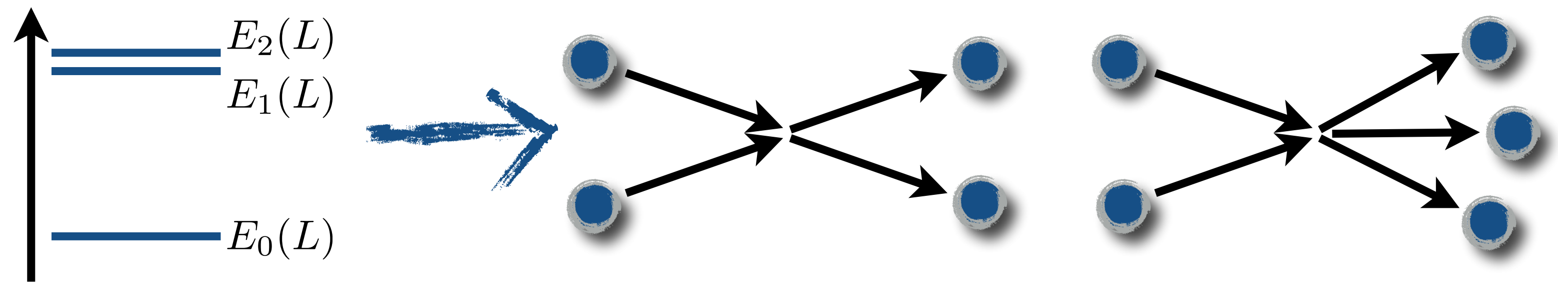
Not possible to directly calculate scattering observables to all orders

But it is possible to derive *general, all-orders relations* to finite-volume quantities

Assume lattice effects are small and accommodated elsewhere

Work in continuum field theory throughout

Finite-volume correlators



Convenient to work with momentum space, finite-volume correlators

$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

Total 4-momentum

$$P = (E, \vec{P}) = (E, 2\pi\vec{n}/L)$$

c.m. frame energy: $E^{*2} = E^2 - \vec{P}^2$

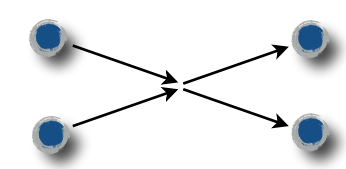
n-particle interpolator

e.g. $\pi(\mathbf{p})\pi(-\mathbf{p})$ or $\bar{q}\Gamma q$

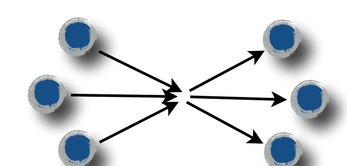
(only quantum numbers relevant)

Focus on a widow of energies to isolate particular on-shell states

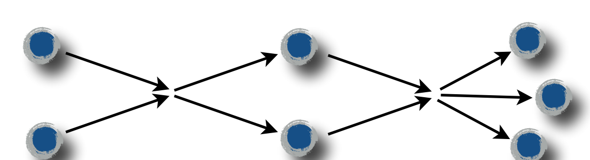
- **Two particles:** $\boxed{\mathbb{Z}_2 \checkmark}$ $0 < E^* < 4m$ $\boxed{\cancel{\mathbb{Z}_2}}$ $m < E^* < 3m$



- **Three particles** $\boxed{\mathbb{Z}_2 \checkmark}$ $m < E^* < 5m$



- **Two and three particles** $\boxed{\cancel{\mathbb{Z}_2}}$ $m < E^* < 4m$



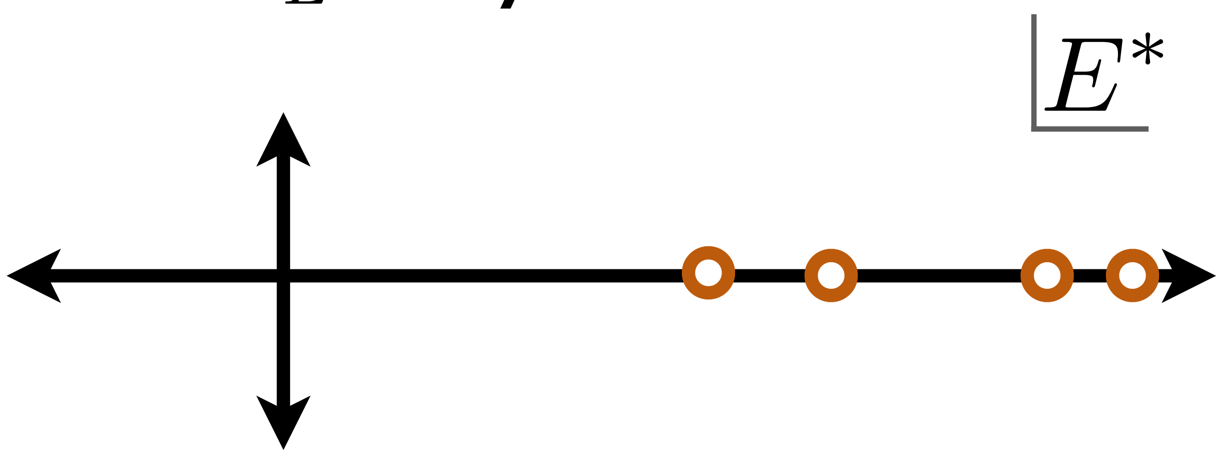
Of poles and branch cuts

$$C_L(P) \equiv \int_L d^4x\, e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

At fixed $L, \vec{P}$, poles in C_L give the finite-volume spectrum

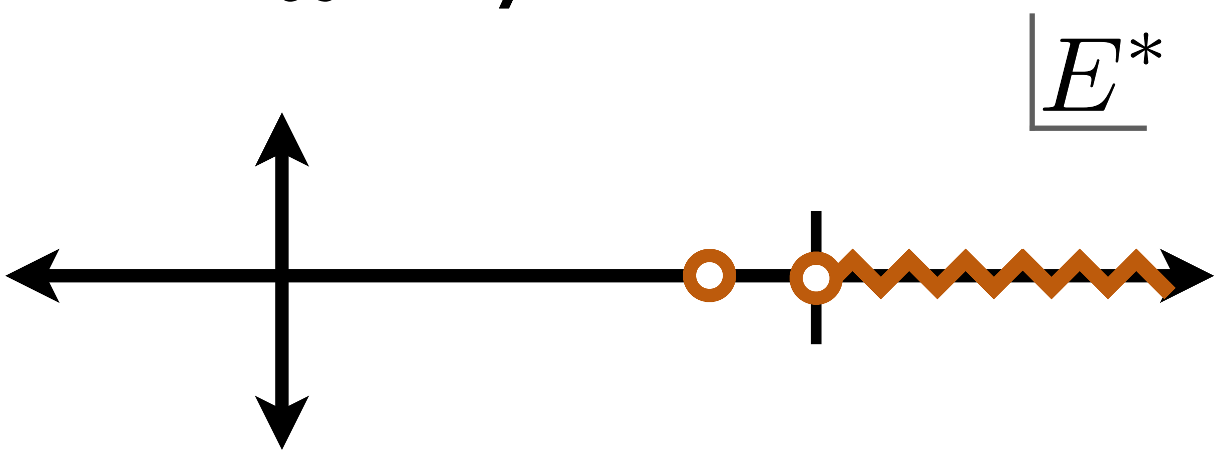
---

C_L analytic structure



- Real function (unitarity hidden)
- No cuts $\longrightarrow$ only one sheet
- No resonance poles

C_∞ analytic structure

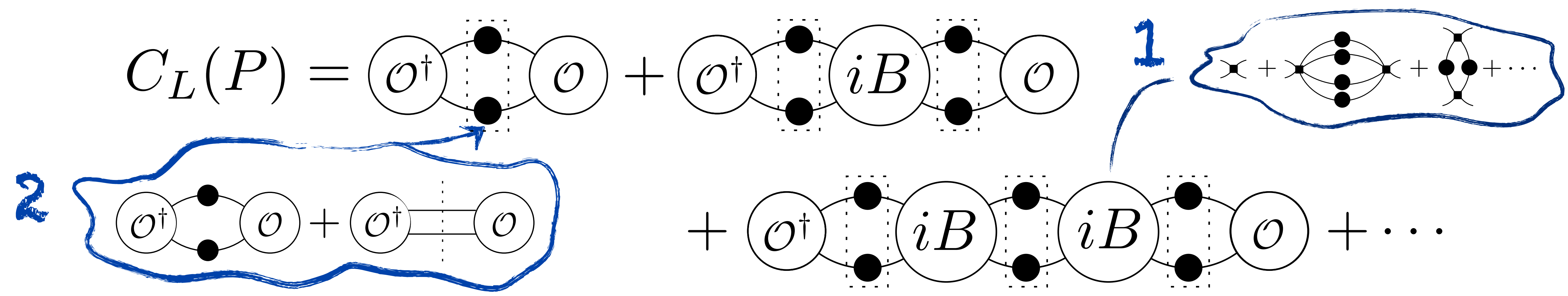


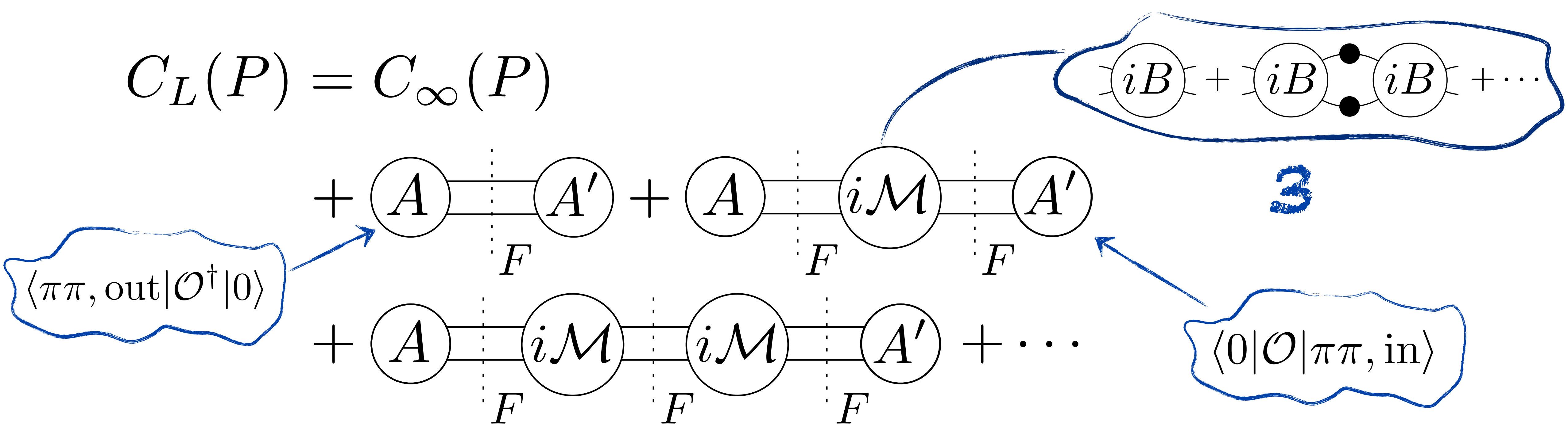
- Complex function (constrained by unitarity)
- Scattering states form cut
- Resonance poles on unphysical sheets

Want to relate $C_L \longleftrightarrow C_\infty$ ($\mathcal{M}_{n\to m}$ is just a specific choice of C_∞)

The idea is to "reach in" and correct the singularity structure

Two-to-two review *(Here with identical, scalar, Z_2 symmetry)*

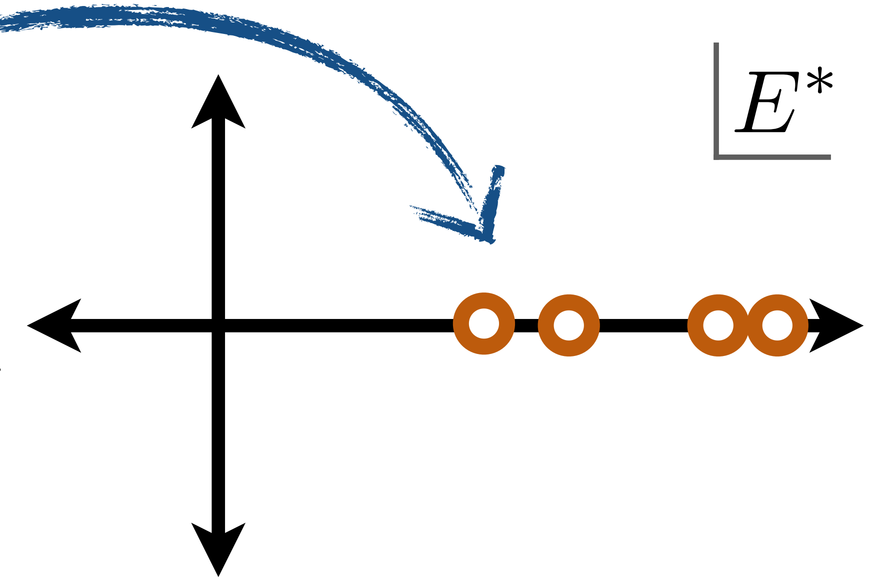




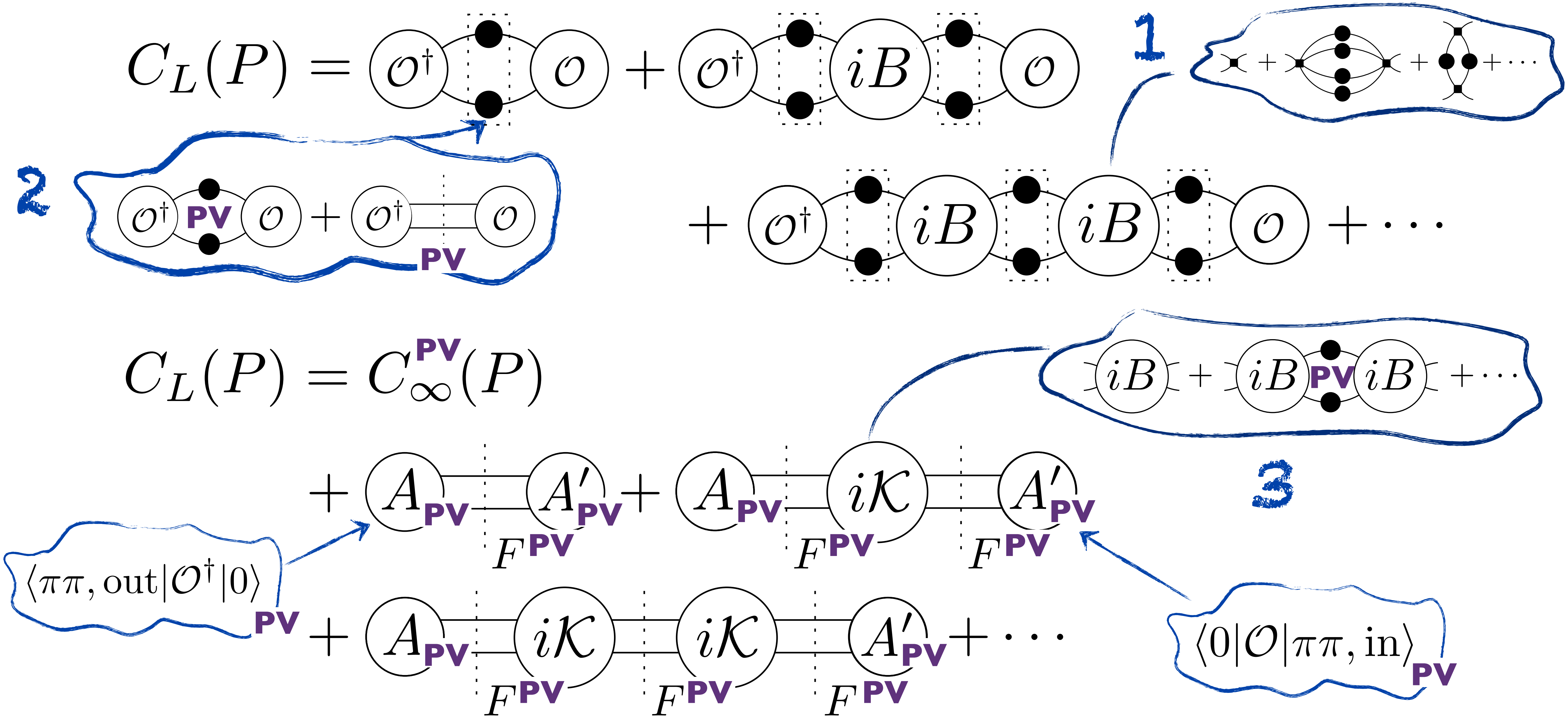
We deduce...

poles are in here

$$C_L(P) = C_\infty(P) - A' F \boxed{\frac{1}{1 + \mathcal{M}_{2\to 2} F}} A$$



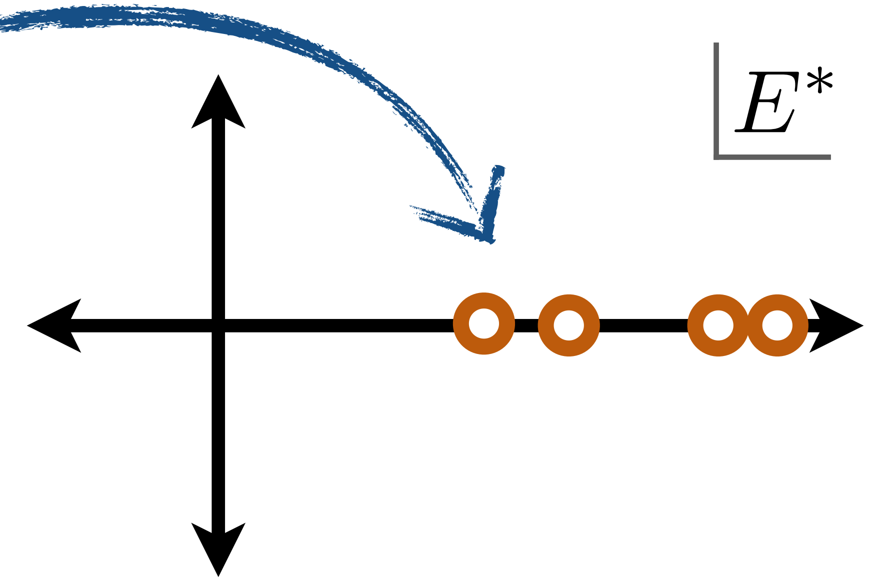
Two-to-two review *(Here with identical, scalar, Z_2 symmetry)*



We deduce...

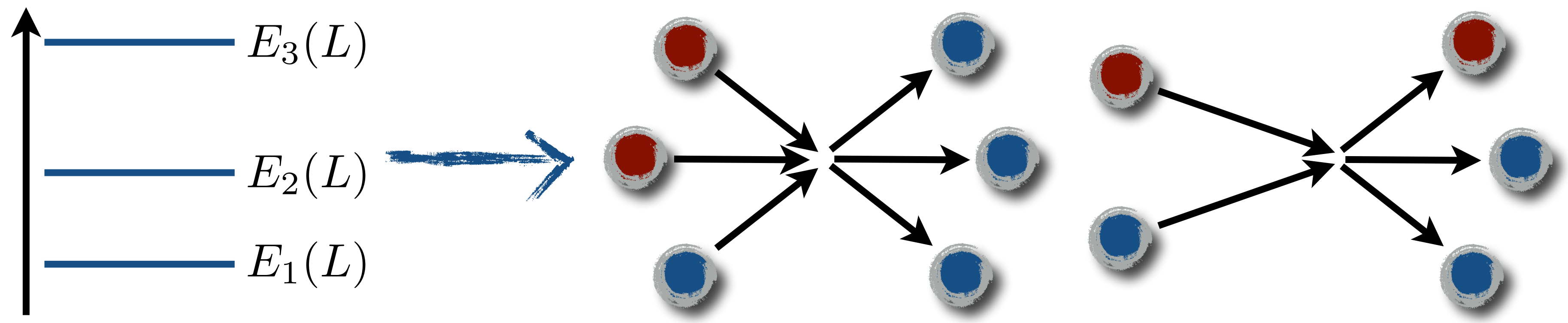
poles are in here

$$C_L(P) = C_\infty^{\mathrm{PV}}(P) - A'_{\mathrm{PV}} \boxed{F^{\mathrm{PV}} \frac{1}{1 + \mathcal{K}_2 F^{\mathrm{PV}}}} A_{\mathrm{PV}}$$



Our aim is to extend the derivation for arbitrary

relativistic two- and three-particle systems

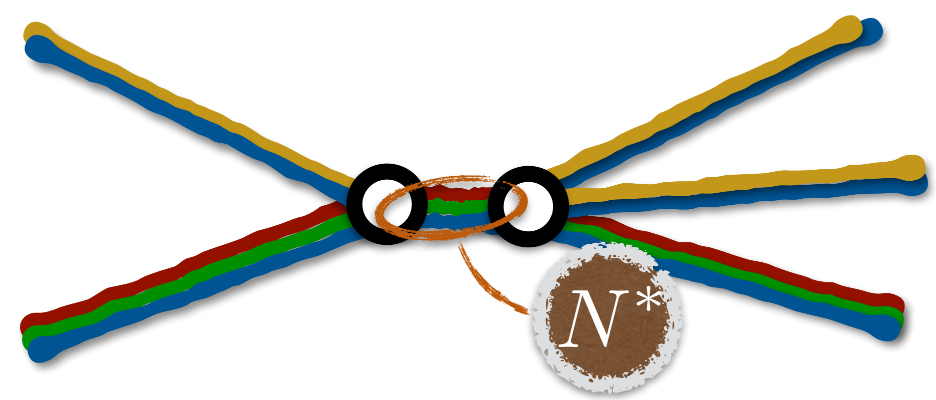


Potential applications...

Studying three-particle resonances

$$\omega(782) \to \pi\pi\pi$$

$$N(1440) \to N\pi, N\pi\pi$$

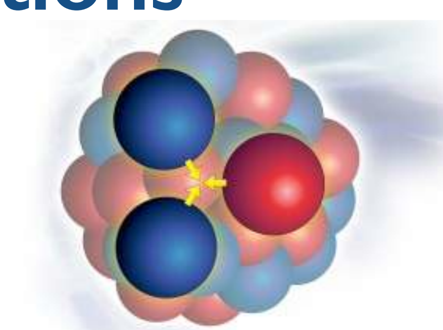


Calculating weak decay amplitudes and form factors

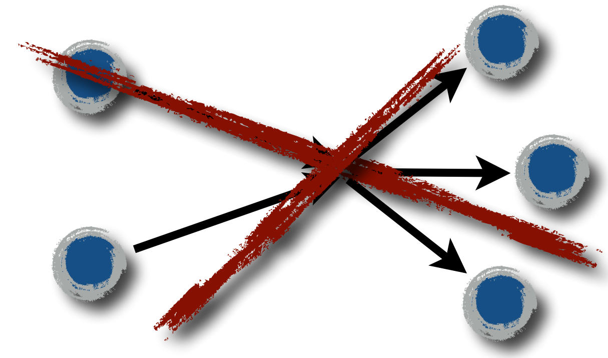
$$K \to \pi\pi\pi$$

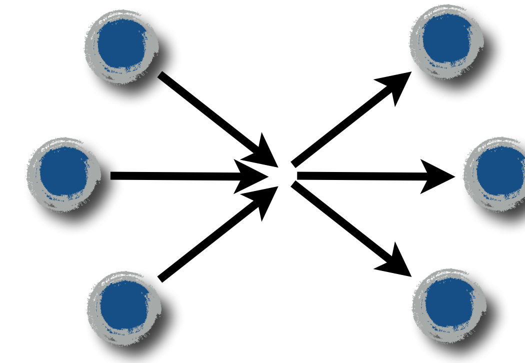
Determining three-body interactions

NNN three-body forces needed as EFT input for studying larger nuclei and nuclear matter



We begin by considering identical scalar particles



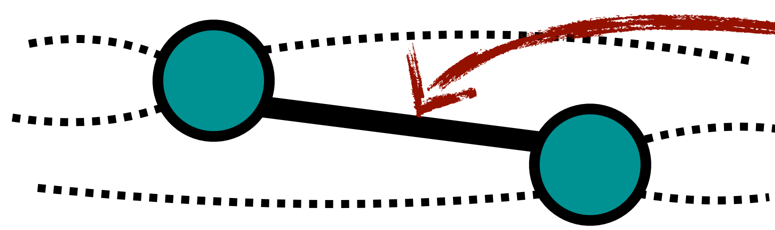


For now we turn off **two-to-three scattering** using a symmetry

Three-to-three amplitude has kinematic singularities

$$i\mathcal{M}_{3\to 3} \equiv$$ fully connected correlator with six external legs amputated and projected on shell

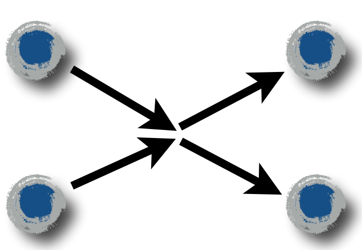
$$=$$



$$+\cdots$$

Certain external momenta put this on-shell!

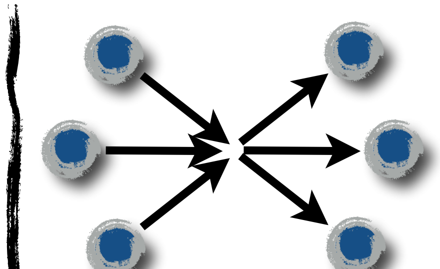
Three-to-three amplitude has more degrees of freedom



12 momentum components

-10 Poincaré generators

2 degrees of freedom



18 momentum components

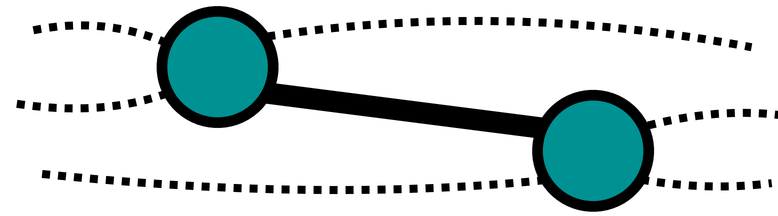
-10 Poincaré generators

8 degrees of freedom

How can we extract a singular, eight-coordinate function using finite-volume energies?

Spectrum depends on a modified quantity with singularities removed

$$\mathcal{K}_{\text{df},3} \not\ni$$

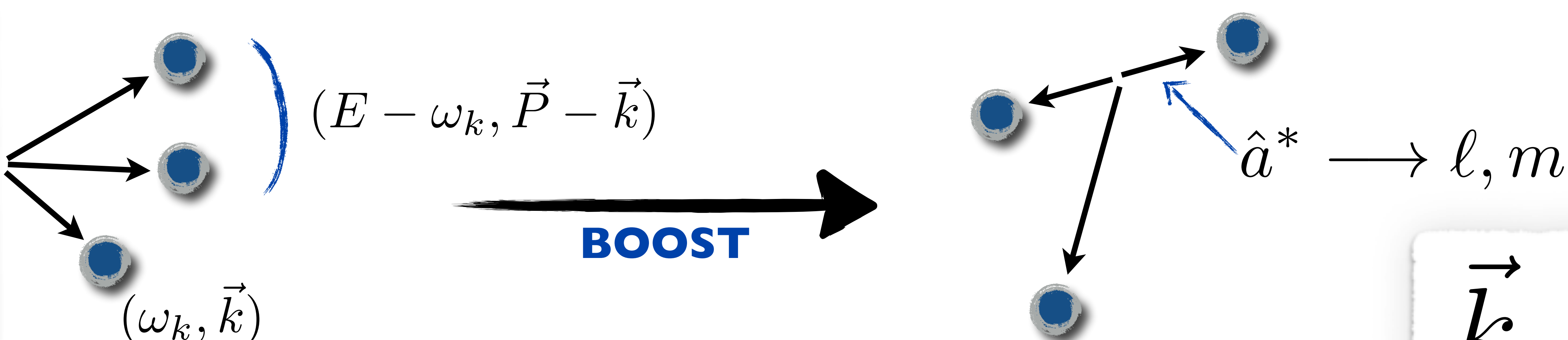


df stands for "divergence free"

Same degrees of freedom as $\mathcal{M}_3$; Smooth, real function (easier to extract)

Relation to $\mathcal{M}_3$ is known (depends only on on-shell $\mathcal{M}_2$)

Degrees of freedom encoded in an extended matrix space

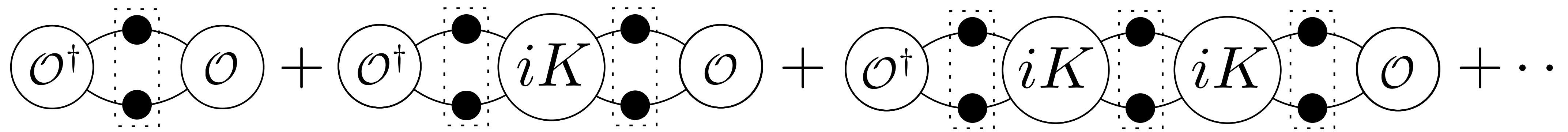


$\vec{k}, \ell, m$

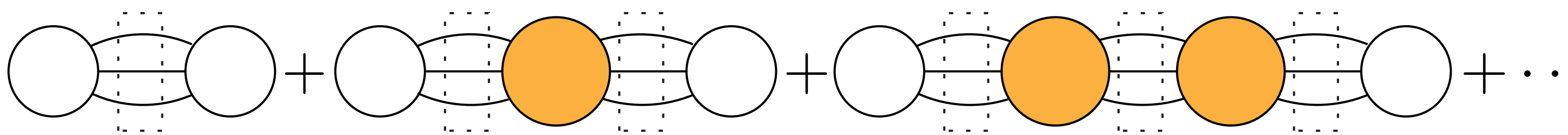
($\vec{k}$ is restricted to finite-volume momenta)

New skeleton expansion

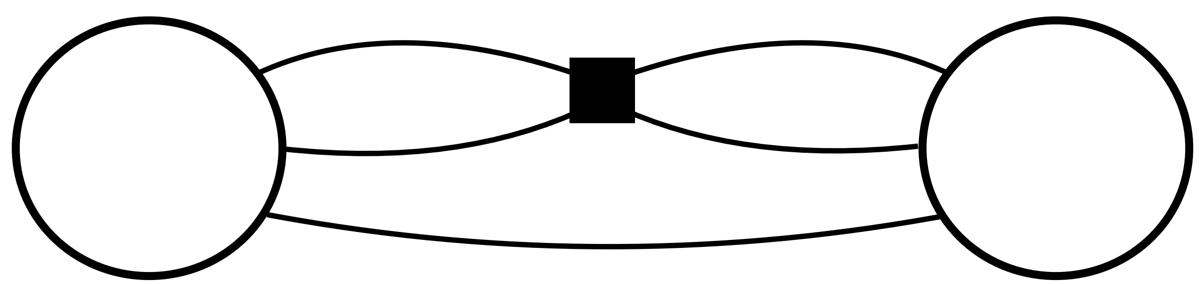
Recall for two particles we started with a "skeleton expansion"

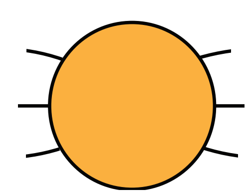
$C_L(P) =$ 

So now we need the same for three…

$C_L(E,\vec{P}) \stackrel{?}{=}$ 

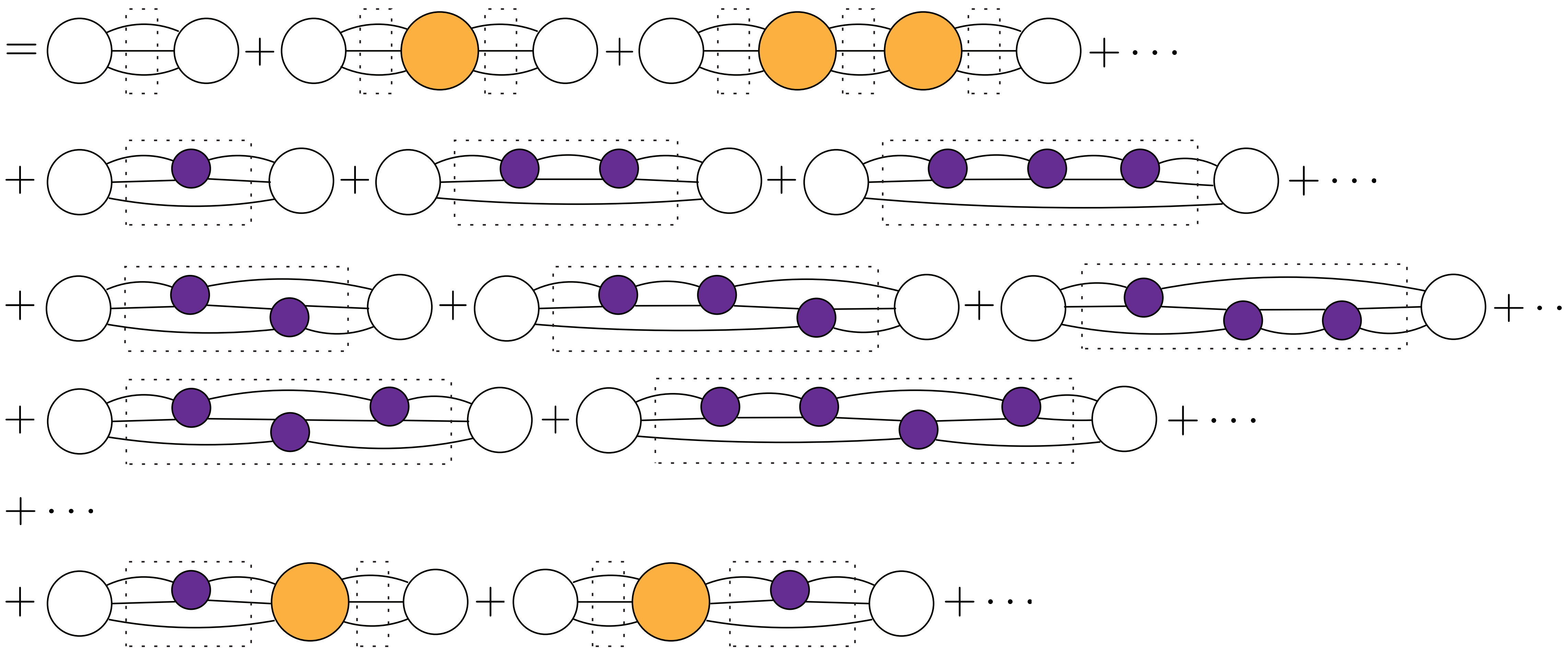
No!… We must also accommodate diagrams like



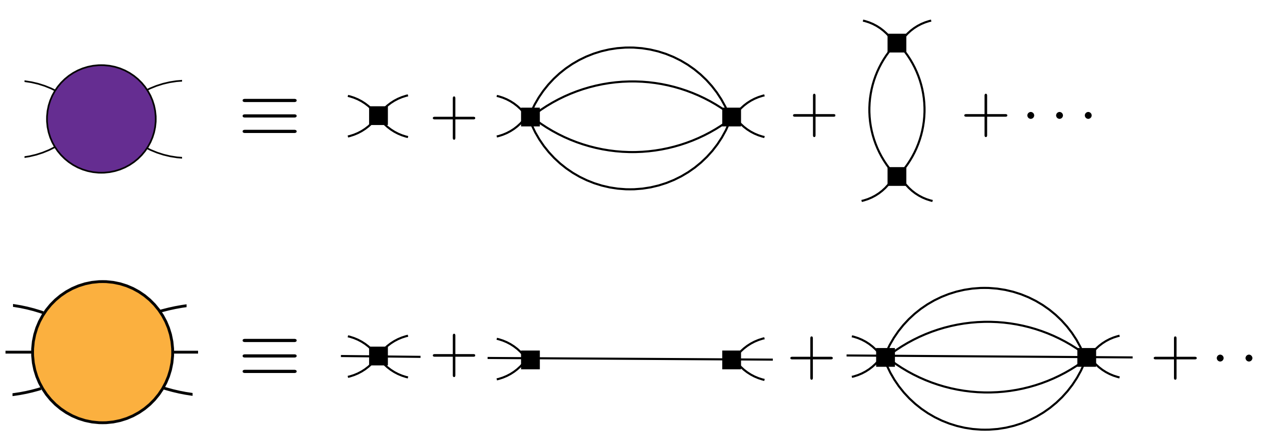
Disconnected diagrams in  **lead to singularities that invalidate the derivation**

New skeleton expansion

$C_L(E, \vec{P}) =$



Kernel definitions:

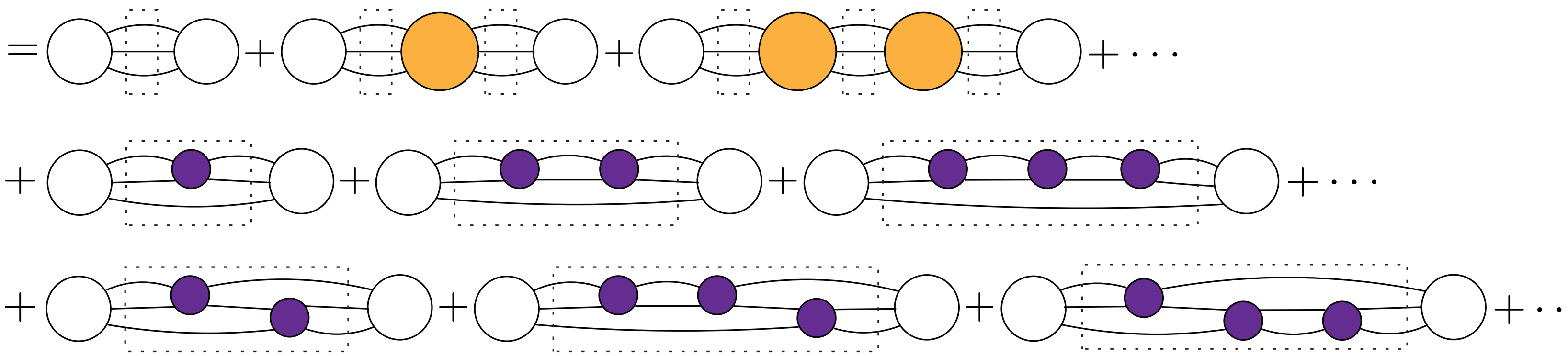


- All lines are fully dressed propagators
- Boxes represent sums over finite-volume momenta
- Kernels may contain fixed poles

Basic approach

1. Work out the three particle skeleton expansion

$C_L(E, \vec{P}) =$



2. Break diagrams into finite- and infinite-volume parts

3. Organize and sum terms to identify
infinite-volume observables

Result

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) - A' F_3 \frac{1}{1 + \mathcal{K}_{\mathrm{df},3} F_3} A$$

- Looks similar to the two-particle case
- All quantities defined with PV-pole prescription
- F_3 depends on finite-volume and two-to-two scattering

Quantization condition

At fixed $(L, \vec{P})$, finite-volume energies are solutions to

$$\det_{k,\ell,m}\left[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3\right] = 0$$

$F_3 \equiv$ **matrix that depends on geometric functions and** $\mathcal{M}_{2\to2}$.

MTH and Sharpe (2014)

(1). Use two-particle q.c. to constrain $\mathcal{M}_2$ and determine $F_3(E, \vec{P}, L)$.

$$\det[\mathcal{M}_2^{-1} + F_2] = 0 \longrightarrow \mathcal{M}_2 \longrightarrow F_3(E, \vec{P}, L)$$

(2). Use decomposition + parametrization to express $\mathcal{K}_{\mathrm{df},3}(E^*)$ in terms of α_i.

$$\mathcal{K}_{\mathrm{df},3}(E^*, \Omega_3', \Omega_3) \approx \mathcal{K}_{\mathrm{df},3}[\alpha_1, \cdots, \alpha_N] \longleftarrow$$ Recall, this is a real, smooth function

(3). Use three-particle q.c. with finite-volume energies to determine $\mathcal{K}_{\mathrm{df},3}(E^*)$.

$$\det[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3] = 0 \longrightarrow \boxed{\mathcal{K}_{\mathrm{df},3}(E^*)\ \checkmark}$$

Quantization condition

At fixed $(L, \vec{P})$, finite-volume energies are solutions to

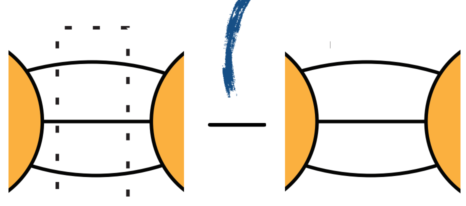
$$\det_{k,\ell,m}\left[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3\right] = 0$$

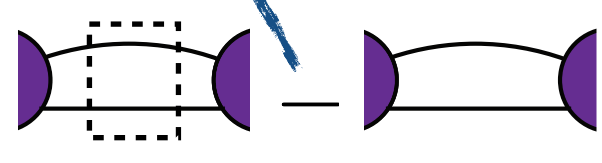
$F_3 \equiv$ **matrix that depends on geometric functions and** $\mathcal{M}_{2\to 2}$.

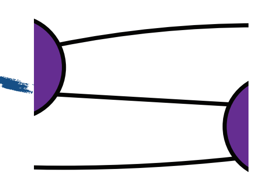
MTH and Sharpe (2014)

All of the complication is buried inside F_3

$$F_3 = \frac{F}{6\omega L^3} - \frac{F}{2\omega L^3}\frac{1}{1+\mathcal{M}_{2,L}G}\mathcal{M}_{2,L}F$$







These are all matrices with indices

momentum of one particle

$$\vec{k} = \frac{2\pi\vec{n}}{L}$$

$\otimes$

angular momentum of the other two

$$\ell, m$$

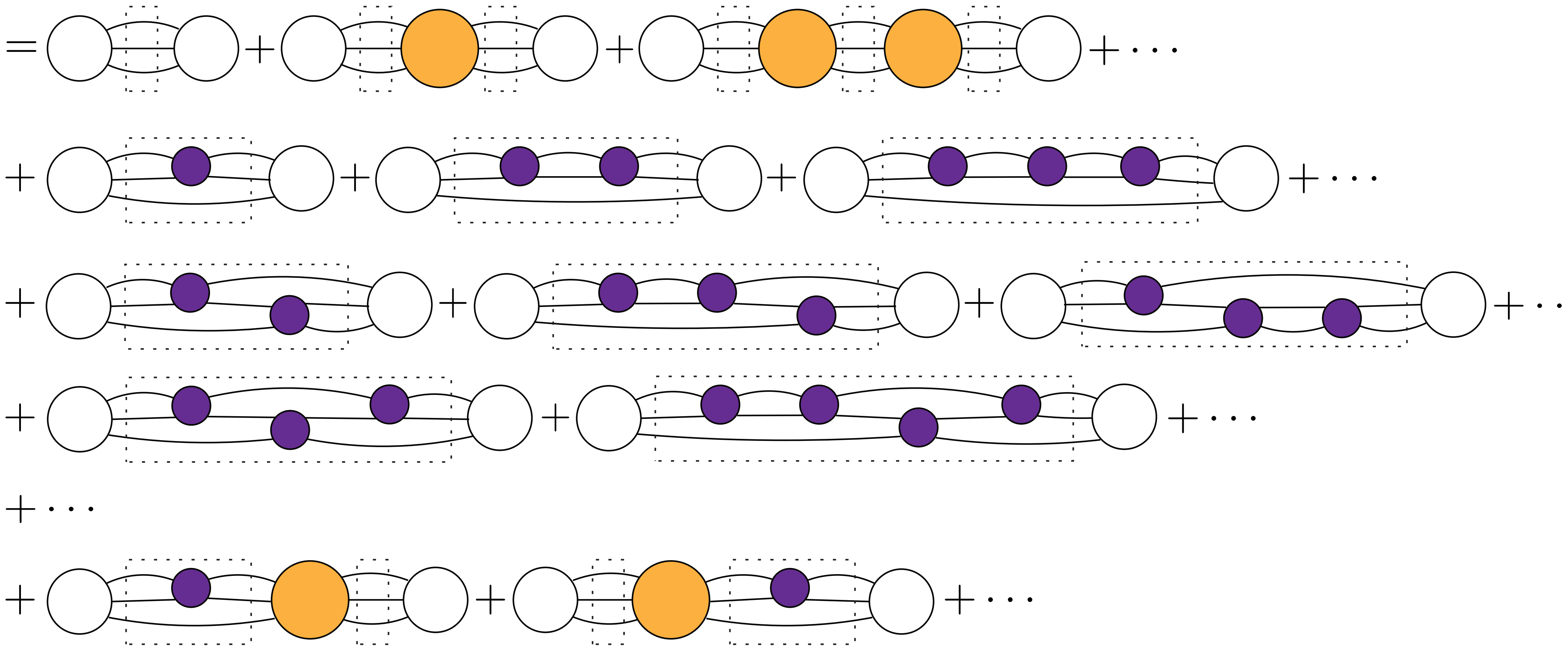
F and G are geometric functions

$\mathcal{M}_{2,L}$ **depends on F and** $\mathcal{M}_2$

Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

First we modify $C_L(E,\vec{P})$ to define $i\mathcal{M}_{L,3}$

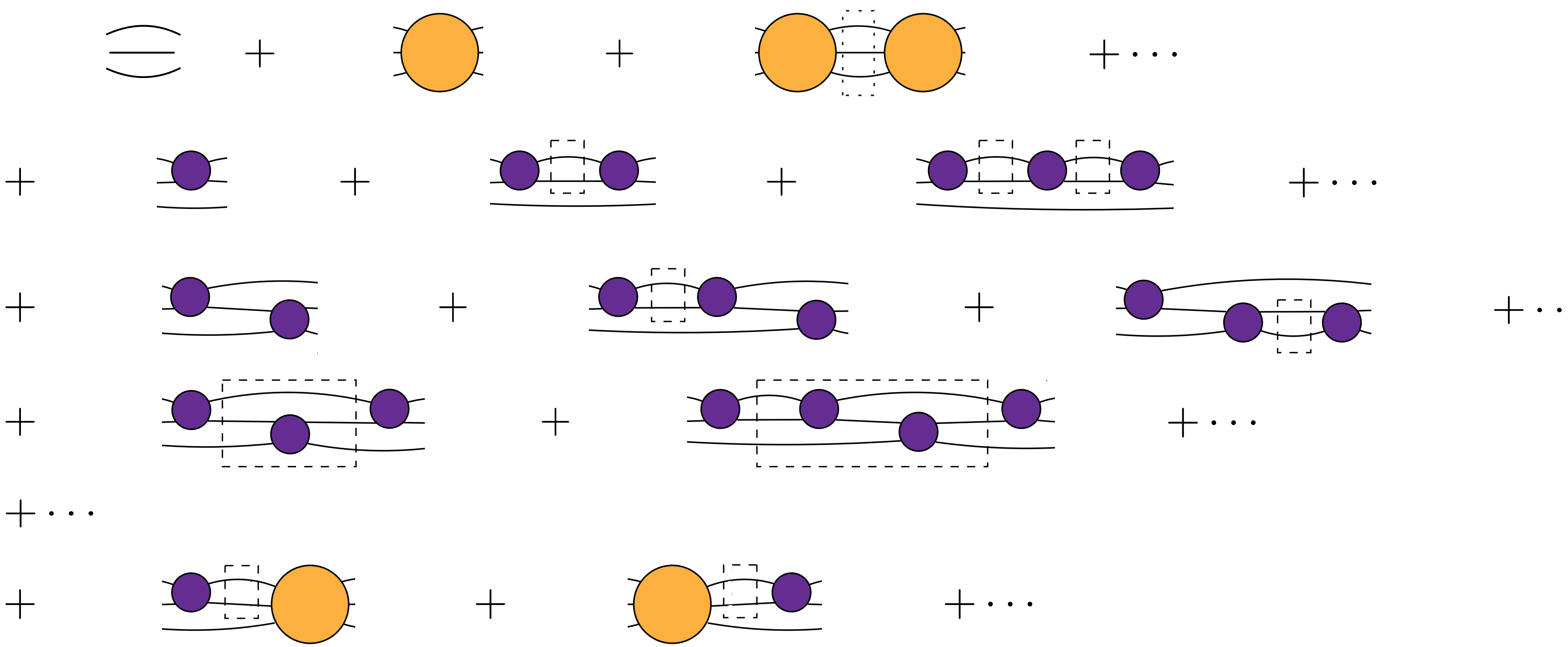
$C_L(E,\vec{P}) =$



Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

First we modify $C_L(E,\vec{P})$ to define $i\mathcal{M}_{L,3}$

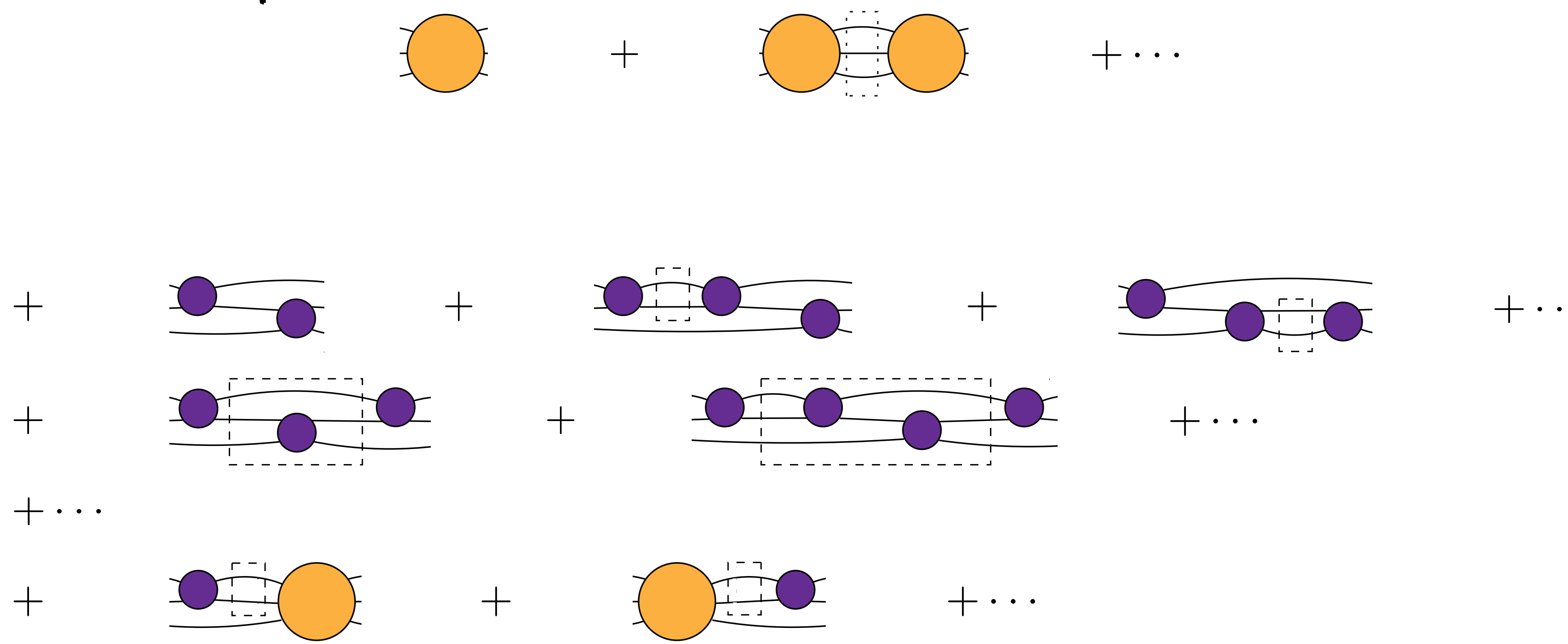
1. Amputate interpolating fields



Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

First we modify $C_L(E,\vec{P})$ to define $i\mathcal{M}_{L,3}$

1. Amputate interpolating fields
2. Drop disconnected diagrams

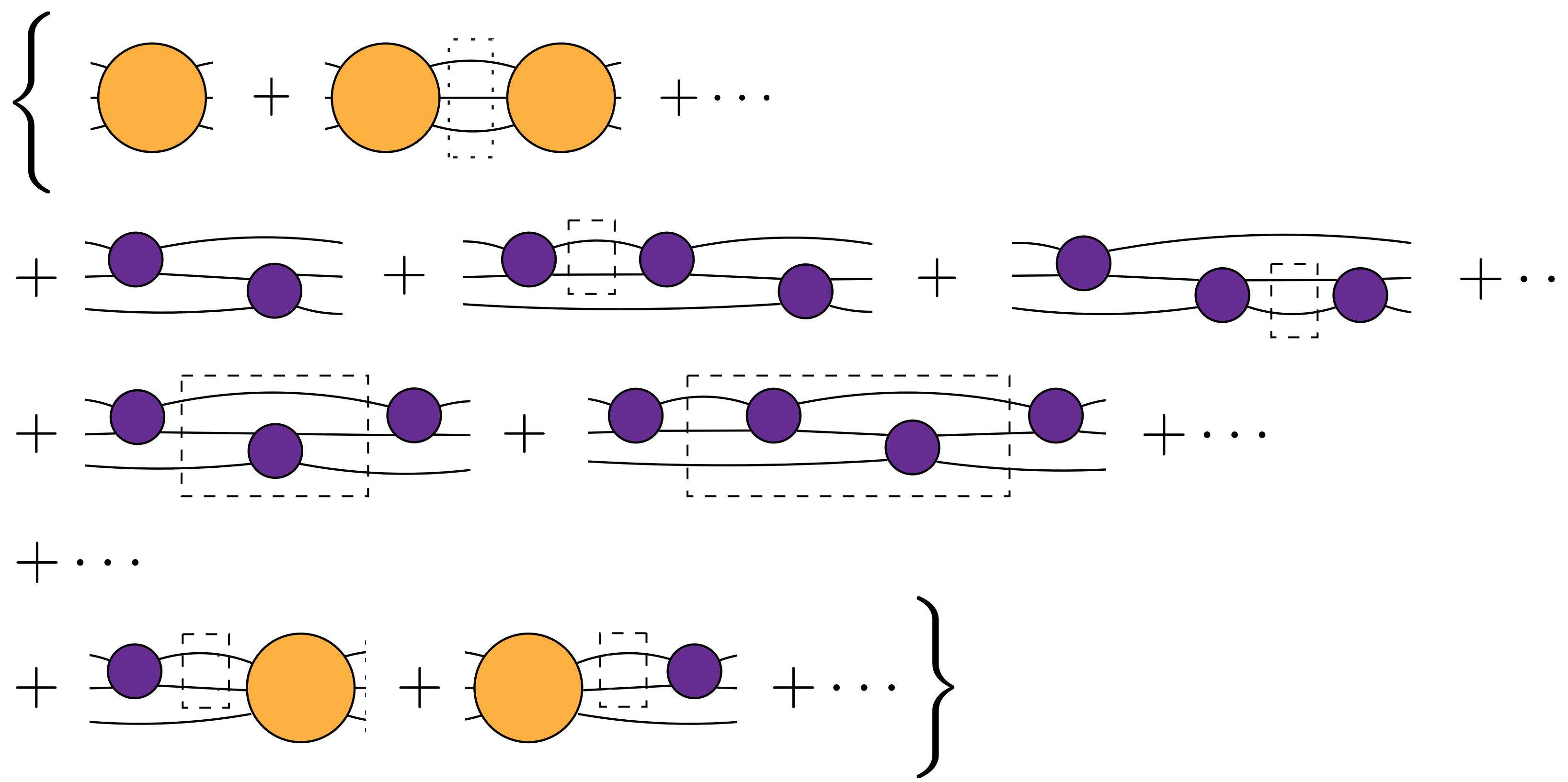


Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

First we modify $C_L(E, \vec{P})$ to define $i\mathcal{M}_{L,3}$

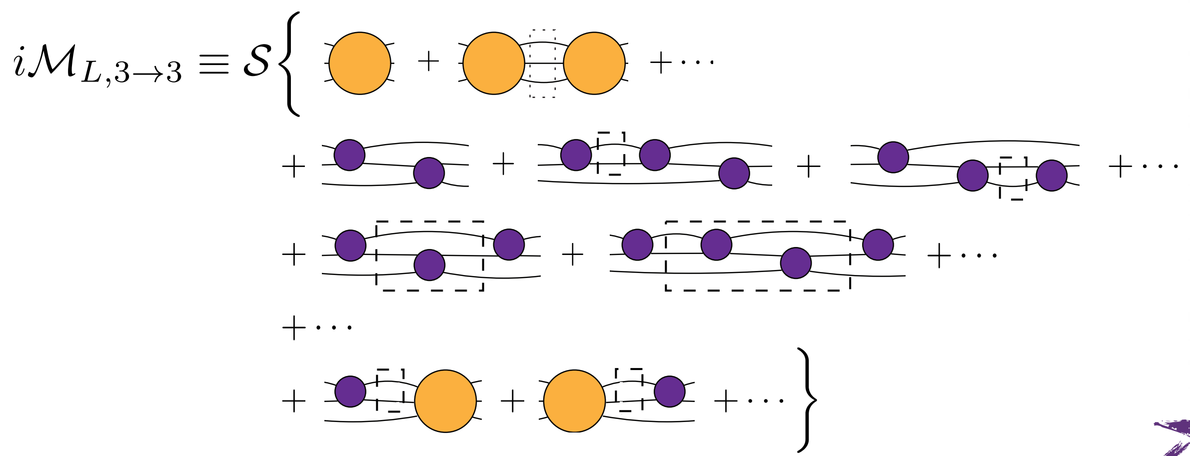
1. Amputate interpolating fields
2. Drop disconnected diagrams
3. Symmetrize

$$i\mathcal{M}_{L,3\to3} \equiv \mathcal{S}$$



Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

Combined with our earlier analysis
this gives a matrix equation



$$\mathcal{M}_{L,3} = \mathcal{S}\left[\mathcal{D}_L + \mathcal{L}_L \frac{1}{\mathcal{K}_{\mathrm{df},3}^{-1} + F_3}\mathcal{R}_L\right]$$

$$\mathcal{L}_L = \mathcal{X} F_3, \quad \mathcal{R}_L = F_3 \mathcal{X},$$

$$\mathcal{D}_L = -\mathcal{X}\left[F_3 - F_3\big|_{G\to0}\right]\mathcal{X}$$

with the "amputation matrix" $\mathcal{X} = \left(\frac{F}{2\omega L^3}\right)^{-1}$

With this analytic relation in hand we can…

(a) Set $E \to E + i\epsilon$, **(b) Send** $L \to \infty$, **(c) Send** $\epsilon \to 0^+$.

Leads to an integral equation for the scattering amplitude

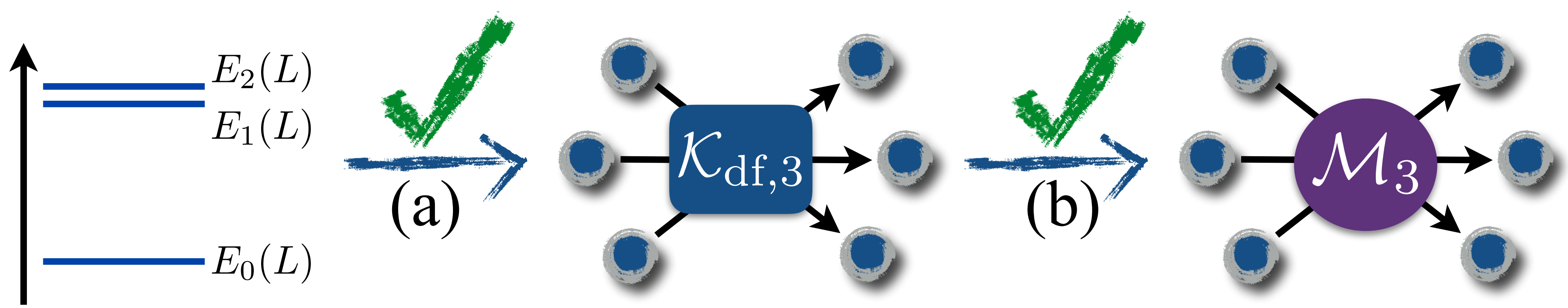
$$\mathcal{M}_3(E^*) = \mathcal{I}\left[\mathcal{K}_{\mathrm{df},3}(E^*), \mathcal{M}_2\right]$$

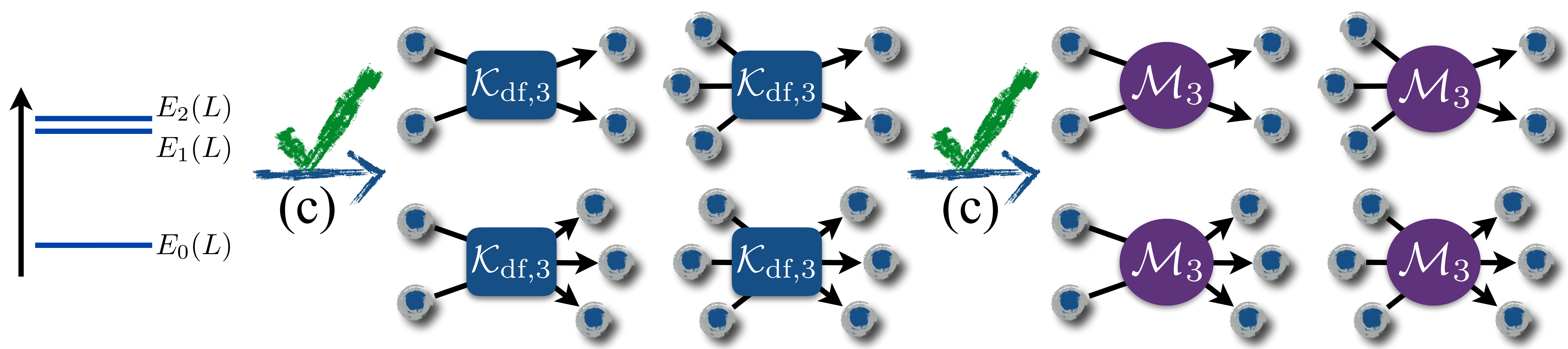
Fixed total energy, manifestly convergent, on-shell only, no reference to EFT,
takes care of unitarity and singularities, useful independent of finite-volume physics?

MTH and Sharpe (2015)

Current status

Model- & EFT-independent relation between
finite-volume energies and relativistic two-and-three particle scattering





(a),(b) ***MTH and Sharpe (2015),(2016)***
(c) ***Briceño, MTH, Sharpe (2017)***

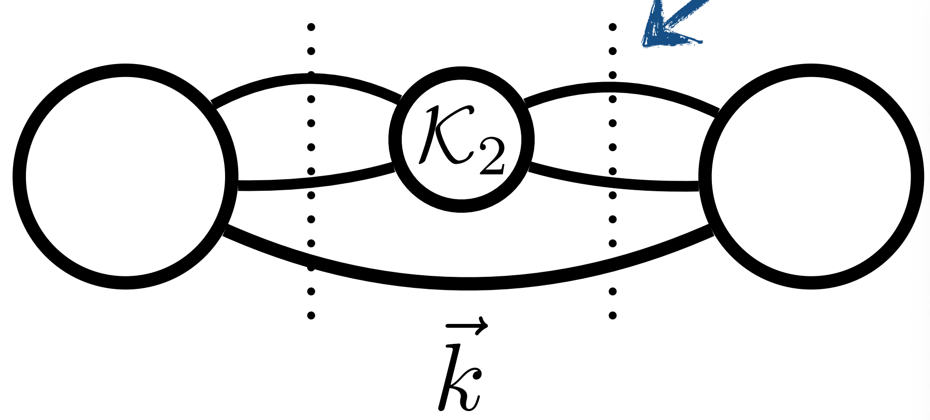
Smooth cutoff function

$\mathcal{K}_{\mathrm{df},3}$ **and** F_3 **depend on a smooth cutoff function**

To see why, consider one of the contributions to C_L…

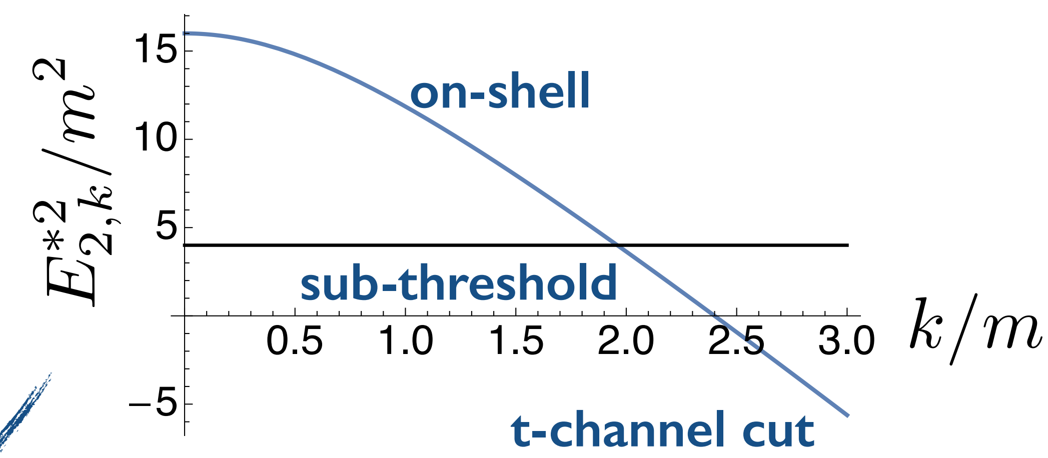
How do we define this on-shell cut?

$$C_L(E,\vec{P}) \supset$$



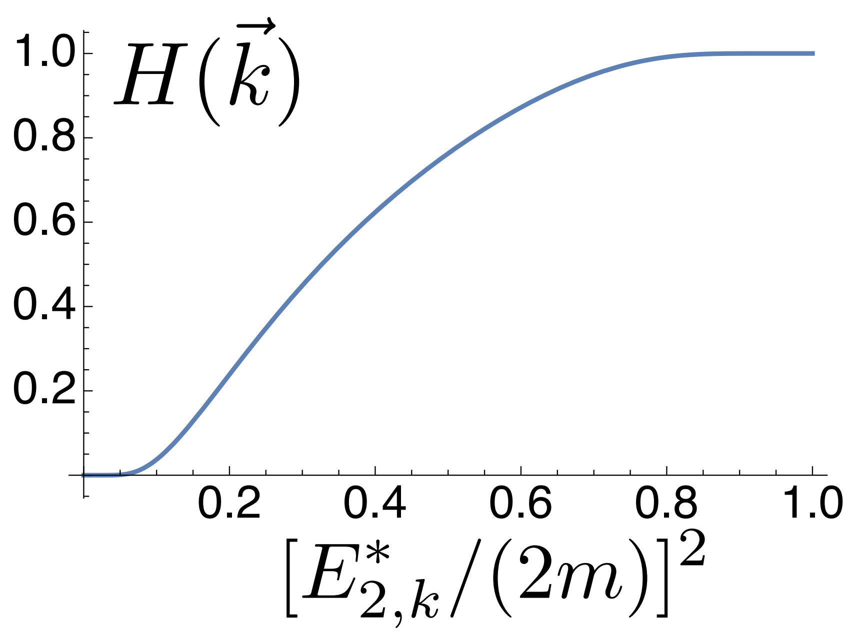
Energy of top two particles is:

$$E_{2,k}^{*2} = (E-\omega_k)^2 - (\vec{P}-\vec{k})^2$$



To keep on-shell states and avoid spurious off-shell contributions…

Cuts are defined with



🤔 **Is this really necessary?** 🤔

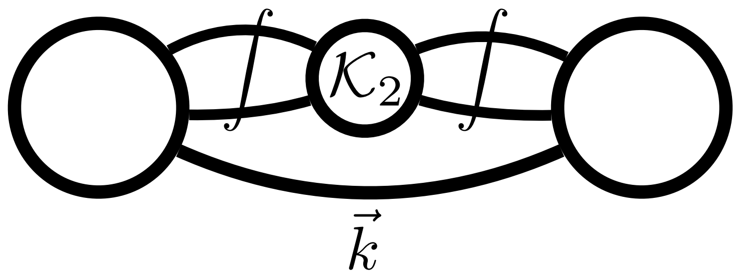
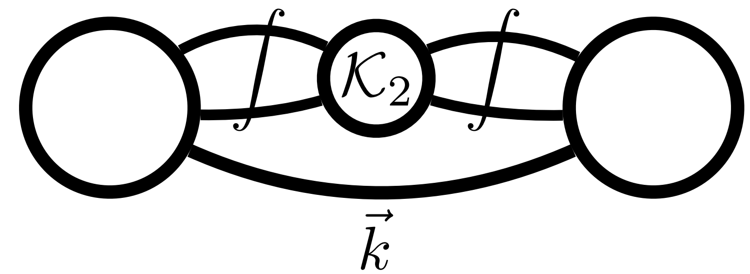
We choose to sum subdiagrams into $\mathcal{K}_2$

On-shell $\mathcal{K}_2$ gives important volume effects
No important effects far below threshold

Must connect the two regions

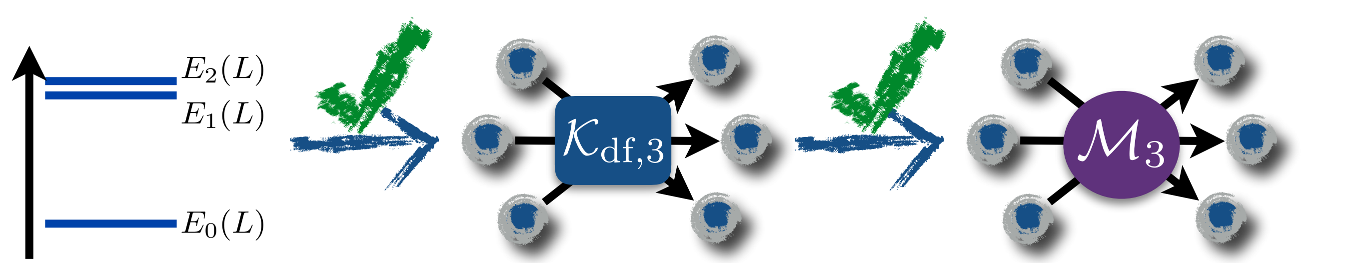
Important limitation

Current formalism requires no poles in $\mathcal{K}_2$ … Derivation assumes

$\frac{1}{L^3}\sum_{\vec k}$  $= \int_{\vec k}$ 

Given that we are seeking an EFT-independent mapping…

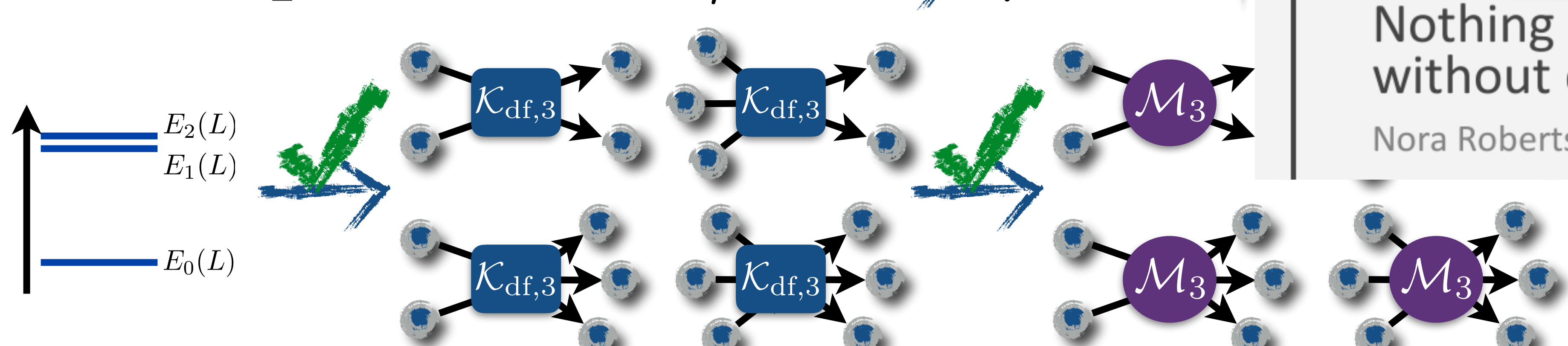
Is it intuitive that $\mathcal{K}_2$ poles need special treatment?



$$M_\rho > E_2^* > 2M_\pi \longrightarrow \pi\pi\pi \to \pi\pi\pi$$

[Need to bridge the gap]

$$E_2^* > 2M_\pi \gg M_\rho \longrightarrow \rho\pi \to$$



Update

We now of a complete derivation of formalism that includes K_2 poles.

Needs further checks.

Briceño, MTH, Sharpe (underway)

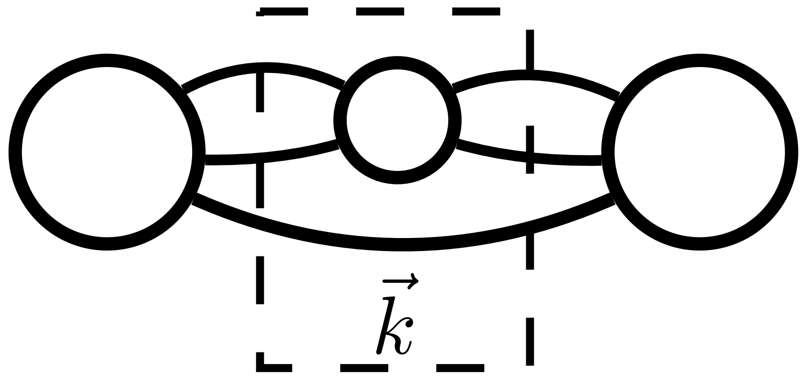
Nothing worthwhile is ever without complications.

Nora Roberts

The most technical detail of all…

Far below threshold there is no ambiguity about which two-to-two scattering quantity appears in C_L

$C_L(E,\vec{P}) \supset$



$= \mathcal{M}_2$

Reason: $$\frac{1}{L^3}\sum_{\vec{k}} \frac{1}{(2\omega_k)^2(E_{\rm sub}-2\omega_k)} = \int_{\vec{k}} \frac{1}{(2\omega_k)^2(E_{\rm sub}-2\omega_k)} = \text{Analytic Continuation}\left[\int_{\vec{k}} \frac{1}{(2\omega_k)^2(E-2\omega_k+i\epsilon)}\right]$$

Upshot is that our subthreshold $\mathcal{K}_2$ is non-standard

$$\mathcal{K}_2^{-1} \propto p^* \cot\delta(p^*) + [1-H(\vec{k})]\kappa(p^*)$$

K matrix above threshold, smooth at threshold, interpolates to the amplitude below threshold

Why are you telling me this?

It is important because our formalism breaks down when there are poles in this definition of $\mathcal{K}_2$.

Testing the formalism

MTH and Sharpe (2016,2017)

Weak interactions: **Expand the threshold energy in powers of inverse box length**

$$E = 3m + \frac{12\pi a}{mL^3}\left(1 + c_4 \frac{a}{L} + \cdots\right) - \frac{\mathcal{M}_{\rm thr}}{48 m^3 L^6} + \cdots$$

We reproduce known results through $1/L^5$ and derive a relation at $1/L^6$

Huang and Yang (1957); Beane, Detmold, Savage, (2007); Tan(2007); Sharpe 2017

Note: Relativistic effects enter at $1/L^6$, same order as three-to-three

Testing the formalism

MTH and Sharpe (2016,2017)

Weak interactions: Expand the threshold energy in powers of inverse box length

$$E = 3m + \frac{12\pi a}{mL^3}\left(1 + c_4 \frac{a}{L} + \cdots\right) - \frac{\mathcal{M}_{\rm thr}}{48m^3L^6} + \cdots$$

We reproduce known results through $1/L^5$ and derive a relation at $1/L^6$

Huang and Yang (1957); Beane, Detmold, Savage, (2007); Tan(2007); Sharpe 2017

Note: Relativistic effects enter at $1/L^6$, same order as three-to-three

Strong interactions (unitary limit, $P=0$, s-wave only):

The infinite-volume energy, $E_B \equiv 3m - \frac{\kappa^2}{m}$, is shifted by

$$\Delta E(L) = c|A|^2 \frac{\kappa^2}{m} \frac{1}{(\kappa L)^{3/2}} e^{-2\kappa L/\sqrt{3}} + \cdots$$

$c = -96.351\cdots$

geometric constant

"normalization correction factor"

Meißner, Rìos and Rusetsky, (2015)

Our formalism gives a relation between scattering and energies. So we substitute…

$$\mathcal{M}_3 \sim -\frac{\Gamma\,\overline{\Gamma}}{E^2 - E_B^2} \qquad \mathcal{M}_2 = -\frac{16\pi E_2^*}{ip^*}$$

and study the lowest level

We reproduce the exponent, leading power and overall constant using our relativistic formalism

Usability?

"Despite this success, the quantization condition in these papers is not yet given in a form suitable for the analysis of the real lattice data"

Hammer, Pang and Rusetsky (2017)

We were motivated to challenge this claim…

We find that the "degree of usability" is comparable between the two approaches, provided one applies similar approximations.

How do we make the two-particle formalism usable?

Truncate partial waves

$$\mathcal{M}_2(E_2^*, \theta^*) \approx \sum_{\ell=0}^{N} P_\ell(\cos\theta^*) \mathcal{M}_{2,\ell}(E_2^*) \longrightarrow$$

Single partial wave

$$\mathcal{M}_2(E_2^*, \theta^*) \approx \mathcal{M}_{2,s}(E_2^*) \propto \frac{1}{p^* \cot\delta_0(p^*) - ip^*}$$

Is there a three-particle analog?

$$\mathcal{K}_{\mathrm{df},3}(E^*, \Omega_3', \Omega_3) \approx \sum_{n=0}^{N} \mathcal{P}_n(\Omega_3', \Omega_3) \mathcal{K}_{\mathrm{df},3,n}(E^*) \longrightarrow$$

$$\mathcal{K}_{\mathrm{df},3}(E^*, \Omega_3', \Omega_3) \approx \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E^*) \in \mathbb{R}$$

At fixed energy $\mathcal{M}_2(E_2^*, \theta^*)$, $\mathcal{K}_{\mathrm{df},3}(E^*, \Omega_3', \Omega_3)$ **is a smooth function on a compact space.**

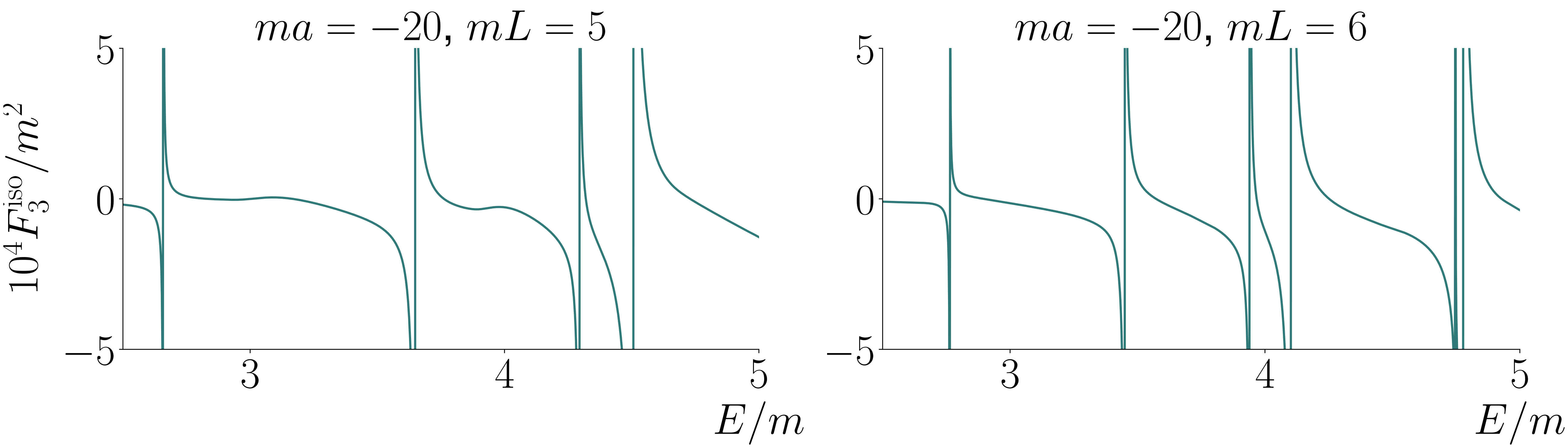
Further investigation is needed to understand suppression of higher $\mathcal{K}_{\mathrm{df},3,n}(E^*)$.

Numerics (keeping only s-wave and $\mathcal{K}_{\mathrm{df},3}(E^*,\Omega_3',\Omega_3) \approx \mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3}(E^*)$)

$$1/\mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3}(E^*) = -F_3^{\mathrm{iso}}[E,\vec{P},L,\mathcal{M}_2^s] \qquad \mathcal{M}_3(E^*,\Omega_3',\Omega_3) = \mathcal{S}\left[\mathcal{D} + \mathcal{L}\frac{1}{1/\mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3} + F^{\mathrm{iso}}_{3,\infty}}\mathcal{R}\right]$$

For the numerical approach we restrict attention to… $p^*\cot\delta_0(p^*) = -\dfrac{1}{a}$, $\vec{P}=0$

Then the quantization condition is based on $F_3^{\mathrm{iso}}(E,L,a)$



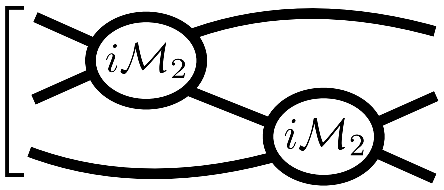
Finite-volume energies wherever these curves intersect $-1/\mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3}(E)$

Briceño, Hansen and Sharpe (to appear)

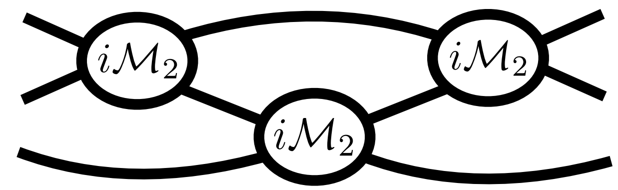
$\mathcal{K}_{\rm df,3}^{\rm iso}(E)=0$ solutions

- **Provides a useful benchmark: Deviations measure three-particle physics**
- **Meaning for three-to-three scattering is clear**

$i\mathcal{M}_3 = \mathcal{S}\Big[$



$+$

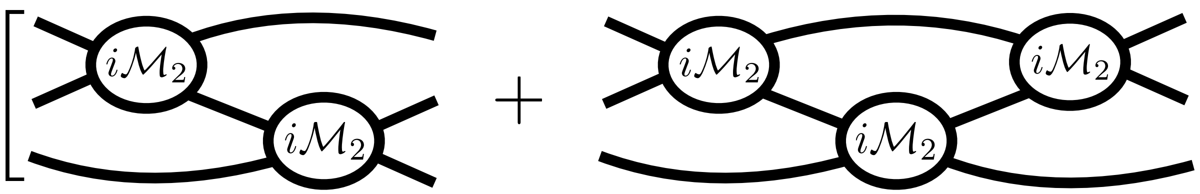


$+\cdots\Big]$

$\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E) = 0$ solutions

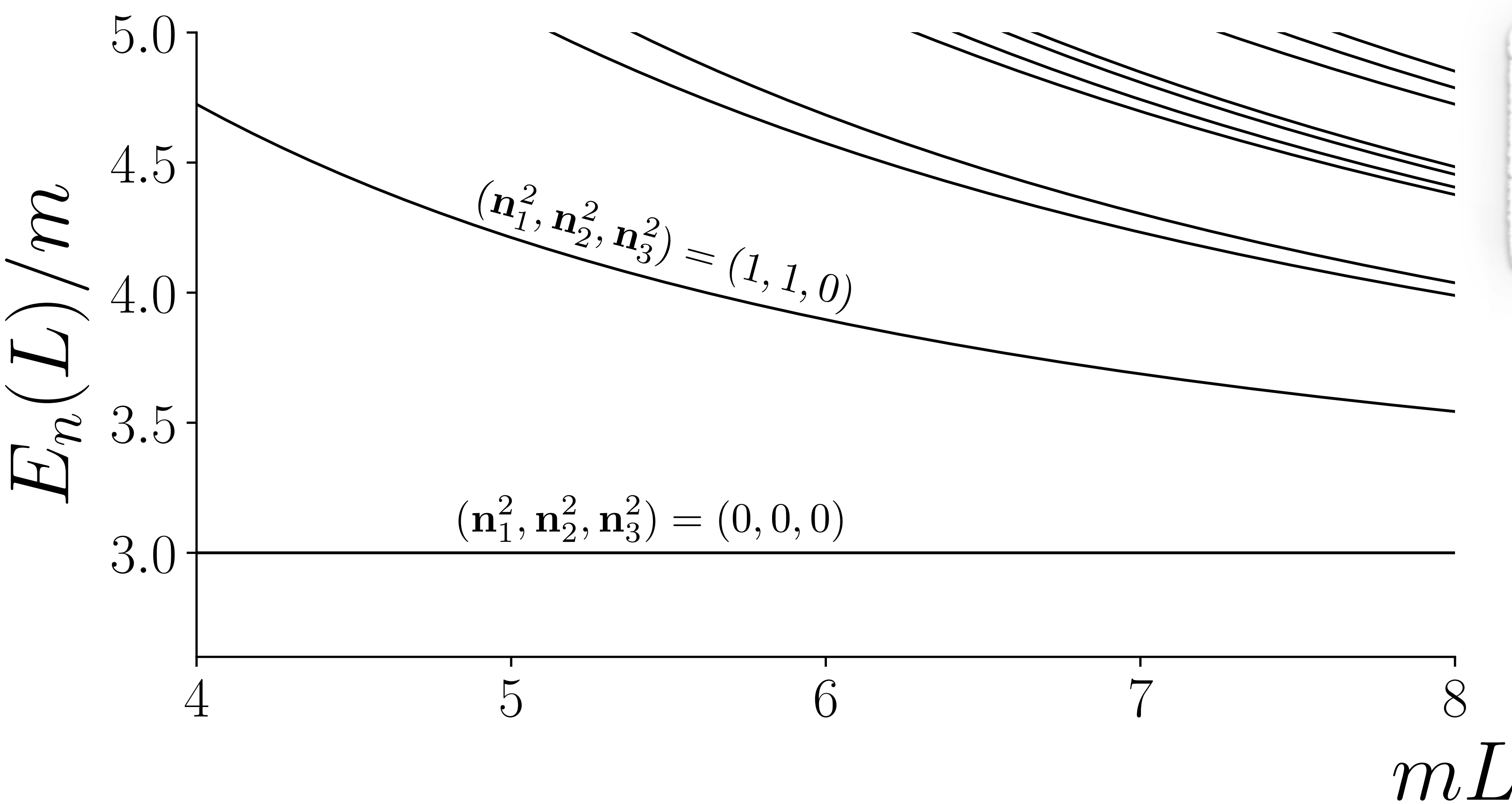
- **Provides a useful benchmark: Deviations measure three-particle physics**
- **Meaning for three-to-three scattering is clear**

$$i\mathcal{M}_3 = \mathcal{S}\Big[$$



$$+\cdots\Big]$$

Aside: Non-interacting states ($\mathcal{M}_2 = \mathcal{M}_3 = 0$)

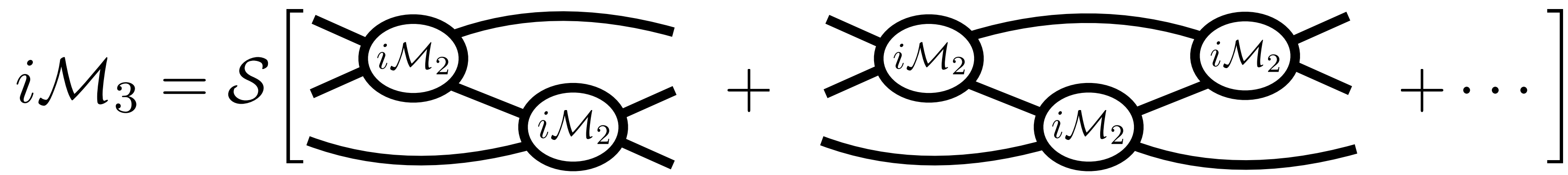


$$E_n^{\text{non-int}}(L) = \omega_1 + \omega_2 + \omega_3$$

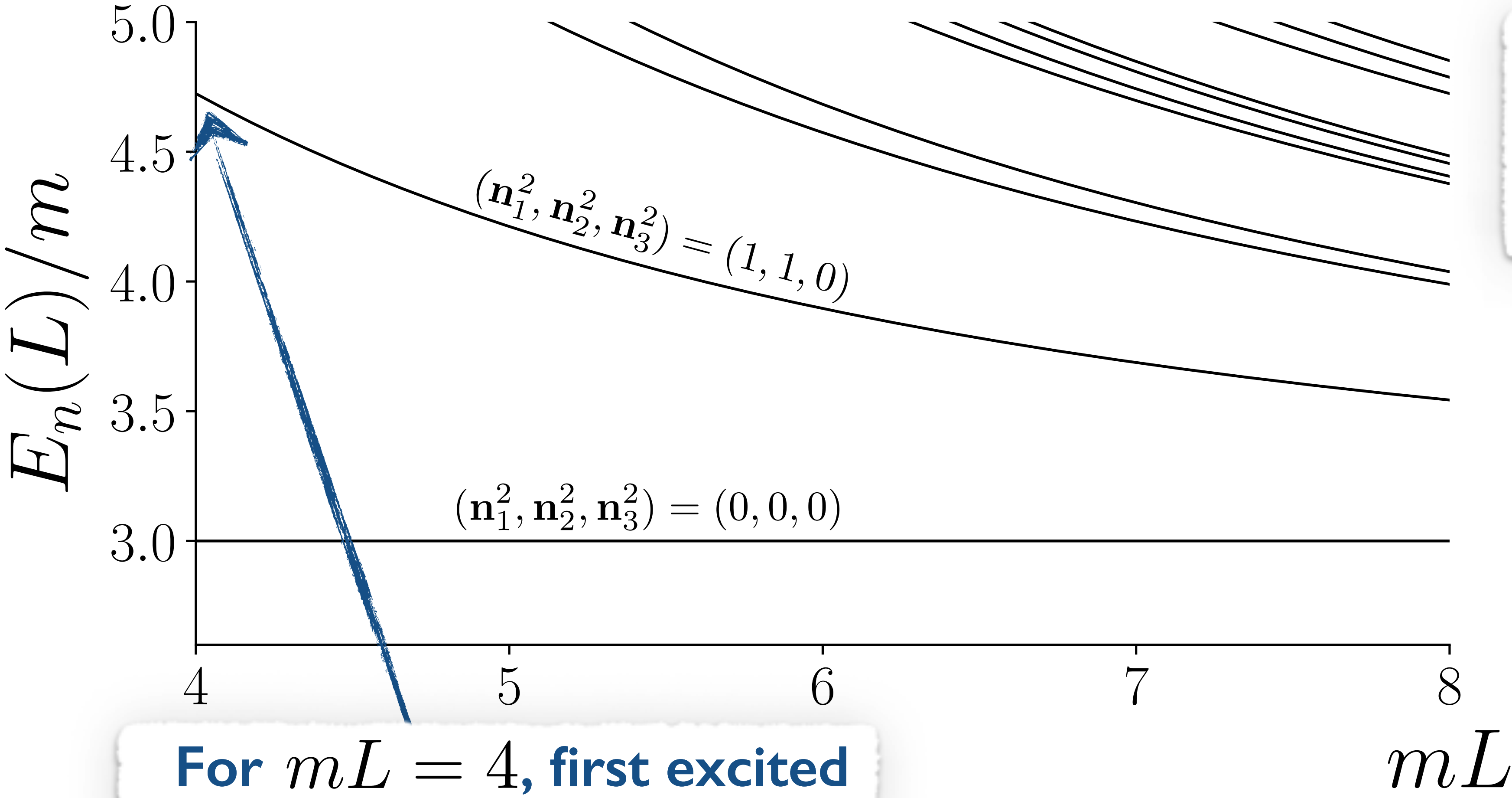
$$\omega_i = \sqrt{m^2 + 4\pi^2 \mathbf{n}_i^2 / L^2}$$

$\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E) = 0$ solutions

- **Provides a useful benchmark: Deviations measure three-particle physics**
- **Meaning for three-to-three scattering is clear**



Aside: Non-interacting states ($\mathcal{M}_2 = \mathcal{M}_3 = 0$)



$$E_n^{\text{non-int}}(L) = \omega_1 + \omega_2 + \omega_3$$

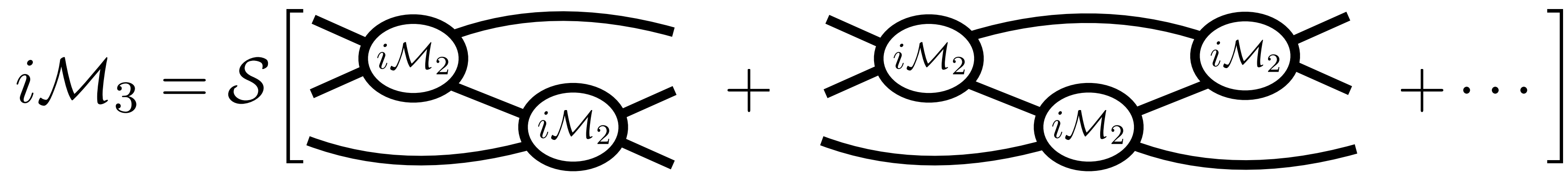
$$\omega_i = \sqrt{m^2 + 4\pi^2 \mathbf{n}_i^2 / L^2}$$

For $mL = 4$, first excited state is already relativistic

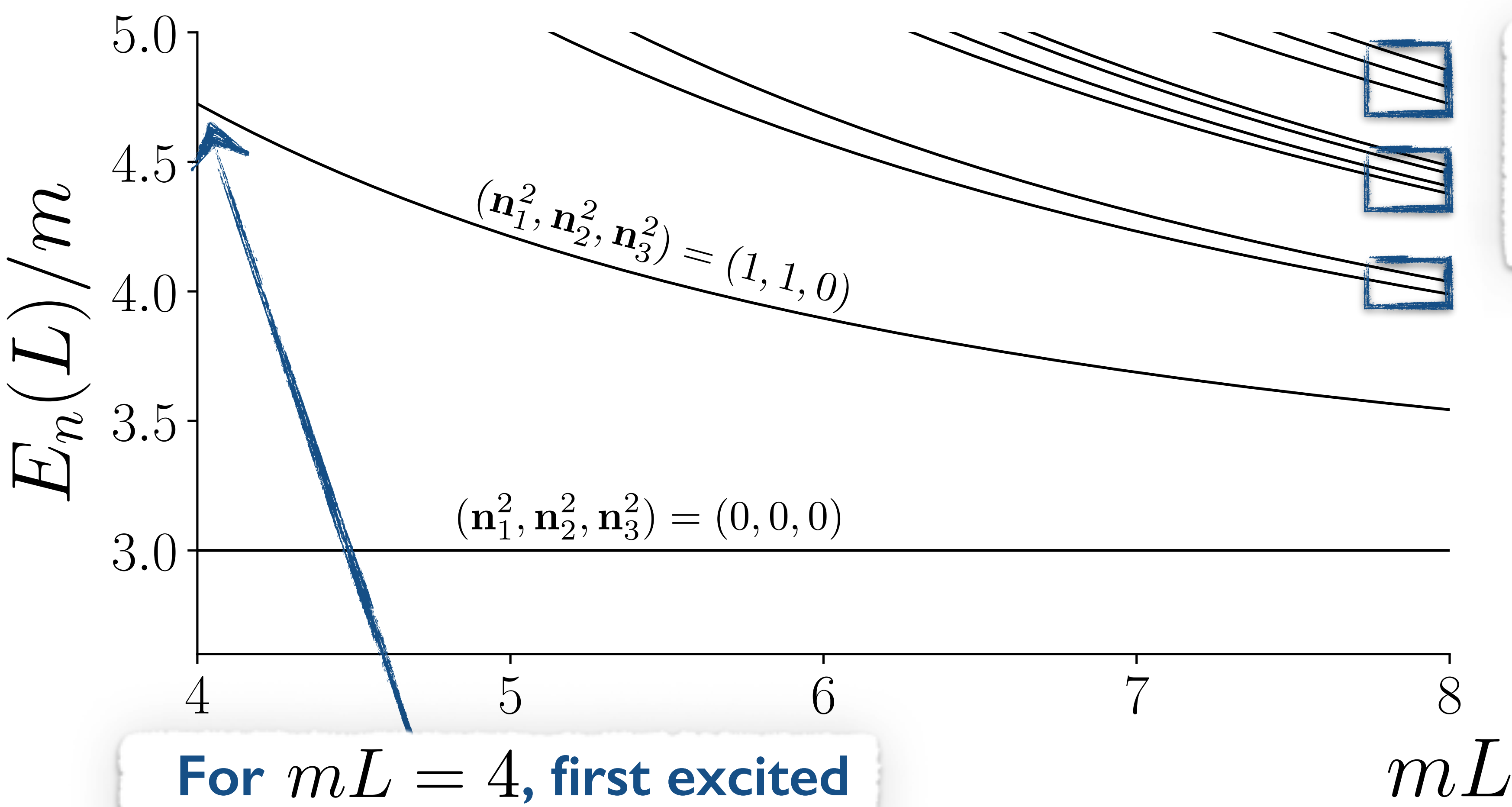
$$\frac{p^2}{m^2} = \left(\frac{2\pi}{mL}\right)^2 \approx 2.46$$

$\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E) = 0$ solutions

- **Provides a useful benchmark: Deviations measure three-particle physics**
- **Meaning for three-to-three scattering is clear**



Aside: Non-interacting states ($\mathcal{M}_2 = \mathcal{M}_3 = 0$)



$$E_n^{\text{non-int}}(L) = \omega_1 + \omega_2 + \omega_3$$

$$\omega_i = \sqrt{m^2 + 4\pi^2 \mathbf{n}_i^2 / L^2}$$

Why are these states clustered?
Accidental NR degeneracy!

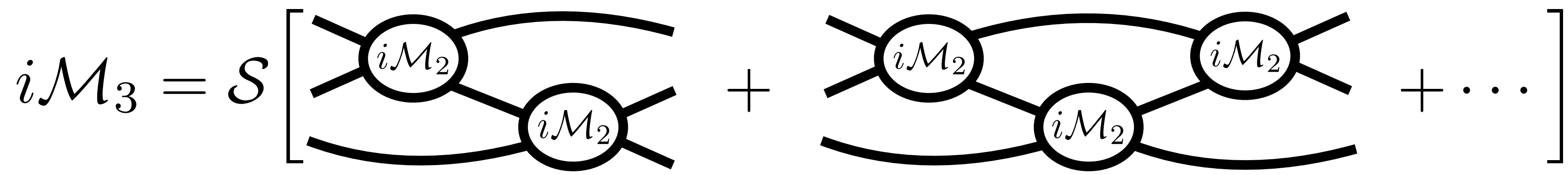
$$E_n^{\mathrm{NR}}(L) = 3m + \frac{2\pi^2}{L^2}(\mathbf{n}_1^2 + \mathbf{n}_2^2 + \mathbf{n}_3^2)$$

For $mL = 4$, first excited state is already relativistic

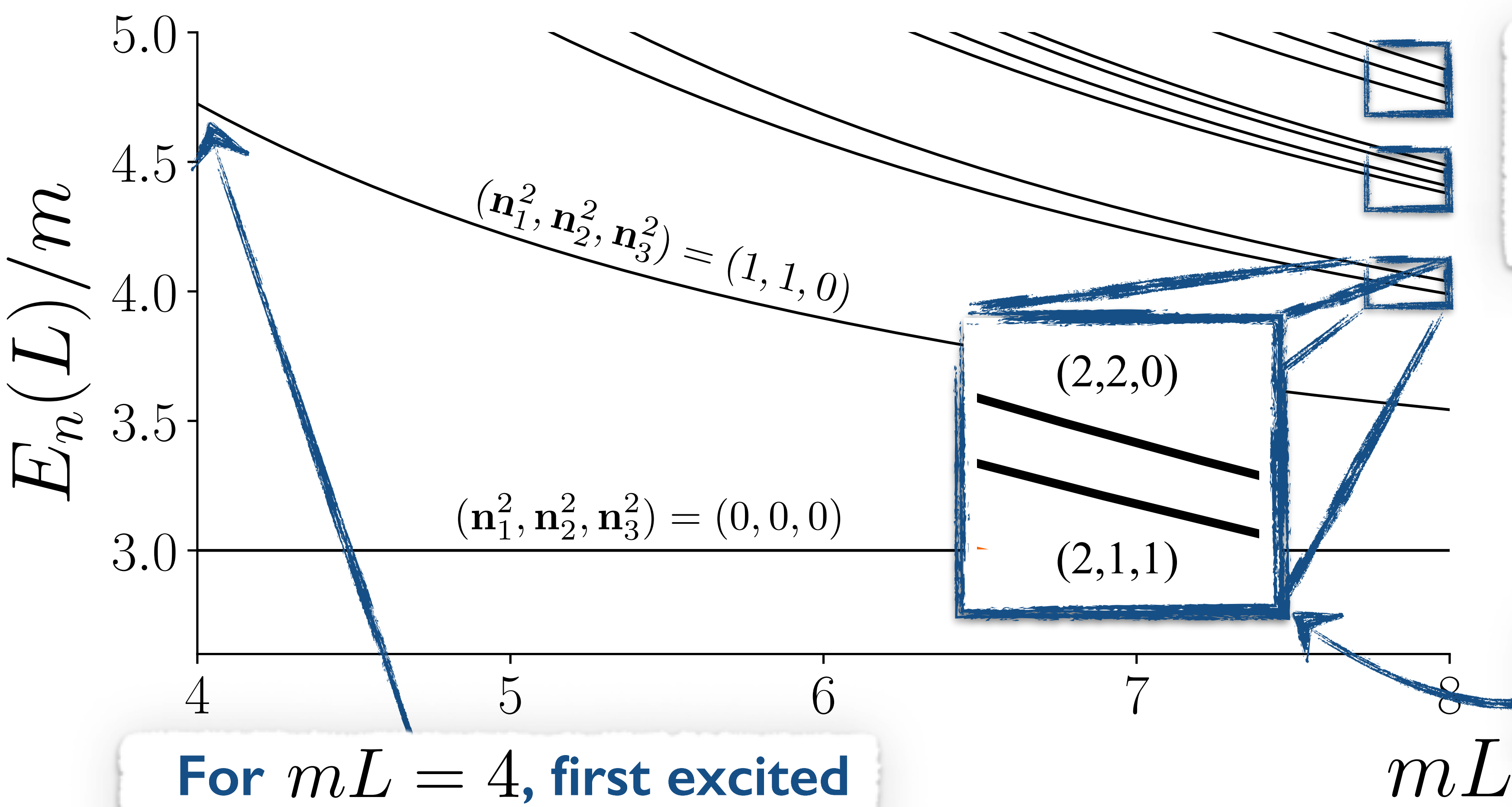
$$\frac{p^2}{m^2} = \left(\frac{2\pi}{mL}\right)^2 \approx 2.46$$

$\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E) = 0$ solutions

- Provides a useful benchmark: Deviations measure three-particle physics
- Meaning for three-to-three scattering is clear



Aside: **Non-interacting states ($\mathcal{M}_2 = \mathcal{M}_3 = 0$)**



$$E_n^{\text{non-int}}(L) = \omega_1 + \omega_2 + \omega_3$$

$$\omega_i = \sqrt{m^2 + 4\pi^2 \mathbf{n}_i^2 / L^2}$$

Why are these states clustered?

Accidental NR degeneracy!

$$E_n^{\mathrm{NR}}(L) = 3m + \frac{2\pi^2}{L^2}(\mathbf{n}_1^2 + \mathbf{n}_2^2 + \mathbf{n}_3^2)$$

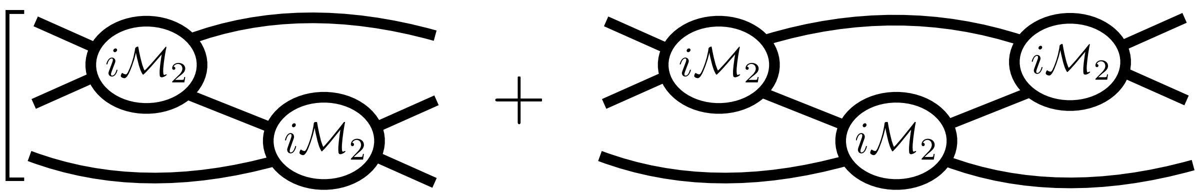
These two states are degenerate in the NR theory

For $mL = 4$, first excited state is already relativistic

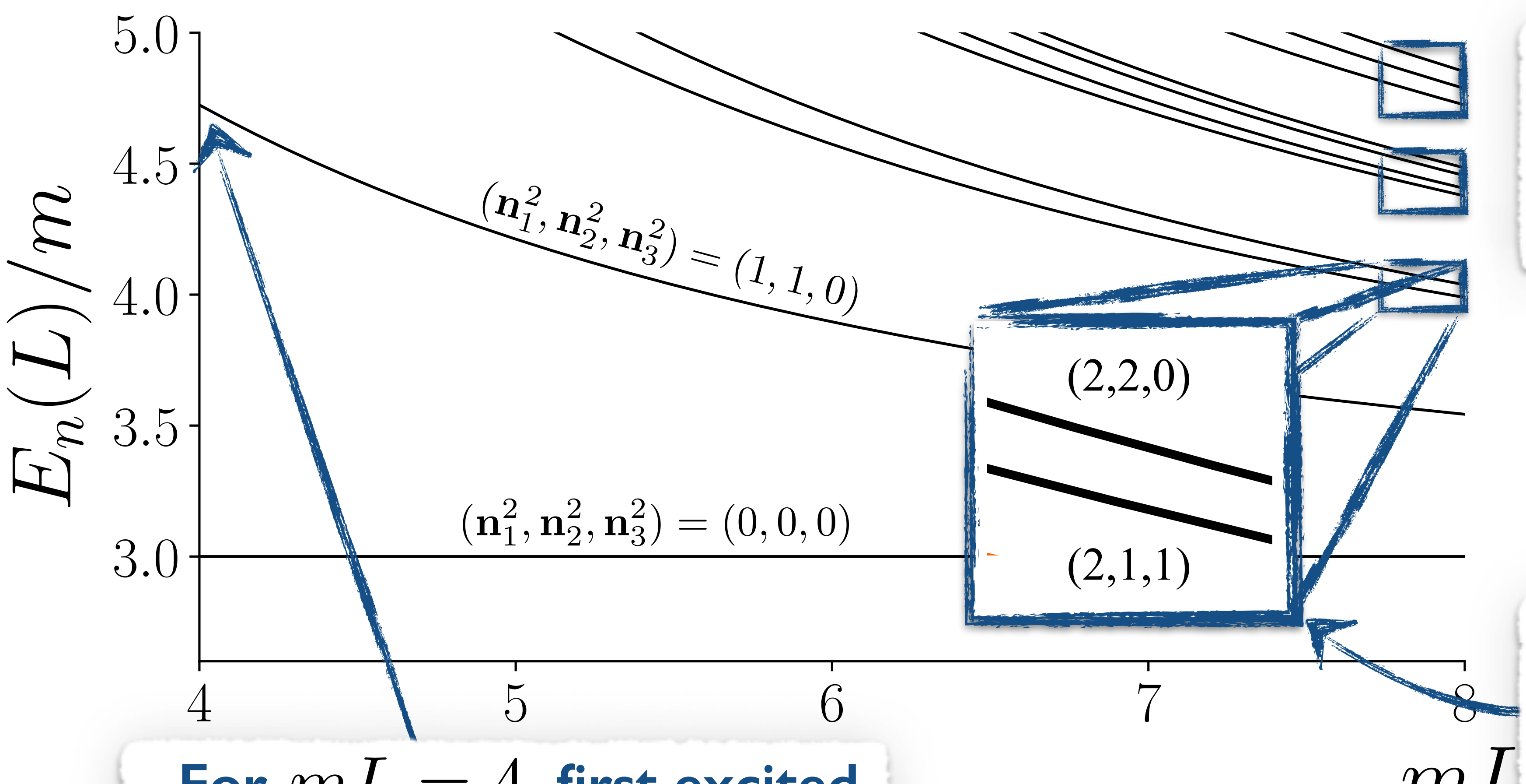
$$\frac{p^2}{m^2} = \left(\frac{2\pi}{mL}\right)^2 \approx 2.46$$

$\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E) = 0$ solutions

- **Provides a useful benchmark: Deviations measure three-particle physics**
- **Meaning for three-to-three scattering is clear**

$i\mathcal{M}_3 = \mathcal{S}\Big[$  $+ \cdots \Big]$

Aside: Non-interacting states ($\mathcal{M}_2 = \mathcal{M}_3 = 0$)



$$E_n^{\text{non-int}}(L) = \omega_1 + \omega_2 + \omega_3$$

$$\omega_i = \sqrt{m^2 + 4\pi^2 \mathbf{n}_i^2 / L^2}$$

Why are these states clustered?

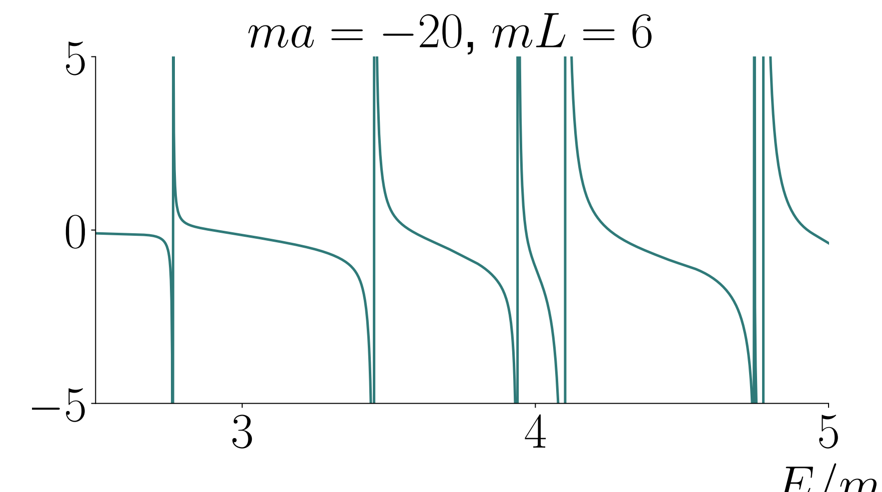
Accidental NR degeneracy!

$$E_n^{\mathrm{NR}}(L) = 3m + \frac{2\pi^2}{L^2}(\mathbf{n}_1^2 + \mathbf{n}_2^2 + \mathbf{n}_3^2)$$

For $mL = 4$, first excited state is already relativistic

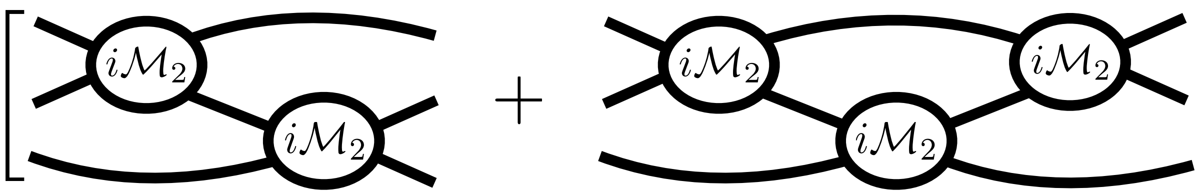
$$\frac{p^2}{m^2} = \left(\frac{2\pi}{mL}\right)^2 \approx 2.46$$

In fact we have already seen these clusters

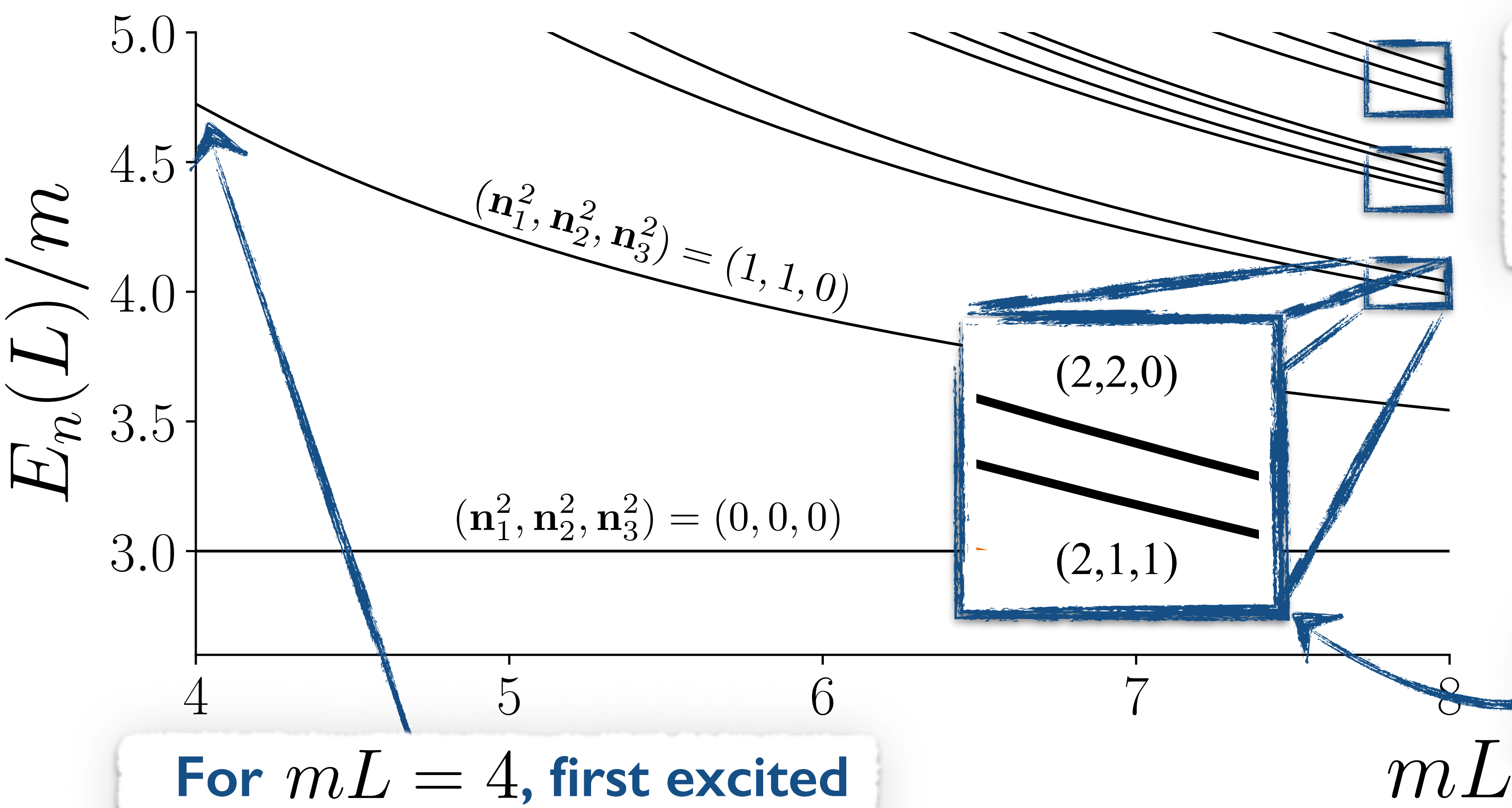


$\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E) = 0$ solutions

- Provides a useful benchmark: Deviations measure three-particle physics
- Meaning for three-to-three scattering is clear

$i\mathcal{M}_3 = \mathcal{S}\Big[$  $+ \cdots \Big]$

Aside: Non-interacting states ($\mathcal{M}_2 = \mathcal{M}_3 = 0$)



$$E_n^{\text{non-int}}(L) = \omega_1 + \omega_2 + \omega_3$$

$$\omega_i = \sqrt{m^2 + 4\pi^2 \mathbf{n}_i^2 / L^2}$$

Why are these states clustered?

Accidental NR degeneracy!

$$E_n^{\mathrm{NR}}(L) = 3m + \frac{2\pi^2}{L^2}(\mathbf{n}_1^2 + \mathbf{n}_2^2 + \mathbf{n}_3^2)$$

These two states are degenerate in the NR theory

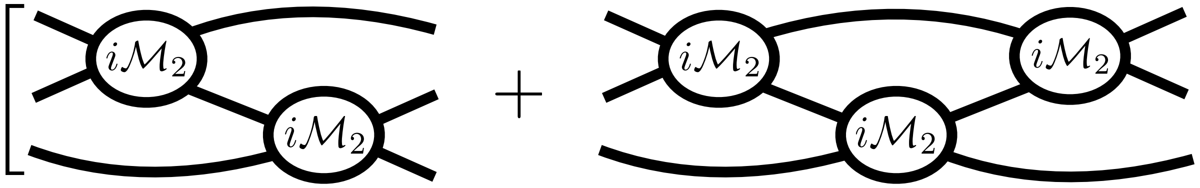
For $mL = 4$, first excited state is already relativistic

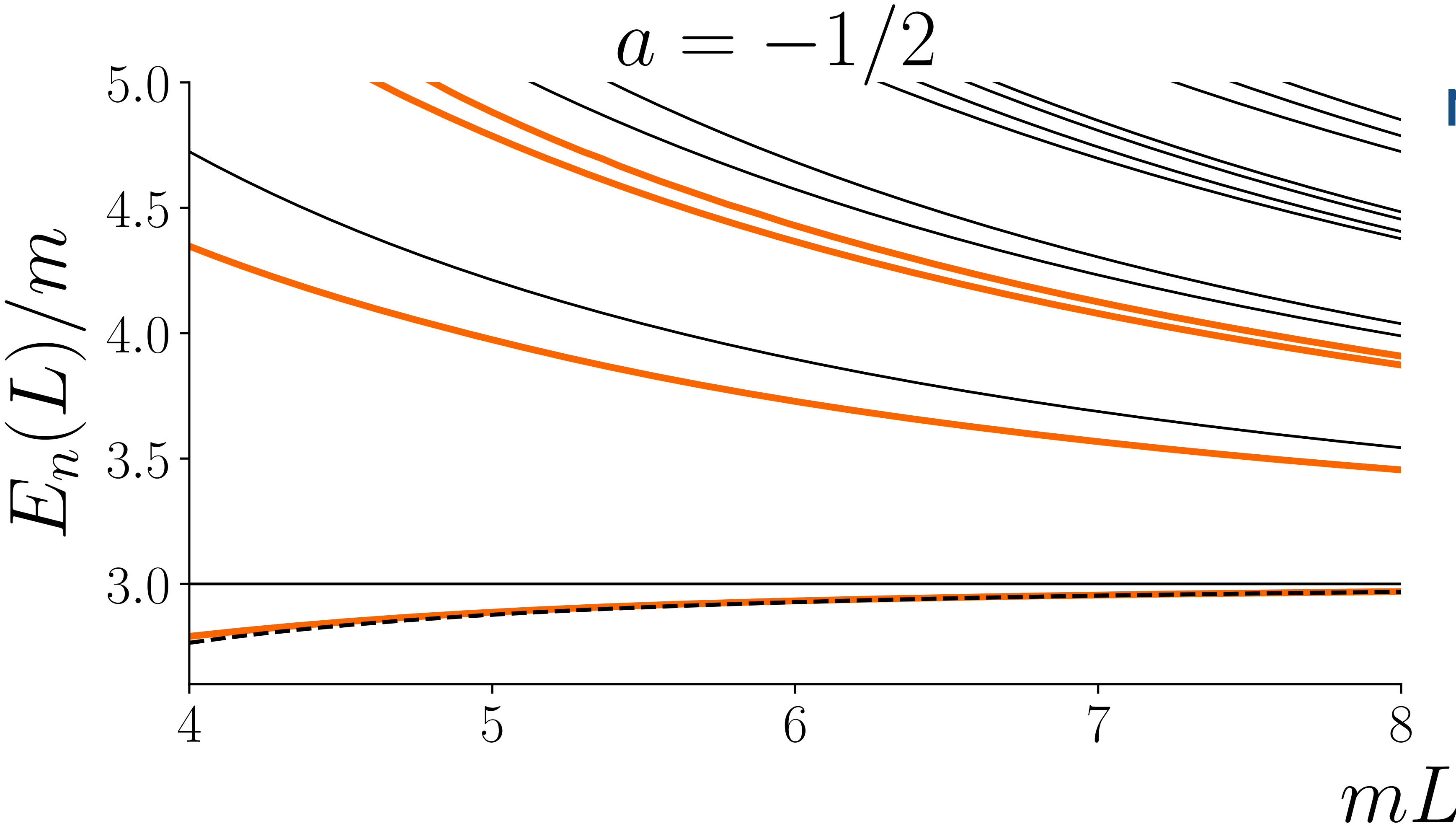
$$\frac{p^2}{m^2} = \left(\frac{2\pi}{mL}\right)^2 \approx 2.46$$

Is it safe to say that we need a relativistic approach?

$\mathcal{K}^{\rm iso}_{\rm df,3}(E) = 0$ solutions

- **Provides a useful benchmark: Deviations measure three-particle physics**
- **Meaning for three-to-three scattering is clear**

$i\mathcal{M}_3 = \mathcal{S}\Bigg[$  $+\cdots\Bigg]$



Now we turn on the interactions

$$1/F_3^{\rm iso}(E,L,a) = 0$$

$$F_3^{\rm iso} = \frac{1}{L^3}\sum_{\vec k,\vec p}\left[\frac{\widetilde F_s}{3} - \widetilde F_s \frac{1}{\mathcal{H}}\widetilde F_s\right]_{kp}$$

$$\mathcal{H} = 1/(2\omega\mathcal{K}_2) + \widetilde F_s + \widetilde G_s$$

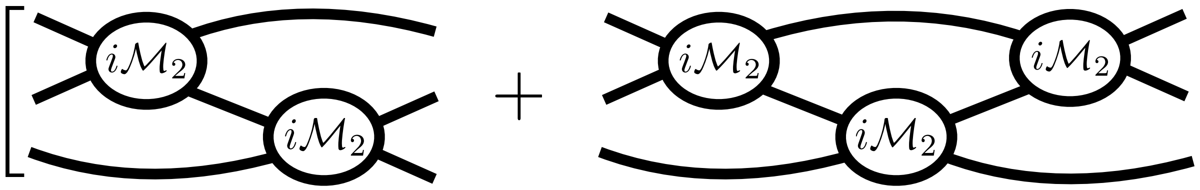
known functions

------ $1/L$ **expansion**

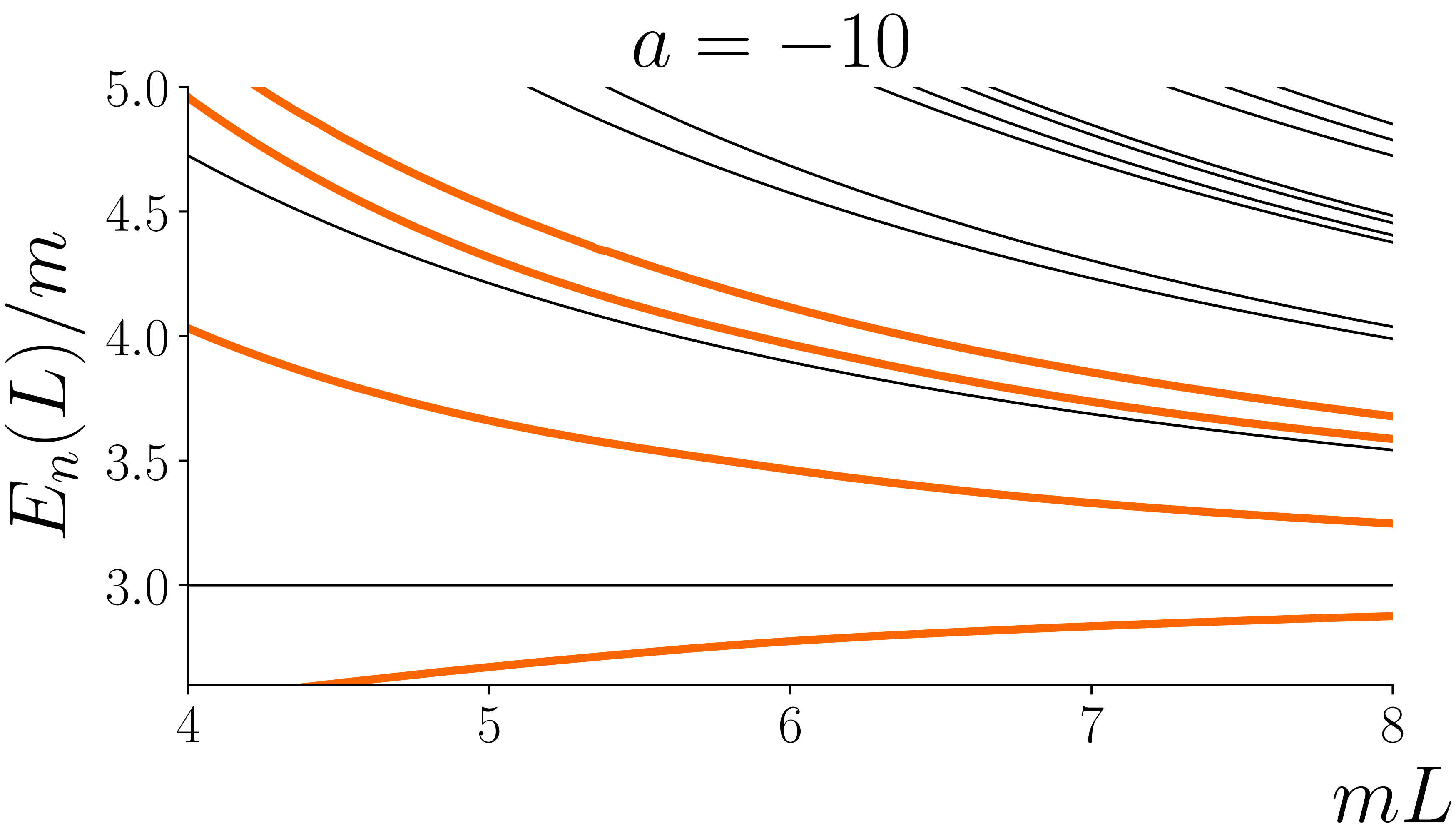
$\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E) = 0$ solutions

- **Provides a useful benchmark: Deviations measure three-particle physics**
- **Meaning for three-to-three scattering is clear**

$$i\mathcal{M}_3 = \mathcal{S}\Bigg[$$



$$+\cdots\Bigg]$$



Can also accommodate large *a*

$$1/F_3^{\mathrm{iso}}(E, L, a) = 0$$

$$F_3^{\mathrm{iso}} = \frac{1}{L^3} \sum_{\vec{k},\vec{p}} \left[\frac{\widetilde{F}_s}{3} - \widetilde{F}_s \frac{1}{\mathcal{H}} \widetilde{F}_s \right]_{kp}$$

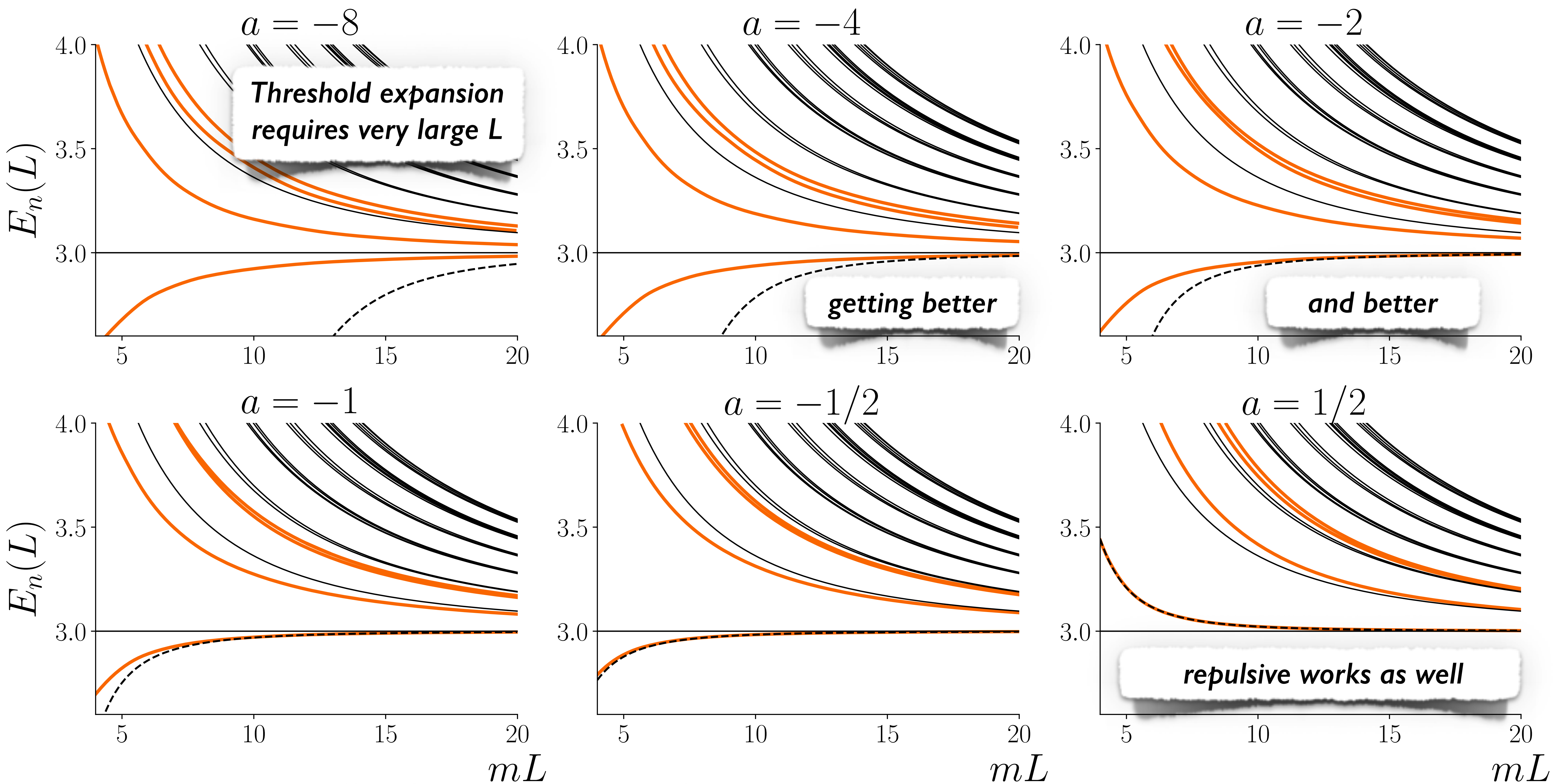
$$\mathcal{H} = 1/(2\omega\mathcal{K}_2) + \widetilde{F}_s + \widetilde{G}_s$$

known functions

------- $1/L$ **expansion**

$\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E) = 0$ **solutions**

Straightforward to vary *a* and to study large volumes



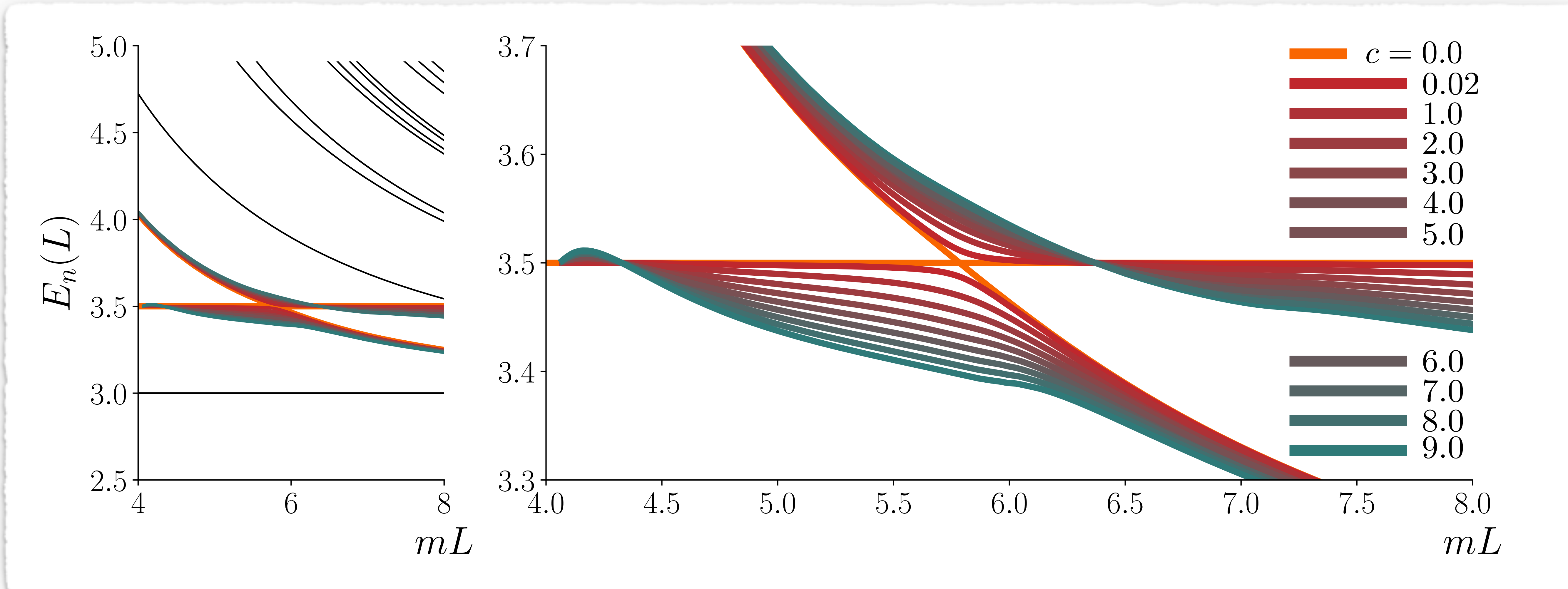
But, to avoid poles in $\mathcal{K}_2$, we must require $a < 1/m$

Non-zero $\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E)$: Toy resonance

Here we consider a fun example for non-zero $\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}$

$$a = -10 \qquad \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E) = -\frac{c \times 10^3}{E^2 - M_R^2}$$

For small *c* we expect a narrow avoided level crossing, as *c* increases the gap grows



Further investigation is needed to see if this gives a physical resonance description

Non-zero $\mathcal{K}_{\mathrm{df},3}(E)$: Unitary bound state

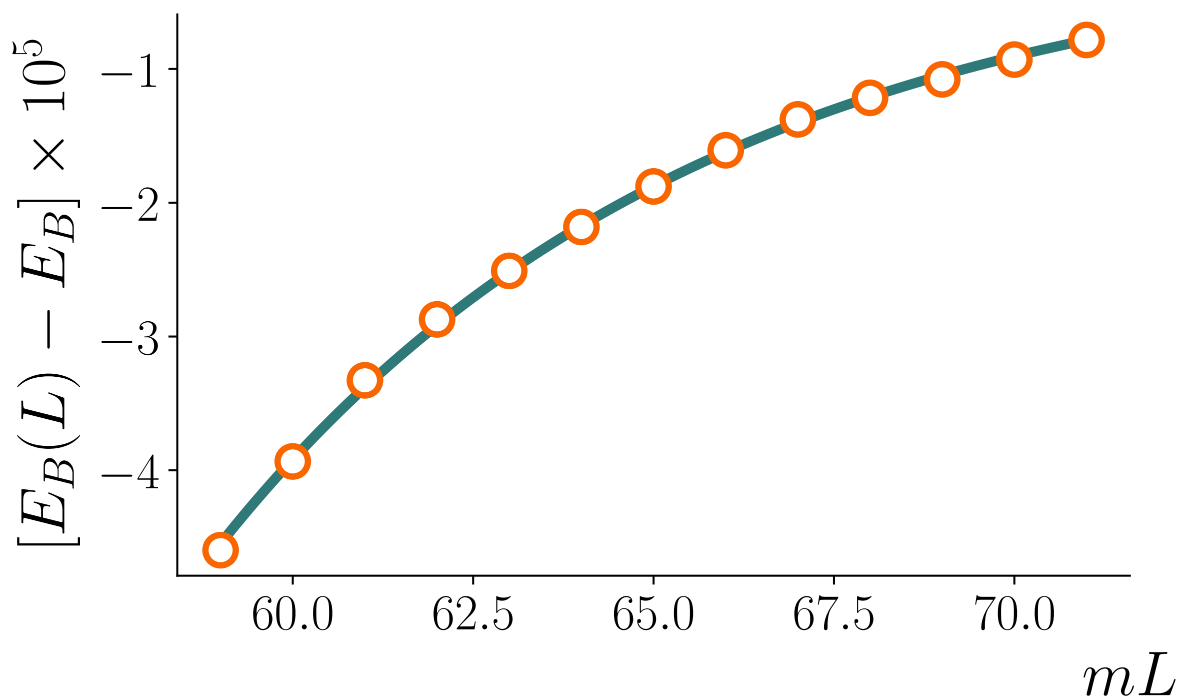
The parameters $a = -10^4, \ \mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3}(E) = 2500$ lead to a shallow bound state

$\kappa \approx 0.1m$ where $E_B = 3m - \kappa^2/m$

Finite-volume behavior of this state has a known asymptotic form

Meißner, Rios, Rusetsky (2015)

$$E_B(L) = 3m - \frac{\kappa^2}{m} - (98.35\cdots)|A|^2 \frac{\kappa^2}{m} \frac{e^{-2\kappa L/\sqrt{3}}}{(\kappa L)^{3/2}} \left[1 + \mathcal{O}\left(\frac{1}{\kappa L}, \frac{\kappa^2}{m^2}, e^{-\alpha\kappa L}\right)\right]$$



This describes the bound state for

$\kappa L \approx 0.1 mL \gg 1$

We fit our q.c. data over $60 < mL < 70$

$\longrightarrow \kappa = 0.1068, \ |A|^2 = 0.948$

Close to one ✔

Non-zero $\mathcal{K}_{\mathrm{df},3}(E)$: Unitary bound state

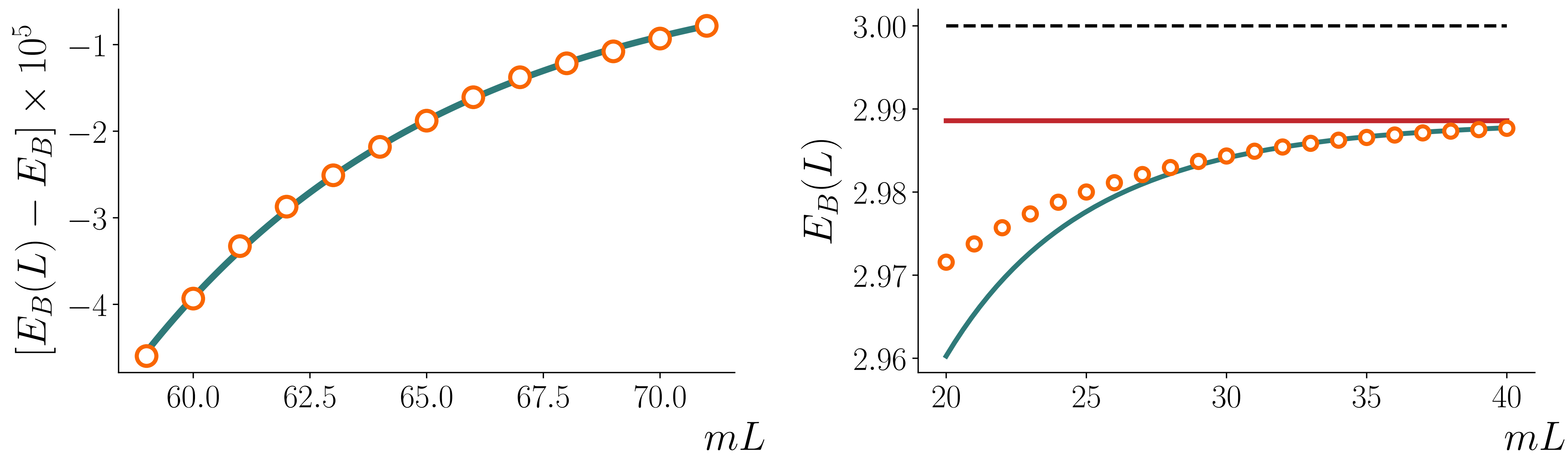
The parameters $a = -10^4$, $\mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3}(E) = 2500$ lead to a shallow bound state

$\kappa \approx 0.1m$ **where** $E_B = 3m - \kappa^2/m$

Finite-volume behavior of this state has a known asymptotic form

Meißner, Rios, Rusetsky (2015)

$$E_B(L) = 3m - \frac{\kappa^2}{m} - (98.35\cdots)|A|^2 \frac{\kappa^2}{m} \frac{e^{-2\kappa L/\sqrt{3}}}{(\kappa L)^{3/2}} \left[1 + \mathcal{O}\left(\frac{1}{\kappa L}, \frac{\kappa^2}{m^2}, e^{-\alpha \kappa L}\right)\right]$$



Non-zero $\mathcal{K}_{\mathrm{df},3}(E)$: Unitary bound state

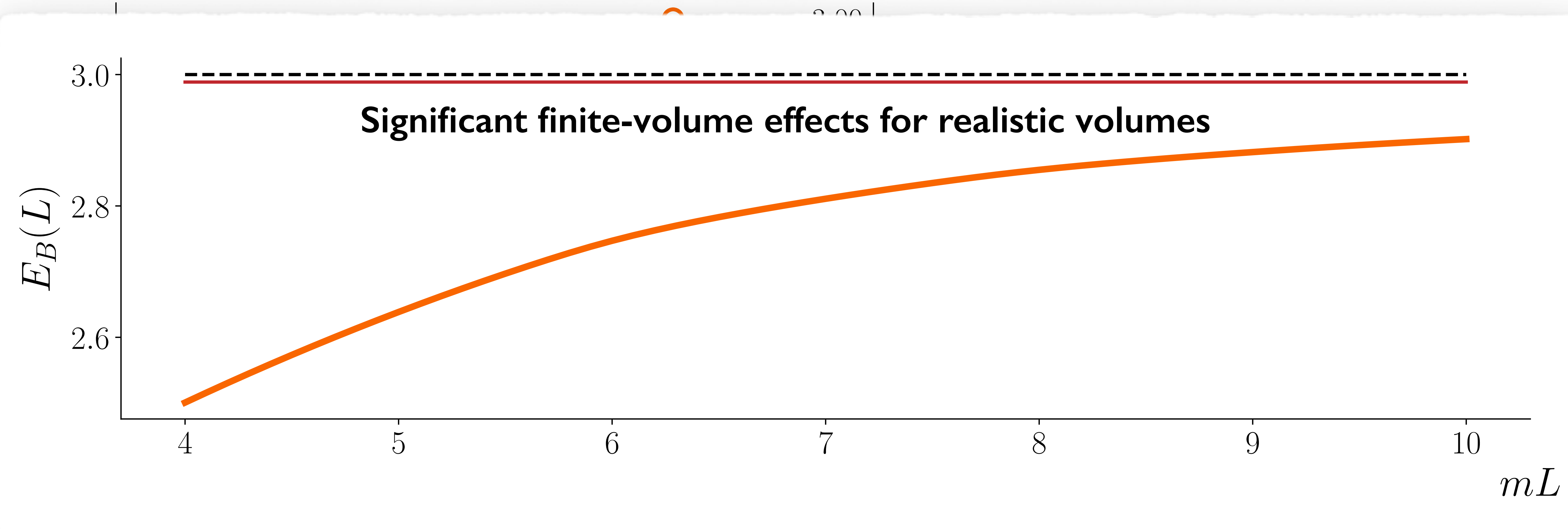
The parameters $a = -10^4$, $\mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3}(E) = 2500$ lead to a shallow bound state

$\kappa \approx 0.1m$ **where** $E_B = 3m - \kappa^2/m$

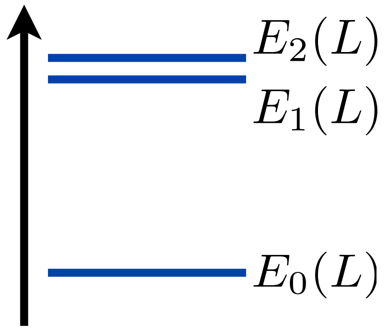
Finite-volume behavior of this state has a known asymptotic form

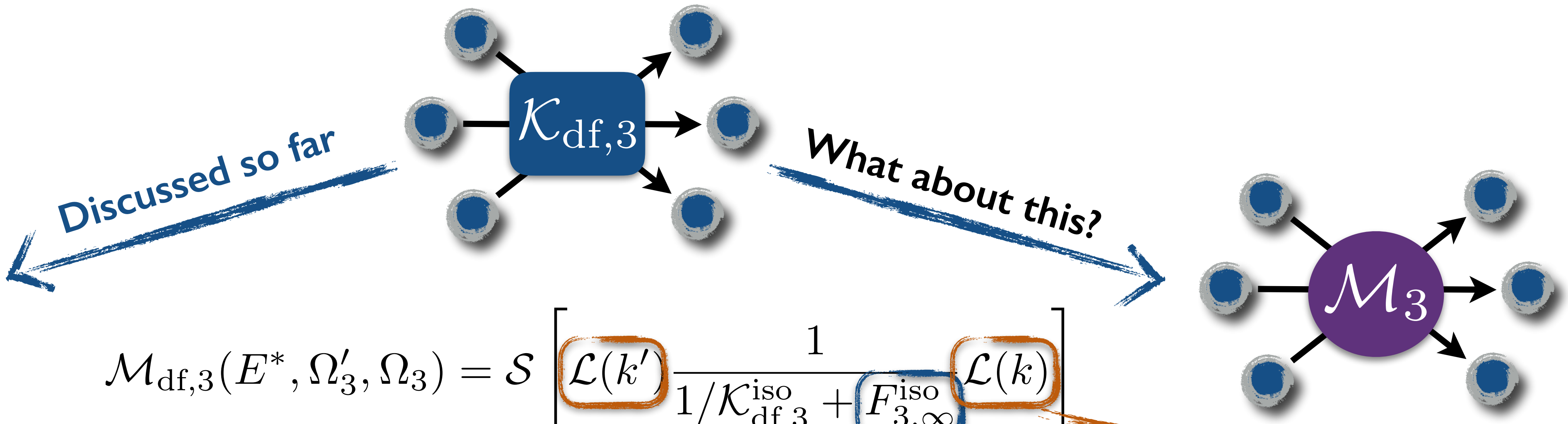
Meißner, Rios, Rusetsky (2015)

$$E_B(L) = 3m - \frac{\kappa^2}{m} - (98.35\cdots)|A|^2 \frac{\kappa^2}{m} \frac{e^{-2\kappa L/\sqrt{3}}}{(\kappa L)^{3/2}} \left[1 + \mathcal{O}\left(\frac{1}{\kappa L}, \frac{\kappa^2}{m^2}, e^{-\alpha \kappa L}\right)\right]$$



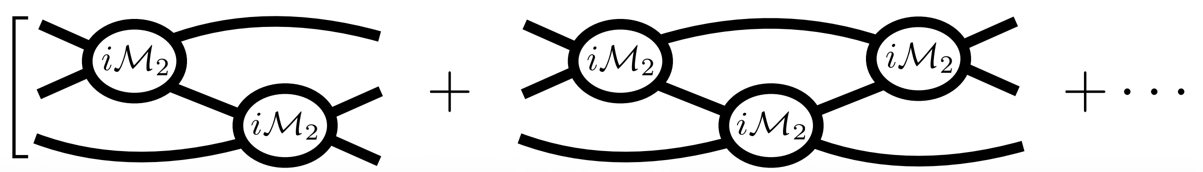
Converting to scattering amplitudes



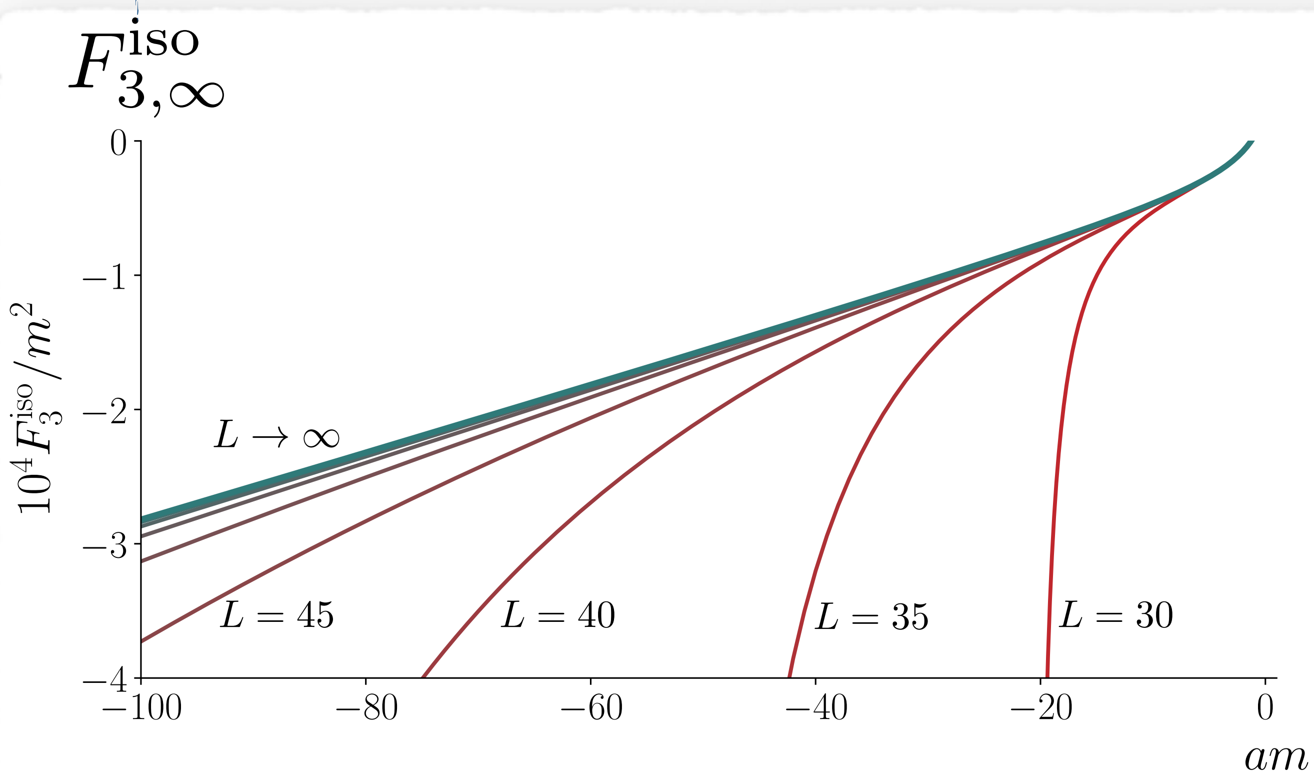


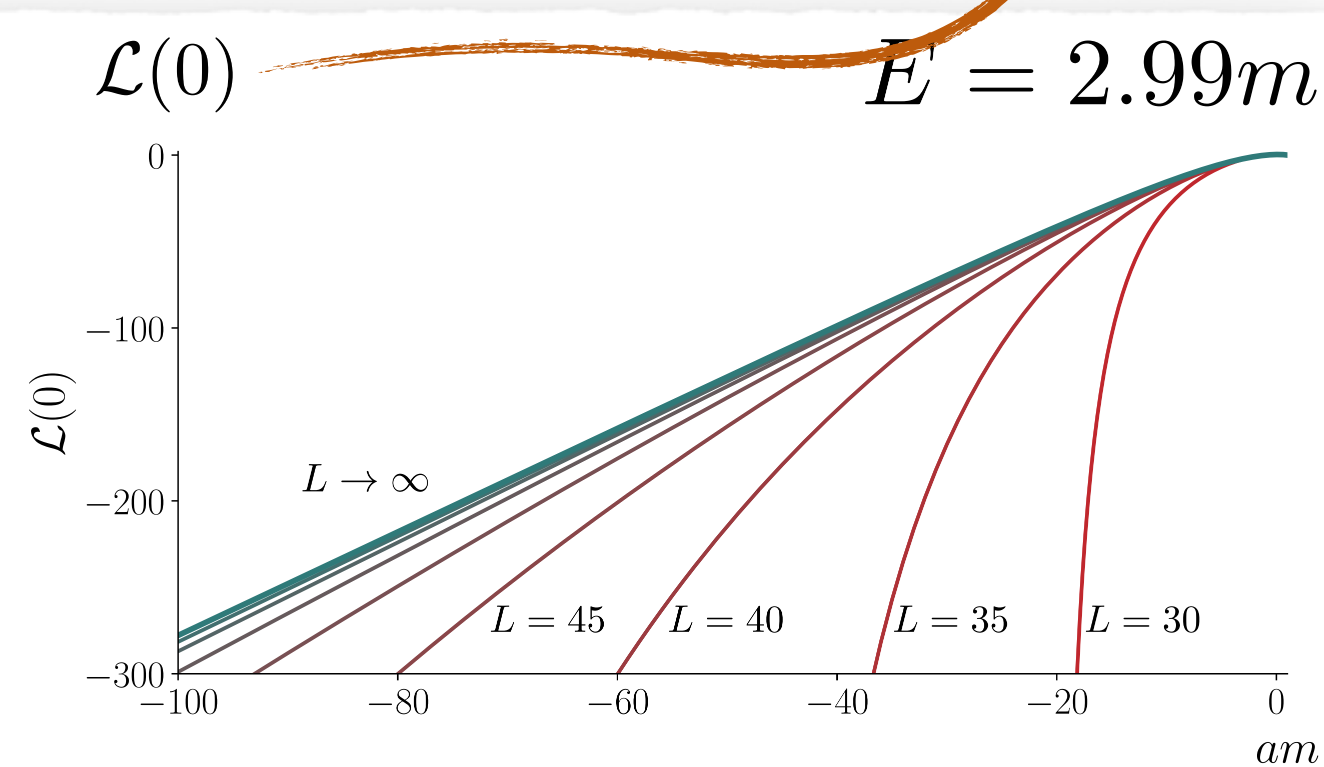
$$\mathcal{M}_{\mathrm{df},3}(E^*,\Omega_3',\Omega_3) = \mathcal{S}\left[\mathcal{L}(k')\frac{1}{1/\mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3} + F^{\mathrm{iso}}_{3,\infty}}\mathcal{L}(k)\right]$$

$$i\mathcal{M}_{\mathrm{df},3} = i\mathcal{M}_3 - \mathcal{S}\Big[$$



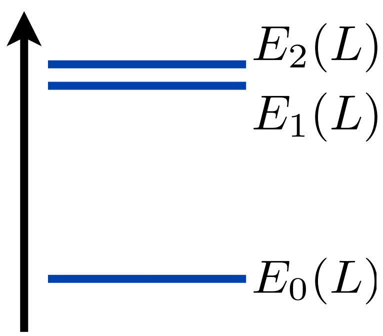
$$\Big]$$



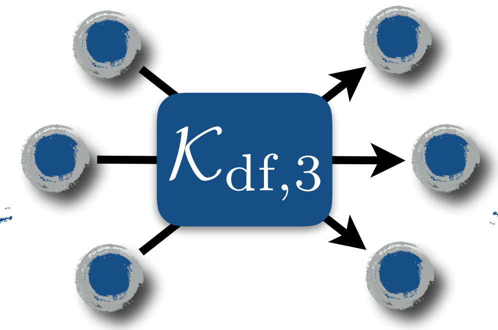


This only works below threshold… Relation above threshold crucially needed (Gernot?)

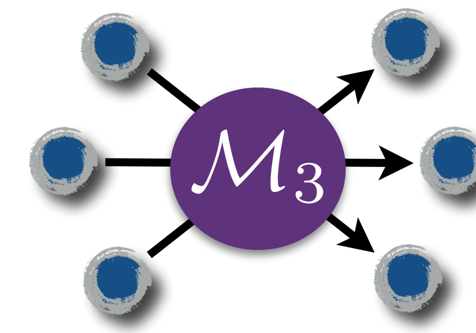
Converting to scattering amplitudes



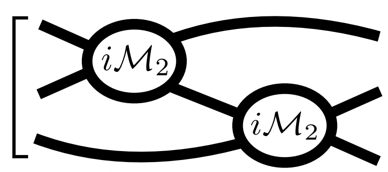
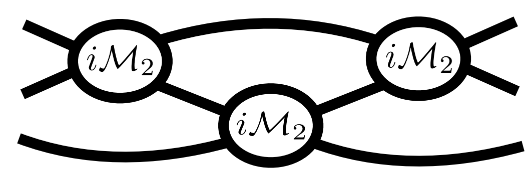
Discussed so far

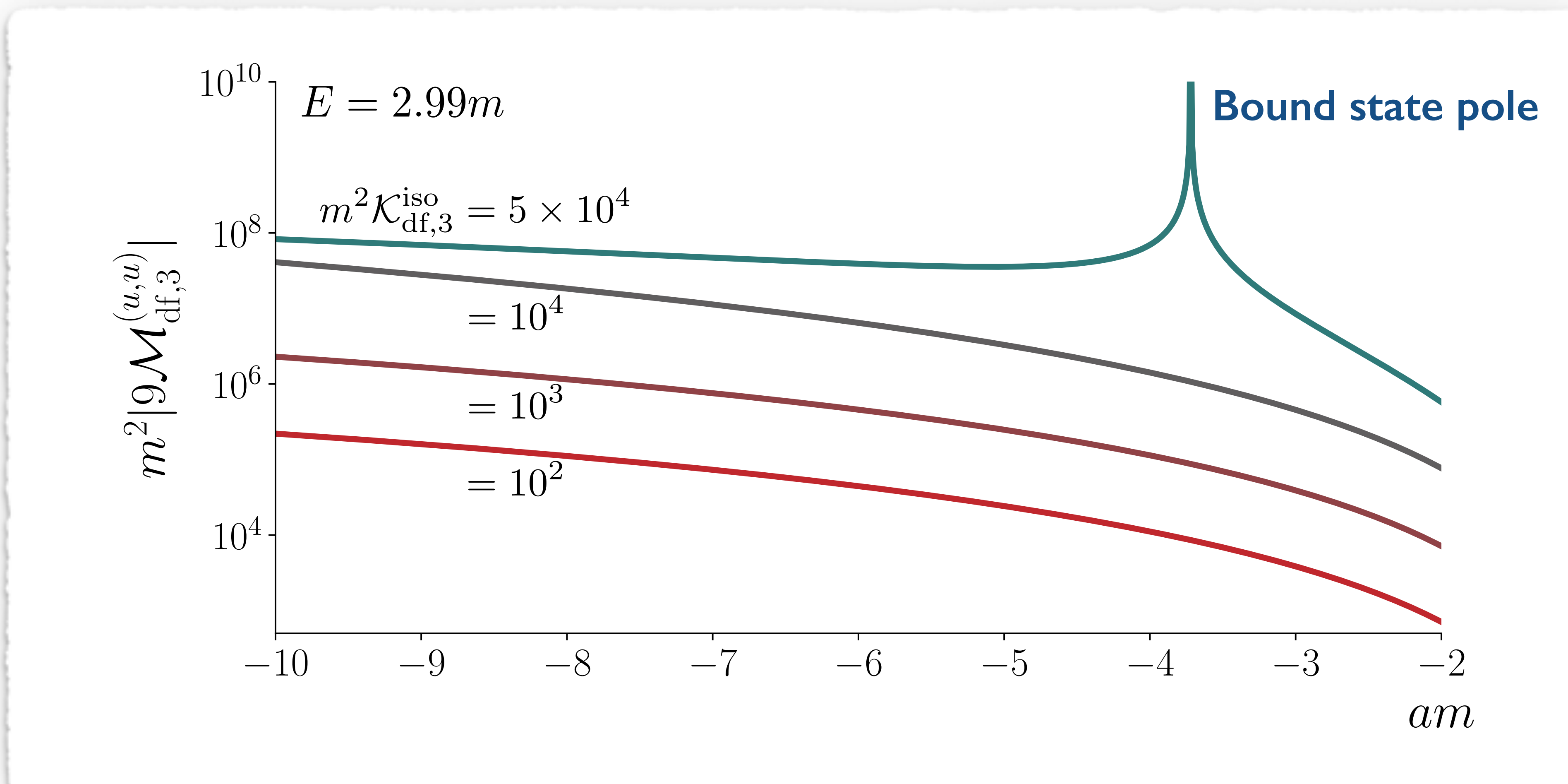


What about this?

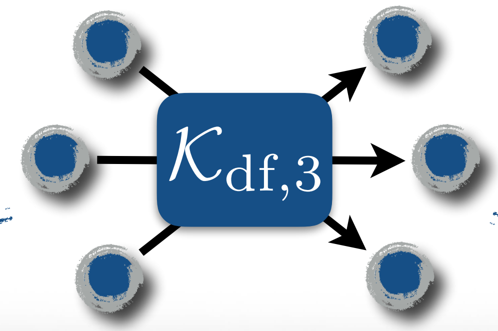


$$\mathcal{M}_{\mathrm{df},3}(E^*, \Omega_3', \Omega_3) = \mathcal{S}\left[\mathcal{L}(k')\frac{1}{1/\mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3} + F^{\mathrm{iso}}_{3,\infty}}\mathcal{L}(k)\right]$$

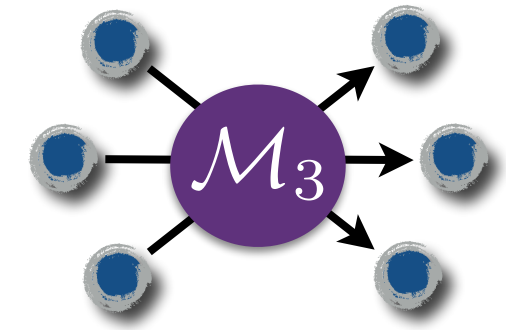
$i\mathcal{M}_{\mathrm{df},3} = i\mathcal{M}_3 - \mathcal{S}\Big[$  $+$  $+\cdots\Big]$



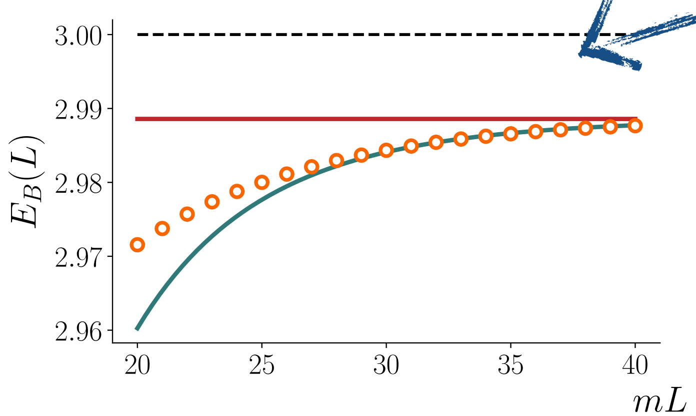
Back to the bound state



Discussed so far

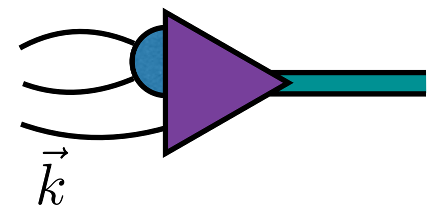


What about this?



It's convenient to consider the unsymmetrized version

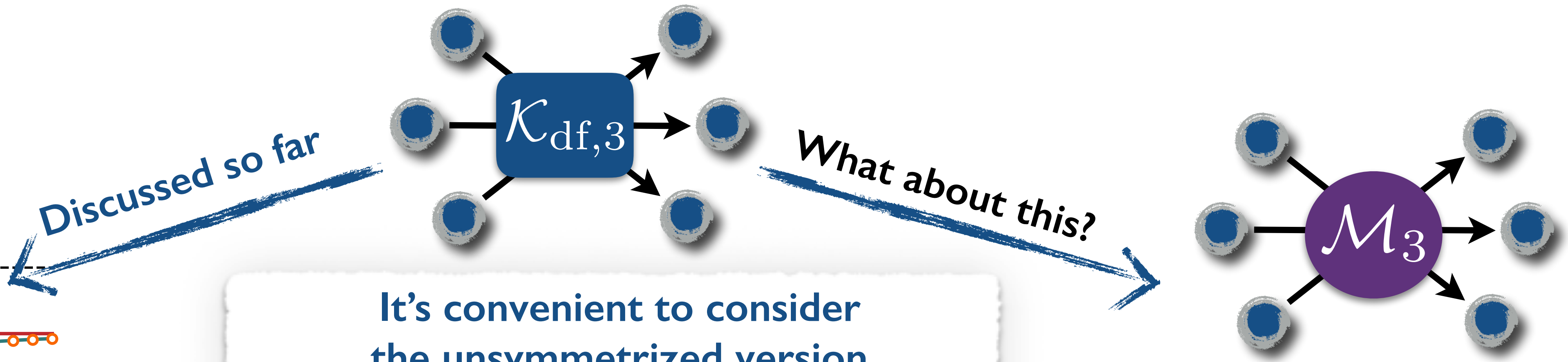
$$\Gamma^u(k) \equiv$$

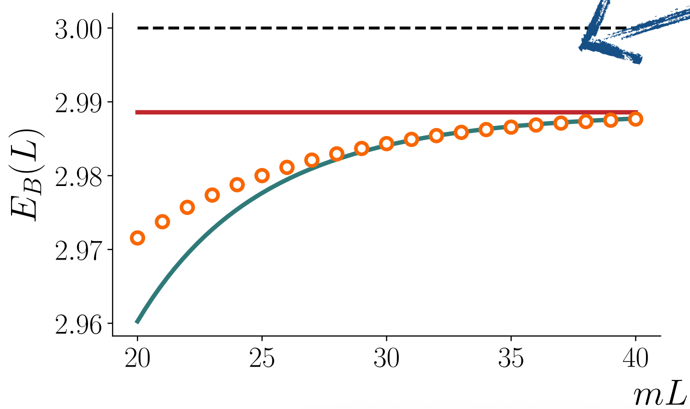


$$\mathcal{S}[\Gamma^u(k)] \equiv \langle \pi(k)\pi\pi | E_B \rangle$$

$$\mathcal{M}_3 \sim -\frac{\Gamma(\Omega_3')\overline{\Gamma}(\Omega_3)}{E^2 - M_B^2}$$

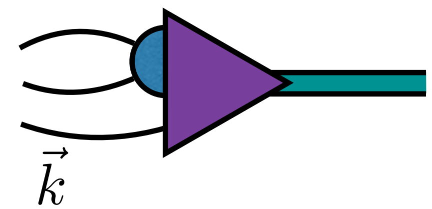
Back to the bound state



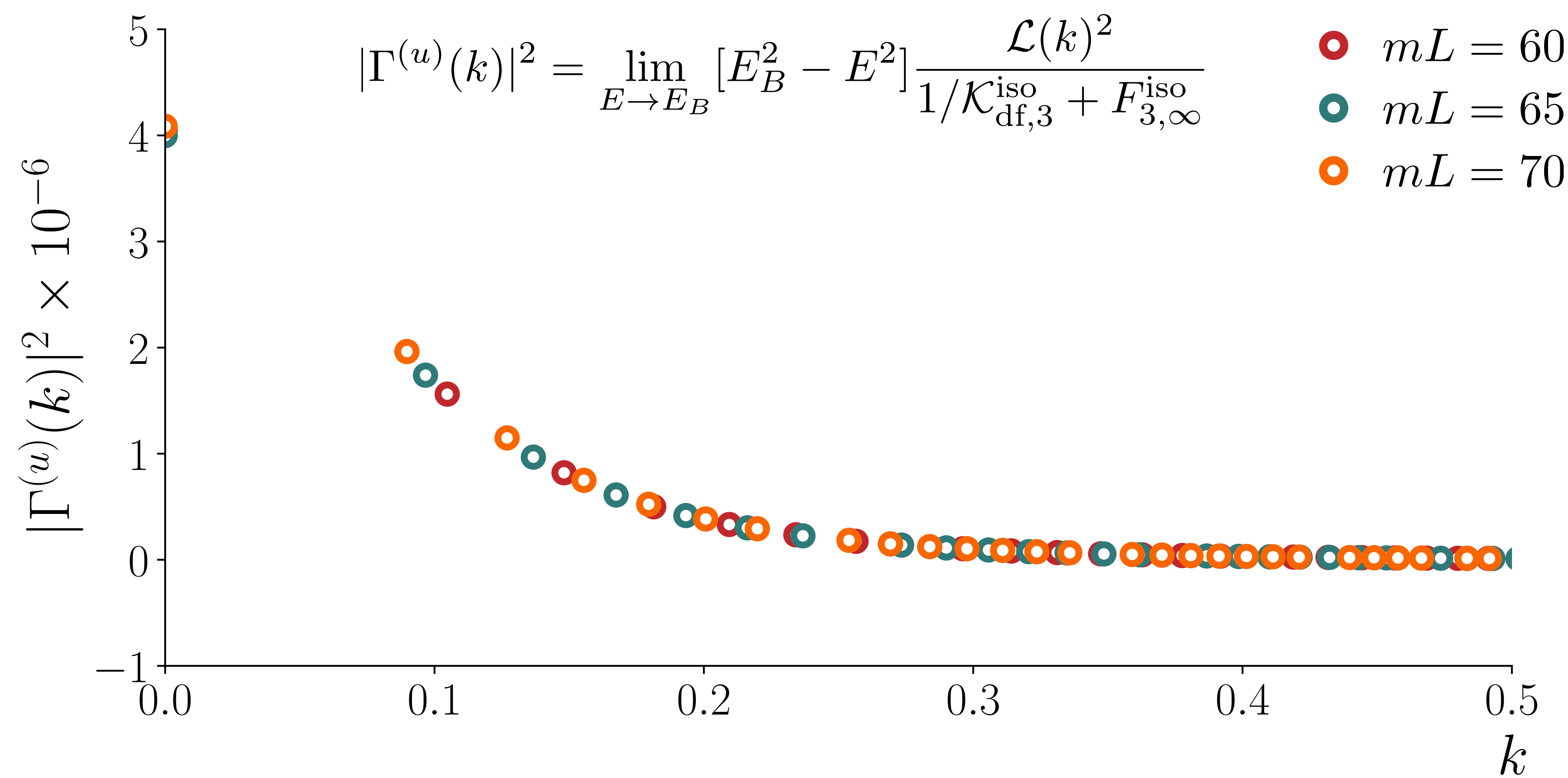


It's convenient to consider the unsymmetrized version

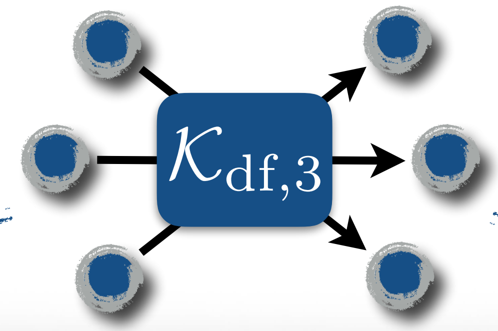
$$\Gamma^u(k) \equiv$$



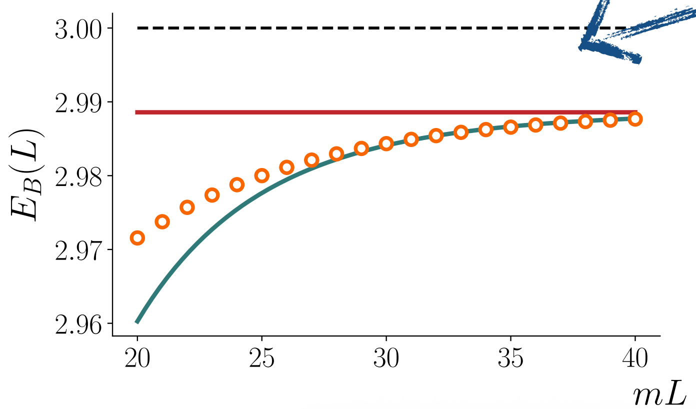
$$\mathcal{M}_3 \sim -\frac{\Gamma(\Omega_3')\overline{\Gamma}(\Omega_3)}{E^2 - M_B^2}$$



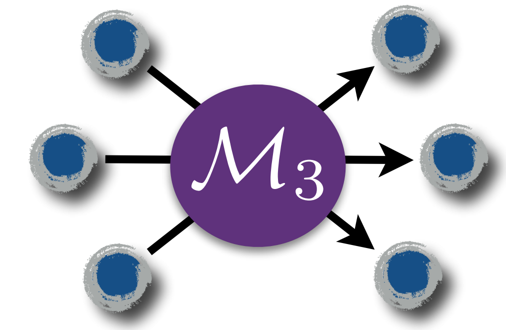
Back to the bound state



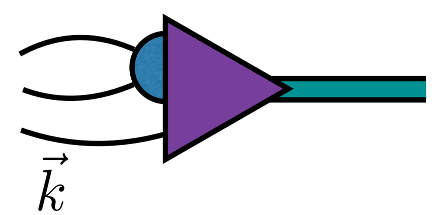
Discussed so far



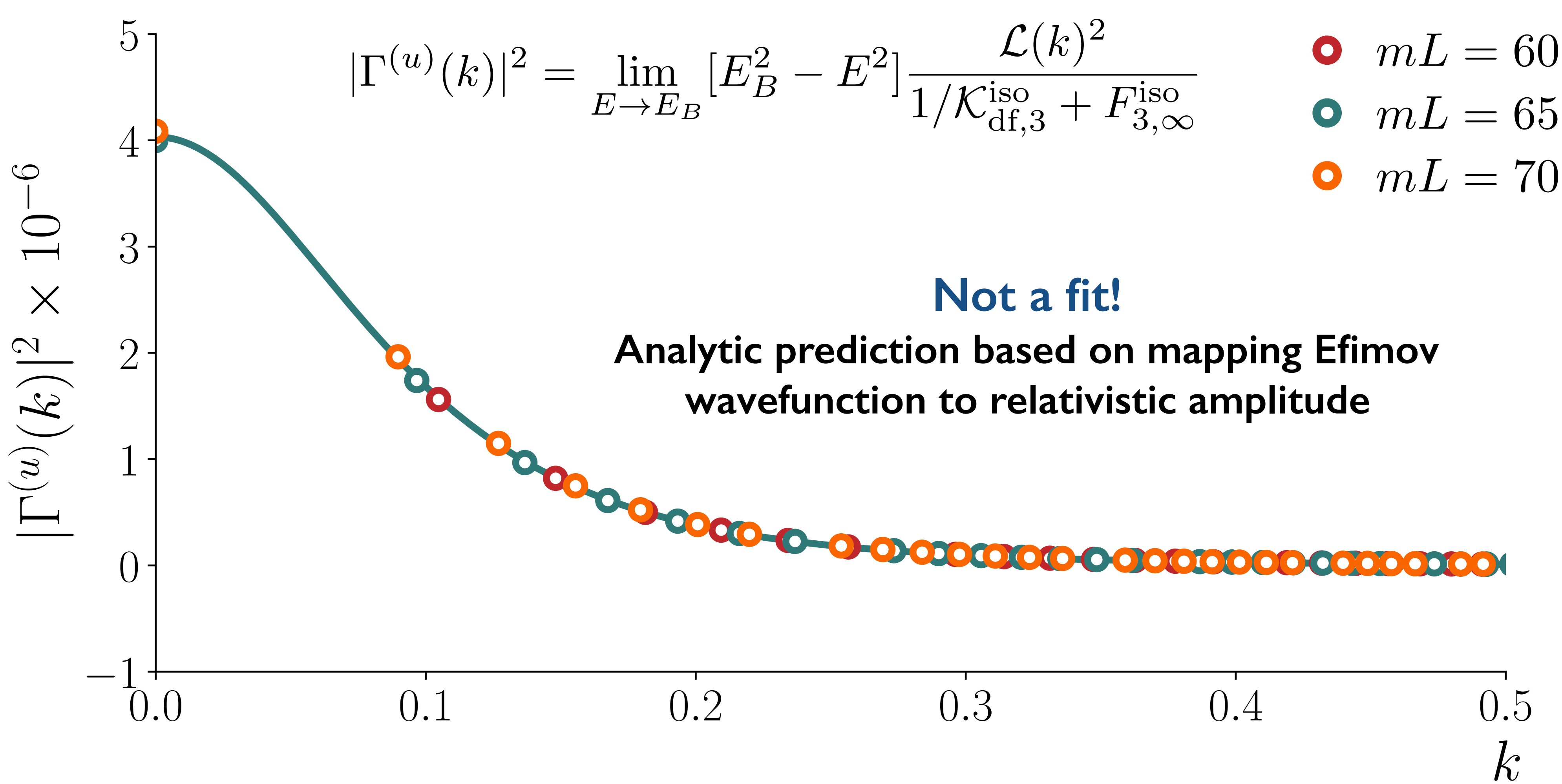
What about this?



It's convenient to consider the unsymmetrized version

$\Gamma^u(k) \equiv$ 

$$\mathcal{M}_3 \sim -\frac{\Gamma(\Omega_3')\overline{\Gamma}(\Omega_3)}{E^2 - M_B^2}$$



Not a fit!

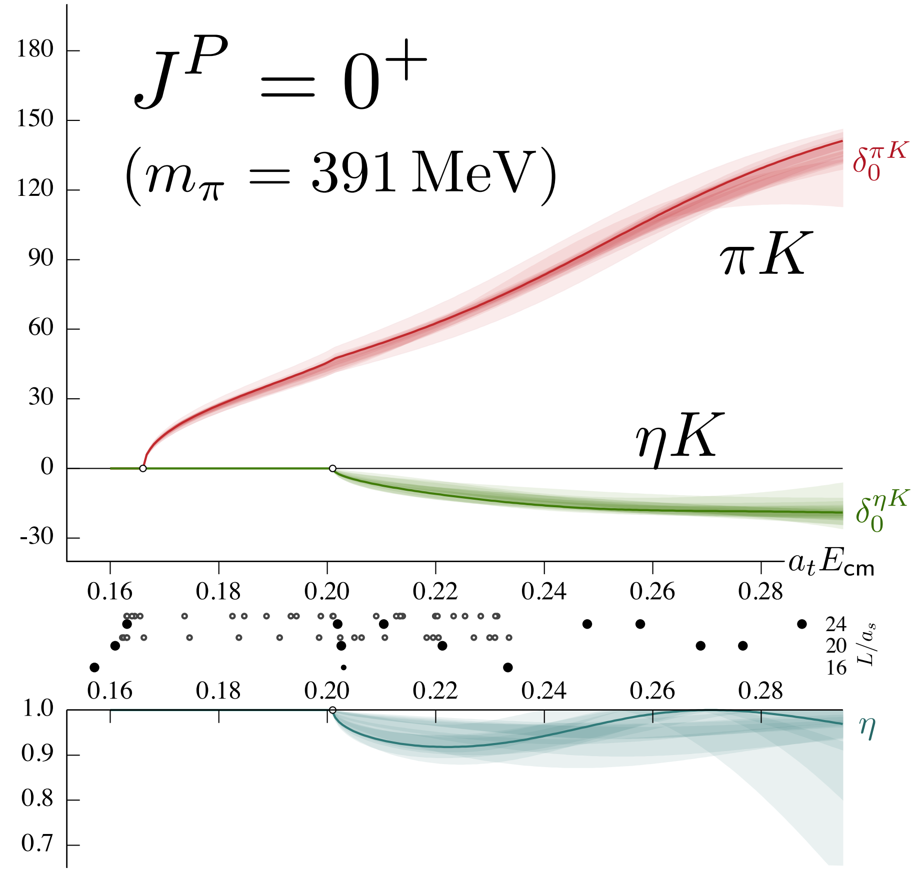
Analytic prediction based on mapping Efimov wavefunction to relativistic amplitude

Questions for the “competitors”

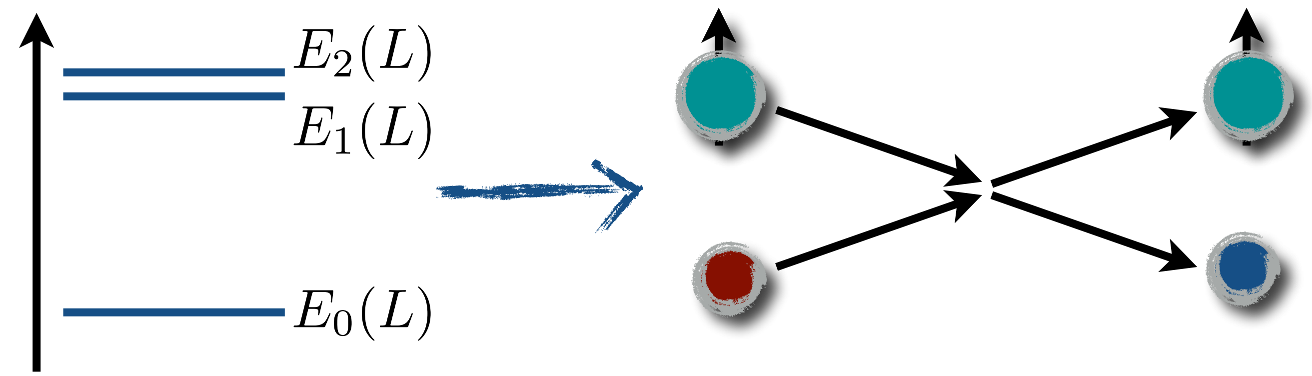
- Are the quantization conditions equivalent?
- Is the formalism limited by 5 particle threshold?
- What are the prospects for two-to-three? multiple-channels?
- Is it clear that all relativistic effects are captured by relativistic kinematics?
- What are the general work flows for each method?

The Big Picture

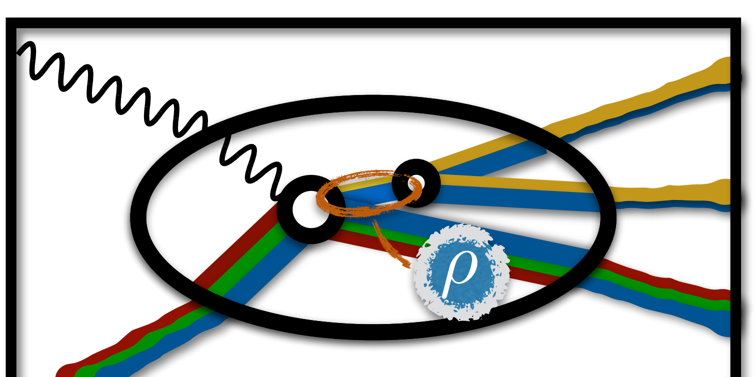
Two-to-two scattering from LQCD is reaching maturity



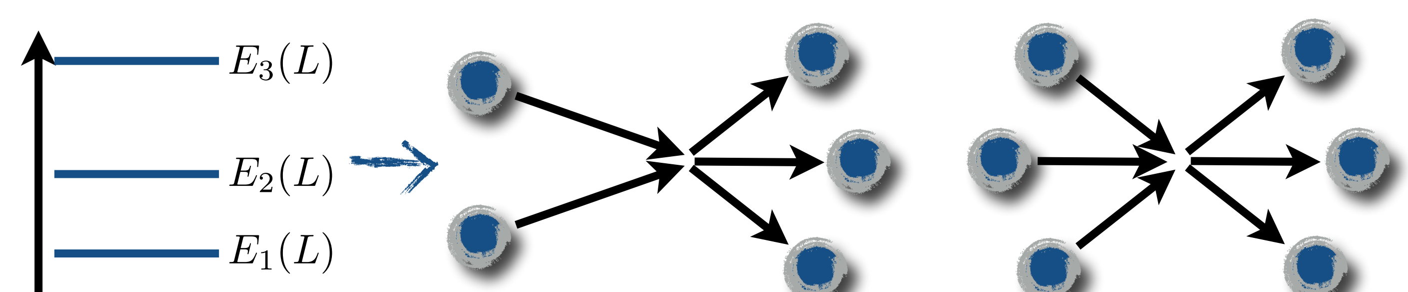
Wilson et. al., *Phys. Rev.* D 91, 054008 (2015)



Stay tuned for three-particle observables from LQCD



$p\gamma \to N\rho \to N\pi\pi$



Thanks for listening!