



Update on the relativistic three-particle finite-volume formalism

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A good time for three particles in a box!

- 🎤 At this workshop, **three** competing groups are presenting finite-volume **three**-particle formalism
- 🎤 Not just that!...
All three groups are presenting examples of **numerical implementation**

- 🎤 In a nutshell...

Raul, Steve and I

Relativistic, EFT-independent, follows the approach of Lüscher (ala Kim, Sachrajda and Sharpe) to the extent possible

Akaki, (H.W. Hammer, J.-Y. Pang)

Non-relativistic, EFT based, focuses on extracting LECs, simpler derivation and formulae

Maxim and Michael

Relativistic, built on unitary constraints + replacing integrals with sums over shells

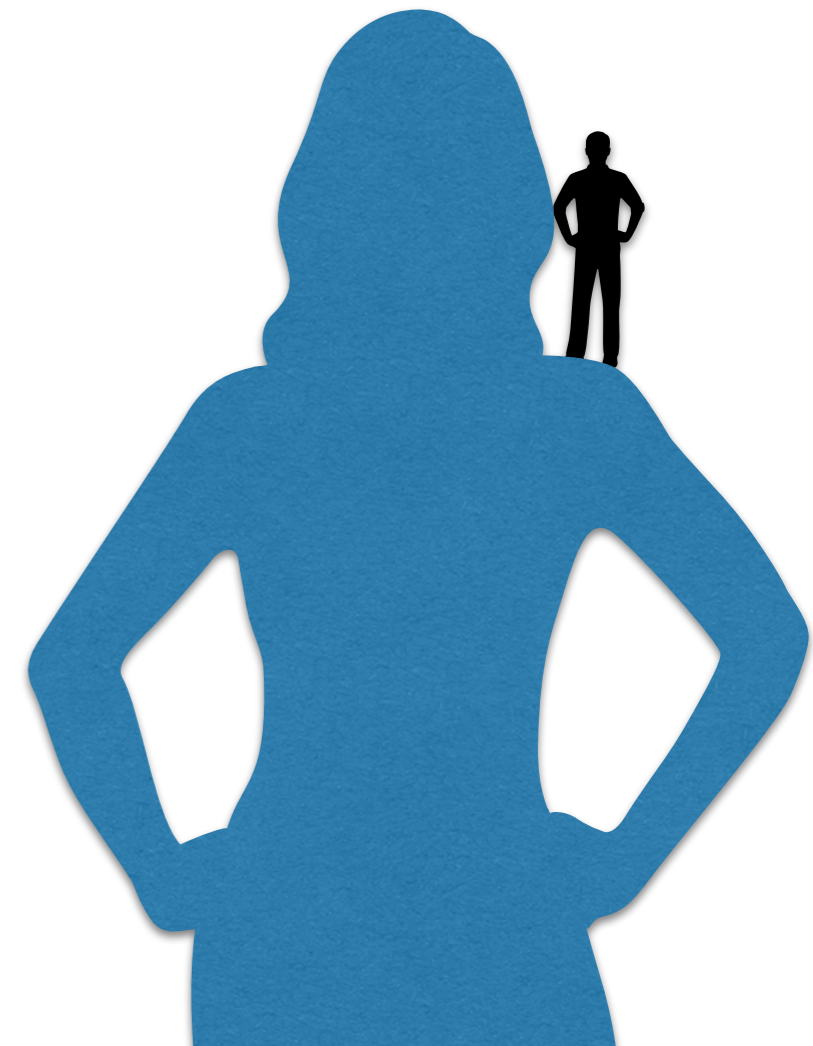
- 🎤 Aims for today:

- 🎤 Better understand, compare and contrast the methods
- 🎤 Understand when (if) each method is useful or even best
- 🎤 Discuss if ideas can be combined to reach an optimal approach

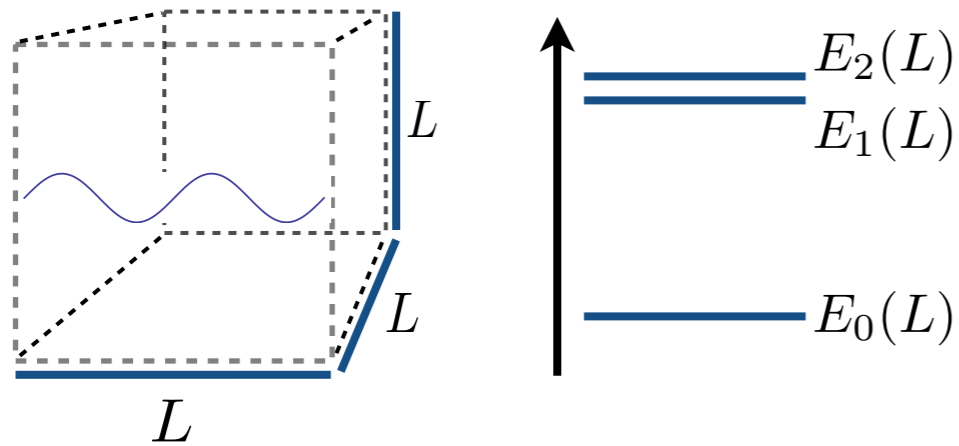
If I have seen further...

- 🌐 These ideas build on a great deal of earlier work
Agadjanov, Beane, Bernard, Briceño, Christ, Davoudi, Detmold, Döring, Fu, Guo, Huang, Kim, Lellouch, Leskovec, Lüscher, Luu, Mai, Meißner, Meyer, Oset, Polejaeva, Prelovsek, Rios, Rusetsky, Savage, Sachrajda, Sharpe, Tan, Walker-Loud, Yamazaki, Yang
- 🌐 To go forward we will undoubtedly need input from non-finite-volume experts here and elsewhere
- 🌐 Most importantly, this would all be quite useless without the remarkable numerical progress in this field

Let's jump in!



Basic set-up



cubic, spatial volume (extent L)

periodic boundary conditions

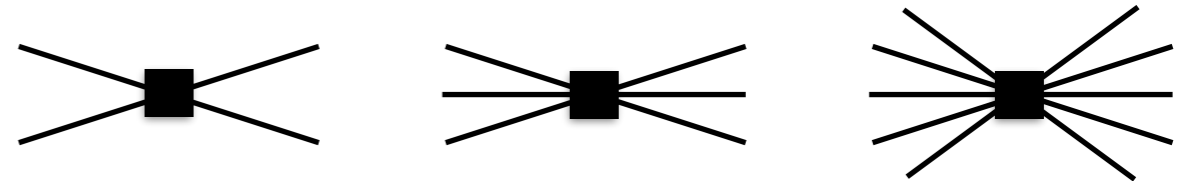
$$\vec{p} \in (2\pi/L)\mathbb{Z}^3$$

time direction **infinite**

L large enough to ignore e^{-mL}

Generic relativistic QFT

1. Include all interactions



2. no power-counting scheme

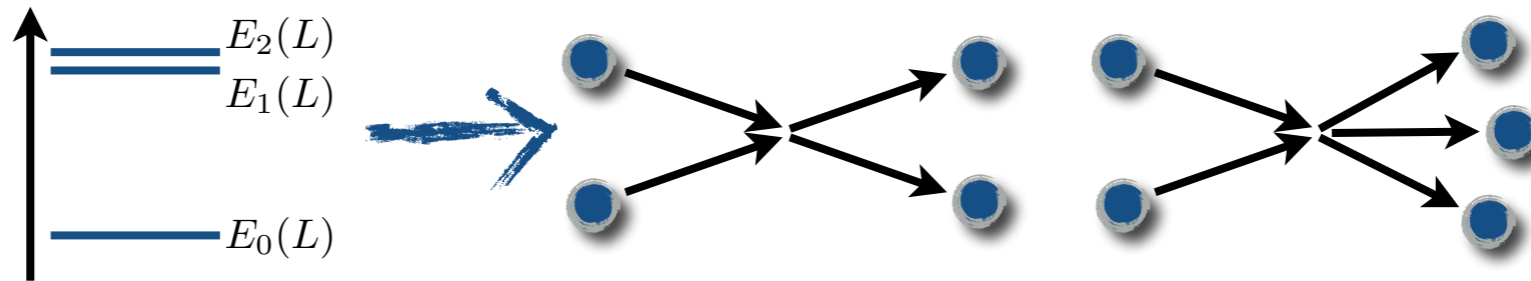
Not possible to directly calculate scattering observables to all orders

But it is possible to derive general, all-orders relations to finite-volume quantities

Assume lattice effects are small and accommodated elsewhere

Work in continuum field theory throughout

Finite-volume correlators



Convenient to work with momentum space, finite-volume correlators

$$C_L(P) \equiv \int_L d^4x e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

Total 4-momentum

$$P = (E, \vec{P}) = (E, 2\pi\vec{n}/L)$$

c.m. frame energy: $E^{*2} = E^2 - \vec{P}^2$

n-particle interpolator

e.g. $\pi(\mathbf{p})\pi(-\mathbf{p})$ or $\bar{q}\Gamma q$
(only quantum numbers relevant)

Focus on a window of energies to isolate particular on-shell states

- Two particles: \mathbb{Z}_2 ✓ $0 < E^* < 4m$ \mathbb{Z}_2 ✗ $m < E^* < 3m$

- Three particles \mathbb{Z}_2 ✓ $m < E^* < 5m$

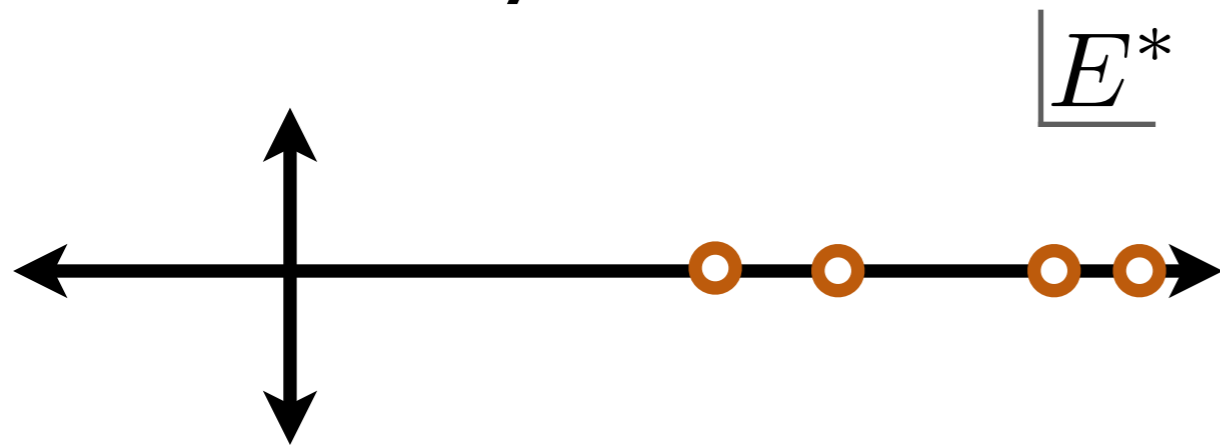
- Two and three particles \mathbb{Z}_2 ✗ $m < E^* < 4m$

Of poles and branch cuts

$$C_L(P) \equiv \int_L d^4x e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

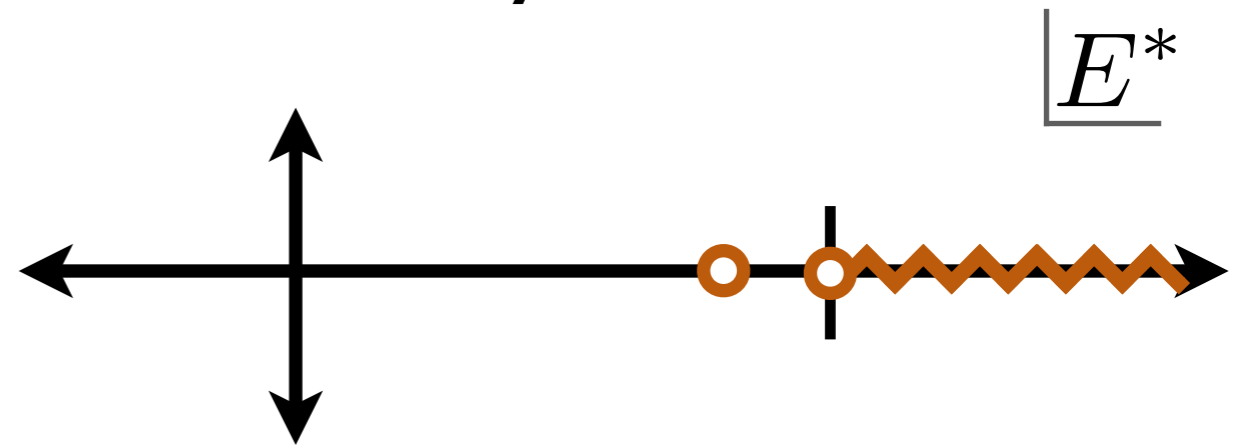
At fixed L, \vec{P} , poles in C_L give **the finite-volume spectrum**

C_L analytic structure



- Real function (unitarity hidden)
- No cuts \rightarrow only one sheet
- No resonance poles

C_∞ analytic structure



- Complex function (constrained by unitarity)
- Scattering states form cut
- Resonance poles on unphysical sheets

Want to relate $C_L \leftrightarrow C_\infty$ ($\mathcal{M}_{n \rightarrow m}$ is just a specific choice of C_∞)

The idea is to “reach in” and correct the singularity structure

Two-to-two review (Here with identical, scalar, Z_2 symmetry)

$$C_L(P) = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots$$

Diagram 1: \mathcal{O}^\dagger and \mathcal{O} connected by two lines, each with a black dot. A blue bracket labeled '1' groups this and the next two diagrams.

Diagram 2: \mathcal{O}^\dagger and \mathcal{O} connected by two lines, each with a black dot, and a central circle labeled iB with two black dots. A blue bracket labeled '2' groups this and the next diagram.

Diagram 3: \mathcal{O}^\dagger and \mathcal{O} connected by two lines, each with a black dot, and two central circles labeled iB with two black dots each. A blue bracket labeled '3' groups this and the next diagram.

$$C_L(P) = C_\infty(P)$$

$$+ \text{diagram 4} + \text{diagram 5} + \dots$$

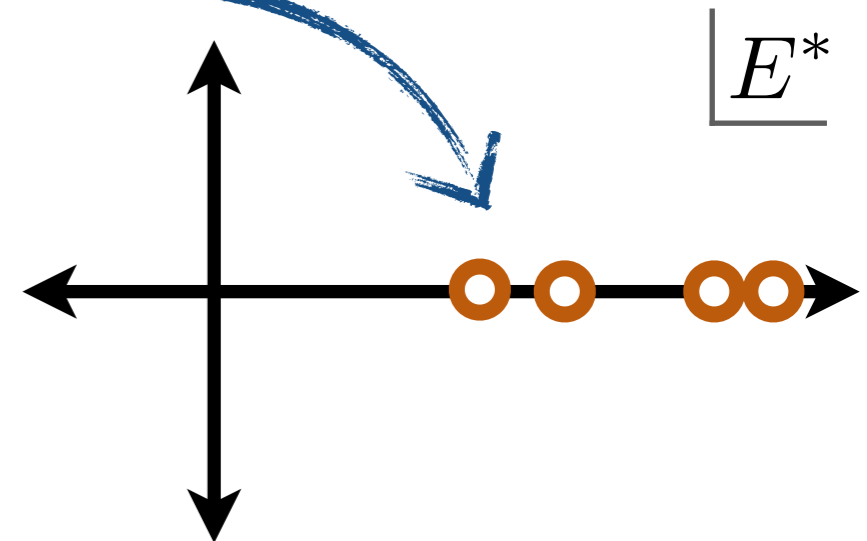
Diagram 4: A and A' connected by a horizontal line with a vertical dashed line labeled F in the middle. A blue cloud labeled $\langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$ points to this diagram.

Diagram 5: A and A' connected by a horizontal line with two vertical dashed lines labeled F and a central circle labeled $i\mathcal{M}$. A blue cloud labeled $\langle 0 | \mathcal{O} | \pi\pi, \text{in} \rangle$ points to this diagram.

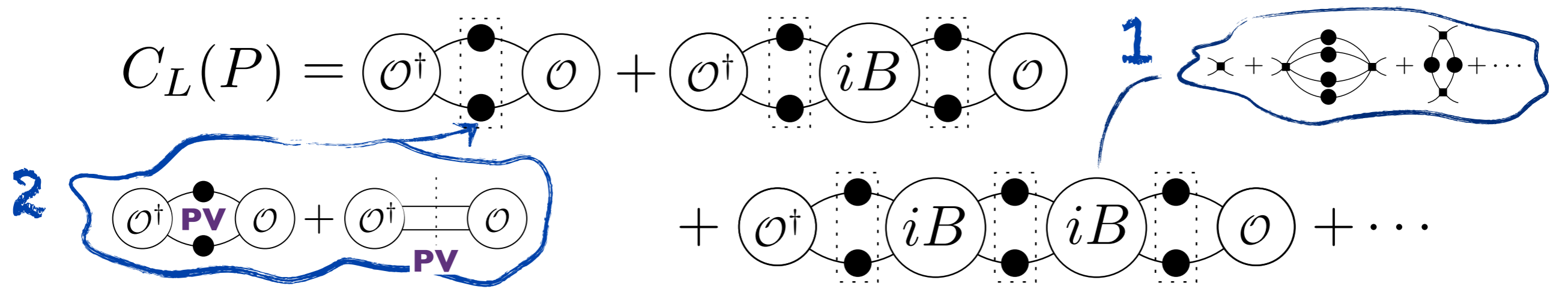
We deduce...

$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2 \rightarrow 2} F} A$$

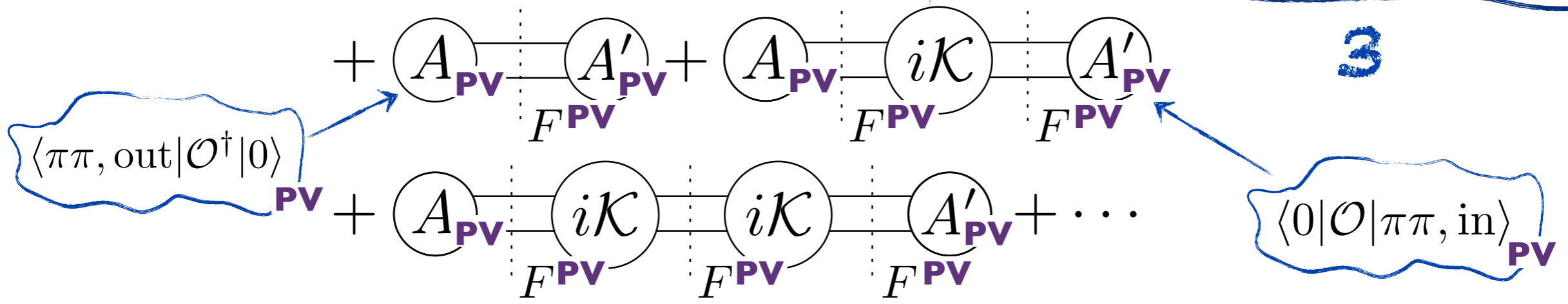
poles are in here



Two-to-two review (Here with identical, scalar, Z_2 symmetry)

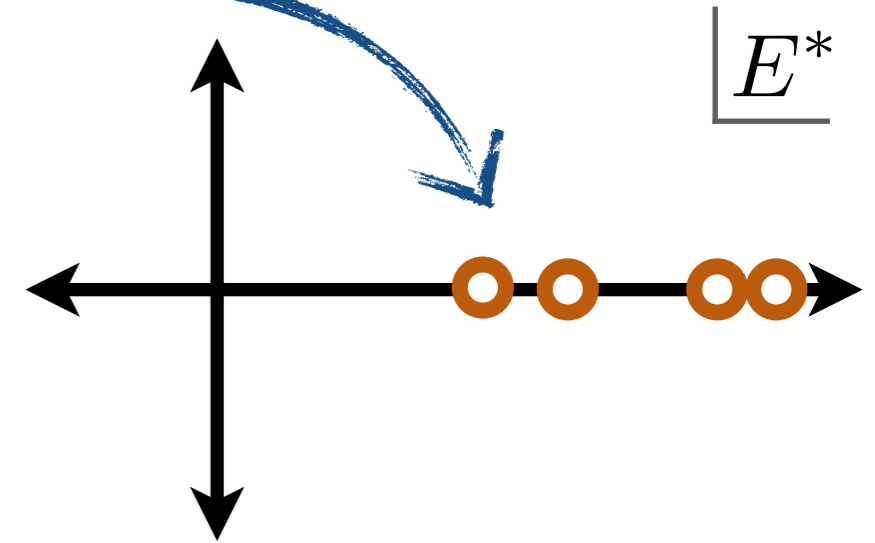


$$C_L(P) = C_\infty^{\text{PV}}(P)$$

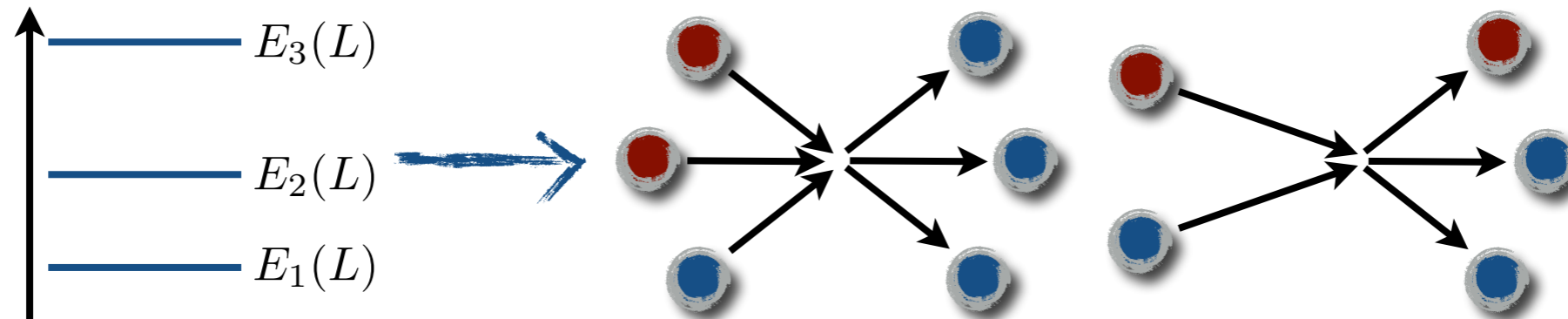


We deduce...

$$C_L(P) = C_\infty^{\text{PV}}(P) - A'_{\text{PV}} \boxed{F \frac{1}{1 + \mathcal{K}_2 F}} A_{\text{PV}}$$



Our aim is to extend the derivation for arbitrary relativistic two- and three-particle systems

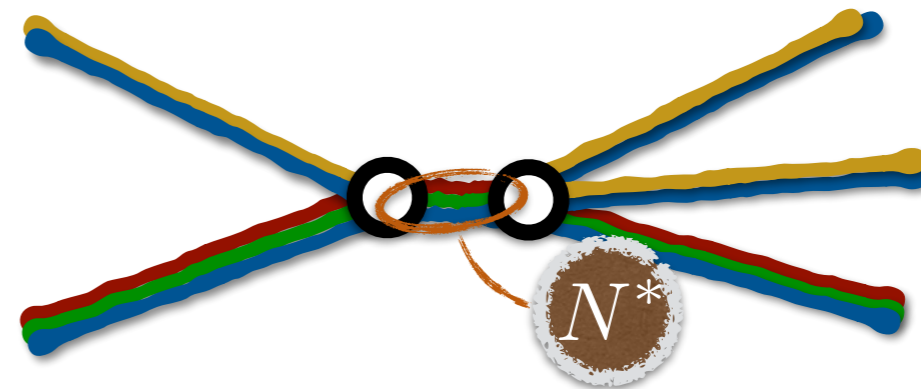


Potential applications...

Studying three-particle resonances

$$\omega(782) \rightarrow \pi\pi\pi$$

$$N(1440) \rightarrow N\pi, N\pi\pi$$

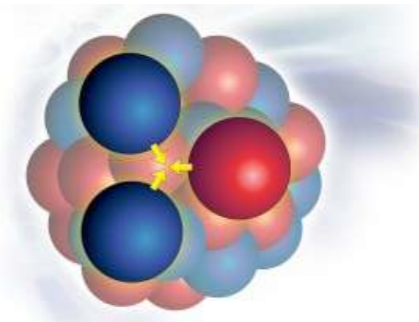


Calculating weak decay amplitudes and form factors

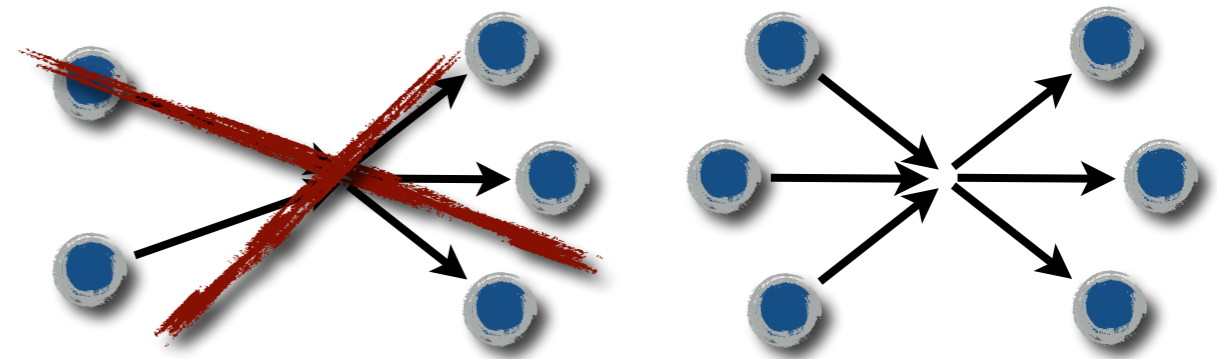
$$K \rightarrow \pi\pi\pi$$

Determining three-body interactions

NNN three-body forces needed as EFT input for studying larger nuclei and nuclear matter



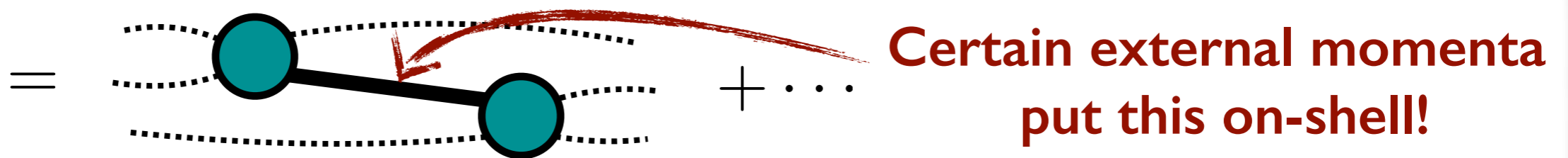
We begin by considering identical scalar particles



For now we turn off two-to-three scattering using a symmetry

Three-to-three amplitude has kinematic singularities

$i\mathcal{M}_{3\rightarrow 3} \equiv$ fully connected correlator with six external legs amputated and projected on shell



Three-to-three amplitude has more degrees of freedom

12 momentum components
-10 Poincaré generators

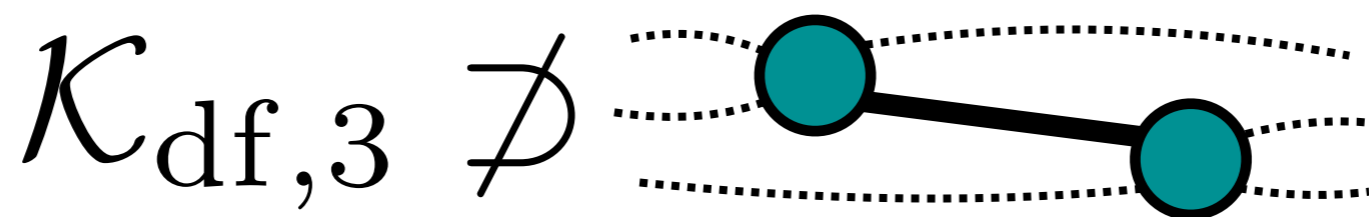
2 degrees of freedom

18 momentum components
-10 Poincaré generators

8 degrees of freedom

How can we extract a singular, eight-coordinate function using finite-volume energies?

Spectrum depends on a modified quantity with singularities removed

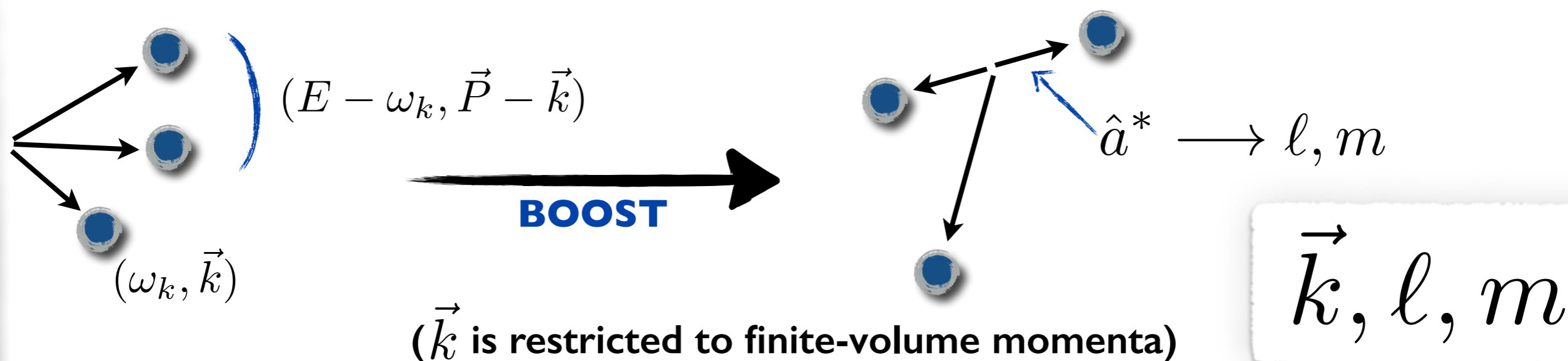


df stands for “divergence free”

Same degrees of freedom as \mathcal{M}_3 } Smooth, real function (easier to extract)

Relation to \mathcal{M}_3 is known (depends only on on-shell \mathcal{M}_2)

Degrees of freedom encoded in an extended matrix space



New skeleton expansion

Recall for two particles we started with a “skeleton expansion”

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

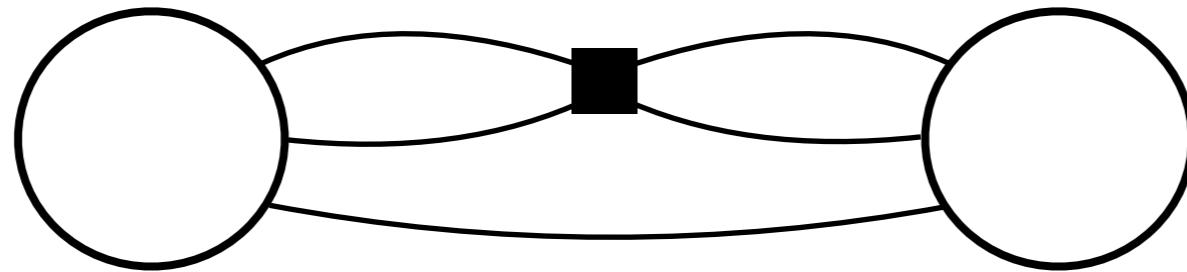
The diagrammatic expansion for $C_L(P)$ consists of three terms. Each term shows a sequence of circles representing particles. The first term has two circles, \mathcal{O}^\dagger and \mathcal{O} , connected by two arcs with two black dots in the middle. The second term has three circles, \mathcal{O}^\dagger , iK , and \mathcal{O} , with the same two-dot structure between \mathcal{O}^\dagger and iK , and between iK and \mathcal{O} . The third term has four circles, \mathcal{O}^\dagger , iK , iK , and \mathcal{O} , with the same two-dot structure between adjacent circles. Dotted lines enclose the two-dot structures in each diagram.

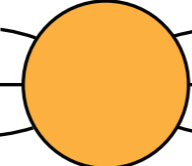
So now we need the same for three...

$$C_L(E, \vec{P}) \stackrel{?}{=} \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The diagrammatic expansion for $C_L(E, \vec{P})$ is shown with a question mark. It consists of three terms. Each term shows a sequence of circles. The first term has two white circles connected by two arcs. The second term has three circles: two white circles and one orange circle in the middle, all connected by two arcs. The third term has four circles: two white circles and two orange circles in the middle, all connected by two arcs. Dotted lines enclose the two arcs in each diagram.

No!... We must also accommodate diagrams like

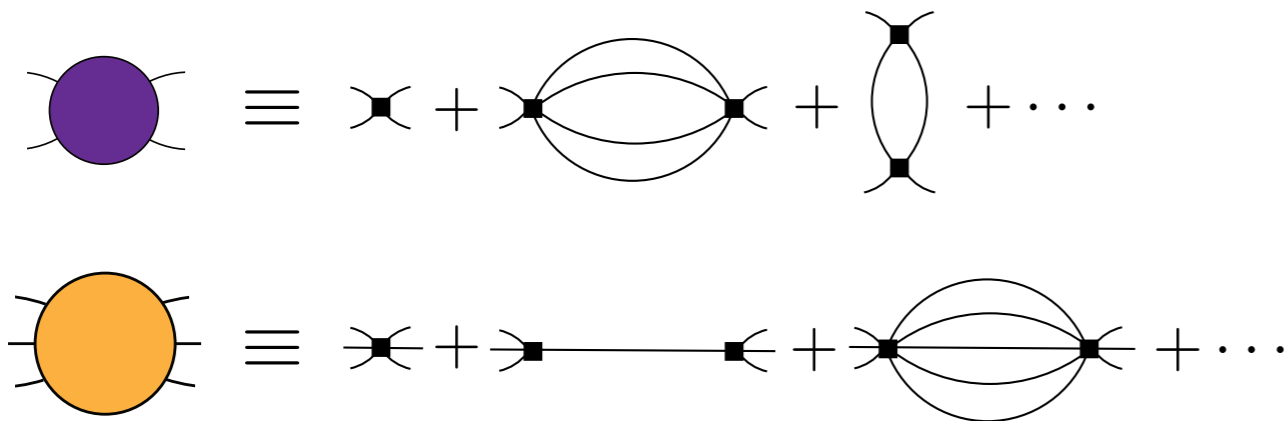


Disconnected diagrams in  lead to singularities that invalidate the derivation

New skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots \\
 & + \dots \\
 & + \text{Diagram 12} + \text{Diagram 13} + \dots
 \end{aligned}$$

Kernel definitions:



- All lines are fully dressed propagators
- Boxes represent sums over finite-volume momenta
- Kernels may contain fixed poles

Basic approach

1. Work out the three particle skeleton expansion

$$C_L(E, \vec{P}) = \begin{array}{l} \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\ \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\ \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \end{array}$$

2. Break diagrams into finite- and infinite-volume parts

3. Organize and sum terms to identify
infinite-volume observables

Result

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) - A' F_3 \frac{1}{1 + \mathcal{K}_{\text{df},3} F_3} A$$

- Looks similar to the two-particle case
- All quantities defined with PV-pole prescription
- F_3 depends on finite-volume and two-to-two scattering

Quantization condition

At fixed (L, \vec{P}) , finite-volume energies are solutions to

$$\det_{k,\ell,m} \left[\mathcal{K}_{\text{df},3}^{-1} + F_3 \right] = 0$$

$F_3 \equiv$ matrix that depends on geometric functions and $\mathcal{M}_{2 \rightarrow 2}$.

MTH and Sharpe (2014)

(1). Use two-particle q.c. to constrain \mathcal{M}_2 and determine $F_3(E, \vec{P}, L)$.

$$\det[\mathcal{M}_2^{-1} + F_2] = 0 \longrightarrow \mathcal{M}_2 \longrightarrow F_3(E, \vec{P}, L)$$

(2). Use decomposition + parametrization to express $\mathcal{K}_{\text{df},3}(E^*)$ in terms of α_i .

$$\mathcal{K}_{\text{df},3}(E^*, \Omega'_3, \Omega_3) \approx \mathcal{K}_{\text{df},3}[\alpha_1, \dots, \alpha_N] \longleftarrow \text{Recall, this is a real, smooth function}$$

(3). Use three-particle q.c. with finite-volume energies to determine $\mathcal{K}_{\text{df},3}(E^*)$.

$$\det[\mathcal{K}_{\text{df},3}^{-1} + F_3] = 0 \longrightarrow \mathcal{K}_{\text{df},3}(E^*) \checkmark$$

Quantization condition

At fixed (L, \vec{P}) , finite-volume energies are solutions to

$$\det_{k,\ell,m} \left[\mathcal{K}_{\text{df},3}^{-1} + F_3 \right] = 0$$

$F_3 \equiv$ matrix that depends on **geometric functions** and $\mathcal{M}_{2 \rightarrow 2}$.
MTH and Sharpe (2014)

All of the complication is buried inside F_3

$$F_3 = \frac{F}{6\omega L^3} - \frac{F}{2\omega L^3} \frac{1}{1 + \mathcal{M}_{2,L} G} \mathcal{M}_{2,L} F$$

These are all matrices with indices

momentum of
one particle

$$\vec{k} = \frac{2\pi\vec{n}}{L}$$



angular momentum
of the other two

$$\ell, m$$

F and **G** are geometric functions

$\mathcal{M}_{2,L}$ depends on **F** and \mathcal{M}_2

Relating $\mathcal{K}_{\text{df},3}$ to \mathcal{M}_3

First we modify $C_L(E, \vec{P})$ to define $i\mathcal{M}_{L,3}$

$$\begin{aligned} C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\ & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\ & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\ & + \text{Diagram 10} + \text{Diagram 11} + \dots \\ & + \dots \\ & + \text{Diagram 12} + \text{Diagram 13} + \dots \end{aligned}$$

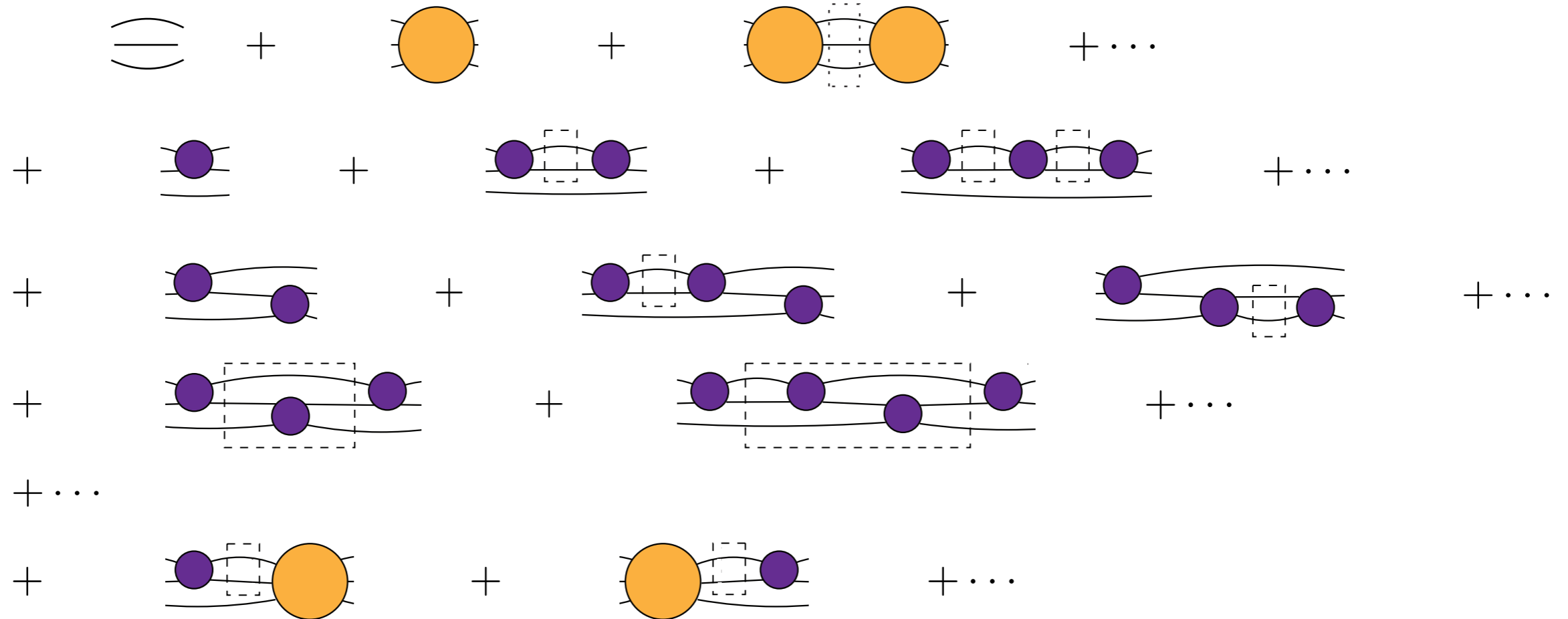
The diagrams in the equation represent terms in a series expansion of $C_L(E, \vec{P})$. Each diagram consists of two external white circles connected by two lines. The internal structure is as follows:

- Diagram 1:** Two white circles connected by two lines, with a dashed box around the entire structure.
- Diagram 2:** Two white circles connected by two lines, with a single orange circle in the middle. A dashed box is around the orange circle.
- Diagram 3:** Two white circles connected by two lines, with two orange circles in the middle. A dashed box is around the second orange circle.
- Diagram 4:** Two white circles connected by two lines, with one purple circle in the middle. A dashed box is around the purple circle.
- Diagram 5:** Two white circles connected by two lines, with two purple circles in the middle. A dashed box is around the second purple circle.
- Diagram 6:** Two white circles connected by two lines, with three purple circles in the middle. A dashed box is around the second purple circle.
- Diagram 7:** Two white circles connected by two lines, with two purple circles in the middle. A dashed box is around the second purple circle.
- Diagram 8:** Two white circles connected by two lines, with three purple circles in the middle. A dashed box is around the second purple circle.
- Diagram 9:** Two white circles connected by two lines, with four purple circles in the middle. A dashed box is around the second purple circle.
- Diagram 10:** Two white circles connected by two lines, with three purple circles in the middle. A dashed box is around the second purple circle.
- Diagram 11:** Two white circles connected by two lines, with four purple circles in the middle. A dashed box is around the second purple circle.
- Diagram 12:** Two white circles connected by two lines, with one purple circle in the middle and one orange circle to its right. A dashed box is around the purple circle.
- Diagram 13:** Two white circles connected by two lines, with one orange circle in the middle and one purple circle to its right. A dashed box is around the orange circle.

Relating $\mathcal{K}_{\text{df},3}$ to \mathcal{M}_3

First we modify $C_L(E, \vec{P})$ to define $i\mathcal{M}_{L,3}$

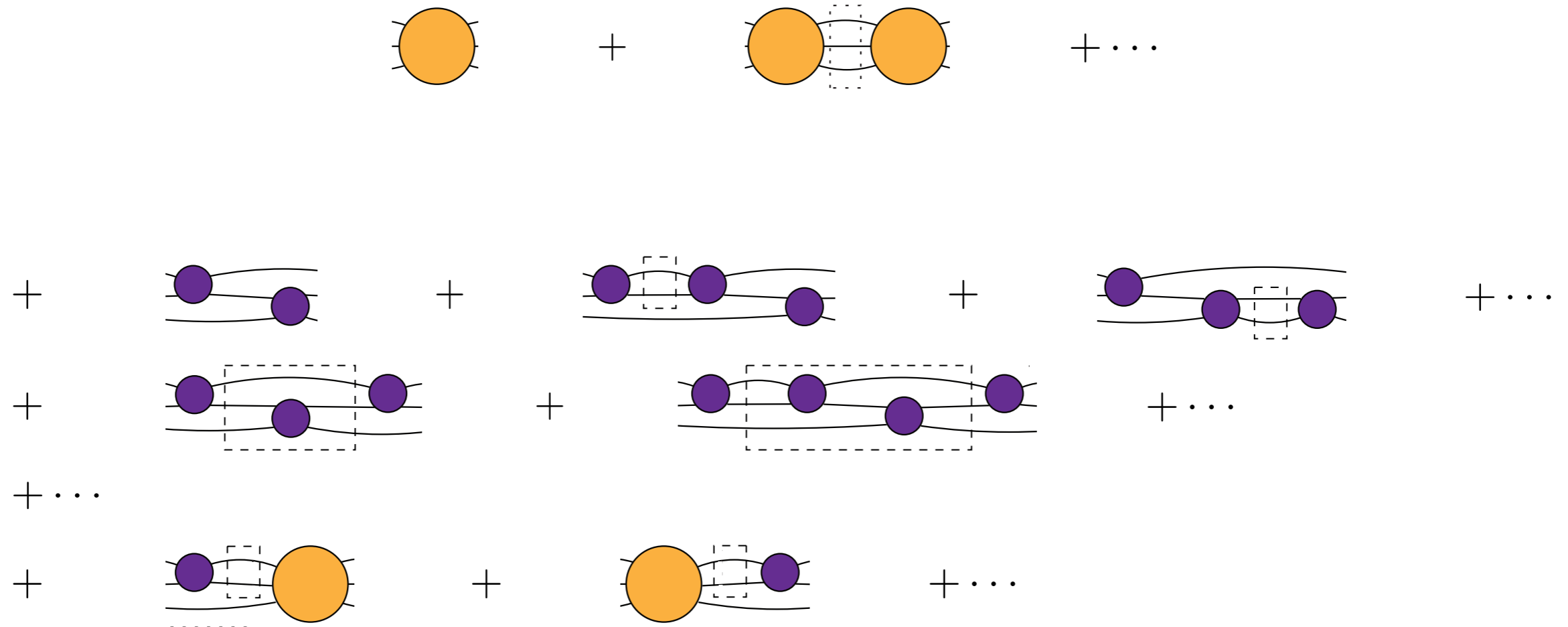
1. Amputate interpolating fields



Relating $\mathcal{K}_{\text{df},3}$ to \mathcal{M}_3

First we modify $C_L(E, \vec{P})$ to define $i\mathcal{M}_{L,3}$

1. Amputate interpolating fields
2. Drop disconnected diagrams



Relating $\mathcal{K}_{\text{df},3}$ to \mathcal{M}_3

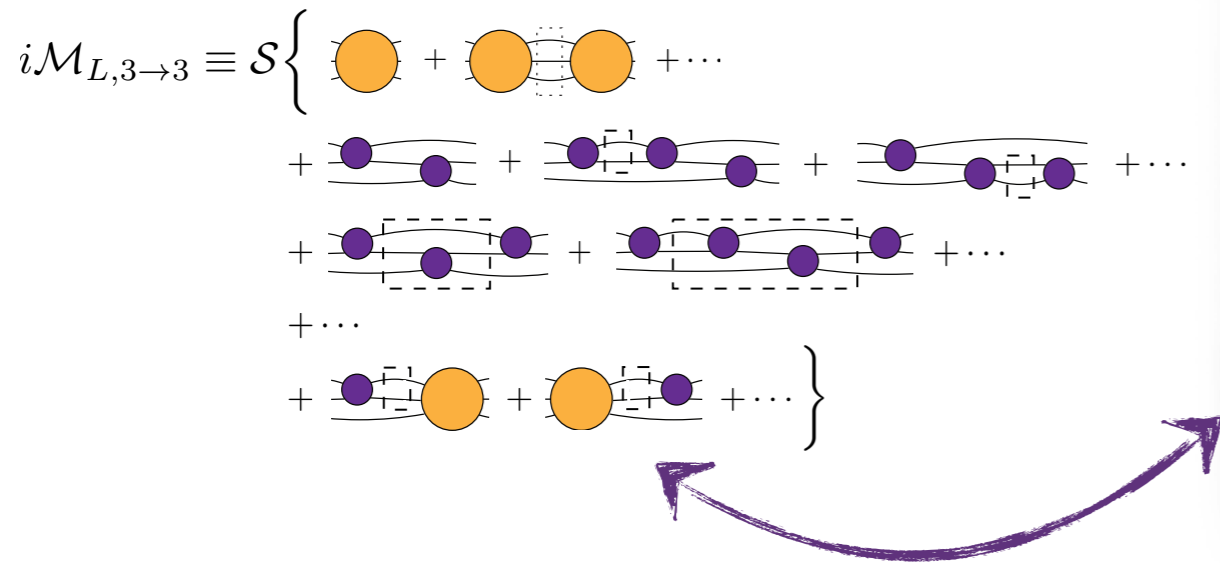
First we modify $C_L(E, \vec{P})$ to define $i\mathcal{M}_{L,3}$

1. Amputate interpolating fields
2. Drop disconnected diagrams
3. Symmetrize

$$i\mathcal{M}_{L,3 \rightarrow 3} \equiv \mathcal{S} \left\{ \begin{array}{l} \text{Orange circle} + \text{Orange circle} \text{---} \text{Orange circle} + \dots \\ + \text{Purple circle} \text{---} \text{Purple circle} + \text{Purple circle} \text{---} \text{Purple circle} \text{---} \text{Purple circle} + \dots \\ + \text{Purple circle} \text{---} \text{Purple circle} \text{---} \text{Purple circle} + \text{Purple circle} \text{---} \text{Purple circle} \text{---} \text{Purple circle} \text{---} \text{Purple circle} + \dots \\ + \dots \\ + \text{Purple circle} \text{---} \text{Orange circle} + \text{Orange circle} \text{---} \text{Purple circle} + \dots \end{array} \right\}$$

Relating $\mathcal{K}_{\text{df},3}$ to \mathcal{M}_3

Combined with our earlier analysis
this gives a matrix equation



$$\mathcal{M}_{L,3} = \mathcal{S} \left[\mathcal{D}_L + \mathcal{L}_L \frac{1}{\mathcal{K}_{\text{df},3}^{-1} + F_3} \mathcal{R}_L \right]$$

$$\mathcal{L}_L = \mathcal{X} F_3, \quad \mathcal{R}_L = F_3 \mathcal{X},$$

$$\mathcal{D}_L = -\mathcal{X} [F_3 - F_3|_{G \rightarrow 0}] \mathcal{X}$$

with the “amputation matrix” $\mathcal{X} = \left(\frac{F}{2\omega L^3} \right)^{-1}$

With this analytic relation in hand we can...

(a) Set $E \rightarrow E + i\epsilon$, (b) Send $L \rightarrow \infty$, (c) Send $\epsilon \rightarrow 0^+$.

Leads to an integral equation for the scattering amplitude

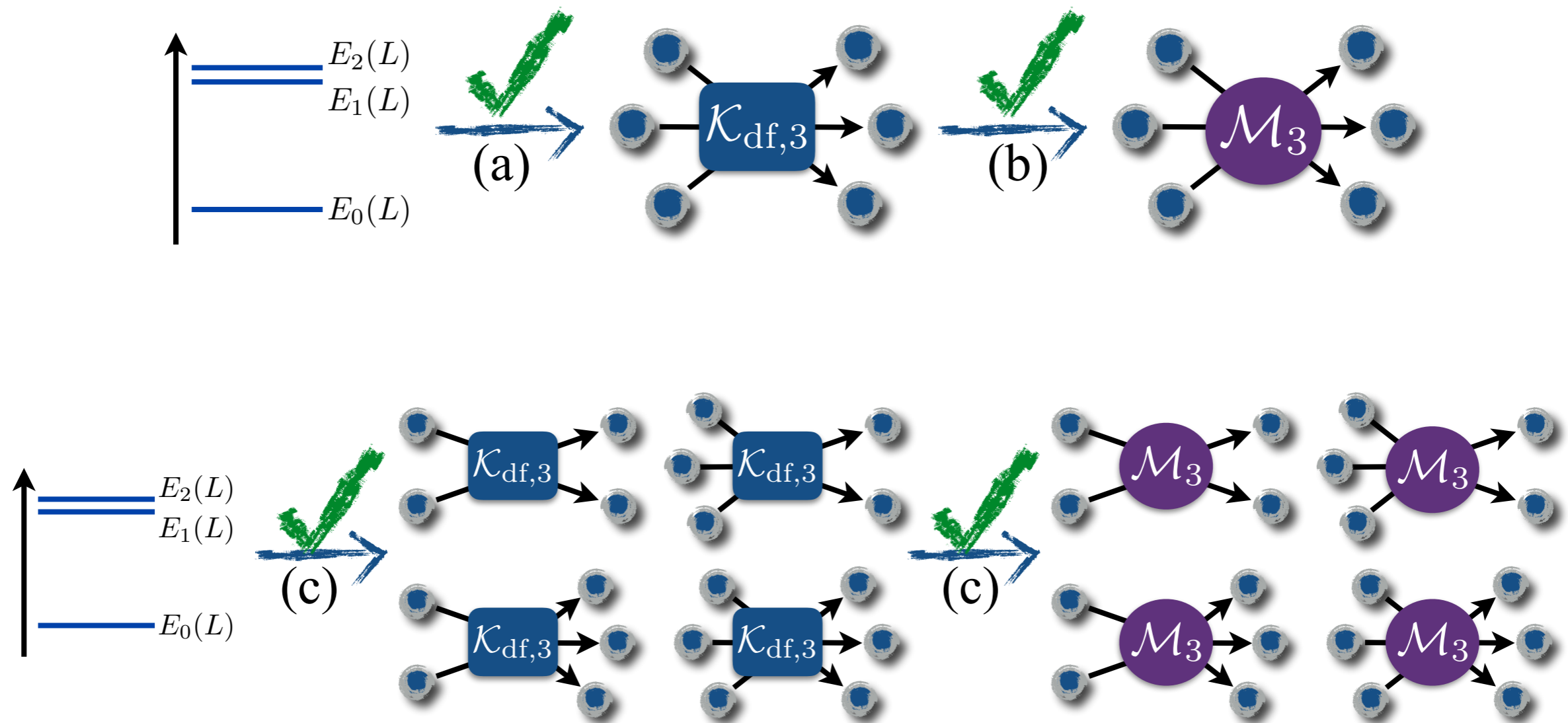
$$\mathcal{M}_3(E^*) = \mathcal{I} [\mathcal{K}_{\text{df},3}(E^*), \mathcal{M}_2]$$

Fixed total energy, manifestly convergent, on-shell only, no reference to EFT,
takes care of unitarity and singularities, useful independent of finite-volume physics?

MTH and Sharpe (2015)

Current status

Model- & EFT-independent relation between
finite-volume energies and **relativistic** two-and-three particle scattering



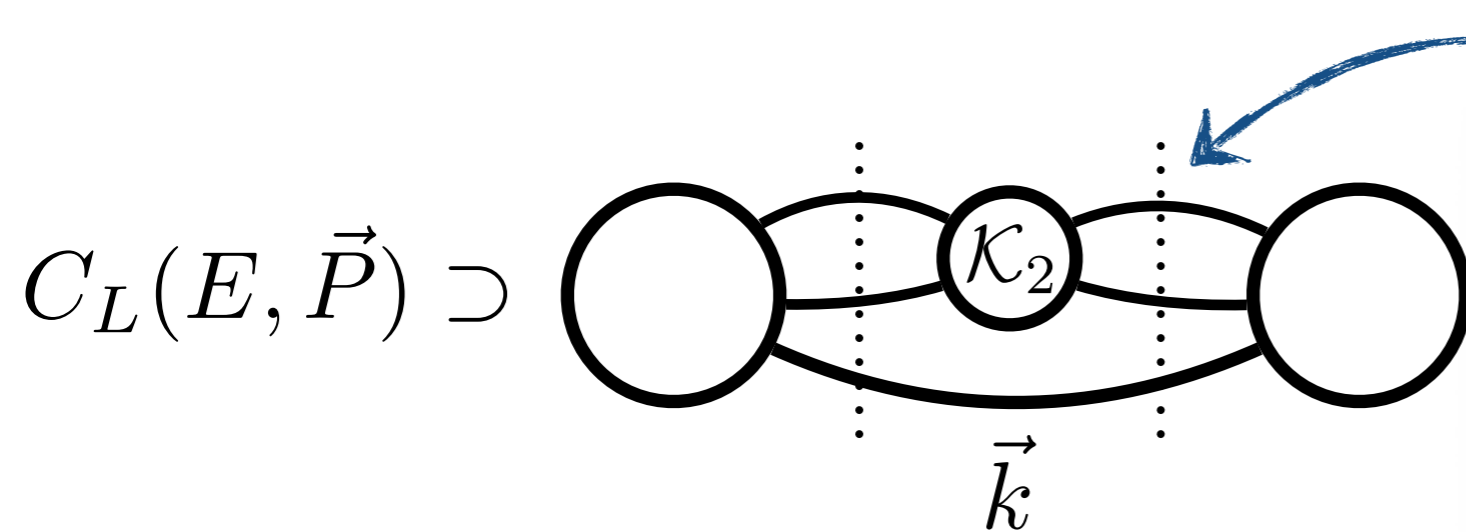
(a),(b) *MTH and Sharpe (2015),(2016)*

(c) *Briceño, MTH, Sharpe (2017)*

Smooth cutoff function

$\mathcal{K}_{\text{df},3}$ and F_3 depend on a smooth cutoff function

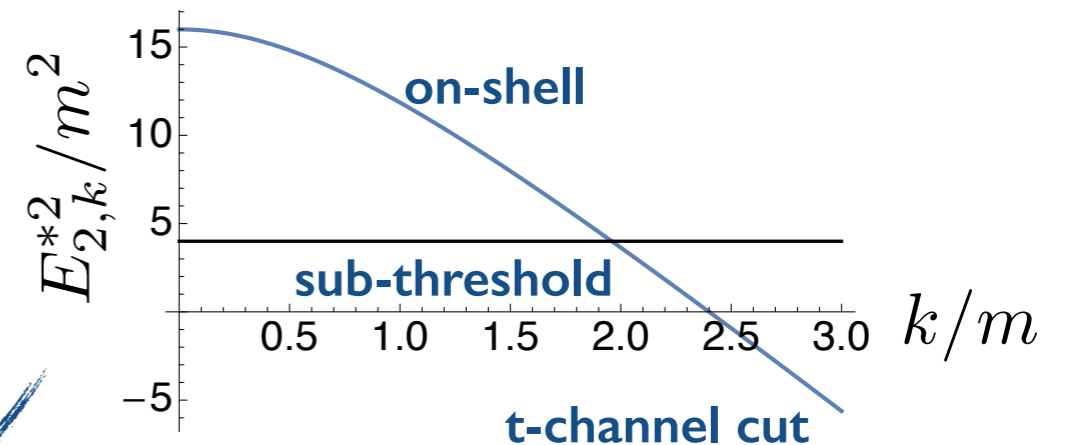
To see why, consider one of the contributions to C_L ...



How do we define this on-shell cut?

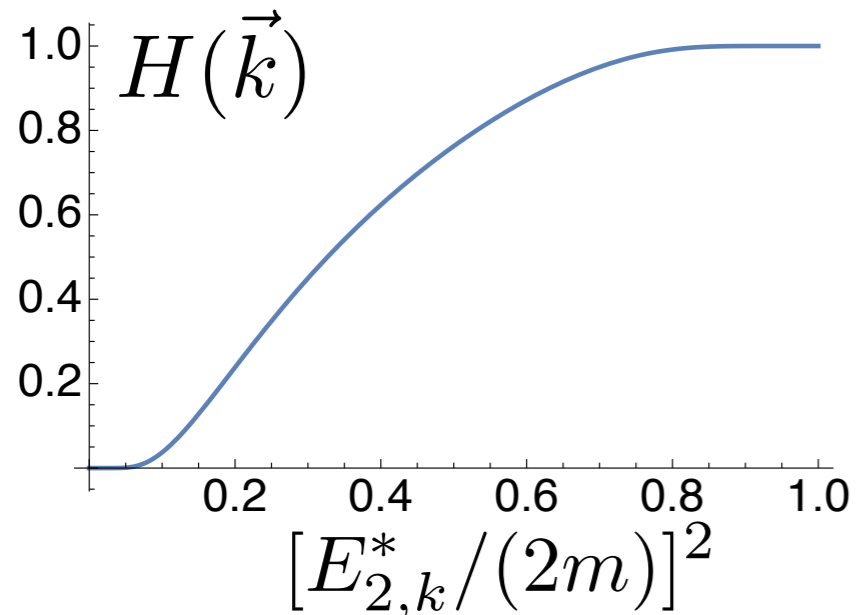
Energy of top two particles is:

$$E_{2,k}^{*2} = (E - \omega_k)^2 - (\vec{P} - \vec{k})^2$$



To keep on-shell states and avoid spurious off-shell contributions...

Cuts are defined with



🤔 Is this really necessary? 🤔

We choose to sum subdiagrams into \mathcal{K}_2

On-shell \mathcal{K}_2 gives important volume effects
No important effects far below threshold

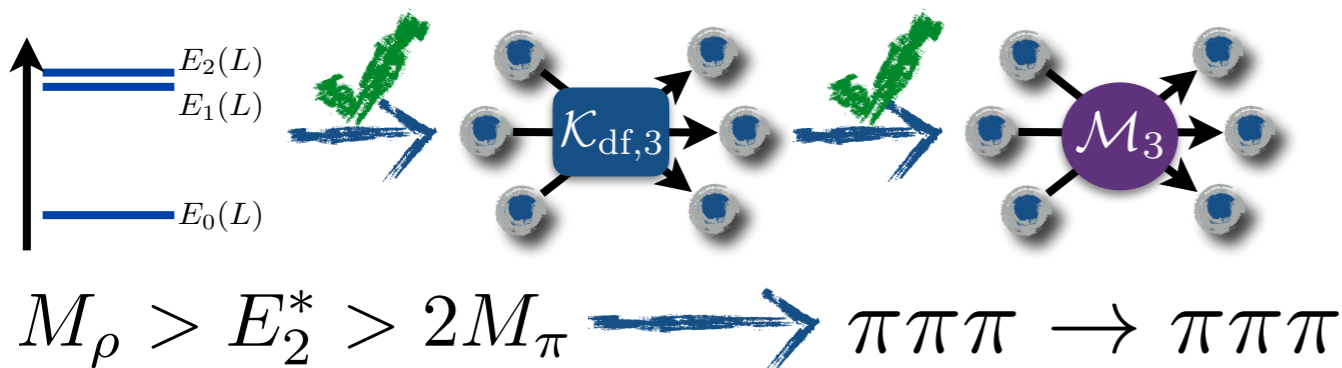
Must connect the two regions

Important limitation

Current formalism requires no poles in $\mathcal{K}_2 \dots$ **Derivation assumes**

$$\frac{1}{L^3} \sum_{\vec{k}} \text{Diagram} = \int_{\vec{k}} \text{Diagram}$$

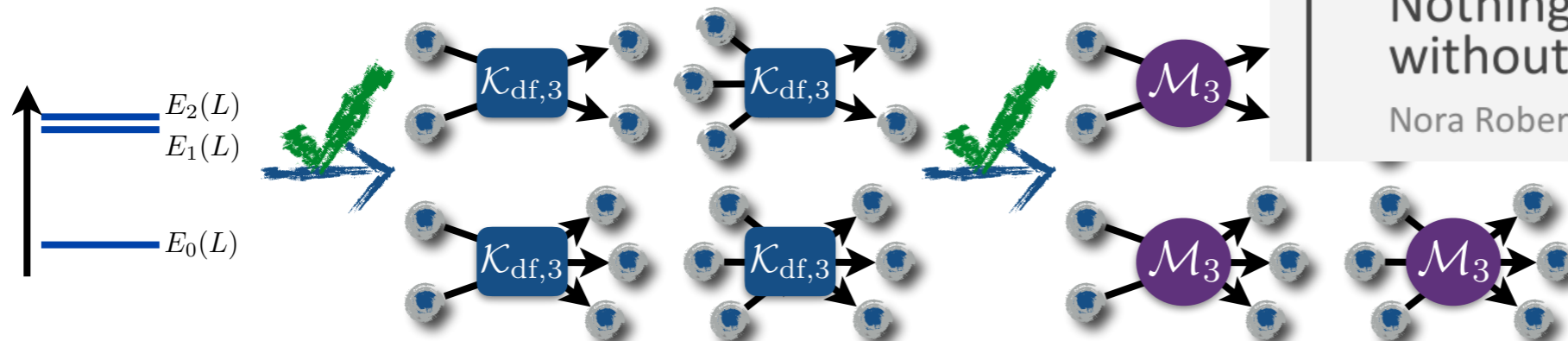
Given that we are seeking an EFT-independent mapping...
Is it intuitive that \mathcal{K}_2 poles need special treatment?



Need to bridge the gap

Update
We now have a complete derivation of formalism that includes K_2 poles.
Needs further checks.
Briceño, MTH, Sharpe (underway)

$$E_2^* > 2M_\pi \gg M_\rho \longrightarrow \rho\pi \longrightarrow$$

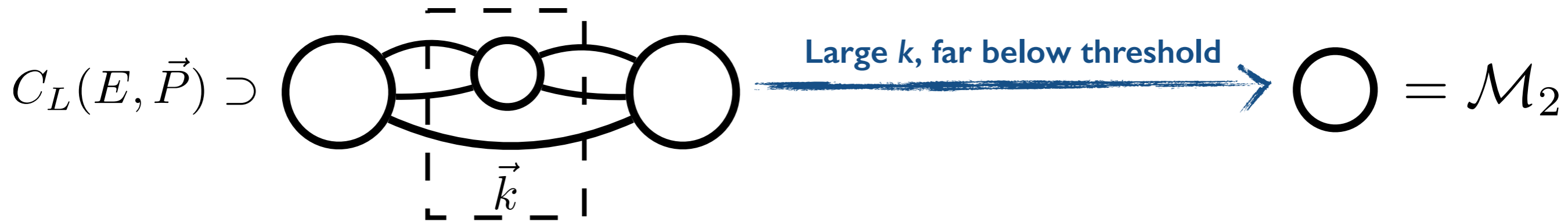


Nothing worthwhile is ever without complications.

Nora Roberts

The most technical detail of all...

Far below threshold there is no ambiguity about which two-to-two scattering quantity appears in C_L



Reason: $\frac{1}{L^3} \sum_{\vec{k}} \frac{1}{(2\omega_k)^2 (E_{\text{sub}} - 2\omega_k)} = \int_{\vec{k}} \frac{1}{(2\omega_k)^2 (E_{\text{sub}} - 2\omega_k)} = \text{Analytic Continuation} \left[\int_{\vec{k}} \frac{1}{(2\omega_k)^2 (E - 2\omega_k + i\epsilon)} \right]$

Upshot is that our subthreshold \mathcal{K}_2 is non-standard

$$\mathcal{K}_2^{-1} \propto p^* \cot \delta(p^*) + [1 - H(\vec{k})] \kappa(p^*)$$

K matrix above threshold, smooth at threshold, interpolates to the amplitude below threshold

 *Why are you telling me this?* 

It is important because our formalism breaks down when there are poles in this definition of \mathcal{K}_2 .

Testing the formalism

Weak interactions: Expand the threshold energy in powers of inverse box length

$$E = 3m + \frac{12\pi a}{mL^3} \left(1 + c_4 \frac{a}{L} + \dots \right) - \frac{\mathcal{M}_{\text{thr}}}{48m^3 L^6} + \dots$$

We reproduce known results through $1/L^5$ and derive a relation at $1/L^6$

Huang and Yang (1957); Beane, Detmold, Savage, (2007); Tan(2007); Sharpe 2017

Note: Relativistic effects enter at $1/L^6$, same order as three-to-three

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Huang and Yang (1957); Beane, Detmold, Savage, (2007); Tan(2007); Sharpe 2017

Note: Relativistic effects enter at $1/L^6$, same order as three-to-three

Strong interactions (unitary limit, $P=0$, s-wave only):

The infinite-volume energy, $E_B \equiv 3m - \frac{\kappa^2}{m}$, is shifted by

$$\Delta E(L) = c |A|^2 \frac{\kappa^2}{m} \frac{1}{(\kappa L)^{3/2}} e^{-2\kappa L/\sqrt{3}} + \dots \quad c = -96.351 \dots$$

geometric constant

“normalization correction factor”

Meißner, Rios and Rusetsky, (2015)

Our formalism gives a relation between **scattering** and **energies**. So we substitute...

$$\mathcal{M}_3 \sim -\frac{\Gamma \bar{\Gamma}}{E^2 - E_B^2} \quad \mathcal{M}_2 = -\frac{16\pi E_2^*}{ip^*} \quad \text{and study the lowest level}$$

We reproduce the exponent, leading power and overall constant using our relativistic formalism

Usability?

“Despite this success, the quantization condition in these papers is not yet given in a form suitable for the analysis of the real lattice data”

Hammer, Pang and Rusetsky (2017)

We were motivated to challenge this claim...

We find that the “degree of usability” is comparable between the two approaches, provided one applies similar approximations.

How do we make the two-particle formalism usable?

Truncate partial waves

$$\mathcal{M}_2(E_2^*, \theta^*) \approx \sum_{\ell=0}^N P_\ell(\cos \theta^*) \mathcal{M}_{2,\ell}(E_2^*)$$

Single partial wave

$$\mathcal{M}_2(E_2^*, \theta^*) \approx \mathcal{M}_{2,s}(E_2^*) \propto \frac{1}{p^* \cot \delta_0(p^*) - ip^*}$$

Is there a three-particle analog?

$$\mathcal{K}_{\text{df},3}(E^*, \Omega'_3, \Omega_3) \approx \sum_{n=0}^N \mathcal{P}_n(\Omega'_3, \Omega_3) \mathcal{K}_{\text{df},3,n}(E^*)$$

$$\mathcal{K}_{\text{df},3}(E^*, \Omega'_3, \Omega_3) \approx \mathcal{K}_{\text{df},3}^{\text{iso}}(E^*) \in \mathbb{R}$$

At fixed energy $\frac{\mathcal{M}_2(E_2^*, \theta^*)}{\mathcal{K}_{\text{df},3}(E^*, \Omega'_3, \Omega_3)}$ **is a smooth function on a compact space.**

Further investigation is needed to understand suppression of higher $\mathcal{K}_{\text{df},3,n}(E^*)$.

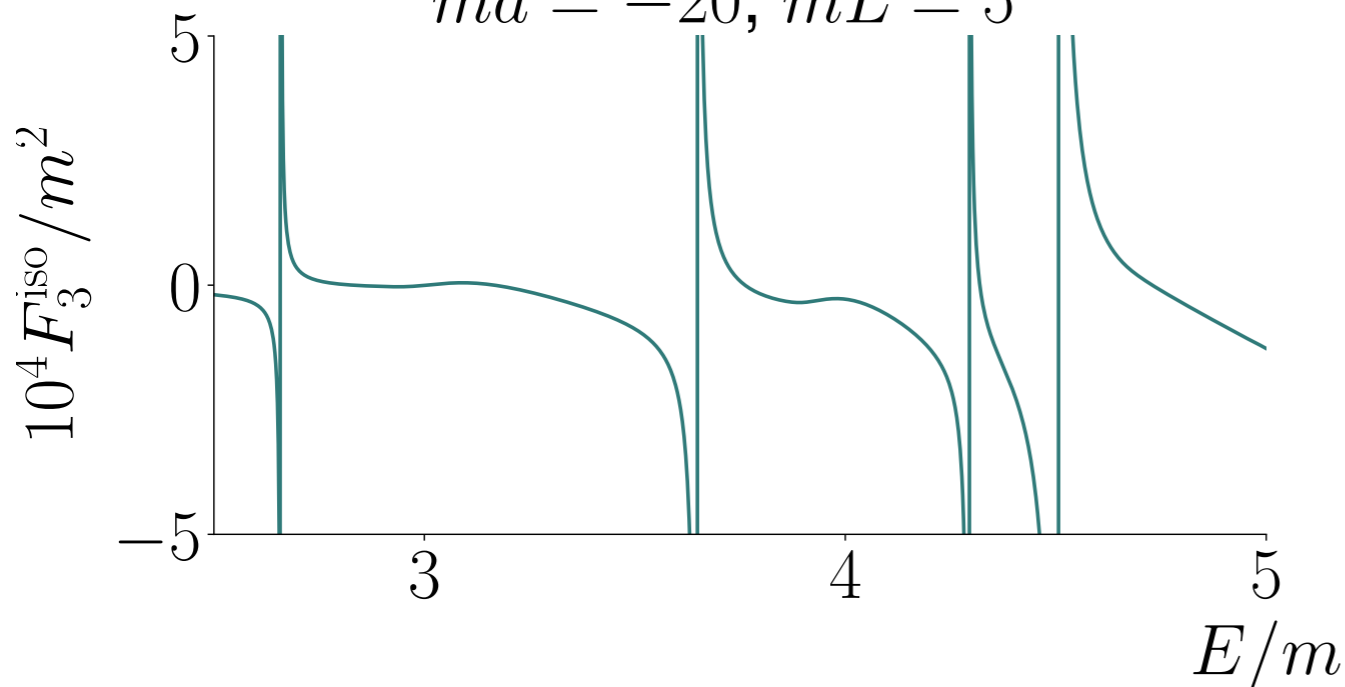
Numerics (keeping only s-wave and $\mathcal{K}_{\text{df},3}(E^*, \Omega'_3, \Omega_3) \approx \mathcal{K}_{\text{df},3}^{\text{iso}}(E^*)$)

$$1/\mathcal{K}_{\text{df},3}^{\text{iso}}(E^*) = -F_3^{\text{iso}}[E, \vec{P}, L, \mathcal{M}_2^s] \quad \mathcal{M}_3(E^*, \Omega'_3, \Omega_3) = \mathcal{S} \left[\mathcal{D} + \mathcal{L} \frac{1}{1/\mathcal{K}_{\text{df},3}^{\text{iso}} + F_{3,\infty}^{\text{iso}}} \mathcal{R} \right]$$

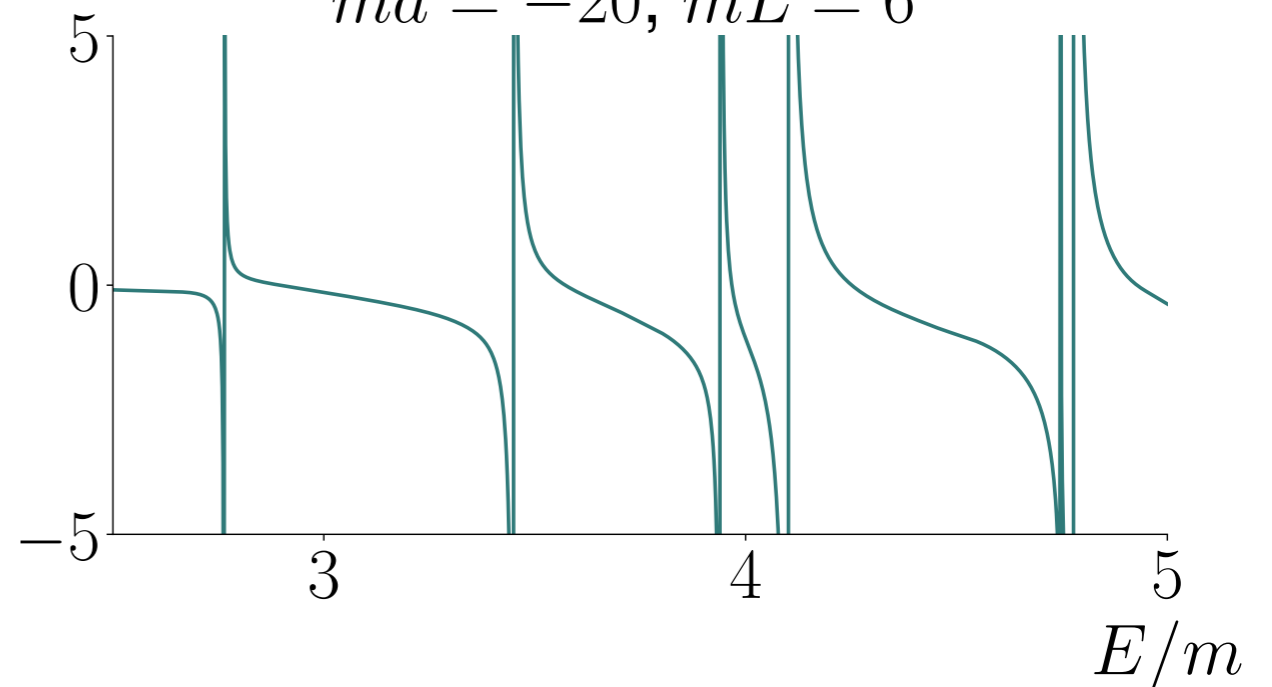
For the numerical approach we restrict attention to... $p^* \cot \delta_0(p^*) = -\frac{1}{a}$, $\vec{P} = 0$

Then the quantization condition is based on $F_3^{\text{iso}}(E, L, a)$

$$ma = -20, mL = 5$$



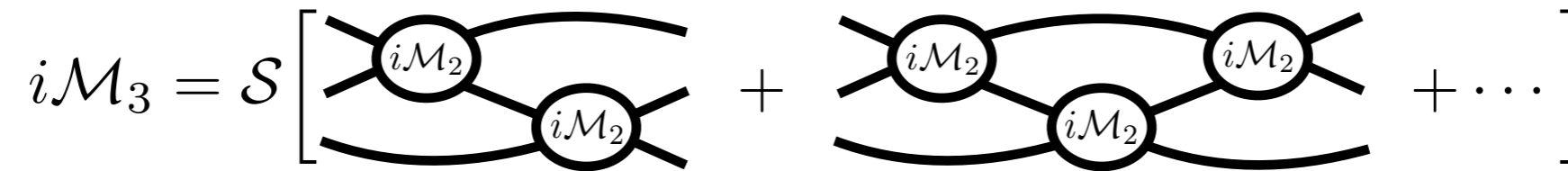
$$ma = -20, mL = 6$$



Finite-volume energies wherever these curves intersect $-1/\mathcal{K}_{\text{df},3}^{\text{iso}}(E)$

$\mathcal{K}_{\text{df},3}^{\text{iso}}(E) = 0$ solutions

- Provides a useful benchmark: Deviations measure three-particle physics
- Meaning for three-to-three scattering is clear

$$i\mathcal{M}_3 = \mathcal{S} \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + \dots \right]$$


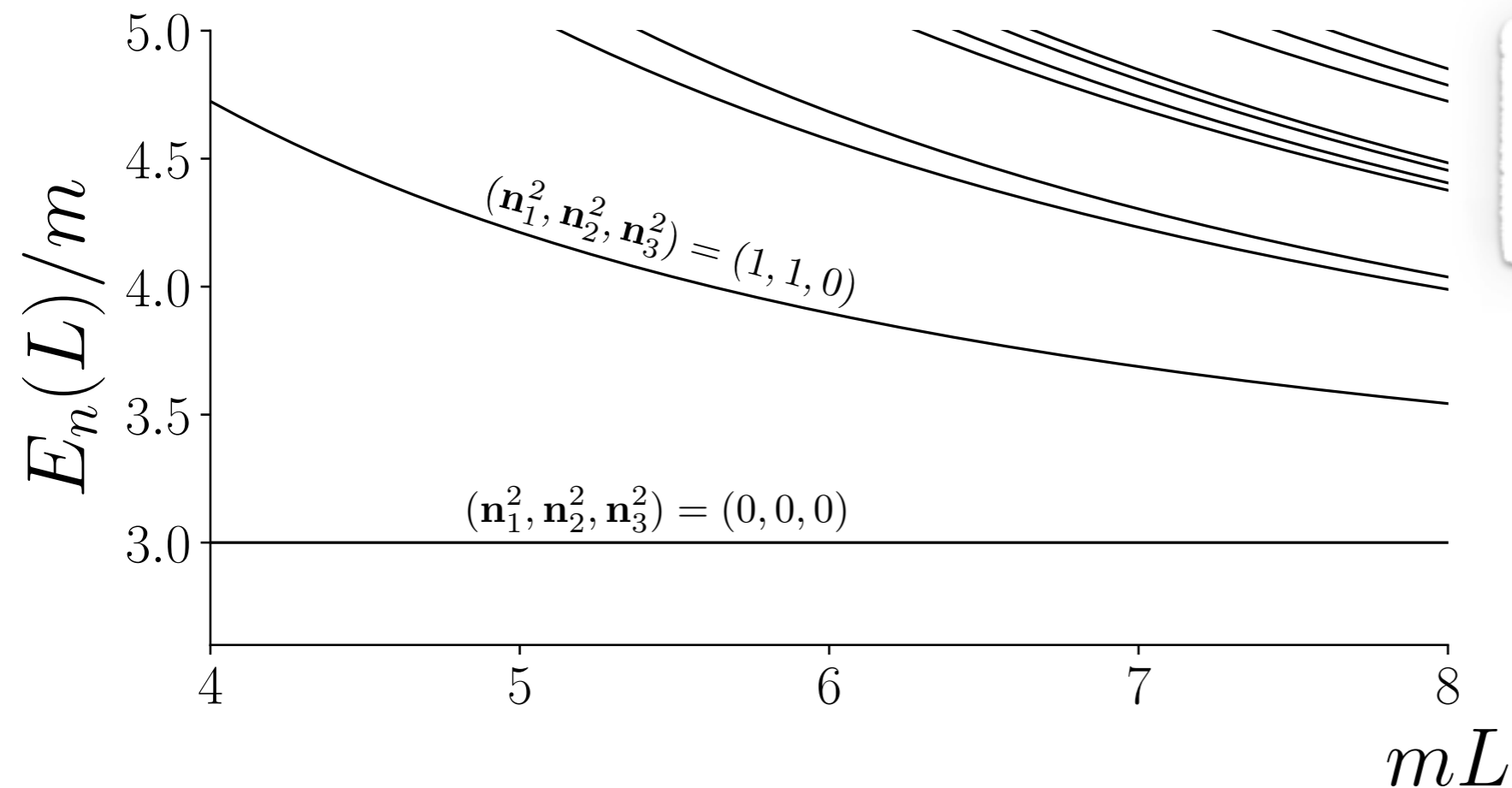
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The diagram shows two Feynman diagrams for three-particle scattering. The first diagram has two external lines on the left and two on the right, with two internal lines forming a loop. The second diagram is similar but with an additional internal line. Both diagrams contain a circle labeled $i\mathcal{M}_2$.

Aside: Non-interacting states ($\mathcal{M}_2 = \mathcal{M}_3 = 0$)



$$E_n^{\text{non-int}}(L) = \omega_1 + \omega_2 + \omega_3$$

$$\omega_i = \sqrt{m^2 + 4\pi^2 \mathbf{n}_i^2 / L^2}$$

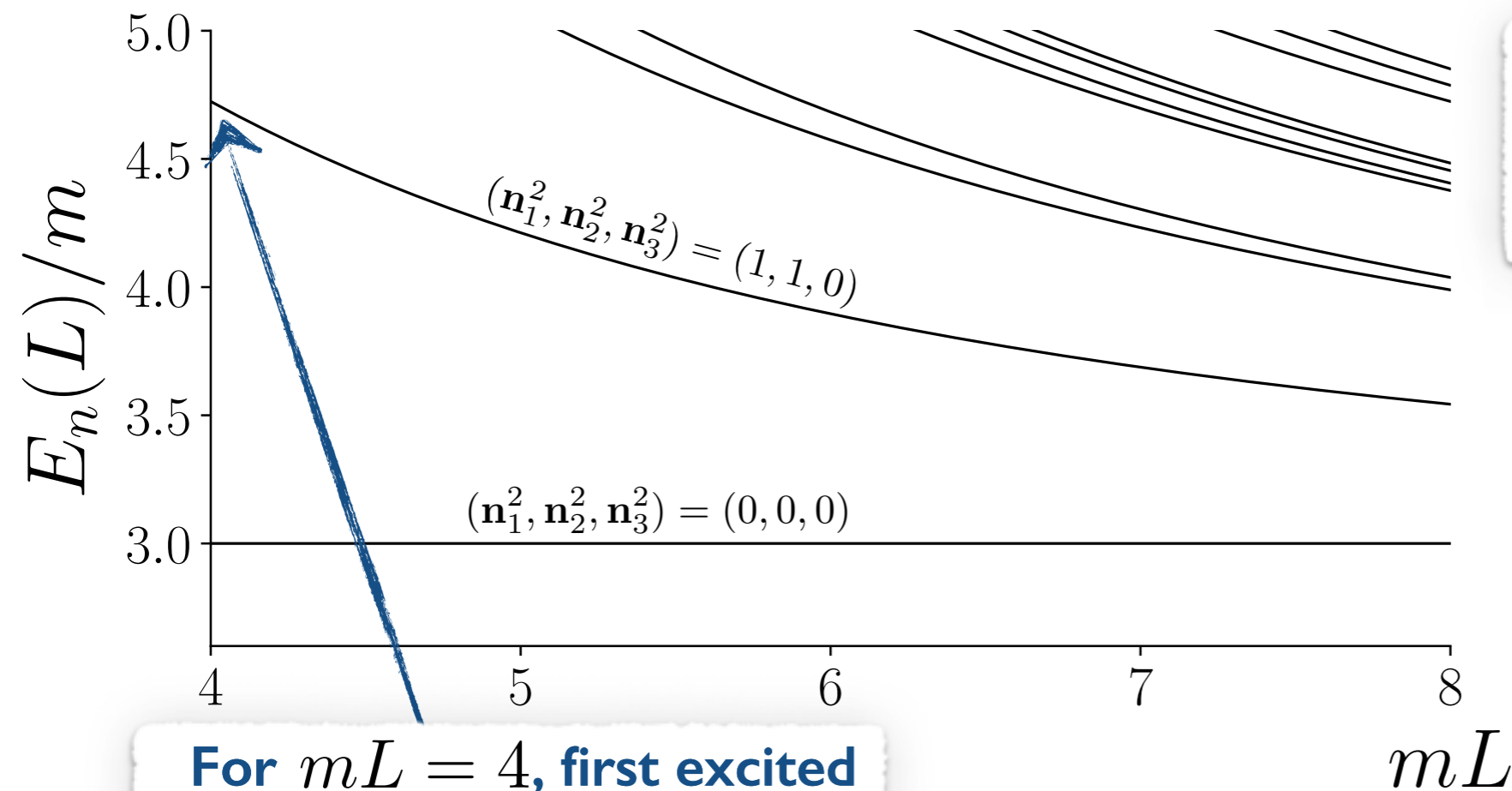
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The diagrams show Feynman-like diagrams for three-particle scattering. The first diagram shows two particles interacting via a vertex $i\mathcal{M}_2$, and then interacting with a third particle via another vertex $i\mathcal{M}_2$. The second diagram shows a similar process with a different internal structure. The diagrams are enclosed in large square brackets with a plus sign and an ellipsis following.

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$$E_n^{\text{non-int}}(L) = \omega_1 + \omega_2 + \omega_3$$

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For $mL = 4$, first excited state is already relativistic

$$\frac{p^2}{m^2} = \left(\frac{2\pi}{mL} \right)^2 \approx 2.46$$

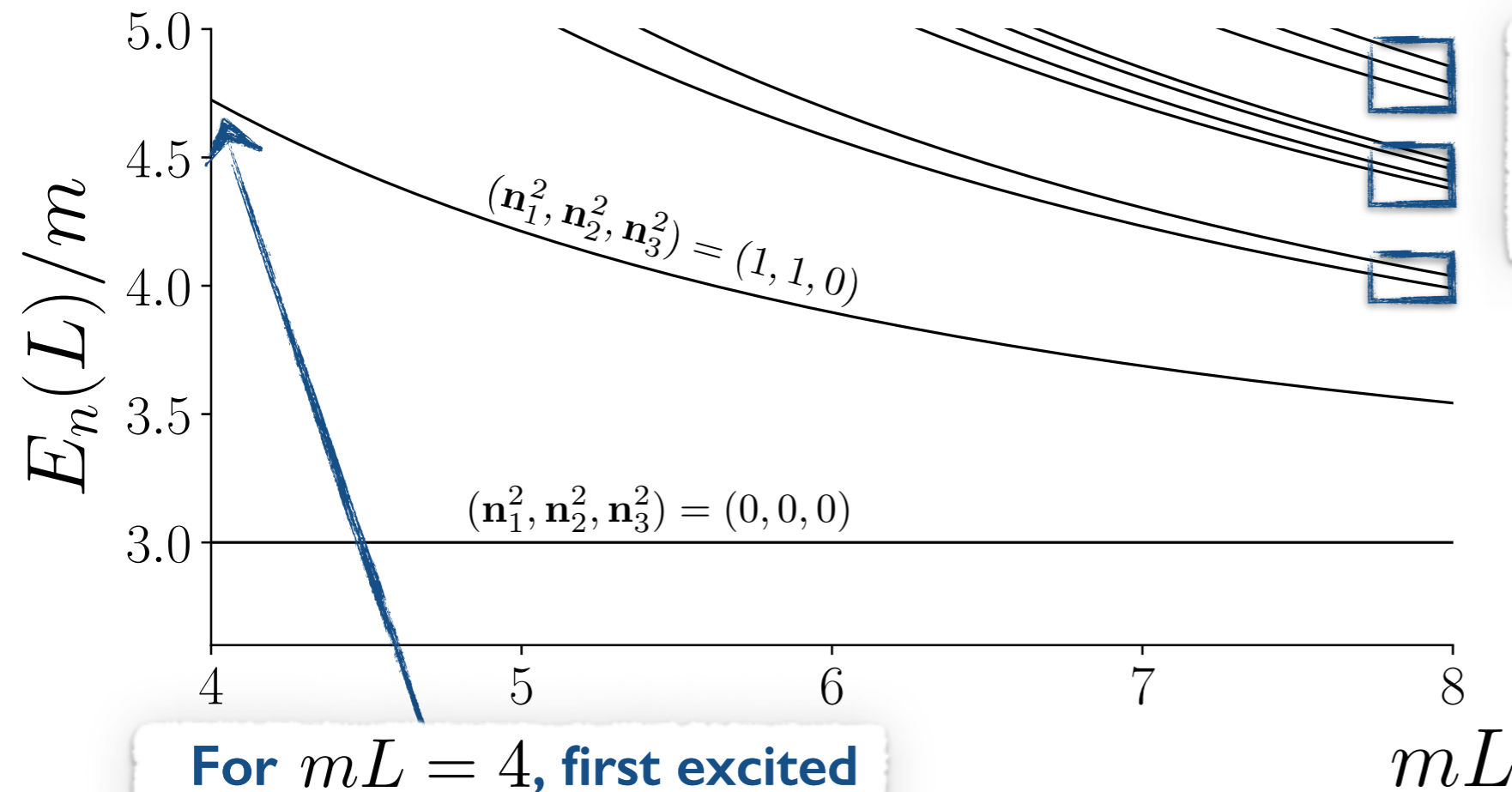
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The diagrams show Feynman-like diagrams for three-particle scattering. The first diagram has two vertices labeled $i\mathcal{M}_2$. The second diagram has three vertices labeled $i\mathcal{M}_2$. The diagrams are connected by plus signs and an ellipsis.

Aside: Non-interacting states ($\mathcal{M}_2 = \mathcal{M}_3 = 0$)



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$$\omega_i = \sqrt{m^2 + 4\pi^2 \mathbf{n}_i^2 / L^2}$$

Why are these states clustered?
Accidental NR degeneracy!

$$E_n^{\text{NR}}(L) = 3m + \frac{2\pi^2}{L^2} (\mathbf{n}_1^2 + \mathbf{n}_2^2 + \mathbf{n}_3^2)$$

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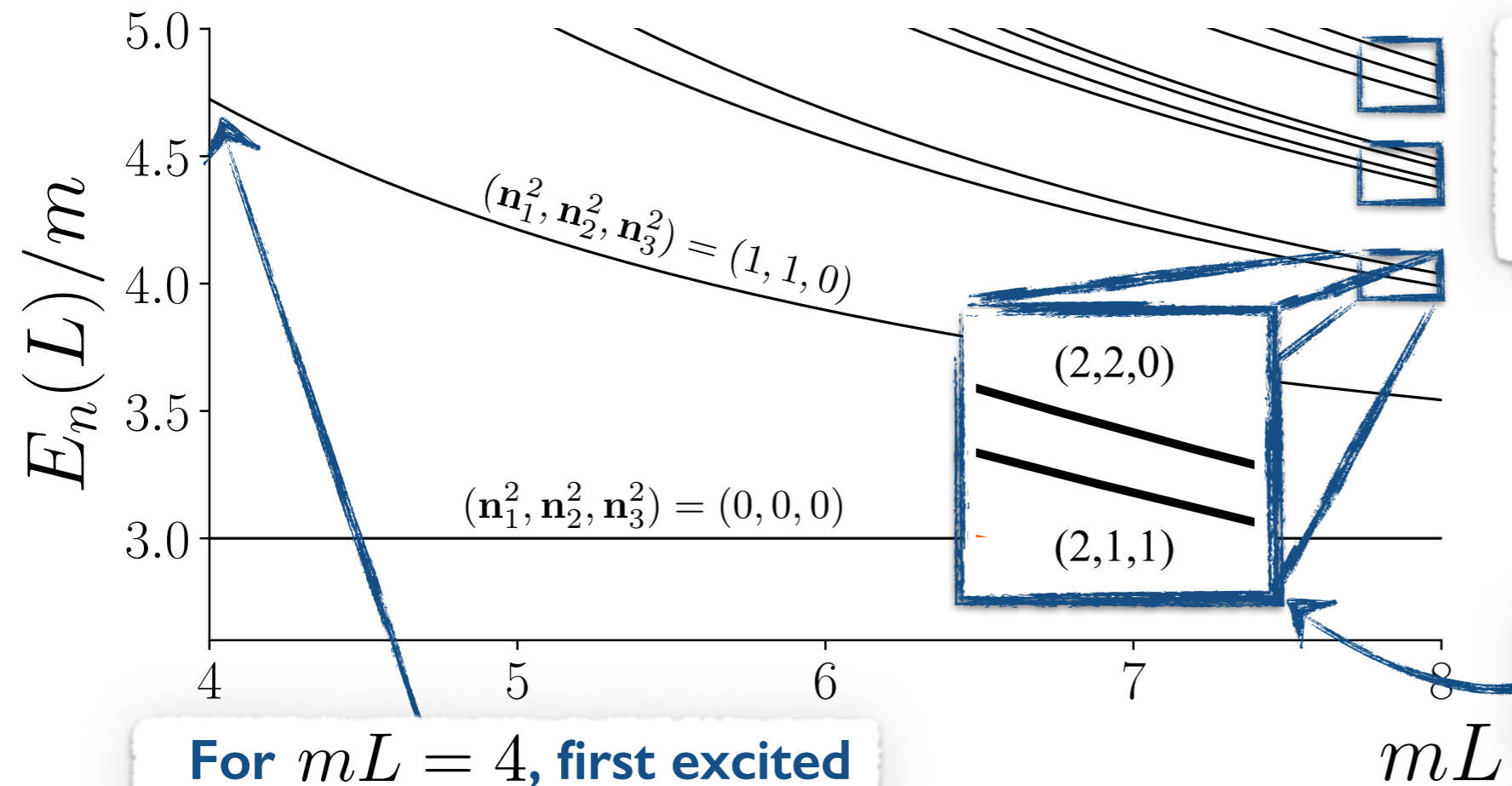
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$$i\mathcal{M}_3 = \mathcal{S} \left[\text{diagram 1} + \text{diagram 2} + \dots \right]$$

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These two states are degenerate in the NR theory

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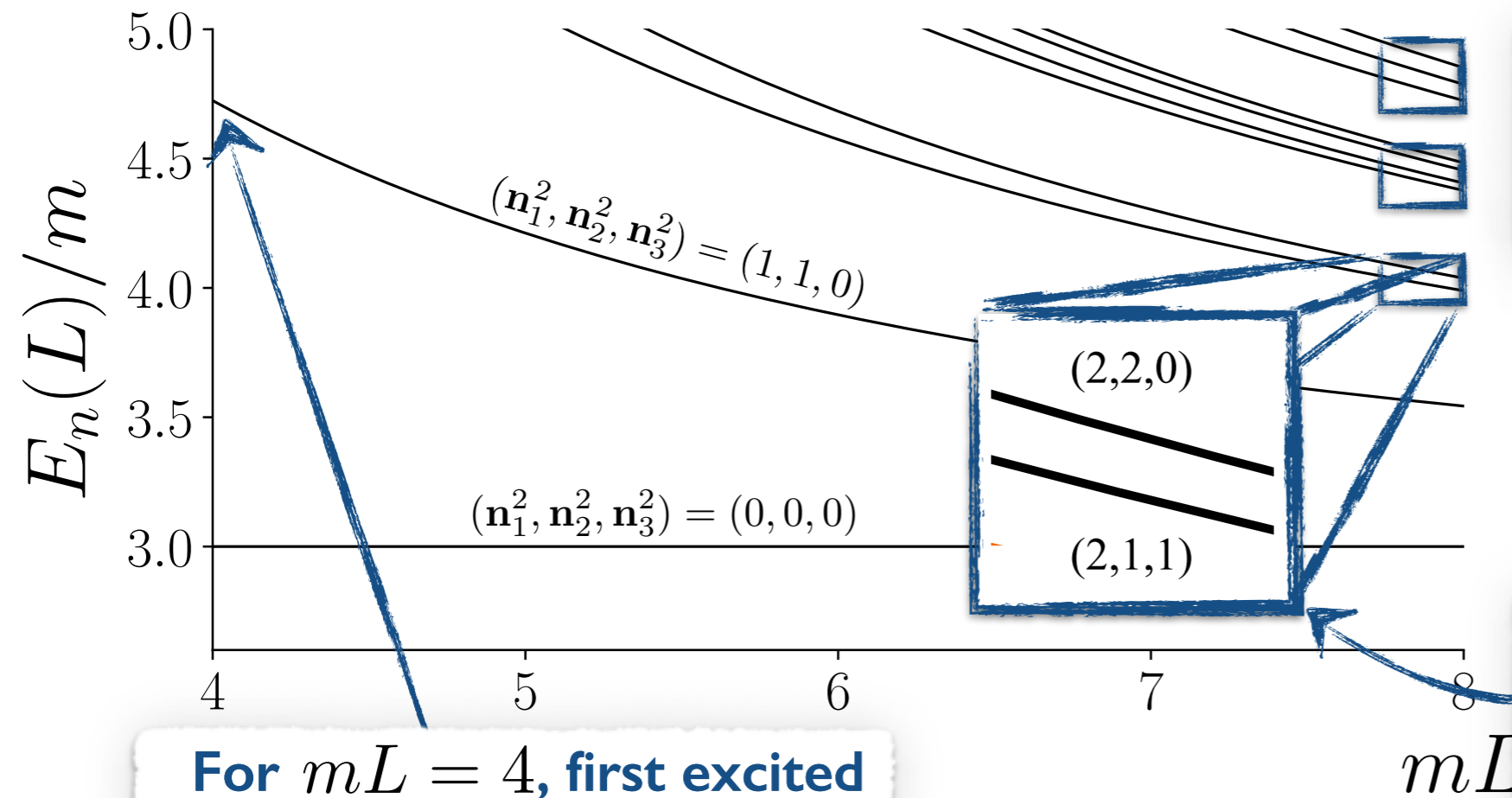
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Aside: Non-interacting states ($\mathcal{M}_2 = \mathcal{M}_3 = 0$)



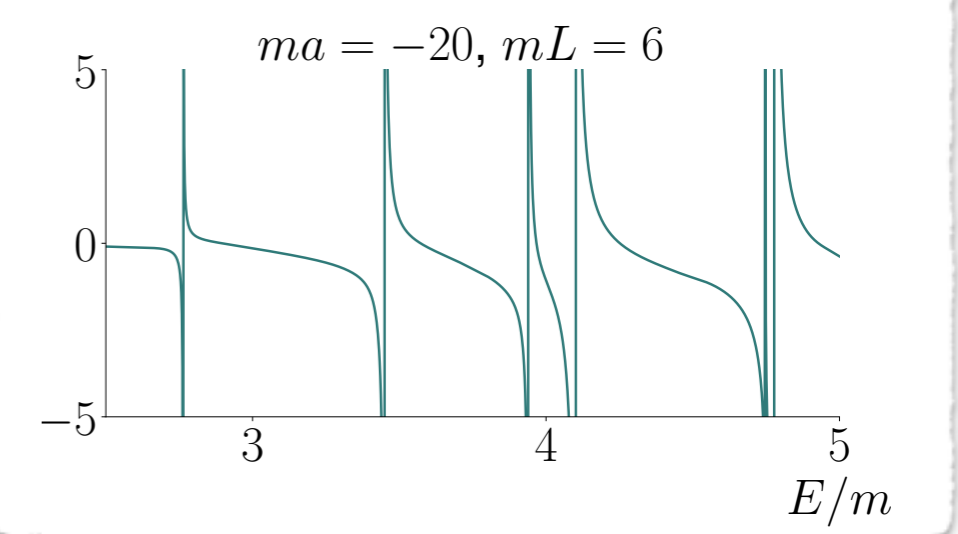
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In fact we have already seen these clusters



For $mL = 4$, first excited state is already relativistic

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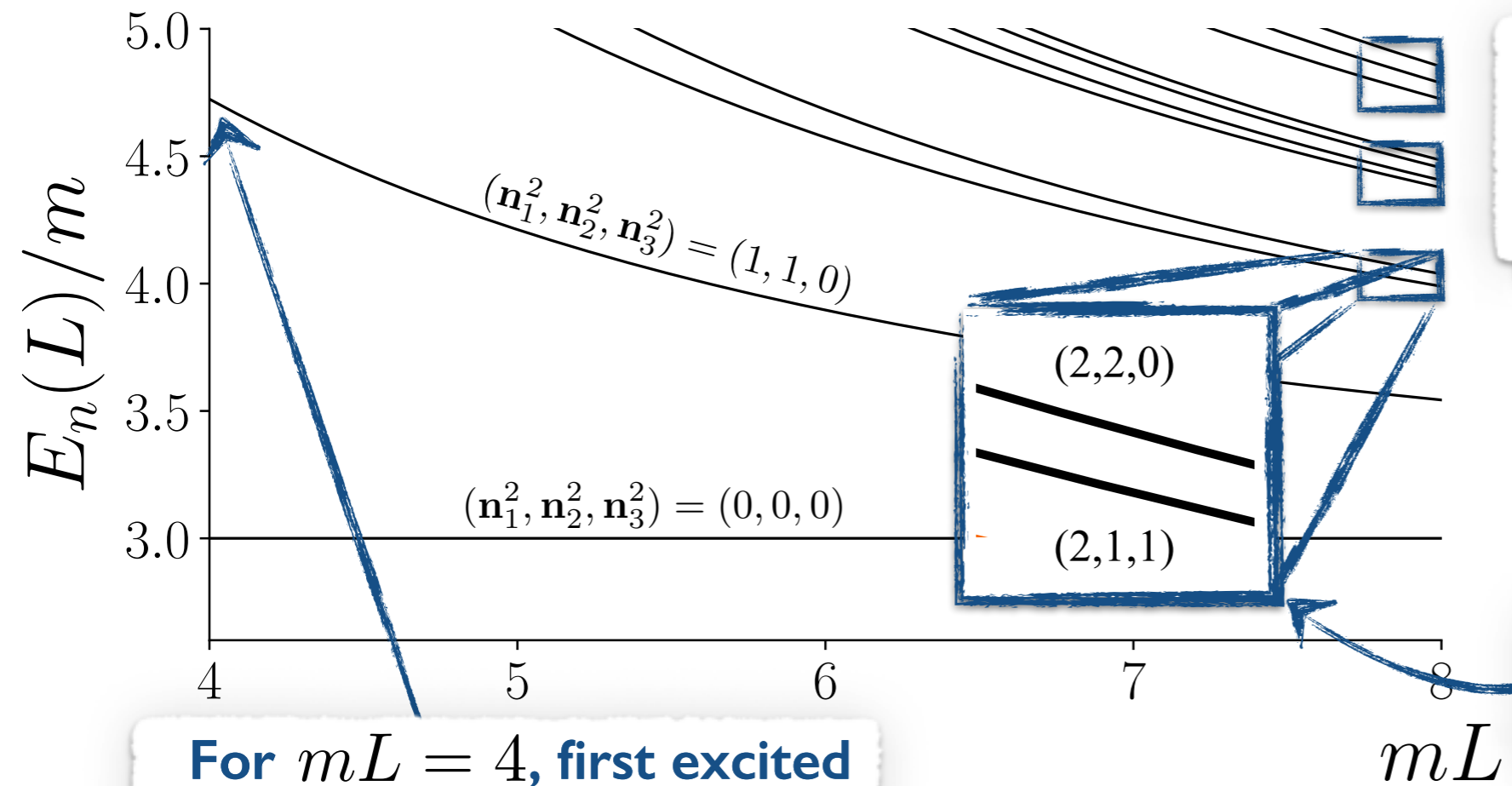
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The diagrams show Feynman-like diagrams for three-particle scattering. The first diagram shows two incoming lines on the left and two outgoing lines on the right, with two interaction vertices labeled $i\mathcal{M}_2$ connected by a propagator. The second diagram is similar but with a different internal propagator structure. Ellipses indicate a series of such diagrams.

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These two states are degenerate in the NR theory

For $mL = 4$, first excited state is already relativistic

$$\frac{p^2}{m^2} = \left(\frac{2\pi}{mL} \right)^2 \approx 2.46$$

Is it safe to say that we need a relativistic approach?

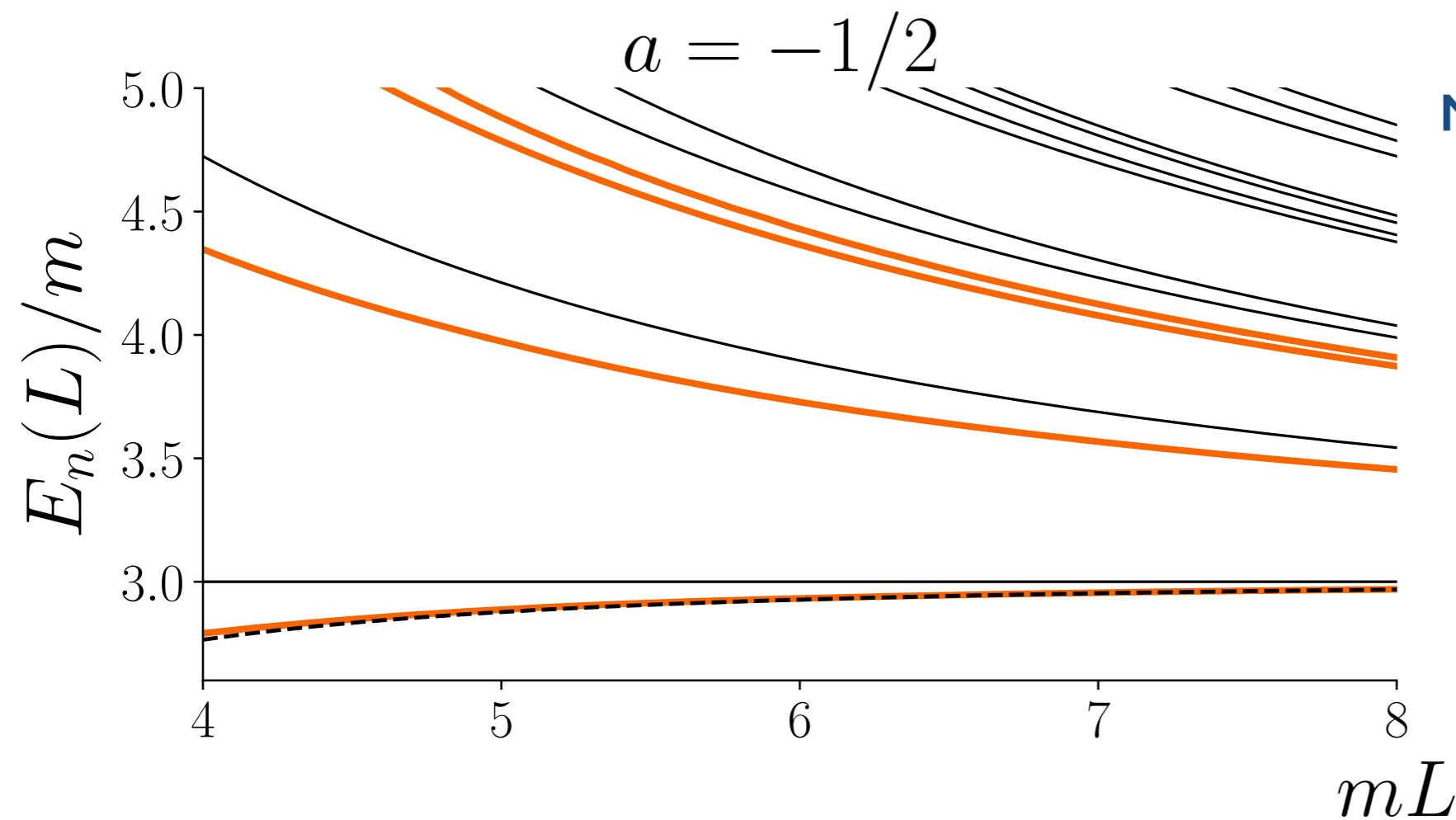


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$$i\mathcal{M}_3 = \mathcal{S} \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + \dots \right]$$

The diagrams show two Feynman-like diagrams for three-particle scattering. The first diagram has two external lines on the left and two on the right, with two internal circles labeled $i\mathcal{M}_2$. The second diagram is similar but with an additional external line on the right.



Now we turn on the interactions

$$1/F_3^{\text{iso}}(E, L, a) = 0$$

$$F_3^{\text{iso}} = \frac{1}{L^3} \sum_{\vec{k}, \vec{p}} \left[\frac{\tilde{F}_s}{3} - \tilde{F}_s \frac{1}{\mathcal{H}} \tilde{F}_s \right]_{kp}$$

$$\mathcal{H} = 1/(2\omega\mathcal{K}_2) + \tilde{F}_s + \tilde{G}_s$$

known functions

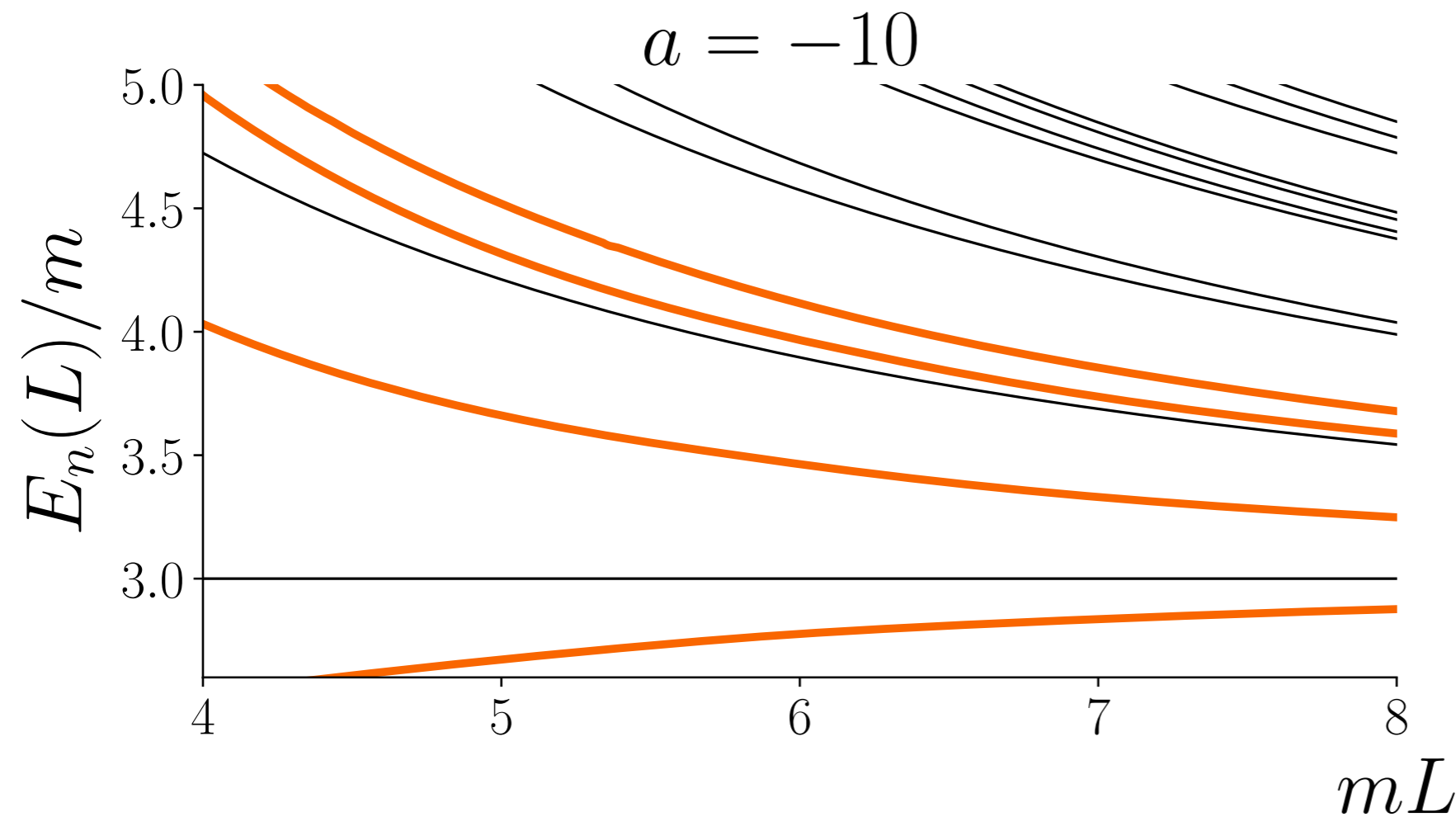
----- $1/L$ expansion

$\mathcal{K}_{\text{df},3}^{\text{iso}}(E) = 0$ solutions

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$$i\mathcal{M}_3 = \mathcal{S} \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + \dots \right]$$

The diagrams show two Feynman-like diagrams for three-particle scattering. The first diagram has two incoming lines on the left and two outgoing lines on the right, with two internal circles labeled $i\mathcal{M}_2$. The second diagram is similar but with a different internal structure. Ellipses indicate a series of such diagrams.



Can also accommodate large a

$$1/F_3^{\text{iso}}(E, L, a) = 0$$

$$F_3^{\text{iso}} = \frac{1}{L^3} \sum_{\vec{k}, \vec{p}} \left[\frac{\tilde{F}_s}{3} - \tilde{F}_s \frac{1}{\mathcal{H}} \tilde{F}_s \right]_{kp}$$

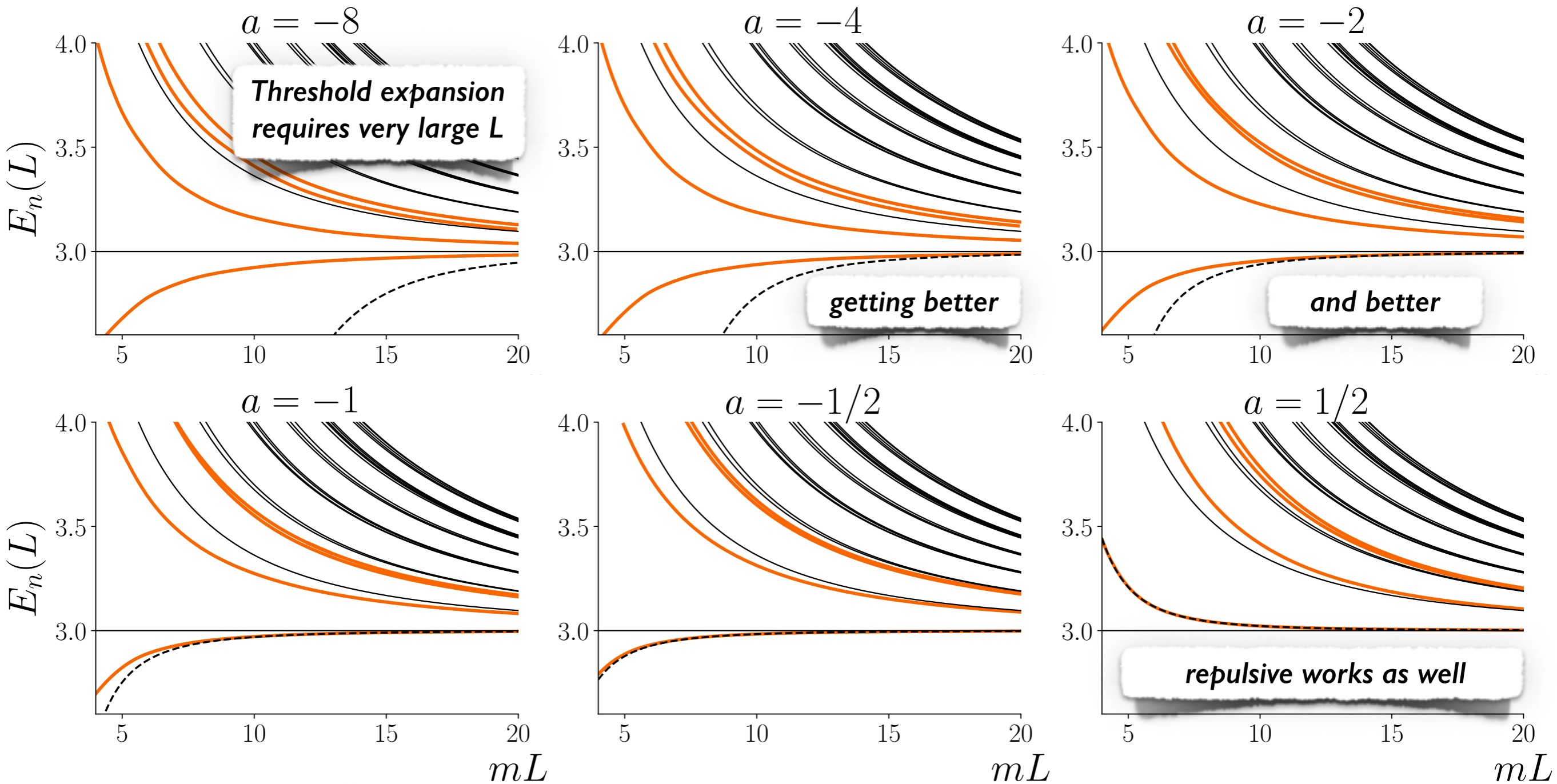
$$\mathcal{H} = 1/(2\omega\mathcal{K}_2) + \tilde{F}_s + \tilde{G}_s$$

known functions

..... $1/L$ expansion

$\mathcal{K}_{\text{df},3}^{\text{iso}}(E) = 0$ solutions

Straightforward to vary a and to study large volumes



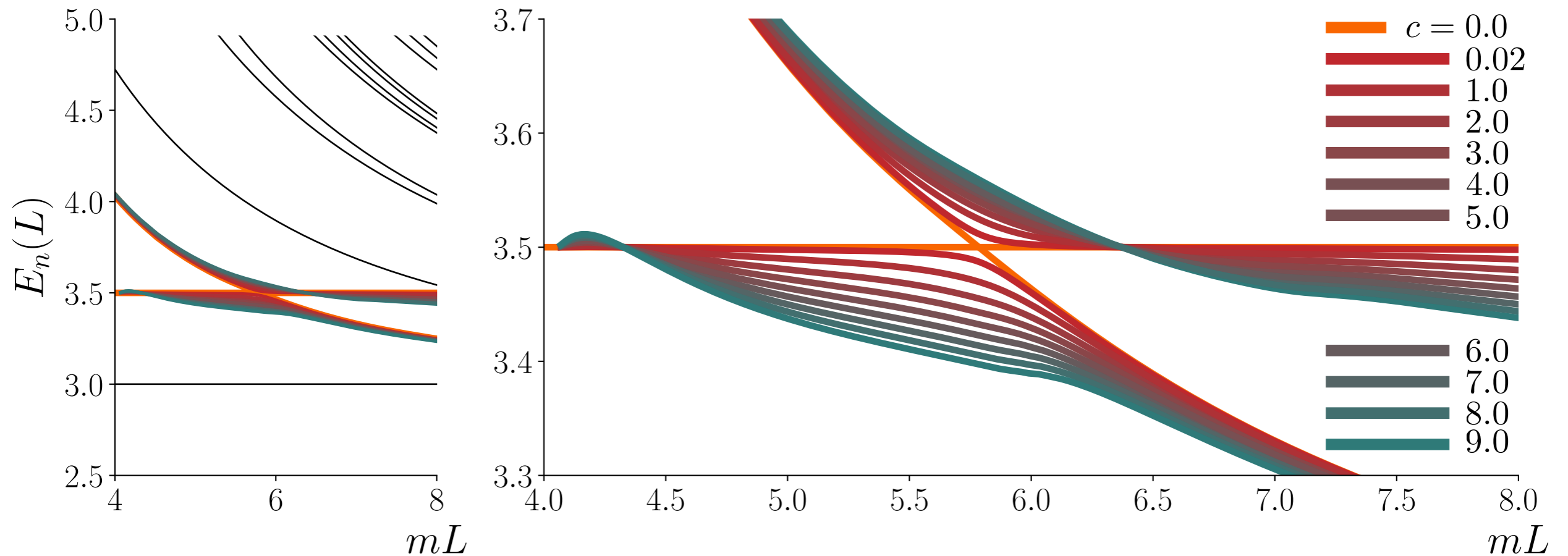
But, to avoid poles in \mathcal{K}_2 , we must require $a < 1/m$

Non-zero $\mathcal{K}_{\text{df},3}^{\text{iso}}(E)$: Toy resonance

Here we consider a fun example for non-zero $\mathcal{K}_{\text{df},3}^{\text{iso}}$

$$a = -10 \quad \mathcal{K}_{\text{df},3}^{\text{iso}}(E) = -\frac{c \times 10^3}{E^2 - M_R^2}$$

For small c we expect a narrow avoided level crossing, as c increases the gap grows



Further investigation is needed to see if this gives a physical resonance description

Non-zero $\mathcal{K}_{\text{df},3}(E)$: Unitary bound state

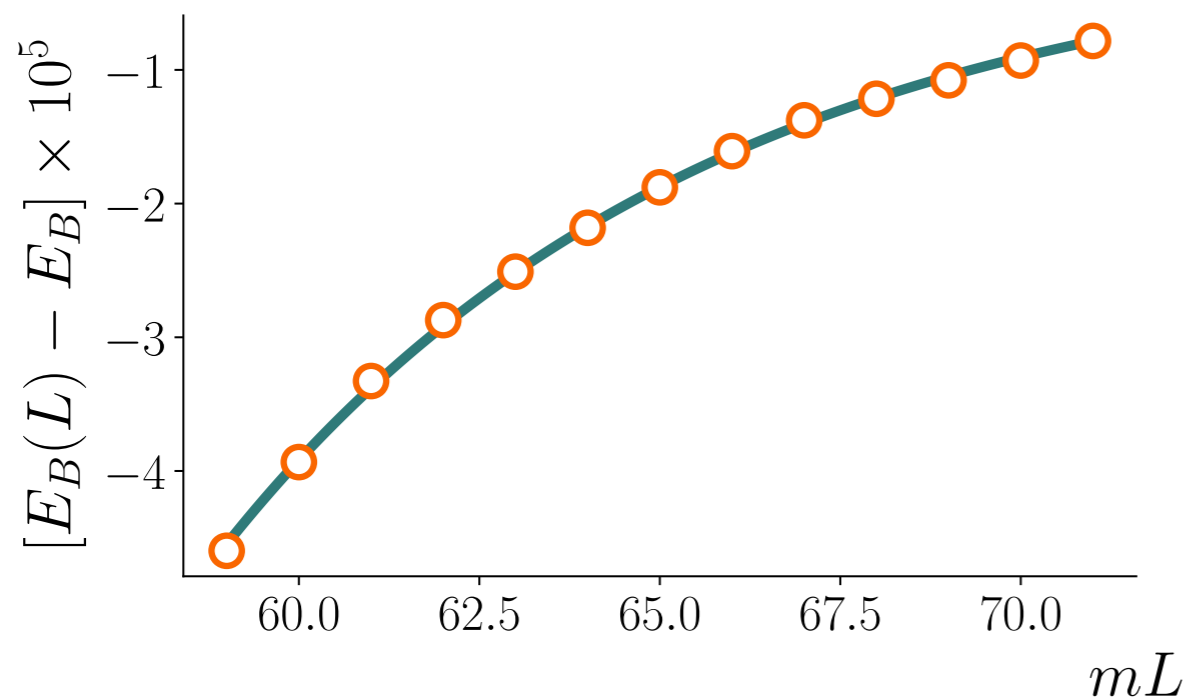
The parameters $a = -10^4$, $\mathcal{K}_{\text{df},3}^{\text{iso}}(E) = 2500$ lead to a shallow bound state

$$\kappa \approx 0.1m \text{ where } E_B = 3m - \kappa^2/m$$

Finite-volume behavior of this state has a known asymptotic form

Meißner, Rios, Rusetsky (2015)

$$E_B(L) = 3m - \frac{\kappa^2}{m} - (98.35 \dots) |A|^2 \frac{\kappa^2}{m} \frac{e^{-2\kappa L/\sqrt{3}}}{(\kappa L)^{3/2}} \left[1 + \mathcal{O}\left(\frac{1}{\kappa L}, \frac{\kappa^2}{m^2}, e^{-\alpha\kappa L}\right) \right]$$



This describes the bound state for

$$\kappa L \approx 0.1mL \gg 1$$

We fit our q.c. data over $60 < mL < 70$

→ $\kappa = 0.1068$, $|A|^2 = 0.948$

Close to one ✓

Non-zero $\mathcal{K}_{\text{df},3}(E)$: Unitary bound state

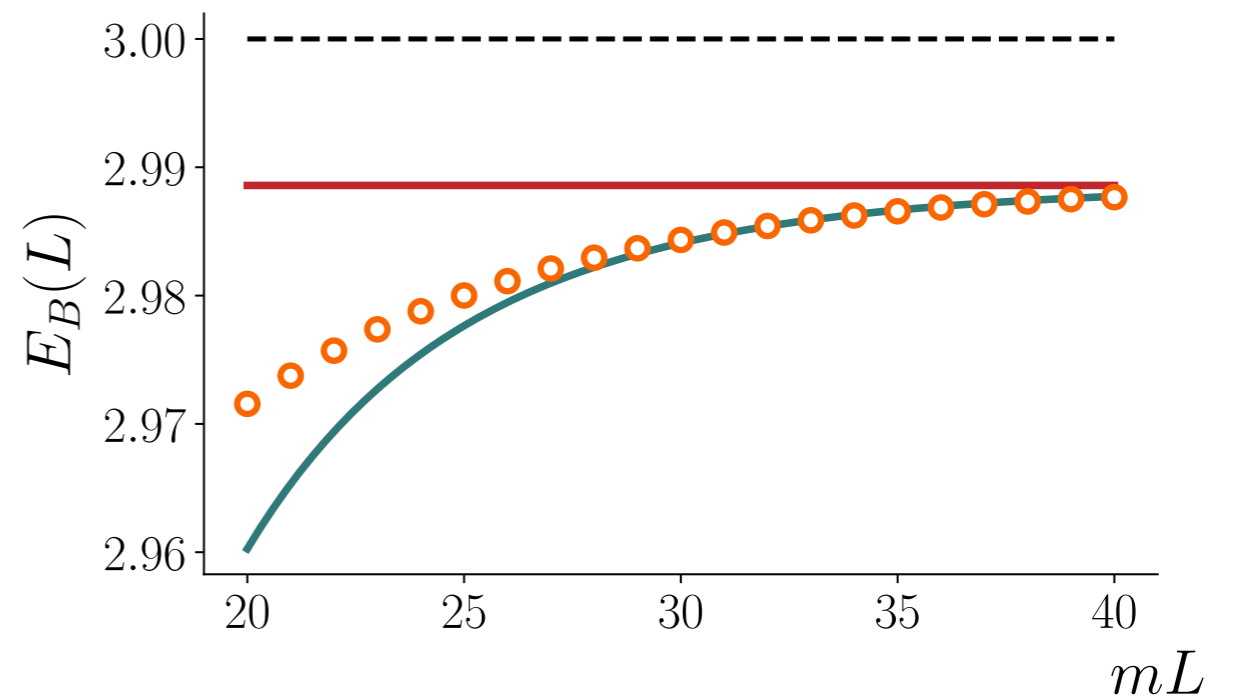
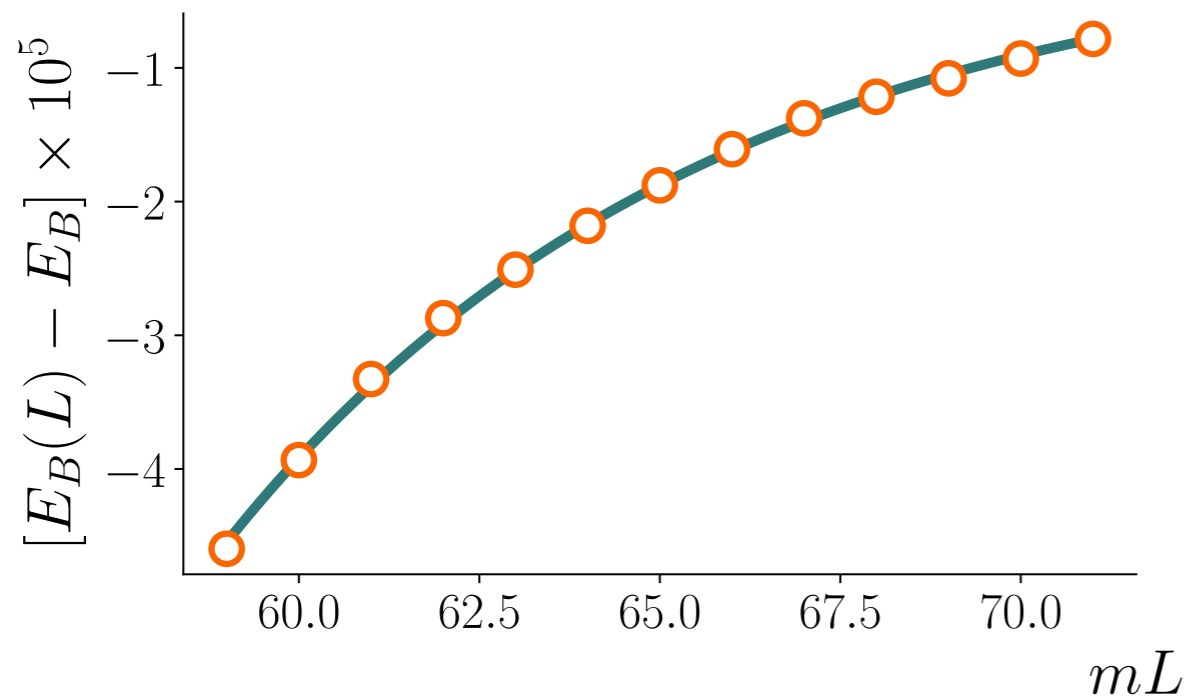
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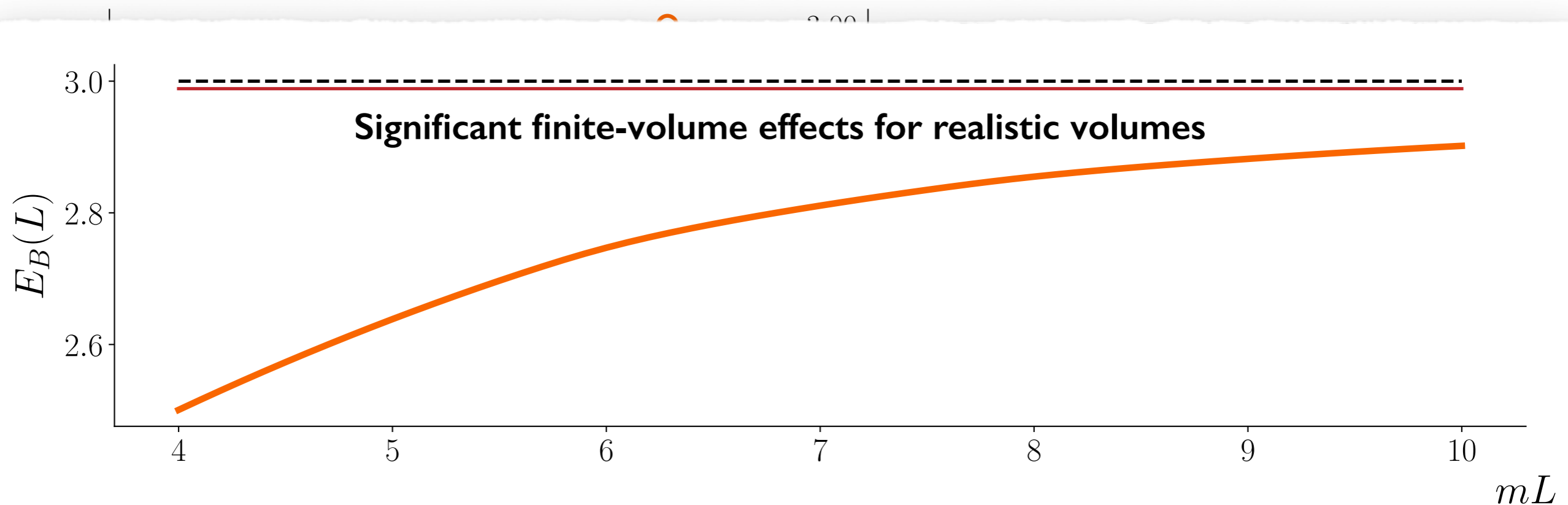
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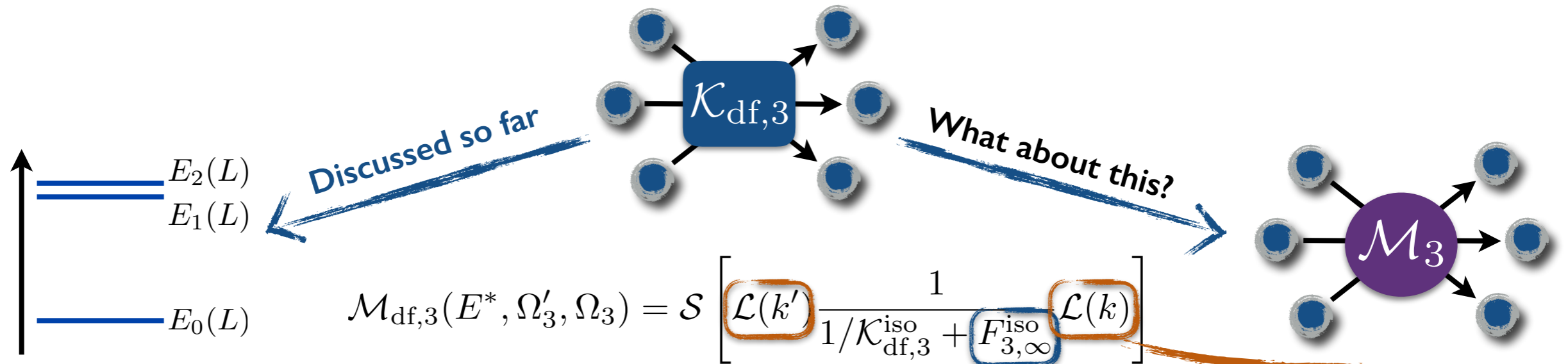
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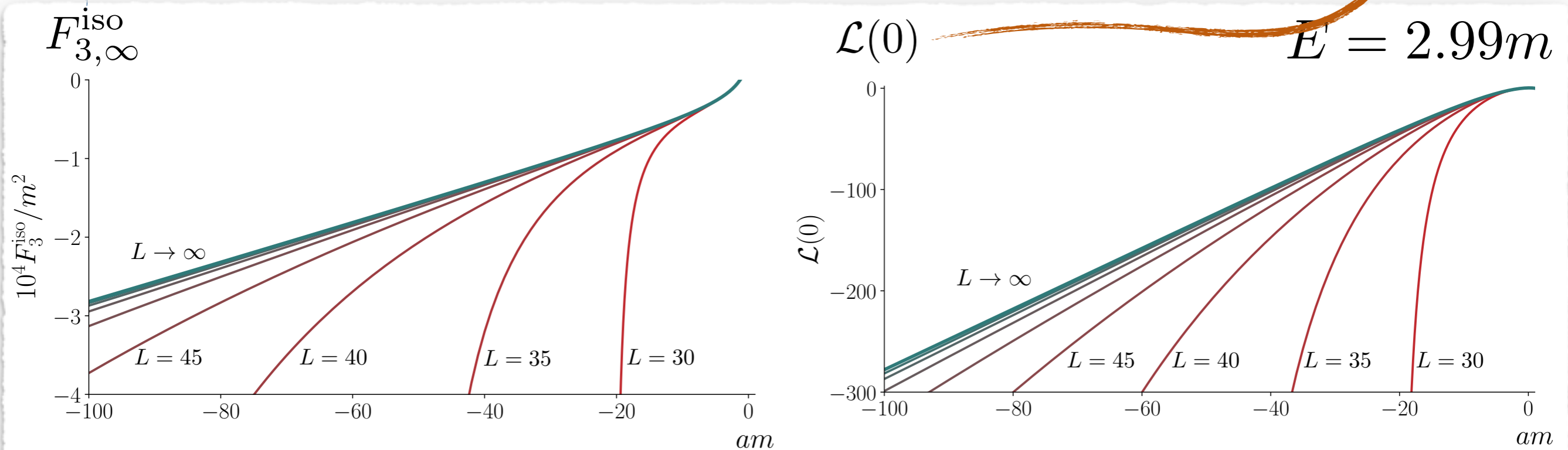
$$E_B(L) = 3m - \frac{\kappa^2}{m} - (98.35 \dots) |A|^2 \frac{\kappa^2}{m} \frac{e^{-2\kappa L/\sqrt{3}}}{(\kappa L)^{3/2}} \left[1 + \mathcal{O}\left(\frac{1}{\kappa L}, \frac{\kappa^2}{m^2}, e^{-\alpha\kappa L}\right) \right]$$



Converting to scattering amplitudes

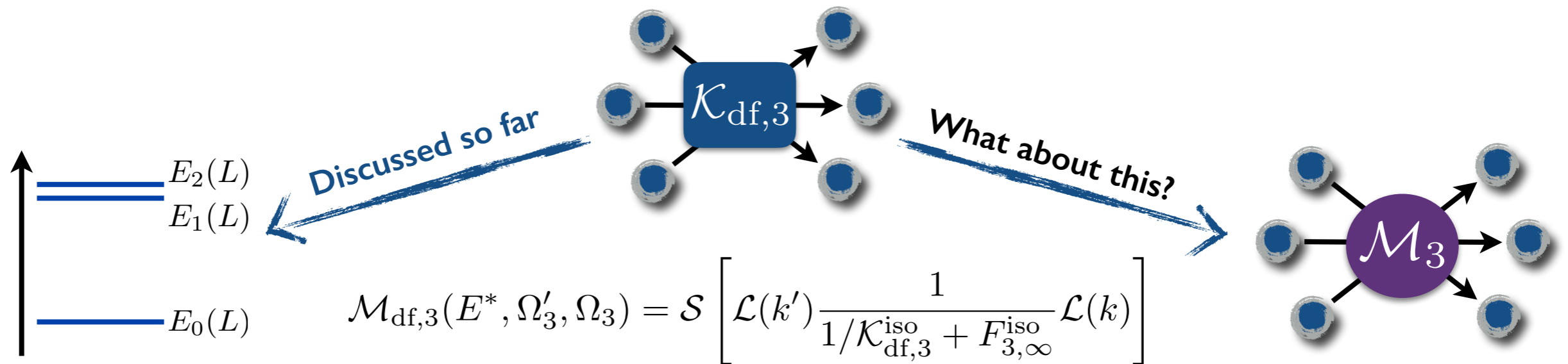


$$i\mathcal{M}_{\text{df},3} = i\mathcal{M}_3 - \mathcal{S} \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \dots \end{array} \right]$$

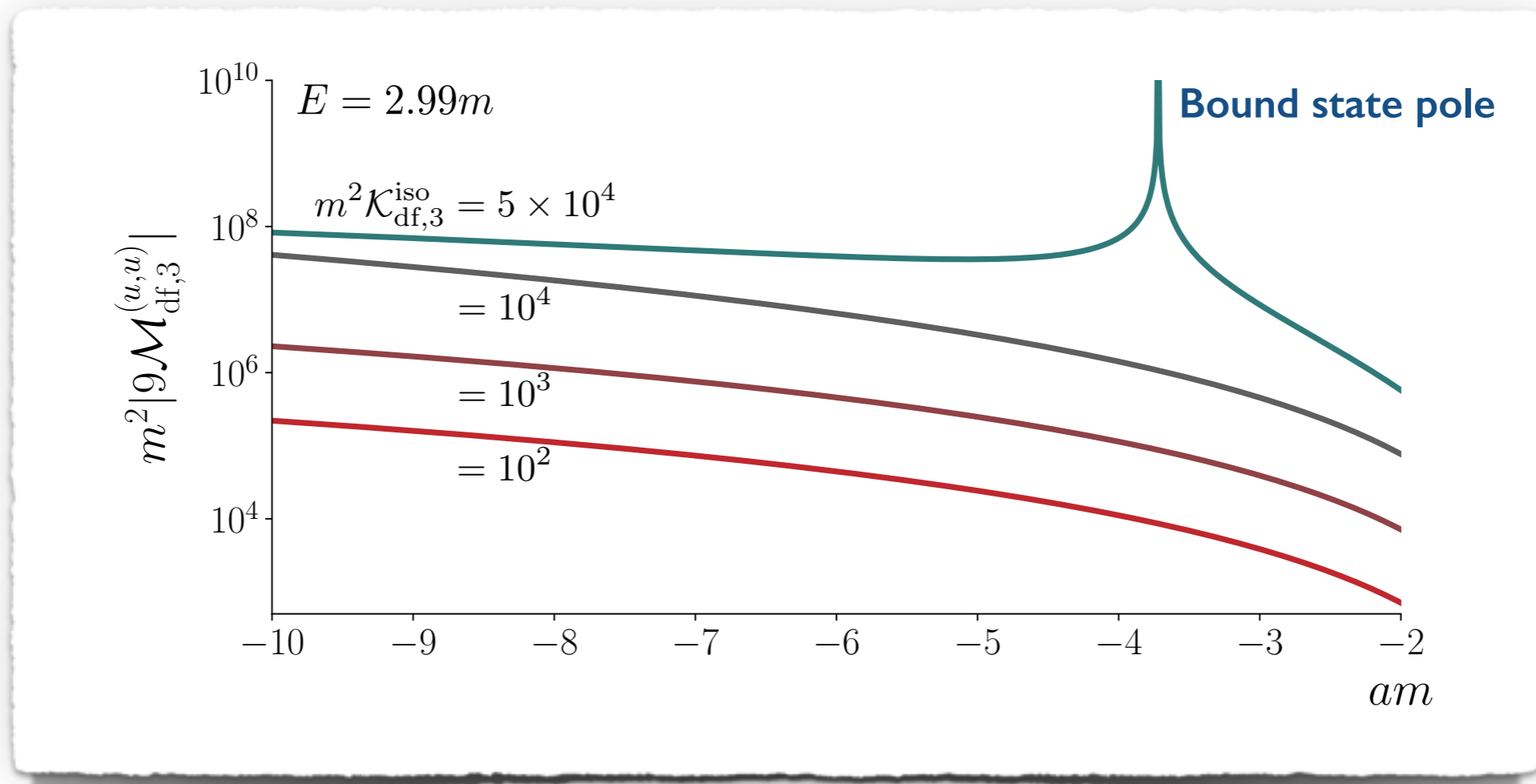


This only works below threshold... Relation above threshold crucially needed (Gernot?)

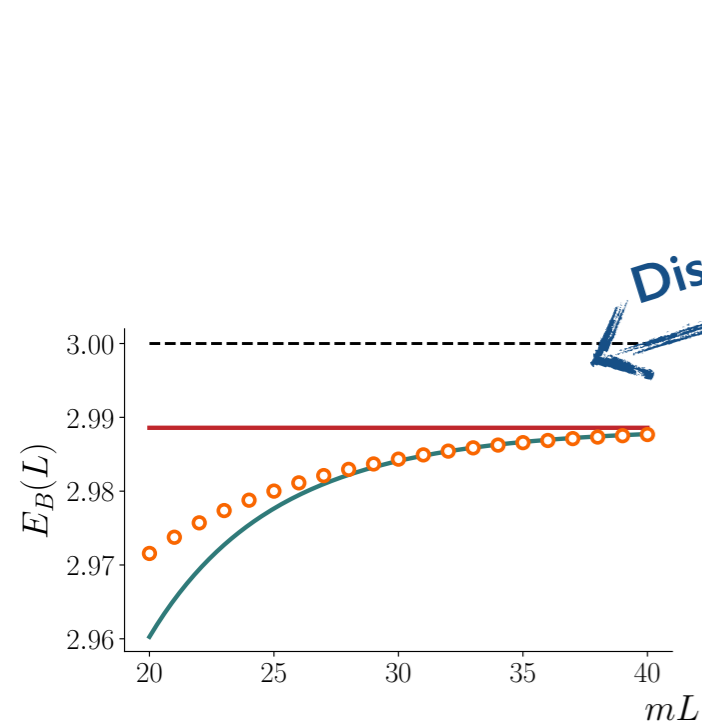
Converting to scattering amplitudes



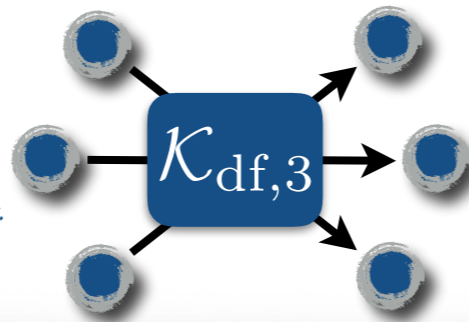
$$i\mathcal{M}_{df,3} = i\mathcal{M}_3 - \mathcal{S} \left[\text{Diagram 1} + \text{Diagram 2} + \dots \right]$$



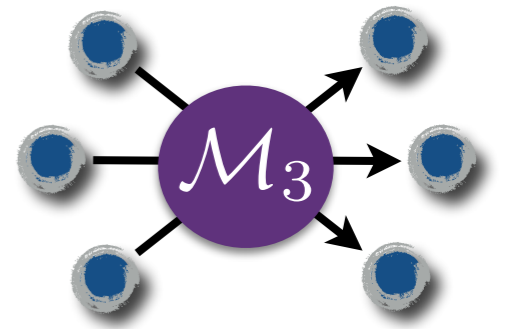
Back to the bound state



Discussed so far



What about this?



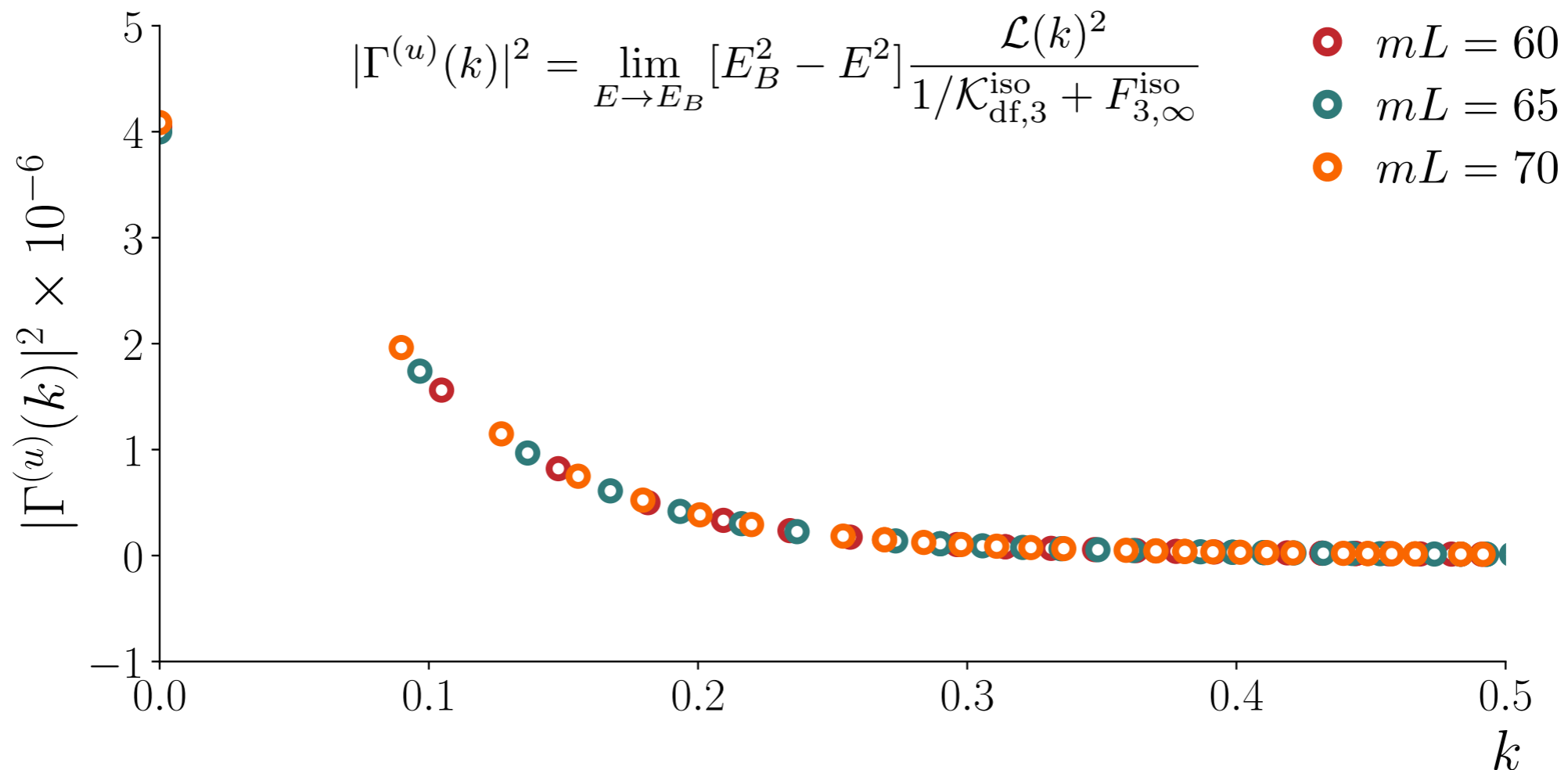
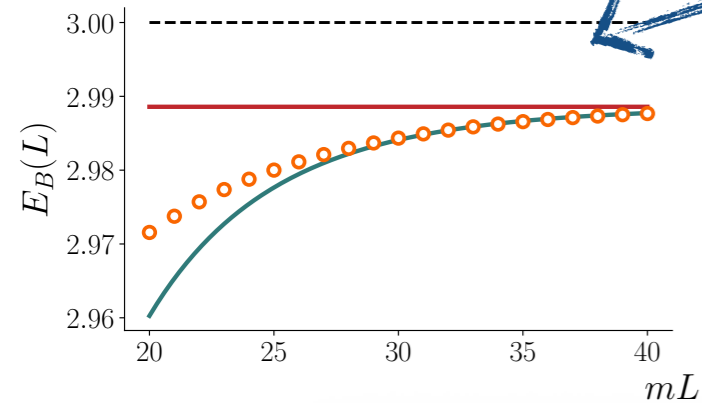
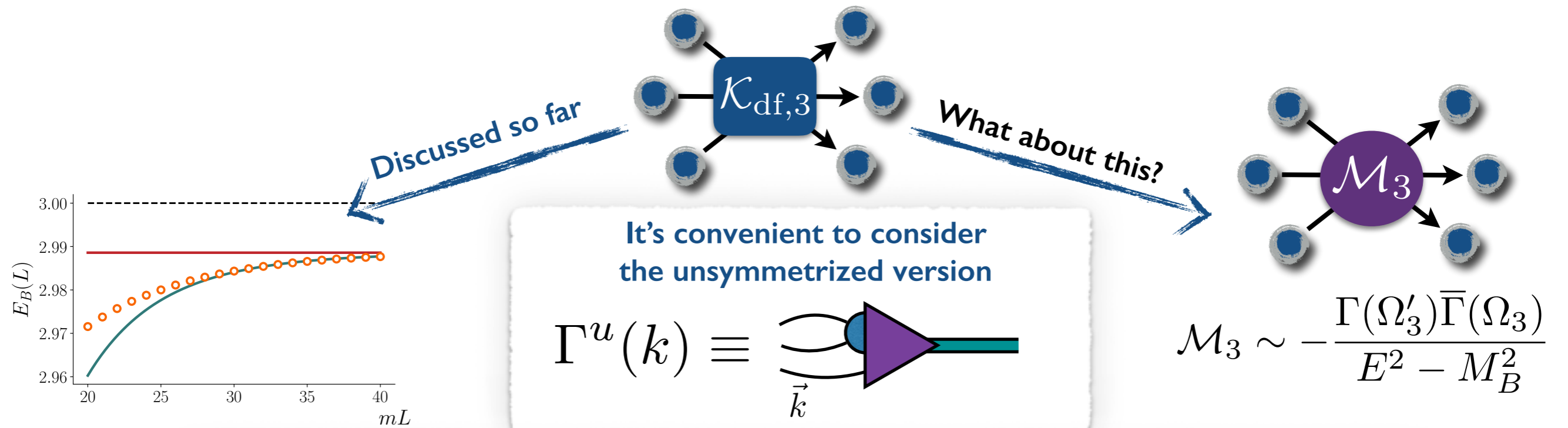
It's convenient to consider the unsymmetrized version

$$\Gamma^u(k) \equiv \text{Diagram of a purple triangle with a blue circle on its left side, connected to a teal line on its right side. Two curved lines labeled \vec{k} enter the blue circle from the left.$$

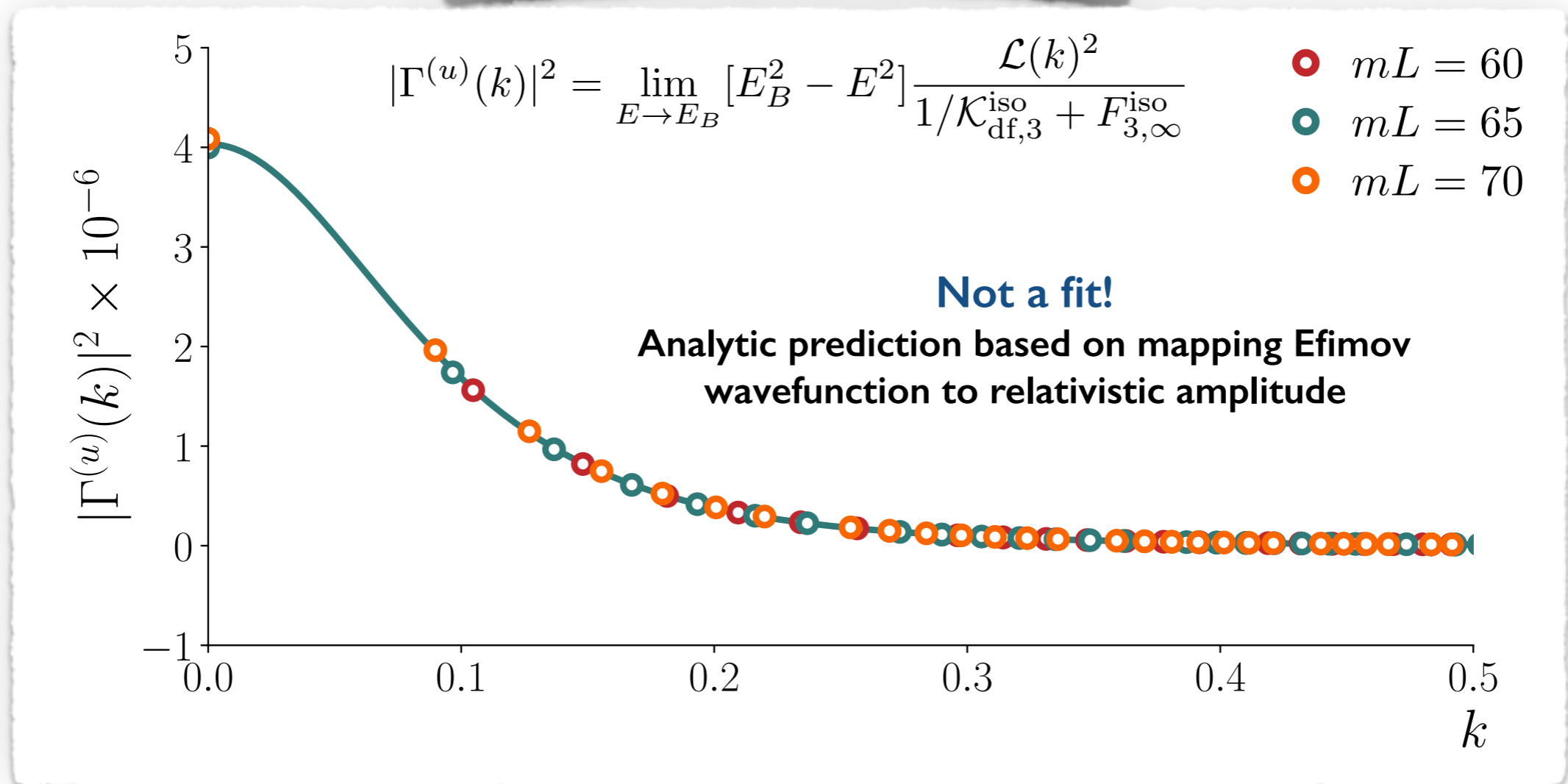
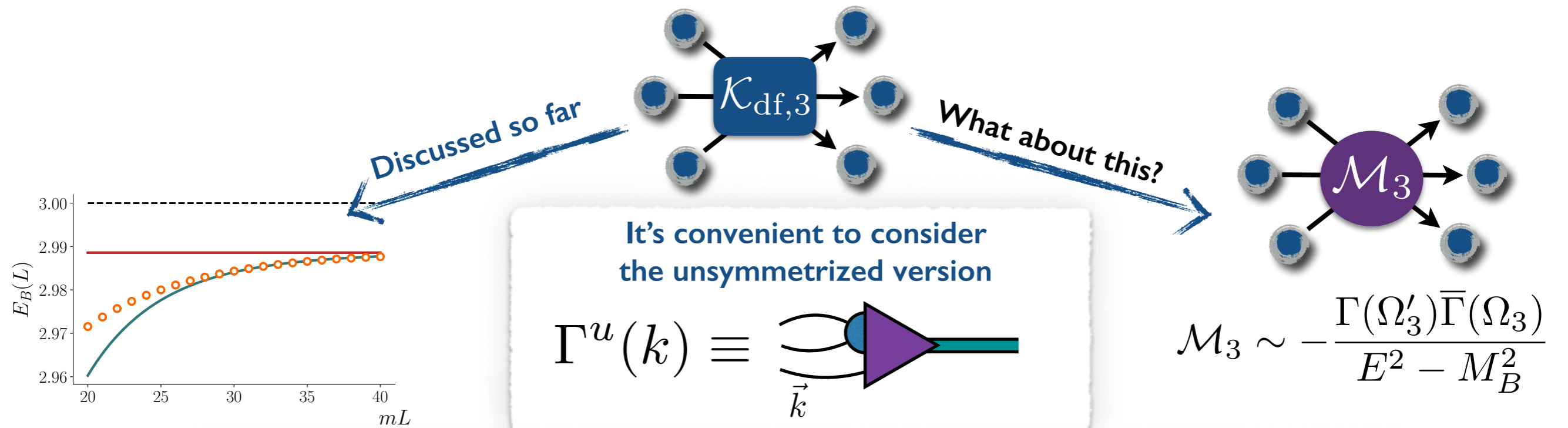
$$\mathcal{S}[\Gamma^u(k)] \equiv \langle \pi(k) \pi \pi | E_B \rangle$$

$$\mathcal{M}_3 \sim -\frac{\Gamma(\Omega'_3) \bar{\Gamma}(\Omega_3)}{E^2 - M_B^2}$$

Back to the bound state



Back to the bound state

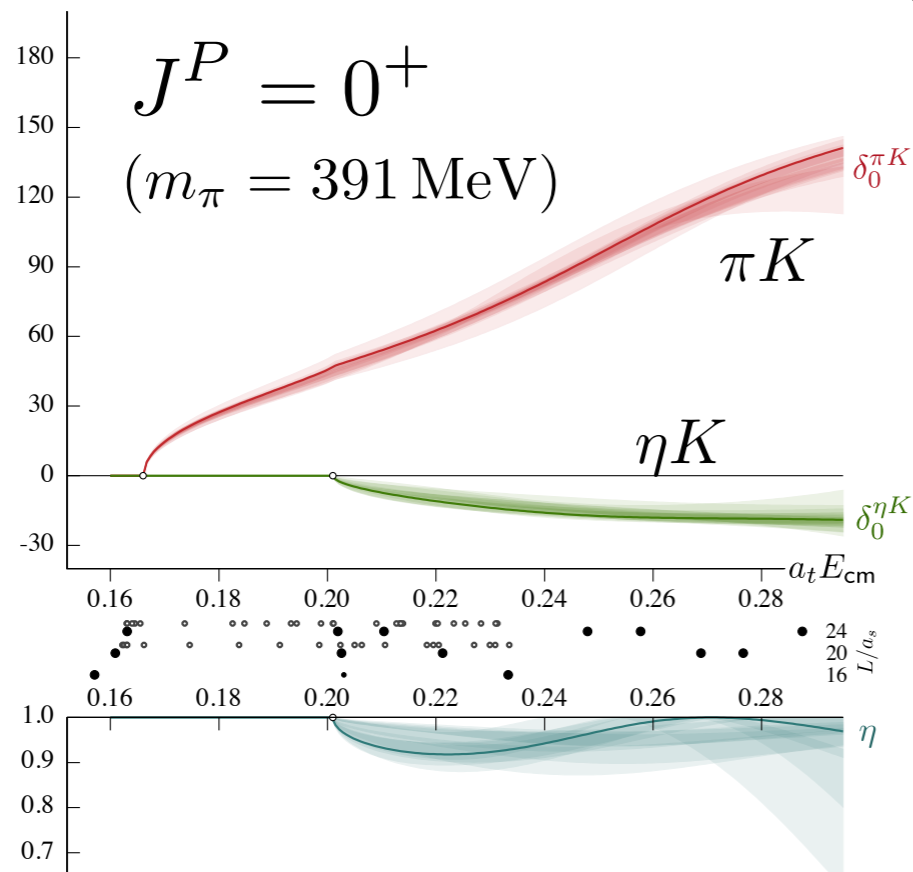


Questions for the “competitors”

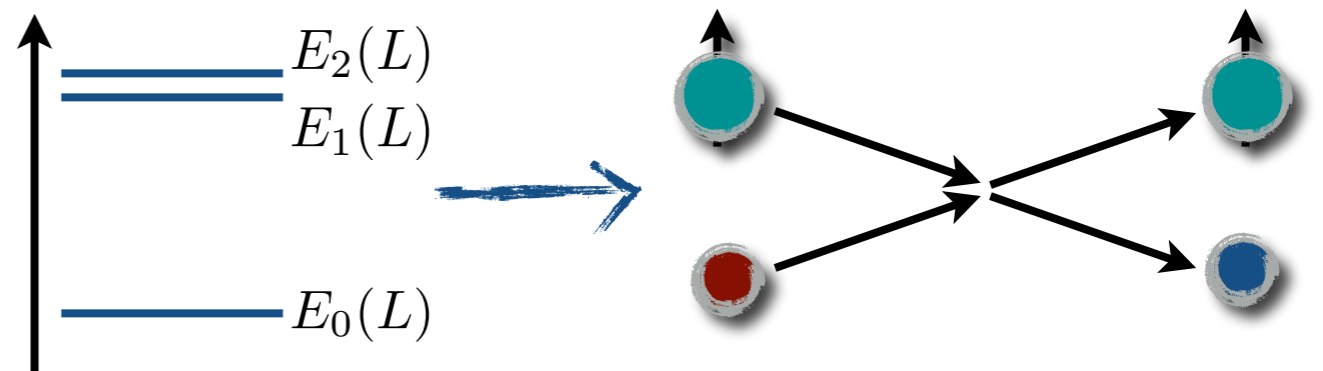
- 🌐 Are the quantization conditions equivalent?
- 🌐 Is the formalism limited by 5 particle threshold?
- 🌐 What are the prospects for two-to-three? multiple-channels?
- 🌐 Is it clear that all relativistic effects are captured by relativistic kinematics?
- 🌐 What are the general work flows for each method?

The Big Picture

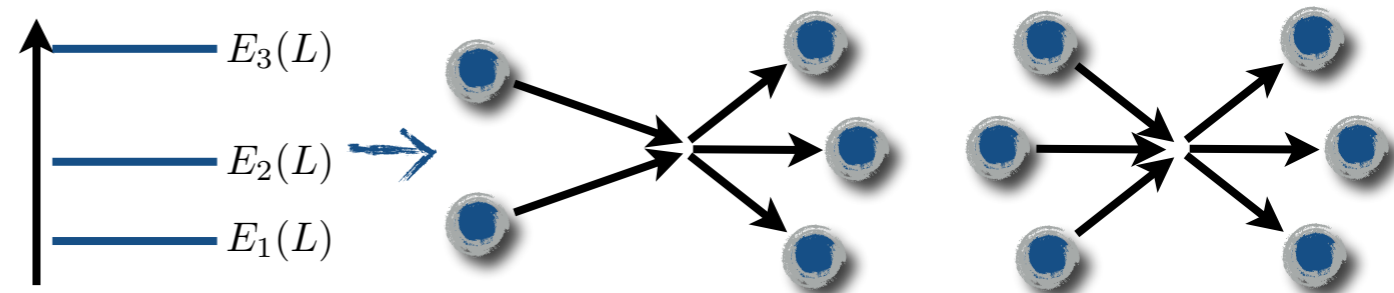
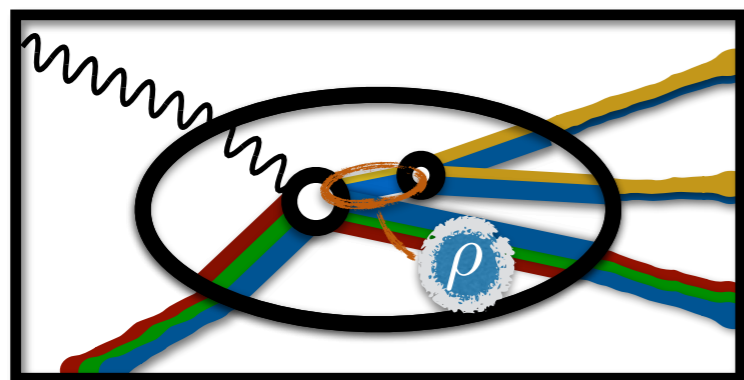
Two-to-two scattering from LQCD is reaching maturity



Wilson et. al., *Phys. Rev. D* 91, 054008 (2015)



Stay tuned for three-particle observables from LQCD



Thanks for listening!